The Surveyor's Area Formula

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Bart Braden studied mathematics and languages at Washington State University, as an undergraduate. After an encounter with abstract 20th-century mathematics in graduate school at Yale and Oregon, which culminated in a doctoral thesis on representations of Lie algebras, he gradually returned to the more concrete parts of classical undergraduate mathematics. Currently, he is Professor of Mathematics at Northern Kentucky University.

A typical survey of a plot of land gives as data the successive displacements required to traverse the boundary of a simple plane polygon. From this data, we wish to determine the area of the plot. In a simple example, such as Figure 1a, one could break up the polygon into triangles whose areas could be laboriously found using trigonometric methods. A better way (Figure 1b) is to introduce rectangular coordinates and change the displacement vectors from polar to rectangular form, so they can be added to give the coordinates of the vertices of the polygon. Then a general formula can be applied, which expresses the area of the polygon as a function of the coordinates of its vertices. Such a polygonal area formula is well known to surveyors but, despite its elementary nature, does not appear in most precalculus or calculus textbooks.

Besides its intrinsic interest, at least two reasons can be advanced for including this surveyor's area formula in the calculus course, when plane vectors are introduced:

1. The derivation provides an excellent opportunity to introduce and use the geometric interpretation of a 2×2 determinant as the oriented area of a parallelogram in R^2. This makes it easier later on for students to understand geometric properties of the cross product of vectors in R^3.

2. The surveyor's formula provides a geometric interpretation of an otherwise mysterious formula in multivariable calculus, expressing the area inside a simple closed curve in parametric form as an integral around its boundary. An elementary derivation of this curvilinear area formula closely resembles that of the arc-length formula, so both can be derived together. An important advantage of the area formula is that the area integrals for many familiar curves are easily evaluated.

The Surveyor's Formula. If the vertices of a simple polygon, listed counterclockwise around the perimeter, are $(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})$, the area of the polygon is

$$A = \frac{1}{2}\left\{\begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \cdots + \begin{vmatrix} x_{n-2} & x_{n-1} \\ y_{n-2} & y_{n-1} \end{vmatrix} + \begin{vmatrix} x_{n-1} & x_0 \\ y_{n-1} & y_0 \end{vmatrix}\right\}.$$

Note that each oriented edge of the polygon corresponds to a 2×2 determinant in the surveyor's formula.

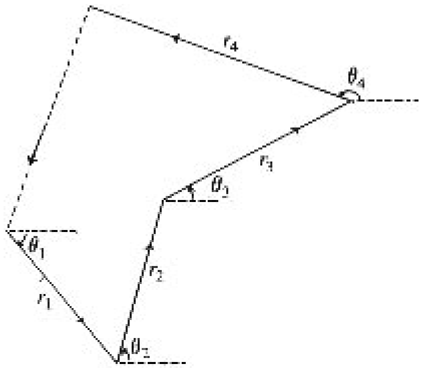


Figure 1a.

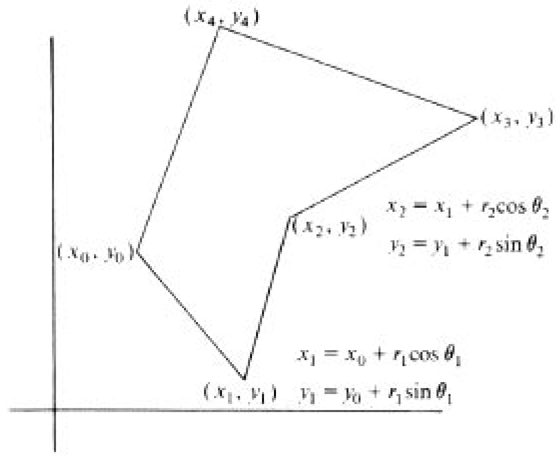


Figure 1b.

Our derivation of the surveyor's formula is based on the geometric interpretation of a 2×2 determinant as the oriented area of the parallelogram whose sides are the vectors comprising the columns of the determinant. We formally state this and provide a proof suitable for a calculus course.

