Mathematical Tripos
Part III Lecture Courses in 2006-2007

Department of Pure Mathematics & Mathematical Statistics

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Department of Applied Mathematics & Theoretical Physics

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Notes and Disclaimers.

- Students may take any combination of lectures that is allowed by the timetable. The examination timetable corresponds to the lecture timetable and it is therefore not possible to take two courses for examination that are lectured in the same timetable slot. There is ***no*** requirement that students study only courses offered by one Department.
- The code in parentheses after each course name indicates the term of the course (M: Michaelmas; L: Lent; E: Easter), and the number of lectures in the course. Unless indicated otherwise, a 16 lecture course is equivalent to 2 credit units, while a 24 lecture course is equivalent to 3 credit units. Please note that certain courses are *non-examinable*. Some of these courses may be the basis for Part III essays.
- At the start of some sections there is a paragraph indicating the desirable previous knowledge for courses in that section. On the one hand such paragraphs are not exhaustive, on the other not all courses require all the pre-requisite material indicated. However you are strongly recommended to read up on the material with which you are unfamiliar if you intend to take a significant number of courses from a particular section.
- The courses described in this document apply only for the academic year indicated. Details for subsequent years are expected to be broadly similar, although *not* identical.
- Please note that while an attempt has been made to ensure that the outlines in this booklet are an accurate indication of the content of courses, the outlines do *not* constitute definitive syllabuses. Each course lecturer has discretion to vary the material covered.
- This document was last updated in September 2004. Further changes to the list of courses will be avoided if at all possible, but may be essential.

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Reading Course on Quantum Groups (Lent)

M. Batchelor

This course will be unusual in that the lectures will be given by the participants. Whether or not it runs will therefore depend on there being a committed group of students to share the responsibility of lecturing. It is anticipated that students will give three or four lectures during the course of the term. Those interested in taking part should contact me during the Michaelmas term to discuss arrangements.

The plan of the course is that the first two thirds of the course will cover material in the first two parts of Kassel Quantum Groups. The direction of the remainder of the course will be decided by those sharing the lecturing.

The approach will be primarily algebraic, although students wishing to may choose to present an account of the applications to physics in the second part of the course. There are no specific prerequisites, although a familiarity with the concepts from the basic Lie algebras course will prove helpful.

Recommended Texts

1. V. Chari and A. Pressley, *A Guide to Quantum Groups*, CUP Press, 1994.
2. C. Kassel, Quantum Groups, *Graduate Texts in Mathematics 155*, Springer Verlag, 1995.

Modular Representations of Finite Groups (L24)

S. Martin

- Background from module theory (projective and free modules, Schur's Lemma, Krull-Schmidt, Wedderburn's Theorem and Jacobson radical)
- Cohomolological language: short exactness and the functor Ext^1
- Review of characteristic 0 theory and comparison with characteristic p; the process of p-modular reduction
- Blocks (as ideals and as idempotents)
- Defect groups, vertices and sources
- The Brauer map
- Brauer's First Main Theorem
- The Green Correspondence
- Brauer's Second Main Theorem
- Clifford Theory and Green's Indecomposability Theorem
- Brauer's Third Main Theorem
- Cyclic Blocks and quivers; Brauer-Dade theory
- Brauer trees
- (if time) Auslander-Reiten sequences: uniqueness and existence

A sheet of examples will be provided backed up by one or two classes.

Desirable Previous Knowledge

Basic group theory; ordinary representations/character theory; Galois Theory; Rings & Modules; Commutative Algebra (completions, local rings, primality); some categorical nonsense.

Introductory Reading

1. J.L. Alperin & Rowen B. Bell, *Groups and Representation Theory* (GTM 162)
2. J.P. Serre, *Linear Representations of Finite Groups* (GTM 42)
3. I.M. Isaacs, *Character Theory of finite groups* (Dover reprint 1994)
4. G.D. James & M.W. Liebeck, *Representations and characters of groups* (CUP, 2nd edn 2001)
5. W. Feit, *The representation theory of finite groups* (North-Holland 1982)

Reading to complement course material

1. J.L. Alperin, *Local Representation theory* (CUP 1986)
2. D.J. Benson, *Representations and cohomology, Vol 1* (CUP 1991)
3. H. Nagao & Y. Tsushima, *Representations of finite groups* (Academic 1988)
4. P. Landrock, *Finite group algebras and their modules* (CUP 1984)
5. F. Digne & J. Michel, *Representations of finite groups of Lie type* (CUP 1991)
6. C.W. Curtis & I. Reiner, *Methods in representation theory, Vol 1* (John Wiley 1981)
7. I. Assem, A. Skowronski & D. Simson, *Elements of the Representation Theory of Associative Algebras* (LMS Student Texts, 2006)
8. B. Külshammer, *Lectures on block theory* (LMS lecture Note Series 161, CUP 1991).
9. B.M. Puttaswamaiah & J.D. Dixon, *Modular representations of finite groups* (Academic Press, 1977)

Finite dimensional Lie algebras and their representations (M24)

I. Grojnowski

This course is an introduction to the basic properties of finite dimensional and affine Lie algebras and of their representations.

Lie algebras are 'infinitesimal symmetries'; linearisations of groups. They are ubiquitous in many branches of mathematics: in topology, in arithmetic and algebraic geometry, and in theoretical physics (string theory, exactly solvable models in statistical mechanics)...

I hope to cover the following topics:

Definitions and basic structure theory: Root systems, Weyl groups, the finite simple Lie algebras.

Classification of finite dimensional representations, Verma modules, Weyl character formula.

Crystals, Littelmann paths.

Affine Lie algebras, the basic representation, Boson-Fermion correspondence, theta functions.

Desirable Previous Knowledge

None, but the part II course on representation theory (or equivalent) will be useful as background.

Reading to complement course material

1. V. Kac, *Infinite dimensional Lie algebras*, Cambridge University Press
2. M. Kashiwara, On crystal bases, in *Representations of groups* (Banff, AB, 1994), 155–197, CMS Conf. Proc., 16.
3. N. Jacobson, *Lie algebras.*

Pro-p groups (L24)

R. Camina

A pro-p group is an inverse limit of finite p-groups. So, although a pro-p group is infinite, much can learnt about the structure of the group by considering its finite quotients. Pro-p groups arise naturally as Galois groups and are thus known to Number Theorists, however this course will take a group theoretic approach. We will consider Lazard's theory of Analytic Pro-p groups as well as the more recent Coclass Conjectures. More obscure examples such as the Nottingham Group and the Grigorchuk Group will also be considered.

Recommended Books:

1. *Analytic Pro-p Groups (2nd edition)*, Dixon, du Sautoy, Mann and Segal, CUP, (1999).
2. *New Horizons in pro-p groups*, editors: du Sautoy, Segal and Shalev, Birkhäuser, (2000).

Noetherian Algebras (M24)

S.J. Wadsley

Noetherian algebras are important in a wide range of Mathematics including Number Theory, both classical and non-commutative Algebraic Geometry, and within Algebra itself. This course is intended to introduce some of the basic theory and techniques, illustrated by a variety of examples. The current plan of the material to be covered is as follows:

- Basic theory;
- Commutative rings: Nullstellensatz, associated primes, localisation;
- Ideal structure: Jacobson radical, prime radical;
- Filtrations, gradations, completions, applications to Noetherian ring theory;
- Artinian rings: Artin-Wedderburn, connections with Noetherian rings;
- Localisation in noncommutative rings, Ore's Theorem, statement of Goldie's Theorem;
- Dimension theory, Hilbert polynomials;
- Homological algebra, applications to Noetherian rings.

