15-396: Science of the Internet

Assignment 5 Solutions

Question 1

Translating slots to actual values, person X's values are (30, 15), person Y's values are (20, 10), and person Z's values are (10, 5). It is clear that giving slot A to person X and slot B to person Y gives social valuation 40 and that no other matching yields higher results.

Each person is charged the 'harm' they cause others. By person X being present, person Y doesn't get slot A and person Z doesn't get slot B. Instead of making 25 in the absence of X, Y and Z together only make 10. Thus, VCG tells us that person X should be charged the difference of 15. If person Y weren't present, the only change that would happen is that person Z would get slot B. Thus, X and Z would make 35 together instead of 30, so person Y is charged 5. Person Z is charge nothing because the social valuation doesn't change whether he is present or absent.

Question 2

Without loss of generality, let's focus on person 1 in group. We want to prove that it's a dominant strategy for person 1 to bid his truthful values. Person 1 has truthful values $(v_1, \ldots, v_m)$ for each of the M items. Suppose that when person 1 submits his truthful values he gets item i in the end. The net value he experiences is $v_i - p_{i,1}$, where $p_{i,1}$ is the price person 1 pays when assigned item i. Rather than providing the truthful values $(v_1, \ldots, v_m)$, what if person 1 provided some other values $(b_1, \ldots, b_m)$ which aren't necessarily truthful? Notice that everyone will still bid the same way; you can imagine person 1 having two envelopes at the ready and picking one of them to hand in. What person 1 hands in doesn't change how the other people bid. Suppose that under the bid $(b_1, \ldots, b_m)$ person 1 is assigned item j, which isn't necessarily an item different from item i. The value person 1 derives is based on his *truthful* values, so his net gain in this case is $v_j - p_{1,j}$.

To show that it's a dominant strategy for person 1 to bid truthfully, it must be the case that

$$v_i - p_{1,i} \geq v_j - p_{1,j}$$

Using the definition of $p_{i,j}$, we have

$$v_i - (V_{N-1}^{M} - V_{N-1}^{M-i}) \geq v_j - (V_{N-1}^{M} - V_{N-1}^{M-j}) \iff v_i + V_{N-1}^{M-i} \geq v_j + V_{N-1}^{M-j}$$

Notice that since person 1 was paired with item i in an unconstrained fashion, that choice was globally optimal. Thus, we see that $v_i + V_{N-1}^{M-i}$ is actually just V_N^M in the case that person 1 submits truthful values. Looking at the right hand side, it has very similar structure to the left hand side. It represents the maximum valuation for the group in the case that person 1 submits truthful values, but for some reason person 1 was arbitrarily forced to receive item j. We'll denote this value as $V_N'^M$ to indicate that in this social valuation we've forced a pairing between person 1 and item j. Clearly $V_N'^M$ cannot be strictly greater than V_N^M. If that were the case then pairing person 1 with item i would not generate the social optimum as we could do better by pairing person 1 with item j. However, person 1 *was* paired with item i under a socially optimal matching. From this we conclude that $V_N^M \geq V_n'^M$, which finishes the proof that the strategy is dominant for person 1. We chose to focus on person 1 arbitrarily at the beginning of the proof, so this argument actually applies to everyone involved in the auction.