Lemma. The absolute value of $\begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix}$ is the area of the parallelogram determined by the vectors $\mathbb{V} = \langle v_1, v_2 \rangle$, $\mathbb{W} = \langle w_1, w_2 \rangle$, and the determinant is positive just if the (shorter) direction of rotation of $\mathbb{V}$ into $\mathbb{W}$ is counterclockwise.

To prove this, let $\mathbb{N}$ denote the vector obtained by rotating $\mathbb{V} = \langle v_1, v_2 \rangle$ counterclockwise by $\pi/2$ radians. If θ is the polar angle of $\mathbb{V}$ (the angle between the positive x-axis and $\mathbb{V}$), then $\theta + (\pi/2)$ is the polar angle of $\mathbb{N}$. Since

$$\mathbb{V} = \langle r \cos\theta, r \sin\theta \rangle,$$

where

$$r = |\mathbb{V}|,$$

we have

$$\mathbb{N} = \langle -r \sin\theta, r \cos\theta \rangle = \langle -v_2, v_1 \rangle.$$

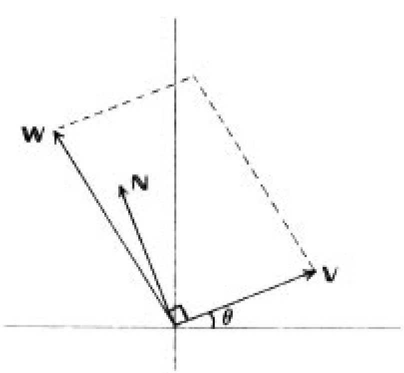


Figure 2.

The altitude of the parallelogram with base $\mathbb{V}$ and adjacent side $\mathbb{W}$ is the absolute value of the component of $\mathbb{W}$ along $\mathbb{N}$; that is, $|\mathbb{W} \cdot \mathbb{N}|/|\mathbb{N}|$. Therefore, the area A of this parallelogram is $|\mathbb{V}||\mathbb{W} \cdot \mathbb{N}|/|\mathbb{N}|$. but $|\mathbb{N}| = |\mathbb{V}|$, so $A = |W \cdot N| = |v_1 w_2 - v_2 w_1|$, the absolute value of the determinant with $\mathbb{V}$ and $\mathbb{W}$ as column vectors. Moreover, this determinant is positive just if the angle between $\mathbb{N}$ and $\mathbb{W}$ is acute; that is, if the angle from $\mathbb{V}$ to $\mathbb{W}$, measured counterclockwise, is between 0 and π.

Now let's turn to the surveyor's formula. The case $n = 3$, when the polygon is a triangle, is the key to our proof.

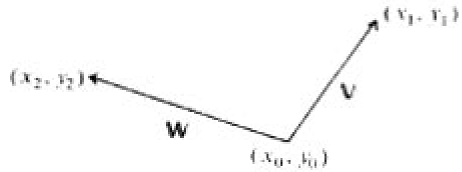


Figure 3.

From the Lemma, we know that the area of a triangle having vertices (x_0, y_0), (x_1, y_1), $(x_{2,} y_2)$ listed counterclockwise (so that the shorter direction of rotation of $\mathbb{V} = \langle x_1 - x_0, y_1 - y_0\rangle$ into $\mathbb{W} = \langle x_2 - x_0, y_2 - y_0\rangle$ is counterclockwise), is

$$A = \frac{1}{2}\begin{vmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{vmatrix}.$$

Now let D be the 3×3 determinant whose rows are $r_1 = (1, 1, 1)$, $r_2 = (x_0, x_1, x_2)$, $r_3 = (y_0, y_1, y_2)$. Comparing the two expansions of D,

$$D = \begin{vmatrix} 1 & 0 & 0 \\ x_0 & x_1 - x_0 & x_2 - x_0 \\ y_0 & y_1 - y_0 & y_2 - y_0 \end{vmatrix} = \begin{vmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{vmatrix} = 2A$$

and

$$\begin{aligned} D &= \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} - \begin{vmatrix} x_0 & x_2 \\ y_0 & y_2 \end{vmatrix} + \begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} \\ &= \begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_0 \\ y_2 & y_0 \end{vmatrix}, \end{aligned}$$

we obtain the surveyor's formula

$$A = \frac{1}{2}\left\{\begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_0 \\ y_2 & y_0 \end{vmatrix}\right\}$$

for the area of a triangle. Note that the determinants appearing in the formula correspond to the three *oriented* edges of the triangle.

