ECE 209: *Circuits and Electronics Laboratory*

Review of Circuits as LTI Systems*

Math Background: ODE's, LTI Systems, and Laplace Transforms

Engineers must have analytical machinery to understand how systems change over time. For example, springs and dampers in car suspension systems absorb kinetic energy after road disturbances and dissipate the energy *gently* over time. Electronic filters protect circuits from large input transients in a similar way. Hence, engineers need to know how to choose components in *dynamical* systems.

LTI Systems and ODEs: The motion of many physical systems can be modeled knowing only a few of its derivatives. For example, a car's position over time can be described well knowing its velocity or acceleration. So we focus on the *linear time-invariant* (LTI) system

$$x(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t)$$

which has input $x(t)$ (e.g., height of terrain under the car) and output $y(t)$ (e.g., height of passenger's seat). By our assumption that all we need are a few derivatives of the input and the output, then a typical *ordinary differential equation* (ODE) that might model such a system is

$$y'' + 3y' + 2y = x'' - x \tag{1}$$

where y' and y'' are the first and second derivatives of signal $y(t)$. Given a known input $x(t)$, we would like to *integrate* this differential equation to find an expression of $y(t)$ that is only in terms of t. Then we can see how to adjust our design parameters so that $y(t)$ has a desirable shape (e.g., even with a bumpy $x(t)$ road surface, the car seat $y(t)$ stays in one place). So we need a convenient way to *solve* this differential equation.

Linear Decomposition: Assume that x can be broken into two parts so that $x(t) = x_1(t) + x_2(t)$. By inspection, if I can solve Equation (1) for the response y_1 to x_1 alone and the response y_2 to x_2 alone, then the response y to x would be $y_1 + y_2$. Further, imagine that I could find a set of functions such that:

- When a function in the set is an input to Equation (1), the output is simple to find.
- Every useful input $x(t)$ can be expressed as a sum (or integral) of functions from this set.

In this case, solving ODE's like Equation (1) would be trivial. Before even knowing the input x, I could find the solution to the ODE for every function in this special set. Then when I finally do know my input, I just sum up the relevant prototypical solutions.

Consider functions of the form $t \mapsto e^{st}$ where $s = \sigma + j\omega$ is a *complex number* with real part σ and imaginary part ω. Notice that for a function $f(t) = Ae^{st}$ (where A is any constant complex number),

$$f'(t) = sAe^{st} = sf(t) \qquad \text{and} \qquad \int f(t)\,\mathrm{d}t = \frac{1}{s}Ae^{st} = \frac{1}{s}f(t). \tag{2}$$

That is, for a *complex exponential*, differentiation is identical to multiplication by s and integration is identical to division by s. So complex exponentials turn ODE calculus into simple algebra.

Stability and the Characteristic Response: Because every $x(t) = x(t) + 0$, then the zero-input case is important to us, and we should handle it first. These zero-input effects will be a part of every solution regardless of the input, and so the output trajectory for zero input is called the *characteristic* or *intrinsic* or *transient* response of the system. We will assume that this output is made up of complex exponentials, and so we let $y(t) \triangleq Ae^{st}$ and $x(t) \triangleq 0$, which turns Equation (1) into

$$Ae^{st}\left(s^2 + 3s + 2\right) = 0, \tag{3}$$

and so we must solve for values of s that make this equation true. Fortunately, $|Ae^{st}| \neq 0$ for all s, and so the roots of the *characteristic polynomial* $s^2 + 3s + 2$,

$$s = -1 \qquad \text{and} \qquad s = -2, \tag{4}$$

*Document from `http://www.tedpavlic.com/teaching/osu/ece209/`. Source code at `http://hg.tedpavlic.com/ece209/`.

both solve Equation (3). Hence, the *zero-input transient response* of Equation (1) is

$$y_0(t) = Ae^{-1t} + Be^{-2t}$$

where A and B are complex numbers that correspond to different initial conditions. Because $e^{\sigma t} \to 0$ as $t \to \infty$ for all $\sigma < 0$, then $y_0(t) \to 0$ as $t \to \infty$. So, as long as the roots in Equation (4) are very negative, then $y_0(t)$ will decay *quickly*, and we can usually *ignore its effect*. In general, the roots of the characteristic polynomial are called *eigenvalues* (i.e., "intrinsic values" or "characteristic values"). When the *real* part of the eigenvalues are *negative*, then the characteristic response will continuously decay. Such LTI systems are called **STABLE** because aspects of the output that are not due to the input will always decay to zero. **Unstable** systems have components that grow and eventually dominate the output.