There will be 4 example sheets.

Prerequisites

Linear algebra (IB), groups rings and modules (IB) or equivalent.

Books

1. M. F. Atiyah, I. M. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley series in Mathematics (1969)

2. K. R. Goodearl, R. B. Warfield, Jr, *An Introduction to Noncommutative Noetherian Rings*, Second Edition, LMS Student Text 61 (2004)

3. J. C. McConnell, J. C. Robson, *Noncommutative Noetherian Rings*, Revised Edition, AMS Graduate Studies in Mathematics volume 30 (2001)

Topics in Group Theory (M24)

R. Lawther

This course aims to introduce the student to (some of) the finite non-abelian simple groups. An initial chapter on series is intended to provide motivation for the study of simple groups; the remaining chapters follow the traditional subdivision into alternating groups, groups of Lie type and sporadic groups, although in each of the latter two it will be possible to treat only a selection of the simple groups concerned.

Series

Composition series. The Jordan-Hölder Theorem.

Soluble and nilpotent groups.

Commutators. Upper and lower central series.

Permutation Groups

Transitivity and multiple transitivity. One-point extensions.

Primitivity.

Regular normal subgroups. Simplicity of A_n for $n \geq 5$.

Automorphisms of S_n.

Groups acting on Vector Spaces

Linear groups.

Groups with BN-pair. Parabolic subgroups and Bruhat decomposition. Simplicity.

Orthogonal and symplectic groups.

Coxeter groups. Brief description of groups of Lie type.

Groups acting on Combinatorial Structures

Steiner systems.

The Mathieu groups and the Golay code.

The Leech lattice and related simple groups.

Desirable Previous Knowledge

I will assume nothing beyond a basic course in Group Theory covering the Fundamental Isomorphism Theorems, the Orbit-Stabilizer Theorem and Sylow theory.

Reading

I am not aware of any one book which covers all the material treated here. I have drawn upon a number of sources, some of which are listed below; on the other hand I do not attempt to cover the entire contents of any of the following.

1. J.L. Alperin and R.B. Bell, *Groups and Representations*, Graduate Texts in Mathematics 162, Springer Verlag, 1995.
2. M. Aschbacher, *Finite Group Theory*, Cambridge studies in advanced mathematics 10, CUP, 1986.
3. M. Aschbacher, *Sporadic Groups*, Cambridge tracts in mathematics 104, CUP, 1994.
4. N.L. Biggs and A.T. White, *Permutation Groups and Combinatorial Structures*, LMS Lecture Note Series 33, CUP, 1979 (out of print).
5. P.J. Cameron, *Permutation Groups*, LMS Student Texts 45, CUP, 1999.
6. R.W. Carter, *Simple Groups of Lie Type*, Wiley, 1989.
7. J.D. Dixon and B. Mortimer, *Permutation Groups*, Graduate Texts in Mathematics 163, Springer Verlag, 1996.
8. J.S. Rose, *A Course on Group Theory*, CUP, 1978.

Analysis

Introduction to Functional Analysis (M24)

T.W. Körner

Fifty years ago Functional analysis was new and sinful. Now it is simply a collection of techniques and ideas which every analyst must know. My course will cover standard topics like the Theorem of Hahn-Banach, elementary Banach Algebras and the spectral Theorem for bounded self adjoint operators. To the limited extent that time permits, I will on showing how they are used rather than how they can be refined (in particular I shall deal with normed spaces and not general topological vector spaces). I shall give a couple of supplementary, non examinable, lectures on topics like the Stone Weierstrass theorem which I assume the majority of my audience will have met before.

In keeping with the status of Functional Analysis as a mature and (at the level of this course) stable subject, there are many excellent text books. There is

W.Rudin *Real and Complex Analysis* (McGraw Hill, 2nd Ed, 1974)

whose spirit I hope to copy, though I can hardly capture its richness, and its sequel

W.Rudin *Functional Analysis* (McGraw Hill 1973).

(Like many sequels this is not quite as good as its predecessor–but it is none the less outstanding.) On a more conventional level, each of the following three texts provides a well written, concise and, most importantly, enthusiastic account of the basic results of the subject.

B.Bollobás *Linear Analysis : an Introductory Course* (CUP 1991)

J.D.Pryce *Basic Methods of Linear Functional Analysis* (Hutchinson 1973)

C.Gofman and G.Pedrick *A First Course in Functional Analysis* (Prentice Hall 1965)

Any of these books would make very good preliminary reading.

Isoperimetry and concentration of measure (L24)

Dr D.J.H. Garling

To begin with, we prove isoperimetric theorems in four classical settings:

- d-dimensional Euclidean space, with its usual measure;
- the surface of a d-dimensional sphere;
- d-dimensional space, with Gaussian measure;
- the d-dimensional hypercube.

We shall also describe, without proof, how these results extend to Riemannian manifolds.

Each of these theorems requires its own technique, and we shall establish all the results that we shall need on the way (the Prékopa-Leindler inequality, the Brunn-Minkowski inequality, the Lévy projection theorem,...). We shall also investigate the many consequences of the isoperimetric theorems, and particularly those that relate to the geometry of Banach spaces (Dvoretzky's Theorem on spherical sections, Gluskin's theorem).

A typical application of the isoperimetric theorems is that in high dimensions, a Lipschitz function takes values near its median with high probability, and the probability of large deviations is small. This is known as the *concentration of measure phenomenon*, or *the theory of large deviations*. In the second part of the course, we shall study this, even in settings where an isoperimetric theorem does not exist.

Desirable Previous Knowledge

In spite of the geometric setting, this will be a course on Analysis and Probability. Attendance at the Part II courses on Probability and Measure, and Linear Analysis, or their equivalents, will be an advantage, as will be attendance at Michaelmas term Analysis courses.

Introductory Reading

1. K.M. Ball. An elementary introduction to modern convex geometry, in *Flavors of Geometry*, edited by Silvio Levy, CUP 1997.

Reading to complement course material

1. K.M. Ball. An elementary introduction to modern convex geometry, in *Flavors of Geometry*, edited by Silvio Levy, CUP 1997.
2. M. Ledoux, *The Concentration of Measure Phenomenon*, AMS 2001.
3. M. Ledoux, *Isoperimetry and Gaussian Analysis*, Saint-Flour Summer School, 1994, Springer Lecture Notes, Volume 1648.
4. V.D. Milman, G. Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces*, Springer Lecture Notes, Volume 1200.

Analysis of Operators (M24)

A. Wassermann

This course aims to provide an introduction to some of the analysis underlying both non-commutative geometry and conformal field theory. It has been planned in conjunction with Alain Connes' College de France course at the Newton Institute. The following is a list of topics which will probably be covered.

The spectral theorem for compact operators. Index of Fredholm operators. Toeplitz operators. Sobolev spaces and elliptic regularity. Young and Hardy-Littlewood-Sobolev inequalities for singular integral operators. Applications of Sobolev spaces to non-linear problems in geometry, e.g. isometric embedding (Nash) and hyperbolic uniformisation (Poincare). Fredholm determinants. Helton-Howe commutator formula. Szego's strong limit theorem. Limit formulas of Hirschman and Widom. Birkhoff decomposition. Random unitary matrices.

The Dirac sea and Jacobi's triple product formula. Positive energy representations. Fermions and the Clifford anticommutation relations. Quantisation. The spin group. Hardy space example. The loop group of $\mathbb{T}$ and the diffeomorphism group of the circle. Projective representations and bicharacters. The Fourier transform. Bosons and the Weyl commutation relations. Stone-von Neumann theorems. The metaplectic group. The boson-fermion correspondence. Second proof of Szego's strong limit theorem. Further applications, e.g. the Ising model, quadratic reciprocity, and the oscillator semigroup.