To establish the surveyor's formula for a polygon with $n > 3$ sides, we use the fact (see [4], p. 286) that any oriented simple polygon can be triangulated; that is, we can add $n - 3$ auxiliary diagonals through the interior to decompose the polygon into $n - 2$ triangles, each diagonal being an edge of two adjacent triangles but inheriting opposite orientations form them (Figure 4). Since the vertices of our polygon are listed counterclockwise, all the triangles inherit this positive orientation; so the oriented area of the polygon is the sum of the areas of the triangles.

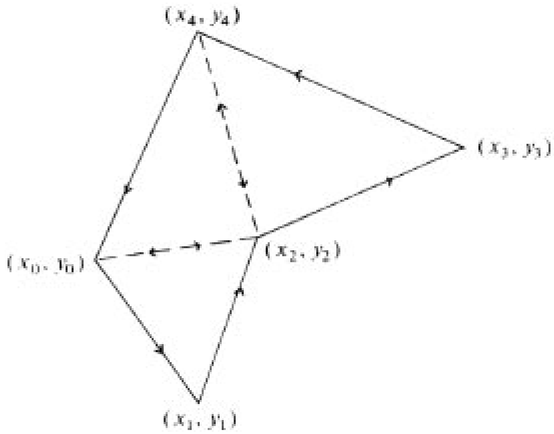


Figure 4.

Applying the surveyor's formula to each triangle and summing gives

$$A = \frac{1}{2}\sum\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix},$$

with one determinant for each oriented edge in our collection of triangles. Since each diagonal occurs twice (as the common edge of adjacent triangles) and with opposite orientations, the two determinants corresponding to each diagonal cancel out $\left(\text{because}\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} + \begin{vmatrix} x_j & x_i \\ y_j & y_i \end{vmatrix} = 0\right)$, and we're left with the sum of the determinants corresponding to the oriented edges of the original polygon. This completes the proof.

Exercises base on a sketch like Figure 1, or on survey data (e.g. [7]) will reinforce the student's grasp of the surveyor's formula.

The Area Inside a Simple Closed Curve. In a typical calculus course, parametric equations of curves are introduced in the 2nd semester and the arc-length formula for a curve in parametric form is derived, but no mention is made of the problem of finding the area inside a simple closed curve. Except for curves in polar coordinates, the student can find area only by breaking up the region into pieces bounded by graphs of functions and lines parallel to the coordinate axes.

Late in the 3rd semester, or perhaps in a multivariable calculus course, the formula

$$A = \frac{1}{2}\oint x\,dy - y\,dx$$

appears as a trivial consequence of Green's theorem. Often no geometric explanation is given, and the formula makes little impression. The surveyor's formula leads naturally to this integral formula, if one thinks of a curve as the limit of inscribed polygons. And since the area formula in polar coordinates is an easy consequence of the general integral formula for parametric curves, class time spent developing the surveyor's formula can be partially regained. The result is an elementary treatment of the calculus of curves, which makes clear the fundamental role of the parametric form.

Example. Let the circle C of radius r, with center at the origin, be given by the parametric equations $x(t) = r\cos t$ and $y(t) = r\sin t$ $(0 \le t \le 2\pi)$. For any natural number n, the points $t_k = (2k\pi)/n$ $(0 \le k \le n)$ form a regular partition of the parameter interval $[0, 2\pi]$, and the corresponding points $(x_k, y_k) = (r\cos t_k, r\sin t_k)$ are the vertices of a positively oriented, regular n-sided polygon inscribed in our circle. (See Figure 5a.) Note that $(x_0, y_0) = (x_n, y_n)$.

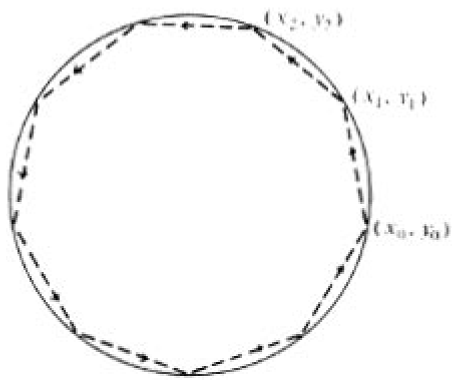


Figure 5a.

