Continuous Random Variables

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1 Introduction

In this section, we consider random variables taking values in a continuum, such as $\mathbb{R}$, or the interval $[a, b]$. This is in part motivated by many applications, in which the (random) numerical outcome of the experiment need not be an integer, or rational number.

Example 1. *The following are examples of random experiments which have numerical outputs that assume values on a continuum.*

1. **Error in precision engineering**. *A mechanical component functions best when it is 153mm in length, and will not function if it is longer than 154.5mm or shorter than 151.7mm. The length of the component (in mm) is X, which can be modelled as a random variable taking values on $(0, \infty)$. The event which represents a component being functional is $\{\omega : 151.7 < X(\omega) < 154.5\}$.*

2. **Waiting for a bus**. *The time (in minutes) which elapses between arriving at a bus stop and a bus arriving can be modelled as a random variable T taking values in $[0, \infty)$. The probability of waiting no more than three minutes is $\mathbb{P}[\omega : T(\omega) \leq 3]$, or, more simply $\mathbb{P}[T \leq 3]$.*

3. **Share price.** *The values of one share of a specific stock at some given future time can be modelled as a random variable S taking values in $[0, \infty)$. If we bought the stock for \$21.30, the event that we make a profit by selling at the future time is given by $\{\omega : S(\omega) > 21.30\}$.*

2   Distribution functions of continuous random variables

How can we calculate the probability of events associated with a random variable X which takes on values on a continuum, such as those in Example 1 above? For discrete random variables, we recall that the probability of all such "interesting" events could be specified **either** through the probability mass function, or the distribution function. Hence, we might think it would **suffice to give** $\mathbb{P}[X = x]$ for all values of x. As we will shortly see, however, this is **not** so. For this type of random variable, we should use the (probability) **distribution function** of X, defined by $F_X(x) = \mathbb{P}[X \le x]$.

Let us list some properties of the distribution function.

Proposition 1. *A function $F : \mathbb{R} \to \mathbb{R}$ is the distribution function of some random variable if and only if it has the following properties:*

  *(i) F is **non-decreasing** (i.e., $F(x) \ge F(y)$ whenever $y > x$).*

  *(ii) F is **right-continuous** i.e.,*

$$\lim_{\varepsilon \downarrow 0} F(x + \varepsilon) = F(x).$$

  *(iii) F is **normalised** i.e.,*

$$\lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to \infty} F(x) = 1.$$

We omit the proof, but leave one part as an exercise.

Exercise 2.1. Prove part (i) of Proposition 1 above.

Hint: Note that $\{X \le x\} \subset \{X \le y\}$ when $y > x$.   □

We focus on part (ii) of the definition: remember that the distribution function of a discrete random variable has "jumps" at each of the values in its range, so the distribution function is **not continuous**. By contrast, we will say that the random variable X is **continuous** if its distribution function is continuous.

Definition 1. *The random variable X is **continuous**, or has **continuous distribution**, if its distribution function F_X is a continuous function.*

Example 2. *Show that $F : [0, \infty) \to \mathbb{R}$ defined by*

$$F(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 - \frac{1}{1+x^2} & \text{for } x \ge 0 \end{cases}$$

is the distribution function of a continuous random variable.

Solution: We check that the properties of Proposition 1 hold. For property (i), we see that $F(x) = 0 = F(y)$ for $0 \geq x > y$, and for $x > y \geq 0$

$$1 - F(x) = \frac{1}{1 + x^2} < \frac{1}{1 + y^2} = 1 - F(y),$$

so $F(x) \geq F(y)$. For property (ii), we see that F is continuous on $(0, \infty)$ and $(-\infty, 0)$, and $\lim_{x \to 0} F(x) = 0 = F(0)$. Therefore F is continuous on $\mathbb{R}$. Finally, $\lim_{x \to -\infty} F(x) = 0$, and

$$\lim_{x \to \infty} F(x) = \lim_{x \to \infty} 1 - \frac{1}{1 + x^2} = 1,$$

so property (iii) holds. The continuity of F ensures that it can be the distribution function of a continuous random variable. $\square$

As for discrete random variables, we can recover the probability of many events from the distribution function. For example, when $a < b$, the probability $\mathbb{P}[a < X \leq b]$ can be obtained from

$$\mathbb{P}[a < X \leq b] = \mathbb{P}[X \leq b] - \mathbb{P}[X \leq a] = F_X(b) - F_X(a),$$

where we exploit the fact that $\{X \leq a\} \cup \{a < X \leq b\}(= \{X \leq b\})$ is a disjoint union of events.