Perron-Frobenius theory in finite and infinite dimensions.

The Weyl-von Neumann theorem for self-adjoint and unitary operators. Essential spectrum. Brown-Douglas-Fillmore theory for unitaries. Representations of classical groups. Frobenius reciprocity. Bott's index theorem and Dirac induction. The Dirac and dual Dirac operators on $\mathbb{C}$. Bott periodicity for K^1 (using Dirac operators) and for K^0 (using Toeplitz operators).

Toeplitz operators on the unit ball in $\mathbb{C}^n$. Index formula of Venugopalkrishna. Cyclic cocycle of Helton-Howe.

Riemann Surfaces and Discrete Groups (M24)

T.K. Carne

This course will study the relationship between discrete groups of Möbius transformations and Riemann surfaces. The key result here is the Riemann mapping theorem, which shows that any simply conected Riemann surface is the unit disc, the complex plane or the Riemann sphere. This leads to the uniformization theorem: any Riemann surface is a quotient of one of the three simply connected surfaces by a discrete group.

I will assume the results from undergraduate complex analysis courses. The undergraduate courses on Riemann surfaces or differential geometry would be useful background but are not essential since I will review the material required.

- Review of Riemann Surfaces. Definition; conformal equivalence; analytic functions; analytic differentials; ∂ and $\overline{\partial}$. The geometric structure of Riemann surfaces.
- The Riemann Mapping Theorem. Harmonic functions and the Green's function on Riemann surfaces. Proof that simply connected Riemann surfaces are conformally equivalent to the Riemann sphere, the complex plane or the unit disc.
- The Uniformisation Theorem. Covering surfaces; Fuchsian groups; the hyperbolic metric. Discrete groups of Möbius transformations and discontinuous group actions.
- Construction of meromorphic functions and differentials on a Riemann surface. The sheaf of meromorphic functions and differentials. Poincaré series. Spaces of analytic functions and differentials.
- One or more of the following topics depending on the time available:
  - Jörgensen's inequality and punctures on Riemann surfaces.
  - Kleinian groups.
  - Compact Riemann surfaces. The Riemann–Roch Theorem.

References

1. A.F. Beardon, *A Primer on Riemann Surfaces*, LMS lecture notes. (This gives a good review of the background material which I will need and a proof of the Riemann Mapping Theorem.)
2. E. Reyssat, *Quelques Aspects des Surfaces de Riemann*, Birkhäuser. (This gives a brief overview of the course and would be good preliminary reading.)
3. O. Forster, *Lectures on Riemann Surfaces*, Springer-Verlag GTM. (This is a good reference for all of the results in the course.)
4. A.F. Beardon, *The Geometry of Discrete Groups*, Springer-Verlag GTM.
5. Farkas and Kra, *Riemann Surfaces*, Springer-Verlag GTM.
6. R.C. Gunning, *Lectures on Riemann Surfaces*, Princeton. (For compact surfaces only.)
7. Griffiths and Harris, *Principles of Algebraic Geometry*, Wiley. (Chapter 2 for compact surfaces.)

Geometric Function Theory (M16)

Non-Examinable (Graduate Level)

A.F. Beardon

This course will contain lectures on the following related topics each of which is connected with the geometric aspects of complex analysis. The time spent on the various topics will be adjusted to suit the interests of the audience.

The topics are:
spaces of holomorphic functions with the compact-open topology,
the conformal automorphism group of a domain,
Prime ends and the boundary correspondence under conformal maps,
the angular derivative of a holomorphic map,
harmonic measure,
the extremal length of curve families,
logarithmic capacity and transfinite diameter.

Texts

(none are suitable for the whole course):

- Ahlfors, L.V., *Conformal invariants, topics in geometric function theory*, McGraw-Hill, 1973.
- Carathéodory, C., *Theory of functions*, Volume II, Chelsea, 1960.
- Cowen, C. and MacCluer, B., *Composition operators on spaces of analytic functions*, CRC Press, 1994.
- Fuchs, W.H.J., *Topics in the theory of functions of one complex variable*, Van Nostrand, 1967.
- Ohtsuka, M., *Dirichlet problem, extremal length and prime ends*, Van Nostrand, 1970.
- Shapiro, J.H., *Composition Operators and classical function theory*, Springer–Verlag, 1993.
- Tsuji, M, *Potential theory in Modern Function Theory*, Maruzen, Tokyo, 1959.
- Remmert, R., *Classical Topics in Complex Function Theory*, Springer, 1998.
- Väisälä, J., *Lectures on n-Dimensional Quasiconformal Mappings*, Springer–Verlag, 1971.
- Vuorinen, M., *Conformal Geometry and Quasiregular Mappings*, Springer–Verlag, 1988.

Combinatorics

Combinatorics (M16)

A. Thomason

The term "combinatorics" is used here in its traditional specific sense of combinatorial set theory, involving the study of families $\mathcal{F}$ of subsets of some finite set. What can be said about $\mathcal{F}$ if its members satisfy some fundamental property? The flavour of the course is given by the outline below of some of the properties to be discussed.

Sperner families. A Sperner family is one containing no pair A, B with $A \subset B$. The LYM inequality bounds the size of such a family. *Shadows.* How should a uniform (that is, all members the same size) family $\mathcal{F}$ be chosen so that $|\{B \subset A : A \in \mathcal{F}\}|$ is as small as possible? The Kruskal-Katona theorem shows the way. *Intersecting systems.* The Erdős-Ko-Rado theorem gives the maximum size of a uniform family that contains no two disjoint members. It is more difficult to determine the effect of requiring that $|A \cap B| \geq t > 1$ for all $A, B \in \mathcal{F}$ but the Ahlswede-Khachatrian theorem gives a full answer. *Exact intersections.* This refers to properties of the kind $|A \cap B| \equiv t \pmod{m}$. Theorems of Fisher, Frankl-Wilson and Grolmusz reveal surprising features and have striking applications. *Shattering.* The family $\mathcal{F}$ shatters a subset Z of its groundset if the projection of $\mathcal{F}$ onto Z is the full power set $\mathcal{P}(Z)$. Shattering families are investigated, beginning with the Sauer-Shelah lemma.

Desirable Previous Knowledge

This is a first course in combinatorics and as such is self-contained, though a familiarity with combinatorial arguments such as are found in the Part II Graph Theory course will be helpful.

Reading to complement course material

1. Bollobás, B. *Combinatorics*, CUP (1986).

Combinatorial Probability (L16)

B. Bollobás

Probabilistic methods are nowadays fundamental to the study of combinatorics, for two reasons. Firstly, probabilistic ideas may often be applied to answer deterministic questions (usually existence questions). Secondly, probabilistic combinatorial objects, such as random graphs, are now a major field of study in their own right, with many fascinating phenomena.

The syllabus is as follows:

Correlation Inequalities. Harris's lemma, the van den Berg–Kesten lemma, the Lovasz local lemma and Shearer's theorem, and their connection to Dobrushin's theorem.

Threshold Functions. The theorem of Kahn, Kalai and Linial about the influence of random variables, and the Friedgut–Kalai theorem about sharp thresholds.

Isoperimetric Inequalities. The basic inequalities on the cube: compressions and extremal sets. Martingale inequalities.

There will be numerous applications of the results to random graphs, percolation, and cellular automata.

These topics are hardly covered in books, so the references will be to the original papers. However, the lectures will be supplemented by a set of fairly detailed printed notes.