Applying the surveyor's formula to each triangle and summing gives

$$A_n = \frac{1}{2}\left\{\begin{vmatrix} r\cos 0 & r\cos\frac{2\pi}{n} \\ r\sin 0 & r\sin\frac{2\pi}{n}\end{vmatrix} + \begin{vmatrix} r\cos\frac{2\pi}{n} & r\cos\frac{4\pi}{n} \\ r\sin\frac{2\pi}{n} & r\sin\frac{4\pi}{n}\end{vmatrix} + \cdots \right.$$

$$\left. \cdots + \begin{vmatrix} r\cos\frac{2(n-1)\pi}{n} & r\cos 2\pi \\ r\sin\frac{2(n-1)\pi}{n} & r\sin 2\pi\end{vmatrix}\right\}$$

$$= \frac{1}{2}r^2\left\{\sin\frac{2\pi}{n} + \sin\frac{2\pi}{n} + \ldots + \sin\frac{2\pi}{n}\right\}$$

$$= \frac{r^2}{2}\cdot n\sin\frac{2\pi}{n} = \pi r^2\left[\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}\right].$$

Thus, the area of this circle is $A = \lim_n A_n = \pi r^2$. Note also that

$$\frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi}[r\cos t(r\cot t) - r\sin t(-r\sin t)]\,dt = \pi r^2.$$

To show that the limit of the areas of inscribed polygons is given by the integral $(1/2)\int_C x\,dy - y\,dx$ for any simple, closed, rectifiable curve C, we modify our notation slightly, writing the surveyor's formula as

$$A = \frac{1}{2}\sum_{i=1}^{n}\begin{vmatrix} x_{i-1} & x_i \\ y_{i-1} & y_i \end{vmatrix}, \tag{1}$$

with the convention $(x_n, y_n) = (x_0, y_0)$. Setting $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, and using the identity

$$\begin{vmatrix} x_{i-1} & x_i \\ y_{i-1} & y_i \end{vmatrix} = \begin{vmatrix} x_{i-1} & \Delta x_i \\ y_{i-1} & \Delta y_i \end{vmatrix},$$

we obtain

$$A = \frac{1}{2}\sum_{i=1}^{n}\begin{vmatrix} x_{i-1} & \Delta x_i \\ y_{i-1} & \Delta y_i \end{vmatrix}. \tag{2}$$

Now suppose C: $t \to C(t) = (x(t), y(t))$ for $t \in [a, b]$ is a simple, closed, smooth, plane curve traversed in the positive (counterclockwise) direction; so that the interior of the curve lies to the left of the moving point $C(t)$. Each partition $a = t_0 < t_1 < \cdots < t_n = b$ of the parameter interval $[a, b]$ determines the vertices $C(t_i)$ of a positively oriented polygon 'inscribed' in the curve. (See Figure 5b).

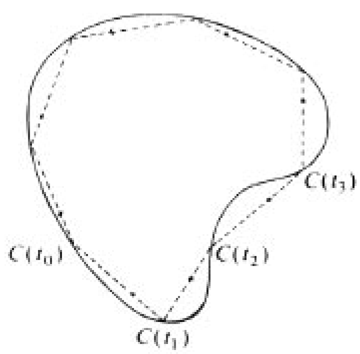


Figure 5b.

Using the Mean Value theorem, the length of the inscribed polygon

$$\sum_{i=1}^{n}|C(t_i) - C(t_{i-1})| = \sum_{i=1}^{n}\{[x(t_i) - x(t_{i-1})]^2 + [y(t_i) - y(t_{i-1})]^2\}^{1/2}$$

can be expressed as

$$\sum_{i=1}^{n}\sqrt{\{x'(u_i)\}^2 + \{y'(v_i)\}^2}(t_i - t_{i-1}),$$

where the derivatives are evaluated at points u_i, v_i in (t_{i-1}, t_i). Similarly, the surveyor's formula (2) for the area of the polygon,

$$\tfrac{1}{2}\sum_{i=1}^{n}\{x(t_{i-1})[y(t_i) - y(t_{i-1})] - y(t_{i-1})[x(t_i) - x(t_{i-1})]\},$$

can be expressed as

$$\tfrac{1}{2}\sum_{i=1}^{n}\{x(t_{i-1})y'(v_i) - y(t_{i-1})x'(u_i)\}(t_i - t_{i-1}).$$