Stable Response to an Input: Now we assume that the LTI system is *stable* (i.e., its eigenvalues have negative real parts) so that we can focus on the system after its transient response has decayed out. So we consider the input $x(t) \triangleq Xe^{st}$ where X is some complex number. It seems reasonable to guess that the output $y(t) = Ye^{st}$ where Y is some complex number. If our guess is wrong, there will be no solution, and we will try something else. So under these assumptions, Equation (1) becomes

$$Ye^{st}(s^2 + 3s + 2) = Xe^{st}(s^2 - 1).$$

Again because $|e^{st}| \neq 0$ for all s,

$$\frac{Y}{X} = \frac{s^2 - 1}{s^2 + 3s + 2}. \tag{5}$$

That is, for a given input Xe^{st}, we can find the output Ye^{st} by scaling X by the ratio in Equation (5). This *transfer function* represents the impact of the system on any individual complex exponential. The denominator of the transfer function is the characteristic polynomial of the system. The system's eigenvalues are called *poles* because the transfer function's denominator is *zero* there, and so the transfer function's magnitude is like a tent being pulled toward the sky by each of its *"poles."* So, given a transfer function, we must make sure its denominator roots (i.e., poles) have negative real parts.

For example, if $x(t) \triangleq e^{-2t}\cos(3t)$, then the input can easily be expressed as a sum of complex exponentials using Euler's formula. That is,

$$x(t) = e^{-2t}\cos(3t) = e^{-2t}\frac{e^{j3t} + e^{-j3t}}{2} = \frac{1}{2}e^{(-2+j3)t} + \frac{1}{2}e^{(-2-j3)t},$$

which is a kind of "generalized Fourier series" of the signal. So the response to $x(t)$ is the sum of $(1/2)e^{(-2+j3)t}$ and $(1/2)e^{(-2-j3)t}$ scaled by the transfer function evaluated at $s = (-2 + j3)$ and $s = (-2 - j3)$.

Laplace Transforms and the Frequency Domain: All that is left is representing any input x as a sum or integral of complex exponentials. If x *can* be expressed this way, then mathematical theory states

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}\,\mathrm{d}s \qquad \text{where} \qquad X(s) = \int_{0-}^{\infty} x(t)e^{-st}\,\mathrm{d}t. \tag{6}$$

That is, the *Laplace transform* $X(s)$ of signal x provides the coefficient for the e^{st} part of the signal, and we can scale $X(s)$ by the transfer function to find how much of e^{st} contributes to our output. Luckily,

(i) Most inputs interesting to engineers *do* have Laplace transforms.

(ii) Many common signals have well-documented Laplace transforms.

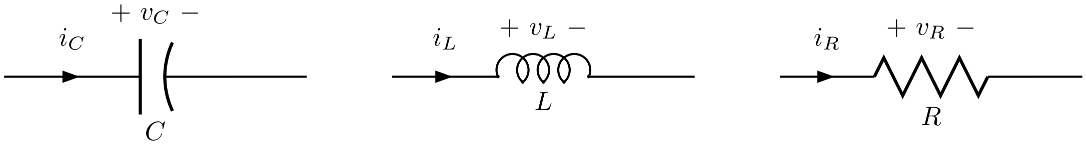
(iii) Operations on Laplace transforms form other Laplace transforms. For example, the signal $g(t) + h(t)$ has the Laplace transform $G(s) + H(s)$. More importantly, the signal $\mathrm{d}g(t)/\mathrm{d}t = sG(s)$.

(iv) As long as each of the *poles* of a *real* system's transfer function has a negative real part, the input $X(s)$ and transfer function only need to be evaluated at $s = j\omega$ for all $\omega > 0$.