Ramsey Theory (M16)

I.B. Leader

Ramsey theory is concerned with the general question of whether, in a large amount of disorder, one can find regions of order. A typical example is van der Waerden's theorem, which states that whenever we partition the natural numbers into finitely many classes there is a class that contains arbitrarily long arithmetic progressions.

The flavour of the course is combinatorial. Ramsey theory is remarkably attractive: we study questions that are very natural and easy to appreciate, but whose answers rely on a great variety of beautiful methods. We shall cover a number of 'classical' Ramsey theorems, such as Gallai's theorem and the Hales-Jewett theorem, as well as some more recent developments, such as Deuber's solution to Rado's partition regularity conjecture. There will also be several indications of open problems.

We hope to cover the following material.

Monochromatic Systems

Ramsey's theorem (finite and infinite). Canonical Ramsey theorems. Colourings of the natural numbers; focusing and van der Waerden's theorem. Combinatorial lines and the Hales-Jewett theorem. Applications, including Gallai's theorem.

Partition Regular Equations

Definitions and examples. The columns property; Rado's theorem. Applications. (m, p, c)-sets and Deuber's theorem. Ultrafilters; the Stone-Čech compactification. Idempotent ultrafilters and Hindman's theorem.

Infinite Ramsey Theory

Basic definitions. Not all sets are Ramsey. Open sets and the Galvin-Prikry lemma. Borel sets are Ramsey. Applications.

Prerequisites

There are almost no prerequisites – the course will start with a review of Ramsey's theorem, so even prior knowledge of this is not essential. At various places we shall make use of some very basic concepts from topology, such as metric spaces and compactness.

Appropriate books

1. B. Bollobás, *Combinatorics*, C.U.P. 1986
2. R. Graham, B. Rothschild and J. Spencer, *Ramsey Theory*, John Wiley 1990

Geometry and Topology

Differential Geometry (M24)

G. P. Paternain

This course aims to provide an introduction to Differential Geometry.

Contents

We will try to cover the following topics (not necessarily in this order):

1. *Differentiable Manifolds.* Definition and examples. Tangent vectors, tangent and cotangent bundles. Smooth maps and the inverse function theorem. Differential forms, Stokes' theorem and de Rham cohomology.
2. *Vector bundles.* Structure group, principal bundles. Connections and curvature.
3. *Riemannian geometry.* Riemannian metrics, Levi-Civita connection. Geodesics, exponential map and Gauss' lemma. The Riemann curvature tensor, sectional curvature, Ricci curvature and scalar curvature. The Hodge star operator and the Laplace-Beltrami operator.

Desirable Previous Knowledge

Familiarity with the classical theory of curves and surfaces will be useful.

References

1. I. Chavel, *Riemannian geometry—a modern introduction,* Cambridge Tracts in Mathematics, 108. Cambridge University Press, 1993.
2. S. Gallot, D. Hulin,J. Lafontaine, *Riemannian geometry,* Universitext, Springer-Verlag, Berlin, 2004.
3. V. Guillemin, A. Pollack, *Differential topology,* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1974.
4. J. Jost, *Riemannian geometry and geometric analysis,* Universitext, Springer-Verlag, Berlin, 2005
5. F.W. Warner, *Foundations of differentiable manifolds and Lie groups,* Graduate Texts in Mathematics, 94. Springer-Verlag, New York-Berlin, 1983.

Algebraic Topology (M24)

B.J. Totaro

Algebraic topology begins with vague ideas such as the 'number of holes' in a space. The more precise versions of the 'number of holes' in a topological space are called homology and cohomology groups. These ideas have turned out to be rich in applications to all areas of mathematics: above all in differential geometry and algebraic geometry, but also in number theory, analysis, physics, and more. This course should be accessible to a broad Part III audience.

Topics: Quick summary of results on the fundamental group and covering spaces. Singular homology groups, computation methods, axioms. Cohomology, the cohomology ring, applications. Manifolds, Poincaré duality, intersections of submanifolds. Basic homological algebra, Ext and Tor. If there is time, cohomology of groups.

Pre-requisite Mathematics

Basic point-set topology is an essential prerequisite (topological spaces, compactness, connectedness, quotient spaces). Some concise references for that material are W. A. Sutherland's "Introduction to metric and topological spaces" or M. A. Armstrong's "Basic topology".

We will begin with a quick summary of results on the fundamental group and covering spaces (from Part II Algebraic Topology). Students who have not seen the fundamental group before should read about it in advance, either in Armstrong or in Hatcher (Chapter 1).

Literature

1. Bott, R. and Tu, L. *Differential forms in algebraic topology.* Springer (1995), £42.50.
2. Hatcher, A. *Algebraic topology.* Cambridge (2002), £20.99.

Toric Geometry (M24)

P.M.H. Wilson

Toric Geometry was introduced some 40 years ago, and has over the intervening period found its way into many areas of mathematics, including algebraic geometry, singularity theory, commutative algebra, algebraic groups, combinatorics, convex body theory, symplectic geometry and superstring theory. Toric varieties are algebraic varieties which can be specified by certain systems (called *fans*) of rational convex polyhedral cones in $\mathbf{R}^n$, with the property that any two cones intersect along a common face. The geometric and topological properties of the toric variety are then reflected in the geometry of these cones. In the projective case, there are other descriptions of toric varieties, which in turn tie in with the concept of a moment map from symplectic geometry. The applications of the theory to combinatorics and convex bodies arise because of the above dictionary, since theorems from algebraic geometry or topology (for instance the Riemann–Roch theorem or the Hard Lefshetz theorem) can yield very non-trivial statements concerning the geometry of cones and polytopes in $\mathbf{R}^n$. The translation is, however, mainly used the other way round in areas such as algebraic geometry or singularity theory, where properties (global or local) of the varieties are proved via combinatorics and geometry of the corresponding fans of cones. Because of the very explicit descriptions involved, toric geometry is a very good way to get one's hands on some interesting and non-trivial examples of varieties.

Pre-requisite Mathematics

There are no formal pre-requisites for this course, but candidates will benefit greatly by having looked beforehand at Miles Reid's book on *Undergraduate Algebraic Geometry* (CUP 1988), in particular §§3-6.

Literature

1. V.I. Danilov. *The Geometry of Toric Varieties.* Russian Math. Surveys **33** (1978), 97-154.
2. W. Fulton. *Introduction to Toric Varieties.* Princeton University Press 1993.

Geometry of Surfaces (L24)

I. Smith

The moduli space of (real, genus g) surfaces is a central object in many parts of geometry; the mapping class group, which is roughly its fundamental group, is also ubiquitous. This course will develop some of the basics of Teichmuller theory, to construct the moduli space, describe some aspects of the mapping class group (generators and relations, links to three- and four-dimensional topology, its cohomology), and then study the surprisingly rich dynamics of surface homeomorphisms. This is probably over-ambitious, so some parts of the theory will be sketched when necessary.

Necessary background

Part III differential geometry & algebraic topology

Relevant books:

1. Fathi, Laudenbach, Poenaru *Les travaux de Thurston sur les surfaces*, Asterisque 66-67, 1979.
2. Imayoshi, Taniguchi *Introduction to Teichmuller Spaces*, Springer, 1992.

Cobordism (L24)

Konstantin Feldman

The idea of a *cobordism* first appeared in works of Poincare at the beginning of the last century. It motivated the rapid development of a new science — *Algebraic Topology* — which is famous for solving many longstanding problems in mathematics. Later Algebraic Topology helped Cobordism to develop into a self-sufficient subject with many beautiful applications.