As the mesh of the partition of $[a, b]$ tends to zero, these 'generalized Riemann sums' converge (see [2], p. 133) to the integrals

$$L = \int_a^b \sqrt{\{x'(t)\}^2 + \{y'(t)\}^2}\, dt$$

and

$$A = \tfrac{1}{2}\int_a^b [x(t)y'(t) - y(t)x'(t)]\, dt = \tfrac{1}{2}\int_a^b \begin{vmatrix} x(t) & x'(t) \\ y(t) & y'(t) \end{vmatrix}\, dt. \quad (3)$$

That the integral L does in fact give the arc-length of C is shown convincingly in [6]. See Remark 2 below for a proof that A is indeed the area inside C.

The connection between the area formula (3) and the surveyor's formula (2) can be displayed most clearly by using the notation for the integral of a differential form over a curve. In this notation ([5], p. 290), the integral (3) over $[a, b]$ is the integral of the form $\omega = \left(\frac{1}{2}\right)(x\, dy - y\, dx)$ over the curve C, denoted

$$\tfrac{1}{2}\int_C x\, dy - y\, dx, \text{ or } \tfrac{1}{2}\int_C \begin{vmatrix} x & dx \\ y & dy \end{vmatrix}.$$

One might simply say that as Δx_i and Δy_i become infinitesimals dx and dy, the summation in the surveyor's formula (2) is transformed into the integral (3).

Remarks. 1. By including in each partition of $[a, b]$ all t-values at which either $C'(t)$ is zero or $C(t)$ fails to be differentiable, the arc-length and area formulas are seen to be valid for piecewise smooth curves, a large enough class to include all curves studied in elementary calculus.

2. For certain pathological curves, the inscribed polygons corresponding to partitions of the parameter interval $[a, b]$ might cross themselves, even for arbitrarily fine partitions. This unfortunate complication could be circumvented either by simply excluding such curves from consideration ([3], p. 187), or better, by broadening the discussion to cover polygons and curves with self-intersections ([2], p. 311).

Better still, by an application of Green's theorem,

$$\int_C f(x, y)\, dx + g(x, y)\, dy = \int\int_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA,$$

which is valid for an arbitrary Jordan region R with boundary curve C ([1], p. 289), the difficulty with pathological curves vanishes. Choosing $f(x, y) = -\left(\frac{1}{2}\right)y$ and $g(x, y) = \left(\frac{1}{2}\right)x$, we get $\left(\frac{1}{2}\right)\int_C x\, dy - y\, dx = \int\int_R 1\, dA$, which (by definition) is the area of the region R. Thus, our formula (3) is valid for any simple, closed, rectifiable, oriented curve. The only point in mentioning the inscribed polygons is to provide a geometric motivation for the line integral formula, and for this purpose we may restrict attention to well-behaved curves.

Variants of the Area Formulas. The formula for the arc-length of a polar curve $r = f(\theta)(a \leq \theta \leq b)$ is usually derived a s a special case of the arc-length formula for curves in parametric form. using the parametrization by polar angle: $x(\theta) = f(\theta)\cos\theta$ and $y(\theta) = f(\theta)\sin\theta$.

Since $\{x'(\theta)\}^2 + \{y'(\theta)\}^2 = \{f(\theta)\}^2 + \{f'(\theta)\}^2$, we have

$$L = \int_a^b \sqrt{\{f(\theta)\}^2 + \{f'(\theta)\}^2}\, d\theta.$$

Similarly, $x(\theta)y'(\theta) - y(\theta)x'(\theta) = \{f(\theta)\}^2$, and this yields the area formula

$$A = \frac{1}{2}\int_a^b \{f(\theta)\}^2\, d\theta$$

for a simple, closed, polar curve.

The usual geometric argument, based on the formula $A = \left(\frac{1}{2}\right)r^2\theta$ for the area of a circular sector, shows that the polar area formula applies not only to simple closed curves but more generally $\left(\frac{1}{2}\right)\int_a^b \{f(\theta)\}^2\, d\theta$ gives the area of the 'sector' bounded by the rays $\theta = a$ and $\theta = b$, and the curve $r = f(\theta)$. (See Figure 6.)