Because of item (iv), design work on *filters* focuses on the *frequency (or **Fourier**) spectrum*. That is, we think of the frequency content (e.g., "fast changing" or "slow changing") of inputs and the ability of systems to shape that content (e.g., "slows the signal" or "removes slow changes"). Because of Euler's formula, we can think of sinusoids with angular frequency ω rather than $e^{j\omega t}$ complex exponentials. Transfer functions then have a *gain magnitude* (e.g., from dissipative attenuation) and *phase shift* (e.g., from delay) for each ω.

Laplace Representations of Circuit Elements

Resistors, capacitors, and inductors are important tools in electronic design. The following summary assumes zero initial conditions.



$$i_C(t) = Cv_C'(t) \qquad v_L(t) = Li_L'(t) \qquad v_R(t) = Ri_R(t)$$

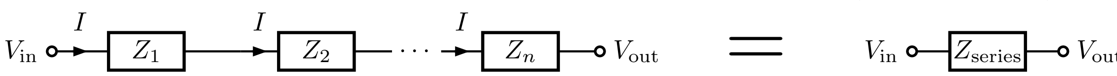
$$I_C(s) = sCV_C(s) \qquad V_L(s) = sLI_L(s) \qquad V_R(s) = RI_R(s)$$

$$Z_C(s) \triangleq \frac{V_C(s)}{I_C(s)} = \frac{1}{sC} \qquad Z_L(s) \triangleq \frac{V_L(s)}{I_L(s)} = sL \qquad Z_R(s) \triangleq \frac{V_R(s)}{I_R(s)} = R$$

The Laplace transformations of the voltage-to-current equations use the fact that derivatives in the *time domain* correspond to multiplication by s in the *Laplace domain*. The *impedance* of a circuit element is its voltage-to-current ratio **at a given frequency.** For example, assume that the voltage driving a capacitor is $v_C \triangleq \sin(2t)$. Then the current $i_C = Cv' = 2C\cos(2t)$. Hence, $\max\{v_C\}/\max\{i_C\} = 1/(2C)$ which matches $|Z_C(j2)| = |1/(j2C)| = 1/(2C)$. The larger the capacitance, the larger the current, and so bigger capacitors provide *less impedance* to high frequency current.

Series Impedance

Reducing a complicated network of impedances to a few equivalent impedances is a common circuit analysis task. Consider the equivalent impedance of n impedances in series. Apply the (Laplace-domain) loop rule.



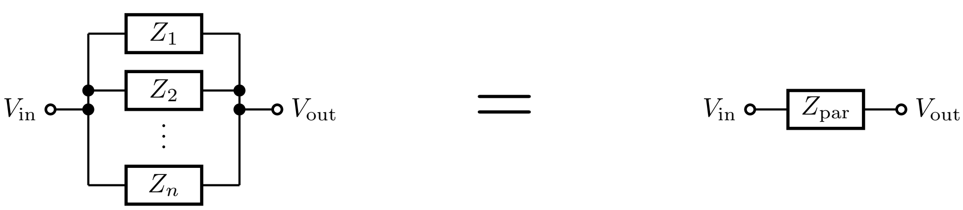
The equivalent impedance Z_{series} is the total voltage drop divided by the series current. That is,

$$\boxed{Z_{\text{series}}(s)} = \frac{V_{\text{in}}(s) - V_{\text{out}}(s)}{I(s)} = \frac{I(s)Z_1(s) + I(s)Z_2(s) + \cdots + I(s)Z_n(s)}{I(s)} = \boxed{Z_1(s) + Z_2(s) + \cdots + Z_n(s),}$$

and so the **equivalent series impedance** is the **simple sum of the constituent impedances.**

Parallel Impedance

Next, consider the equivalent impedance of n impedances in parallel, which is often denoted $Z_1 \| Z_2 \| \cdots \| Z_n$.