We say that two manifolds of the same dimension are *(co)bordant* if there is a manifold whose boundary is diffeomorphic to the disjoint union of the two given manifolds. For example, the two-dimensional torus is bordant to zero (i.e., to the empty set), while the real projective plane is not the boundary of any compact three-manifold. To get more sophisticated information, cobordism theory associates to a manifold its *cobordism ring*.

We will use cobordism theory to prove some of the most striking results in topology: the *Poincare–Hopf Theorem*, the *Frobenius Conjecture on Division Algebras* and the *Hirzebruch Signature Theorem*.

Topics: Topological Spaces and CW Complexes. Smooth Manifolds. Sard's Lemma. Thom Transversality Theorem. Vector Bundles. Cobordism Rings. Poincare Duality. Fixed Points of Group Actions. Homotopy Interpretation of Cobordism Rings. Generalised Cohomology Theories. Pontryagin–Thom Map. Characteristic Classes.

Pre-requisite Mathematics

The course is self-contained. Its advanced level requires familiarity with the material of Part III courses on Algebraic Topology and Differential Geometry.

Literature

1. J. W. Milnor, *Topology From the Differentiable Viewpoint*, Princeton (1997).
2. J. W. Milnor and J. D. Stasheff, *Characteristic Classes*, Princeton (1974).
3. A. T. Fomenko and D. B. Fuks, *A course in homotopic topology*, Nauka (1989).
4. P. E. Conner and E. E. Floyd, *Differentiable Periodic Maps*, Springer (1964).
5. R. E. Stong, *Notes on Cobordism Theory*, Princeton (1968).

Spectral Geometry (L24)

D. Barden

The aim of this course is to give an overview of the work that has blossomed in response to Mark Kac' naive sounding question, posed in 1966: 'Can one hear the shape of a drum?' In other, more general, words can one determine the geometry of a Riemannian manifold from the spectrum, the set of eigenvalues together with their multiplicities, of the Laplacian operator. The answer is

unsurprisingly, no: many pairs, and even continuous families of manifolds, have since been constructed that are isospectral (have the same spectrum) yet are not isometric. BUT

surprisingly, almost yes: these examples are very special, usually highly symmetric, so that it is still possible that generically (a word that may be defined to suit the context) manifolds are spectrally determined. In fact this has already been shown to be the case in certain contexts.

Contents

Chap 1: Definitions and basic results.

Chap 2: Computation of Spectra: flat tori and round spheres. Spectral determination of 2-dimensional flat tori. Examples of isospectral but non-isometric pairs of 4-tori and of planar domains with Dirichlet boundary conditions.

Chap 3: The Heat kernel and some spectrally determined geometric properties.

Chap 4: Sunada's Theorem. The trace formula and the residuality of 'bumpy' metrics.

Chap 5: Riemannian surfaces. The use of Sunada's techniques to construct isospectral non-isometric pairs of Riemann surfaces, with particular attention to those of low genus.

Chap 6: Wolpert's Theorem that generic Riemann surfaces are spectrally determined.

Chap 7: More general examples of isospectral non-isometric manifolds.

Pre-requisite mathematics

Basic Riemannian geometry (as in the Part III Michaelmas course), analysis (including functional analysis) and topology (Part II level) as well as group theory (Part I level). The subject is very much inter-disciplinary. However results that are needed from analysis and algebra will be stated mostly without proof, so the level of knowledge required will be that which is sufficient to understand and apply the statements of the theorems rather than knowing or understanding their proofs.

Introductory reading

Apart from brushing up on the pre-requisites, there is nothing that is truly apposite: the best may be the LMS Student Text no. 31 by S. Rosenberg entitled 'The Laplacian on a Riemannian Manifold'. The book 'Eigenvalues in Riemannian Geometry' by Isaac Chavel is introductory in the sense of showing the historic basis (c.1980) from which the more recent results have flowed. However, being published in 1984, it does not cover those developments and does contain a lot more detailed analysis than it is intended to include in the course. The 'Survey of Isospectral Manifolds' by Carolyn Gordon published in Vol. I of the Handbook of Differential Geometry (published by Elsevier Science in 2000) gives an excellent overview of the subject and the place within it of the topics chosen for the course.

Supporting materials

Fairly complete lecture notes and three example sheets from Lent 2004 will be made available. For further reading the Gordon survey contains an extensive list of references and a further one will be handed out during the course. The course itself should provide some guidance to these.

Birational Geometry (L24)

C. Birkar

Smooth complex algebraic curves, or equivalently compact Riemann surfaces, can be classified according to their genus which can be $0, 1$ or at least 2. These cases correspond to the curvature of the surface being positive, zero or negative. In dimension at least 2, a smooth complex variety (a submanifold of projective space) is usually not of special type (with curvature of a fixed sign). We try to prove that each variety is "close" (that is, birational; two varieties are birational if they have open isomorphic submanifolds) to another variety of special type which we call a "minimal model". This is carried out via an algorithm which is called the minimal model programme. Roughly speaking, it works as follows in the case of algebraic surfaces. Take a smooth surface V. If it is not a minimal model, then we can "contract" a certain curve inside V and get a new V. If the new V is not a minimal model, then we contract another curve and so on. After finitely many steps, we end up with a minimal model.

In higher dimension (that is, at least 3), new features and difficulties arise. For example we need to consider singular varieties, new operations are needed to get rid of "bad" contractions (flippings), and we need to prove that the algorithm stops after finitely many steps. We will discuss these features in some detail and provide concrete examples.

Key words: Singularities of pairs, cone and contraction theorems, flips and termination, Fano varieties and Sarkisov program.

Pre-requisite Mathematics

Familiarity with the basic tools of algebraic geometry (varieties, morphisms between them, sheaves, cohomology of sheaves) is required for this course. We will use Hartshorne's book [1], chapters 1 to 3, as a reference; students should be willing to look things up in there, but we will try to explain the ideas as we go along.

Students must attend at least one of Complex Manifolds or Toric Varieties in Michaelmas term. Attending both would be ideal.

Literature

1. R. Hartshorne; *Algebraic geometry.* Graduate Texts in Mathematics, Springer 1977, £44.
2. J. Kollar; S. Mori; *Birational geometry of algebraic varieties.* Cambridge Tracts in Mathematics, 134. Cambridge University Press, Cambridge, 1998, £45.

Complex Manifolds (L24)

A.G. Kovalev

This course relates to both Differential Geometry (a complex manifold is a differentiable manifold endowed with additional structure) and Algebraic Geometry (a non-singular algebraic variety over **C** gives an instance of a complex manifold). In addition, the subject attracts an increasing interest from theoretical physicists, due to recent developments in String Theory. The course will have a particular emphasis on compact Kähler manifolds.

I hope to cover much of the following:

- *Local analysis and complex structures.* Introduction to several complex variables. Definition and examples of complex manifolds. Holomorphic tangent bundle, (p,q)-forms. Almost complex structures and integrability. De Rham and Dolbeault cohomology. Divisors and line bundles. Blowing up.
- *Hermitian differential geometry.* Hermitian metrics, connections, curvature and Chern classes for complex manifolds. Harmonic forms, Hodge theorem, and Serre duality.
- *Kähler geometry.* Kähler manifolds (from several points of view). Hodge and Lefschetz decompositions, Kodaira–Nakano vanishing, Kodaira embedding. Calculation of basic invariants for hypersurfaces of $\mathbf{CP}^n$. Ricci form and Calabi–Yau manifolds.

The lectures will be supplemented by three example classes.

Desirable Previous Knowledge

Smooth manifolds, tangent bundle, and differential forms. Basic knowledge of Riemannian metrics and curvature (including the concepts of Ricci and scalar curvature). I suppose that all of these will be covered in the course *Differential Geometry* offered in Michaelmas Term which is an ideal pre-requisite.