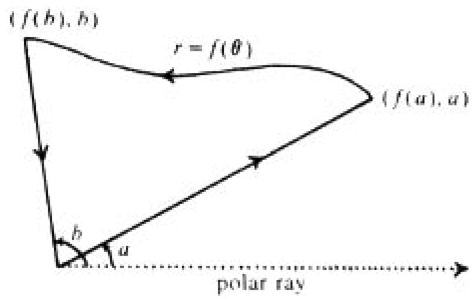


Figure 6.

This more general result can be derived by observing that $\begin{vmatrix} x & x' \\ y & y' \end{vmatrix}$ is identically 0 along any ray through the origin, since along such a ray the tangent vector $\langle x', y'\rangle$ is a multiple of the radius vector $\langle x, y\rangle$. So the integral $\left(\frac{1}{2}\right)\int x\,dy - y\,dx$ along the segments from the pole to the point with polar coordinates $(f(a), a)$ and from $(f(b), b)$ back to the pole will both vanish, leaving $\left(\frac{1}{2}\right)\int_a^b \{f(\theta)\}^2\, d\theta$ as the only nonvanishing part of the integral of $\left(\frac{1}{2}\right)(x\,dy - y\,dx)$ around the boundary of the "sector".

In this connection, another view of the surveyor's area formula should be mentioned. Observe that

$$\frac{1}{2}\begin{vmatrix} x_{i-1} & x_i \\ y_{i-1} & y_i \end{vmatrix}$$

is the area of the triangle having vertices (x_{i-1}, y_{i-1}), (x_i, y_i) and $(0, 0)$, with the orientation determined by the orientation of the ith edge in the original polygon. Thus, the surveyor's formula can be viewed as expressing the area of a polygon as the sum of the oriented areas of the triangles formed by joining successive pairs of vertices to the origin. (See Figure 7.)

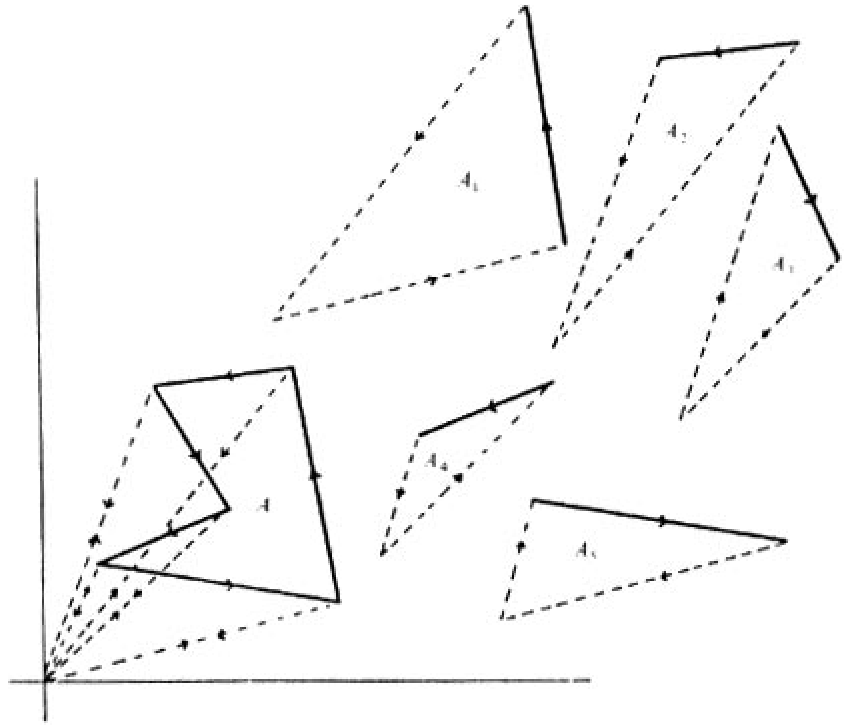


Figure 7. $A = |A_1| + |A_2| - |A_3| + |A_4| - |A_5|$, the sum of the oriented areas of the triangles subtended by the edges of the polygon.