We can also see why we cannot determine a continuous random variable via its "mass function". This is because

$$\mathbb{P}[X = x] = 0, \quad \text{for all } x \in \mathbb{R}. \tag{2.1}$$

To prove (2.1), observe for any $n \in \mathbb{N}$ that

$$\{X = x\} \subset \{x - \frac{1}{n} < X \leq x\},$$

so

$$\mathbb{P}[X = x] \leq \mathbb{P}\left[x - \frac{1}{n} < X \leq x\right] = F_X(x) - F_X(x - \frac{1}{n}).$$

Now, let $n \to \infty$: then the **(left)** continuity of F_X yields $\lim_{n \to \infty} F_X(x - \frac{1}{n}) = F_X(x)$. Hence

$$0 \leq \mathbb{P}[X = x] \leq \lim_{n \to \infty} F_X(x) - F_X(x - \frac{1}{n}) = 0,$$

as required.

This result has an immediate consequence: for a continuous random variable, the probabilities

$$\mathbb{P}[a < X \leq b], \quad \mathbb{P}[a \leq X < b], \quad \mathbb{P}[a \leq X \leq b], \quad \mathbb{P}[a < X < b].$$

are all **equal**.

3   The density of a continuous distribution

Continuous distributions can often be described in terms of another function called the **probability density** of the distribution.

Definition 2. *A continuous distribution function F is said to have a **density** f if there exists a function $f : \mathbb{R} \to \mathbb{R}$ which satisfies $f(x) \geq 0$ for all $x \in \mathbb{R}$ such that*
$$F(x) = \int_{-\infty}^{x} f(u)\,du.$$
*We say that f is the **probability density** associated with f.*

If F is **continuously differentiable** at x, we can recover the probability density at x: it is given by
$$f(x) = F'(x),$$
by the fundamental theorem of calculus.

Exercise 3.1. Consider the distribution function given in Example 2. Show that the function
$$f(x) = \begin{cases} 0 & \text{for } x < 0, \\ \frac{2x}{(1+x^2)^2} & \text{for } x \geq 0, \end{cases}$$
is a probability density associated with F.

This indicates that we can characterise a continuous random variable through its probability density function. In fact, we can show the following.

Proposition 2. *A function $f : \mathbb{R} \to \mathbb{R}$ is the probability density function of a continuous random variable if and only if it satisfies the following two conditions:*

(i) $f(x) \geq 0$, $x \in \mathbb{R}$;

(ii)
$$\int_{-\infty}^{\infty} f(u)\,du = 1.$$

Proof. We prove that if f satisfies (i), (ii), then f is a probability density of a continuous random variable, but omit the proof of the converse result. Suppose that f satisfies (i), (ii). Define
$$F(x) = \int_{-\infty}^{x} f(u)\,du.$$

Then, for $x > y$, we have $F(x) - F(y) = \int_{y}^{x} f(u)\,du \geq 0$, using (i). Taking limits as $x \to \pm\infty$ yields
$$\lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to \infty} F(x) = 1,$$

using property (ii). Since f is bounded on bounded intervals, we get

$$|F(x) - F(y)| = \int_y^x f(u)\, du \leq \sup_{y \leq u \leq x} f(u)(x-y) \to 0, \quad \text{as } y \to x,$$

so F is continuous. Therefore F satisfies the conditions of Proposition 1 and is a continuous function, and is hence a continuous distribution function. $\qquad\square$

Example 3. *Show that f in Exercise 3.1 is the density function of a continuous random variable.*

Solution: Clearly $f(x) \geq 0$ for all $x \in \mathbb{R}$, and

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^{\infty} \frac{2x}{(1+x^2)^2}\, dx = \int_1^{\infty} \frac{1}{u^2}\, du = 1,$$

where we use integration by substitution, with the substitution $u = 1 + x^2$. $\quad\square$

We finally remark that $f(x)$ **is not a probability**, and in particular, is not the probability that $X = x$. Indeed, it is possible that $f(x) > 1$. If we want to obtain a probability from f we must **integrate** it:

$$\mathbb{P}[a < X < b] = F_X(b) - F_X(a) = \int_a^b f(x)\, dx,$$

and more generally:

$$\mathbb{P}[X \in A] = \int_A f(x)\, dx, \quad A \subset \mathbb{R}.$$

These equations explain the reason for the name "density" for f.

Example 4. *Let X be a continuous random variable with density given in Exercise 3.1. Compute $\mathbb{P}[3 \leq X \leq 5]$.*

Solution: The required probability is

$$\mathbb{P}[3 \leq X \leq 5] = \int_3^5 \frac{2x}{(1+x^2)^2}\, dx.$$

Integrating by substitution with the change of variable $u = 1 + x^2$ gives

$$\mathbb{P}[3 \leq X \leq 5] = \int_{10}^{26} \frac{1}{u^2}\, du$$
$$= -\frac{1}{u}\Big|_{u=10}^{26} = \frac{1}{10} - \frac{1}{26} = \frac{4}{65}.$$

$\square$