I will assume basic theory of holomorphic functions in one complex variable. Some familiarity with Riemann surfaces will be useful, but not essential.

Literature

1. P. Griffiths and J. Harris. *Principles of algebraic geometry*, Wiley 1978.

2. D. Huybrechts. *Complex geometry. An introduction.* Springer 2005.

3. R.O. Wells. *Differential analysis on complex manifolds.* Springer 1980.

4. F. Zheng. *Complex differential geometry.* AMS 2000.

The recently published [2] is a bit easier than others; you might like to browse in the first chapter in advance (the author even helps you to recall some necessary linear algebra). A possible alternative is [1], chapters 0 and 1 (further chapters may be of interest if you are taking algebraic geometry courses). On the other hand, [3] elaborates on the analysis side of things. The presentation in [4] is sometimes dense, but includes Riemannian geometry; a good reference source. Both [2] and [4] have a generous collection of examples.

Algebraic Curves (L24)

N.I. Shepherd-Barron

Things to be covered include

(1) The algebraic approach.

Affine and projective space, and affine and projective varieties. Dimension, smoothness. Linear systems and maps to projective space.

Sheaves, schemes and coherent cohomology. Duality. Riemann-Roch.

(2) The transcendental approach.

Periods. Riemann's bilinear relations. The Jacobian variety and Abel's theorem. The Riemann-Kempf singularity theorem.

References:

1. Hartshorne, *Algebraic geometry*.

2. Farkas and Kra, *Riemann surfaces*.

3. Mumford, *Lecture on theta*.

Three-Manifolds (L16)

V.R. Easson

This course will examine the currently very active field of three-dimensional manifolds from several perspectives. We will look at 3-manifolds by taking combinatorial, topological and geometric approaches, and also investigate connections with knot theory, number theory, group theory and dynamical systems.

Pre-requisite Mathematics

This course should be accessible to a broad Part III audience, but you will find it easier if you have attended courses on algebraic topology and differentiable manifolds. You should know about fundamental groups, covering spaces and basic point-set topology.

Literature

Recommended reading and fuller details of the syllabus will be posted on my website:
`http://www.dpmms.cam.ac.uk/~vre20`

Category Theory (M24)

P.T. Johnstone

Category theory begins with the observation (Eilenberg–Mac Lane 1942) that the collection of all mathematical structures of a given type, together with all the maps between them, is itself an instance of a nontrivial structure which can be studied in its own right. In keeping with this idea, the real objects of study are not so much categories themselves as the maps between them—functors, natural transformations and (perhaps most important of all) adjunctions. Category theory has had great success in the unification of ideas from different areas of mathematics; it has now become an indispensable tool for anyone doing research in topology, abstract algebra, mathematical logic or theoretical computer science (to name but a few). This course aims to give a general introduction to the basic grammar of category theory, without any (intentional!) bias in the direction of any particular application. It should therefore be of interest to a large proportion of pure Part III students.

Categories, functors and natural transformations Examples drawn from different areas of mathematics. Comma categories. Faithful and full functors, equivalence of categories, skeletons. [4]
Locally small categories Representable functors, the Yoneda lemma. Generating sets. Projective and injective objects. [2]
Adjunctions Description in terms of comma categories, and by triangular identities. Uniqueness of adjoints. Reflections and coreflections. [4]
Limits Construction of limits from products and equalizers. Preservation and creation of limits. The Adjoint Functor Theorems. [4]
Monads The monad induced by an adjunction. The Eilenberg–Moore and Kleisli categories, and their universal properties. Monadic adjunctions; Beck's Theorem. [4]

The remaining seven lectures of the course will be devoted to topics chosen by the lecturer, probably from among the following.

Filtered colimits Finitary functors, finitely-presentable objects. Applications to universal algebra.

Abelian categories Exact sequences, projective resolutions, derived functors. Introduction to homological algebra.

Monoidal categories Coherence theorems, monoidal closed categories, enriched categories. Weighted limits.

Fibrations Indexed categories, internal categories, definability. The indexed adjoint functor theorem.

Regular categories Embedding theorems. Categories of relations, introduction to allegories.

References

1. F. Borceux *Handbook of Categorical Algebra*, Cambridge U.P., 1994. Three volumes which together provide the best modern account of everything you should know about category theory: volume 1 covers most but not all of the Part III course.

2. S. Mac Lane *Categories for the Working Mathematician*, Springer–Verlag, second edition 1998. Still the best one-volume book on the subject, written by one of its founders.

3. P.J. Freyd and A. Scedrov *Categories, Allegories*, North–Holland, 1990. A radical rethink of category theory 'from the ground up'; it has little in common with the approach adopted in the course, but is worth reading for its style alone

4. J. Adámek, H. Herrlich and G.E. Strecker *Abstract and Concrete Categories*, Wiley–Interscience, 1990. (Not really recommended: it's well-written and quite easy to read, but the emphasis is all wrong.)

5. C. McLarty *Elementary Categories, Elementary Toposes* (chapters 1–12 only), Oxford U.P., 1992. A very gently-paced introduction to categorical ideas, written by a philosopher for those with little mathematical background.

6. S. Awodey *Category Theory*, Cambridge University Press, 2006. A new treatment very much in the spirit of Mac Lane's classic, but rather more gently paced.

Set Theory and Logic (L24)

T. Forster

This course is the sequel to the Part II course with the title 'Logic and Set Theory'. Although it will touch most of the themes in the Part II course, and will have something of the character of a course with a title like "A Twenty-four Lecture Graduate Course in Logic" it will concentrate on Set Theory more than other areas. Topics likely to be covered by other logic-related courses in Part III will be eschewed. At least one topic that has recently been dropped from Part II—namely Recursive Function Theory—will be given a "advanced beginner" treatment. Other topics likely to be covered include: model theory background (products, elementary embeddings, categoricity, saturation, Ehrenfeucht-Mostowski theorem); inner models: the consistency of ZF relative to intuitionistic ZF, (relative) consistency and independence of most of the axioms of ZFC. Forcing. Infinitary combinatorics: the Erdös-Rado theorem on uncountable monochromatic sets, leading to measurable cardinals and other large cardinals; Axiom of Determinacy; Set theory in Analysis (eg Souslin Hypothesis); Fränkel-Mostowski models; Alternative set theories (very briefly); Borel determinacy; WQO and BQO theory at least as far as Kruskal's theorem on wellquasiorderings of trees and possibly connections with Reverse Mathematics.

As in previous years I shall be distributing printed material so that students will not need to wear their wrists out taking notes. The material will contain exercises. Last year's notes are still available on the web on `www.dpmms.cam.ac.uk/~tf/partiii2001.dvi`, and other course materials can be obtained from my home page (see below).

I shall also be giving a "Part IV" course in set theory, starting in the second half of the michaelmas term. It will be a course of lectures aimed specifically at my new students just beginning a Ph.D. in Quine's set theory, but will be open to all members of the university.

Pre-requisite Mathematics

The obvious prerequisite is Prof. Johnstone's Part II Logic and Set Theory, and I am going to assume that everybody coming to my lectures is on top of all the material lectured in that course. Students from other institutions who have moved to Cambridge to study Part III need not fear: supervisions will be provided for students who did not attend the course in the first place and for others who—for whatever reason—feel that they need help.

Literature

These two books below both started life as lecture notes for the Part II Logic and Set Theory course (PTJ and I have both lectured this syllabus), and both cover the background, albeit in completely different ways—complementing each other admirably.