This view leads to considering the differential form $\omega = \left(\frac{1}{2}\right)(x\,dy - y\,dx)$ as the 'radial area element' in rectangular coordinates, just as one speaks of $\left(\frac{1}{2}\right)r^2\,d\theta$ as the area element in polar coordinates. As the parameter t runs from a to b, the ray from the origin to the point $C(t) = (x(t), y(t))$sweeps out an area which is computed by integration of ω over the curve C. Note that this interpretation of ω provides a simple explanation of Kepler's law of equal areas for motion in a central force field. In such a field, $\langle x'', y''\rangle$ is a multiple of $\langle x, y\rangle$, so

$$\frac{d}{dt}\begin{vmatrix} x & x' \\ y & y' \end{vmatrix} = \begin{vmatrix} x & x'' \\ y & y'' \end{vmatrix} = 0.$$

This means $xy' - yx'$ is constant; that is, the area is swept out at a constant rate.

We round out this discussion by briefly considering two useful variants of the integral formula for the area inside a simple, closed curve:

$$A = \int_a^b x(t)y'(t)\,dt \tag{4}$$

and

$$A = -\int_a^b y(t)x'(t)\,dt. \tag{5}$$

Adding these together and dividing by 2 gives the symmetric formula (3). How are (4) and (5) related to the surveyor's formula for the area of a polygon? Defining (x_{n+1}, y_{n+1}) to be (x_1, y_1) and collecting the coefficients of each x_i in the surveyor's formula, one easily verifies the identity

$$\frac{1}{2}\sum_{i=1}^{n}\begin{vmatrix} x_{i-1} & x_i \\ y_{i-1} & y_i \end{vmatrix} = \frac{1}{2}\sum_{i=1}^{n} x_i(y_{i+1} - y_{i-1}).$$

Writing the right side as

$$\frac{1}{2}\sum_{i=1}^{n} x_i(y_{i+1} - y_i) + \frac{1}{2}\sum_{i=1}^{n} x_i(y_i - y_{i-1}),$$

it follows (as noted earlier) that as the mesh of the partition of $[a, b]$ approaches 0, this converges to

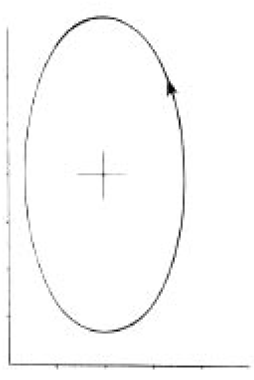
$$\frac{1}{2}\int_a^b x(t)y'(t)\,dt + \frac{1}{2}\int_a^b x(t)y'(t)\,dt = \int_a^b x(t)y'(t)\,dt.$$

A similar argument gives formula (5). As a matter of fact, the surveyor's formula is often expressed in the corresponding forms

$$A = \tfrac{1}{2}\sum x_i(y_{i+1} - y_{i-1}) \quad \text{or} \quad A = \tfrac{1}{2}\sum y_i(x_{i-1} - x_{i+1})$$

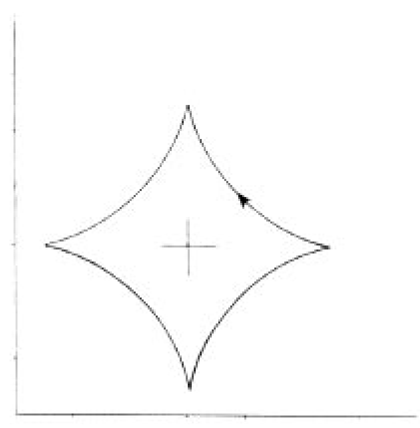
in surveying books. (See [7], p. 202, or [8], p. 483.)

Some Exercises. A drawback of the formula $L = \int_a^b \sqrt{\{x'(t)\}^2 + \{y'(t)\}^2}\,dt$, for classroom purposes, is that this integral is non-elementary for most curves. The area integrals (3), (4), or (5) for many familiar curves, however, are easily evaluated. Indeed, we invite readers to verify the area formulas for the following curves given in parametric form.