The equivalent impedance Z_{parallel} is the voltage drop divided by the total current. That is,

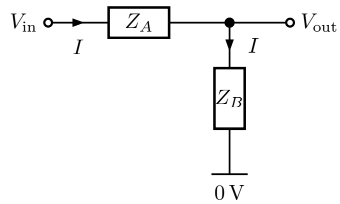
$$\boxed{Z_{\text{par}}(s)} = \frac{V_{\text{in}}(s) - V_{\text{out}}(s)}{\frac{V_{\text{in}}(s)-V_{\text{out}}(s)}{Z_1(s)} + \frac{V_{\text{in}}(s)-V_{\text{out}}(s)}{Z_2(s)} + \cdots + \frac{V_{\text{in}}(s)-V_{\text{out}}(s)}{Z_n(s)}} = \boxed{\left(\frac{1}{Z_1(s)} + \frac{1}{Z_2(s)} + \cdots + \frac{1}{Z_n(s)}\right)^{-1}.}$$

In the **special but *very common* case** of only *two* branches (i.e., $n = 2$),

$$\boxed{Z_{1\|2}(s)} \triangleq Z_1(s) \| Z_2(s) = \left(\frac{1}{Z_1(s)} + \frac{1}{Z_2(s)}\right)^{-1} = \left(\frac{Z_2(s) + Z_1(s)}{Z_1(s)\,Z_2(s)}\right)^{-1} = \boxed{\frac{Z_1(s)\,Z_2(s)}{Z_1(s) + Z_2(s)}.}$$

The Voltage Divider

Most passive electronic circuits can be reduced to a *voltage divider* which splits a driving signal between two impedances.



The (Laplace-domain) current through the entire circuit is

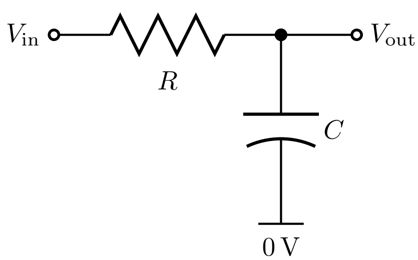
$$I(s) = \frac{V_{\text{in}}(s)}{Z_A(s) + Z_B(s)},$$

and the output $V_{\text{out}}(s) = I(s) \times Z_B(s)$, and

$$\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{Z_B(s)}{Z_A(s) + Z_B(s)}.$$

So at each frequency ω, the output is the same proportion of the input as Z_B is of the total $Z_A + Z_B$. However, the value of impedances Z_A and Z_B vary with frequency.

Example: An RC Low-Pass Filter



$$\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{sRC + 1}$$

The DC (i.e., constant) parts of V_{in} are passed directly through the filter and show up at the output. Faster components of the input induce *displacement current* within the capacitor, and so they get *attenuated* through the resistor. So V_{out} is a "slower" version of V_{in}.

First-Order Filters: Shortcut via Thévenin Equivalent Source

When an RC circuit contains *only* ***one*** *capacitor* and an output *across that capacitor*, it's helpful to convert the rest of the circuit into its Thévenin equivalent; the result is a simple voltage divider. That is,

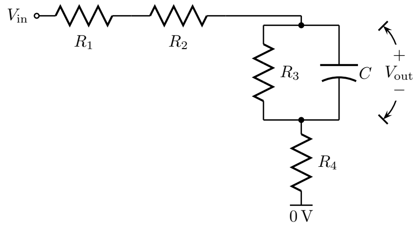
1. Remove the capacitor C from the system (i.e., replace it with an *open circuit*).
2. Find the *constant* gain from input to output. Call it K.
3. Place a *short circuit* on the input and find the equivalent resistance across the *output*. Call it R_{eq}.

Then the transfer function will then be

$$\boxed{\frac{K}{sR_{\text{eq}}C + 1}}$$

(compare to RC divider low-pass example above).

For example,



$$\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{\frac{R_3}{R_1+R_2+R_3+R_4}}{s\left(R_3 \| (R_1+R_2+R_4)\right)C + 1} = \frac{R_3}{sR_3\left(R_1+R_2+R_4\right)C + R_1 + R_2 + R_3 + R_4}$$

Simplifying circuits this way (i.e., by using Thévenin equivalents) is useful in many situations.