P. T. Johnstone: *Notes on Set theory*, CUP

T. E. Forster: *Logic, Induction and Sets*, CUP.
(Errata are on `http://www.dpmms.cam.ac.uk/~tf/typoslis.html`)

Teaching materials will be linked from the page on `http://www.dpmms.cam.ac.uk/~tf/partiii.html`

Drake and Singh, *Intermediate Set Theory* Wiley 1996 is a paperback that may be of use.

T.J. Jech, *Set theory*

K. Kunen, *Set theory*

Dales and Woodin, *An introduction to independence for analysts*, LMS lecture notes **115**. (They mean: forcing independence proofs)

A.J. Dodd, *The Core model*, LMS lecture notes **61**

P. Aczel, *Non-well-founded sets*, Lecture Notes, Number 14, (Center for the Study of Language and Information, 1988). (This is the most widely-read text on the Forti-Honsell antifoundation axiom)

Jon Barwise and Laurence Moss, *Vicious Circles* CLSI. 1996. ISBN.1-57586-009-0 (paper) 202pp. The book is much better than its title. A very good read.

M. Randall Holmes, *Elementary Set Theory with a Universal Set*, volume 10 of the Cahiers du Centre de logique, Academia, Louvain-la-Neuve (Belgium), 241 pages, ISBN 2-87209-488-1.

Forster, T.E. *Set Theory with a Universal Set*, OUP 1994

Sidney Smith's Part III essay: "Hypersets" at `http://ucsu.colorado.edu/~bsid/research.html/`. This exposition of set theory with antifoundation axioms will certainly be at the right level!

Consult the lecturer for a more up-to-date reading list nearer the time.

A Graduate ("Part IV") course in Quine's set theory (M8)

Non-Examinable (Graduate Level)

T. Forster

This will be a course of lectures aimed specifically at my new students just beginning a Ph.D. in Quine's set theory, but will be open to all members of the university. It is likely to be intelligible to anyone who has done part III set theory.

Pre-requisite Mathematics

Part III set theory

Literature

Forster, T.E. *Set Theory with a Universal Set*, OUP 1994

M. Randall Holmes, *Elementary Set Theory with a Universal Set*, volume 10 of the Cahiers du Centre de logique, Academia, Louvain-la-Neuve (Belgium), 241 pages, ISBN 2-87209-488-1.

The first monograph treats the original NF of Quine, and the second monograph treats the amended version of it that allows *urelemente*. This makes more difference than one might expect. However the Holmes volume is well written and it's free!

Drake and Singh, *Intermediate Set Theory*, Wiley 1996 is a paperback that may be of use.

T.J. Jech, *Set theory*

K. Kunen, *Set theory*

Dales and Woodin, *An introduction to independence for analysts*, LMS lecture notes **115**. (They mean: forcing independence proofs)

A.J. Dodd, *The Core model*, LMS lecture notes **61**

P. Aczel, *Non-well-founded sets*, Lecture Notes, Number 14, (Center for the Study of Language and Information, 1988). (This is the most widely-read text on the Forti-Honsell antifoundation axiom)

Jon Barwise and Laurence Moss, *Vicious Circles*, CLSI. 1996. ISBN.1-57586-009-0 (paper) 202pp. The book is much better than its title. A very good read.

Sidney Smith's Part III essay: "Hypersets" at `http://ucsu.colorado.edu/~bsid/research.html/`. This exposition of set theory with antifoundation axioms will certainly be at the right level!

Consult the lecturer for a more up-to-date reading list nearer the time.

Number Theory

Elliptic Curves (L24)

T.A. Fisher

Elliptic curves are the first non-trivial curves, and it is a remarkable fact that they have continuously been at the centre stage of mathematical research for centuries. The course will be aimed at introducing students to the arithmetic aspects of elliptic curves, concentrating on the study of the group of rational points. The following topics will be covered, and possibly others if time is available. There will be examples sheets and examples classes.

Weierstrass equations and the group law. Methods for putting an elliptic curve in Weierstrass form. Definition of the group law in terms of the chord and tangent process. Associativity via identification with the Jacobian. Elliptic curves as group varieties.

Isogenies. Definition and examples. The degree of an isogeny is a quadratic form. The invariant differential and separability. Description of the torsion subgroup over an algebraically closed field.

Elliptic curves over finite fields. Hasse's theorem.

Elliptic curves over local fields. Minimal models. Formal groups. Reduction mod p.

Elliptic curves over number fields. The torsion subgroup. The Lutz-Nagell theorem. The weak Mordell-Weil theorem via Kummer theory. Heights. The Mordell-Weil theorem. Reinterpretation in terms of Galois cohomology. Descent by 2-isogeny. Numerical examples.

Pre-requisite Mathematics

No prior knowledge of subject will be assumed, although it would be useful to have some rudimentary knowledge both of algebraic curves (at the level of Chapters I and II of [2]) and of the field of p-adic numbers.

Books

1. J.W.S. Cassels, *Lectures on Elliptic Curves*, CUP, 1991.
2. J.H. Silverman, *The Arithmetic of Elliptic Curves*, Springer, 1986.
3. J.H. Silverman, J. Tate, *Rational Points on Elliptic Curves*, Springer, 1992.

Local Fields (M16)

T. Dokchitser

The theory of local fields is an interesting blend of algebra and topology, with its main motivation coming from number theory. Topics to be covered in this course are likely to include: absolute values on fields, complete fields, p-adic numbers, ramification theory of local fields, relation to number fields ('passage from local to global'); if time permits, a brief introduction to local class field theory.

Level

Basic

Desirable Previous Knowledge

Basic algebra up to and including Galois theory is essential. Some prior exposure to algebraic number fields is recommended.

Books

1. J.W.S. Cassels, *Local fields*, CUP, 1986.
2. N. Koblitz, *p-adic numbers, p-adic analysis and zeta functions*, Springer 1977.
3. J.-P. Serre, *A course in arithmetic*, Springer, 1973.
4. J.-P. Serre, *Local fields*, Springer, 1979.

Modular Forms (L24)

T. T. Berger

Modular Forms are classical objects that appear in many areas of mathematics, from number theory to representation theory and mathematical physics. Most famous is, of course, the role they played in the proof of Fermat's Last Theorem, through the conjecture of Shimura-Taniyama-Weil that elliptic curves are modular. This course will cover the classical theory of modular forms (modular curves over **C**, Hecke operators, Dirichlet series, theta functions) and some number theoretic applications.

Pre-requisite Mathematics

Prerequisites for the course are fairly modest; some knowledge of quadratic fields and of holomorphic functions in one complex variable (including basic concepts from the theory of Riemann surfaces) will be helpful.

Level

Basic

Books

1. J. P. Serre, *A course in Arithmetic*, Graduate Texts in Maths. **7**, Springer, New York, 1973 (Chapter VII is an easy-going introduction to the subject).
2. D. Bump, *Automorphic forms and representations*, Cambridge Studies in Adv. Maths. **55**, CUP, Cambridge, 1997 (Sections 1.1-1.6 of Chapter I are particularly relevant).
3. F. Diamond, J. Shurman, *A First Course in Modular Forms*, Graduate Texts in Maths. **228**, Springer, New York, 2005 (a good reference providing also an introduction to the algebraic theory of modular forms).
4. J. Milne, *Modular Functions and Modular Forms*, Lecture notes from a course, download available at `http://www.jmilne.org/math`.