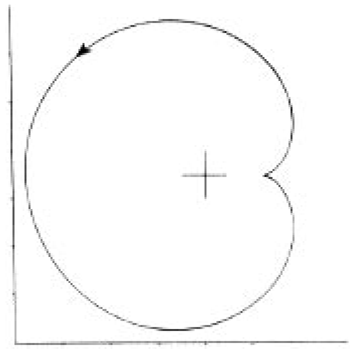
Ellipse: $x = b\cos t$, $y = a\sin t$ $(0 \le t \le 2\pi)$

Area $= \pi ab$



Astroid: $x = a\cos^3 t$, $y = a\sin^3 t$ $(0 \le t \le 2\pi)$

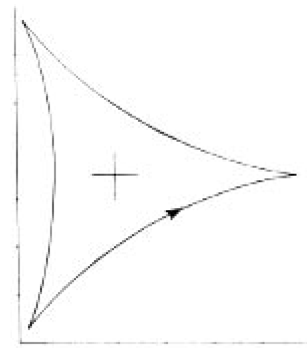
Area $= \frac{3}{8}\pi a^3$



Cardioid:

$x = 2a\cos t - a\cos 2t$
$y = 2a\sin t - a\sin 2t$ $(0 \le t \le 2\pi)$

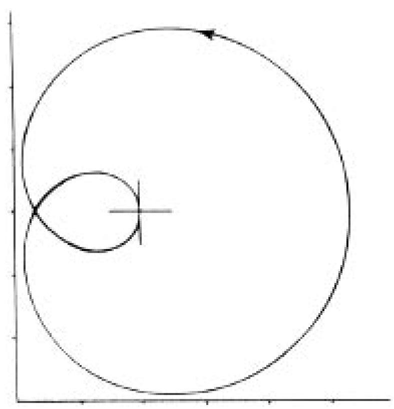
Area $= 6\pi a^2$



Deltoid:

$x = 2a\cos t + a\cos 2t$
$y = 2a\sin t - a\sin 2t$ $(0 \le t \le 2\pi)$

Area $= 2\pi a^2$

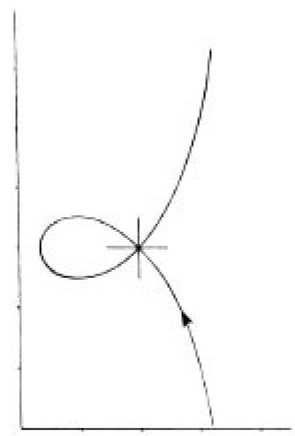


Trisectrix:

$x = a\cos t + a\cos 2t$
$y = a\sin t + a\sin 2t$

inner loop: $(2\pi/3 \le t \le 4\pi/3)$

Area $= a^2\left(\pi - \frac{3\sqrt{3}}{2}\right)$

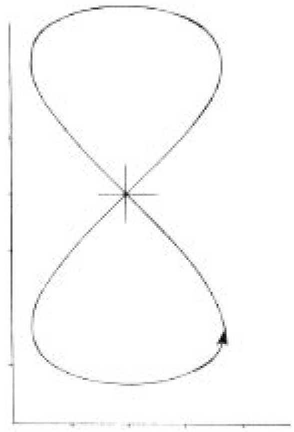


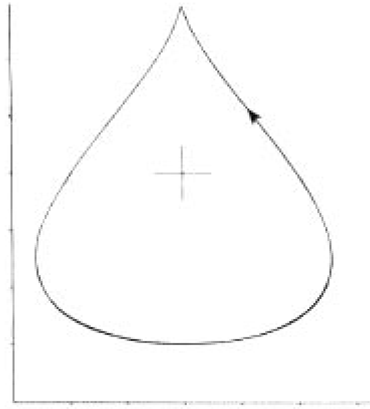
Strophoid:

$x = a(t^2 - 1)/(t^2 + 1)$
$y = at(t^2 - 1)/(t^2 + 1)$

loop: $(-1 \le t \le 1)$

Area $= a^2(4 - \pi)/2$





Hourglass: $x = a \sin 2t$, $y = b \sin t$ One loop: $(0 \le t \le \pi)$

Area $= \frac{3}{4}ab$

Teardrop: $x = 2a \cos t - a \sin 2t$, $y = b \sin t$ $(0 \le t \le 2\pi)$

Area $= 2\pi ab$

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