Additional references for enthusiasts

5. T. Miyake, *Modular Forms*, Springer, Berlin, 1989 (a standard reference for classical theory of modular forms).

6. F. Diamond, J. Im, *Modular forms and modular curves*, in: *Seminar on Fermat's Last Theorem*, CMS Conf. Proc. 17, Amer. Math. Soc., Providence, RI, 1995, 39-133.

7. J. Coates, Shing-Tung Yau, *Elliptic curves, modular forms & Fermat's last theorem- Conference Proceedings*, International Press, Cambridge, MA, 1997 (in particular, the survey article by H. Darmon, F. Diamond, R. Taylor).

8. H. Hida, *Elementary theory of L-functions and Eisenstein series*, London Math. Soc. Student Texts **26**, CUP, Cambridge, 1993 (not so elementary introduction to arithmetic of modular forms).

Additive Number Theory (L24)

B.J. Green

Overview

The aim of the course is to introduce some of the key concepts of the "analytic" side of number theory from a modern perspective. Analytic number theory has a reputation for being a fearsomely technical subject. However by focussing as far as possible on the basic ideas, and by avoiding the temptation to work in too much generality, I hope to make the course fairly accessible. Topics should include the following, and may include other material if time allows.

- The Riemann ζ-function and the prime number theorem (6 lectures). Basic properties of the ζ-function. Zero-free region of Hadamard and de la Vallée Poussin. The method of contour integration and the prime number theorem. *Sketch proof of the Siegel-Walfisz theorem on primes in arithmetic progression*.
- Sieve methods (3 lectures). The Selberg sieve. Upper bounds for twin primes and the Goldbach problem. The large sieve and applications, e.g. to the least quadratic residue mod p on average.
- The Hardy-Littlewood circle method (8 lectures). The basic idea of the circle method. Weyl's inequality and the minor arcs. Solution of Waring's problem using the method of diminishing ranges: proof that every sufficiently large integer can be written as a sum $x_1^k + \cdots + x_s^k$ of kth powers of non-negative integers, where $s \leq (6 + o(1))k \log k$.
- Sums over primes (5 lectures). Vaughan's identity for the Möbius function, and the method of Type I/II sums. Vinogradov's theorem that every sufficiently large odd number is the sum of three primes.
- *Key ideas of additive combinatorics* (2 lectures). An overview of recent developments concerning progressions of primes and related topics.

Desirable Previous Knowledge

I will assume a working knowledge of complex analysis and an undergraduate course in number theory but little more.

Introductory Reading

Two books by H. Davenport are classic reading in the area, and they come highly recommended. The book *Multiplicative Number Theory*, Springer GTM **74**, covers all of the course except for the material on the Hardy-Littlewood method. That material will be found in the book *Analytic methods for Diophantine Equations and Diophantine Inequalities,* recently republished inexpensively by CUP in their series Cambridge Mathematical Library. The techniques used in the course will not follow these books extremely closely, but they are essential reading for aspiring analytic number theorists.

Root numbers (M24)

Non-Examinable (Graduate Level)

J.H. Coates

Root numbers remain one of the great mysteries of the L-functions of arithmetic, most notably in the way they produce zeroes of these L-functions at the centre of their critical strip. The most celebrated example is their prediction that every positive integer which is congruent to 5, 6, or 7 mod 8 should be the area of a right-angled triangle all of whose sides have rational length. In general, they are relatively easy to compute because of their decomposition as a product of local factors (theorem of Tate-Deligne-Langlands). The course will sketch the classical theory of root numbers, and then go on to discuss some of the growing evidence that they are related, at least in some cases, to invariants arising from Iwasawa theory.

L-functions and motives (Michaelmas)

Non-Examinable (Graduate Level)

A.J. Scholl

Probability

Information and Coding (M16)

Y.M. Suhov

The first part of this course discusses properties of the fundamental quantity of entropy, which measures the amount of randomness (uncertainty) present in a system. Subjects discussed will include data compression, maximum entropy problems and reliable transmission through a noisy channel.

The second part considers error-correcting codes, including properties of dual, cyclic and BCH codes, and the construction of the Golay code (a perfect code of length 23).

Pre-requisite Mathematics

The course uses basic ideas from probability and linear algebra, and extends the Part II Coding and Cryptography course. It is intended to serve as an introduction to classical information theory, to benefit students taking the Quantum Information Theory course.

Literature

1. T.M. Cover and J.A. Thomas, *Elements of Information Theory.* Wiley 1991 (£58.50 hardback)
2. C.M. Goldie and R.G.E. Pinch, *Communication Theory.* Cambridge University Press 1991 (£14.95 paperback).
3. D. Welsh, *Codes and Cryptography.* Clarendon Press 1988 (£24.95 paperback).
4. F.D. MacWilliams and N.J.A. Sloane, *The Theory of Error-Correcting Codes.* North-Holland 1978 (out of print)
5. J.H. van Lint, *Introduction to Coding Theory* (3rd edition). Springer 1999 (£46.00 hardback)

Rough Path Theory and Applications (M16)

P.K. Friz

Rough Path Theory is concerned with differential equations driven by signal of little regularity such as Brownian motion. The approach is purely analytic and allows the construction of pathwise solutions which introduces a type stability unknown to the classical martingale-based theory of stochastic differential equations (SDEs). The highlight of the theory is known as universal limit theorem. It establishes continuity of the Ito-map, that is, the path-space transformation which associates a diffusion path to a driving signal. Many deep results from diffusion theory such as Stroock-Varadhan's support description and Freidlin-Wentzell's large deviation result become direct corollaries of statements on Brownian motion and Levy's stochastic area. Beyond such pure interests, rough path applications range from designing numerical algorithms for SDEs to engineering applications such as sound compression.

The theory is young, exciting and provides excellent grounds for new researchers. Topics to be covered in the course incluce:

- Young integration and iterated integrals
- Universal Limit Theorem
- Brownian Rough Path
- Applications: Stratonovich SDEs, Support Theorem, Large Deviations

Pre-requisite mathematics

Good knowledge of analysis and measure theoretic probability. Basics of ODE theory and group theory. Discrete time martingales and Brownian motion at the level "Advanced Probability".

Suggested reading

(it is hoped to provide detailed notes!)

1. T. Lyons, Z. Qian: *Rough Path Theory and System Control*, (OUP, 2003).
2. T. Lyons, *St. Flour notes from 2004* (forthcoming).
3. A. Lejay, *Introduction to Rough Path Theory*, (2004).
4. P. Friz, N. Victoir, *Control of the Brownian Rough Path*, (2003).

Advanced Probability (M24)

G. Miermont

This course aims to cover some advanced topics at the core of research in probability. There is an emphasis on techniques needed for the rigorous analysis of stochastic processes such as Brownian motion. The course finishes with two key structural results – Donsker's invariance principle and the Lévy-Khinchin theorem.

- *Review.* Review of the basics of measure and integration theory, as covered for example in the Part II(B) course Probability and Measure.
- *Conditional expectation.* Discrete case, conditional density functions; existence and uniqueness; basic properties.
- *Martingales.* Discrete parameter martingales, submartingales and supermartingales; optional stopping; Doob's inequalities, upcrossings, convergence theorems, backwards martingales. Applications: sums of independent random variables, strong law of large numbers, Wald's identity; non-negative martingales and changes of measure, Radon-Nikodym theorem, Kakutani's product martingale theorem, consistency of likelihood-ratio tests; Markov chains; stochastic optimal control.
- *Continuous-time random processes.* Kolmogorov's criterion ; continuous-time martingales, path regularization theorem for martingales.
- *Weak convergence in* $\mathbf{R}^n$. Convergence of distribution functions, convergence with respect to continuous bounded functions, Helly's theorem. Characteristic functions, Lévy's continuity theorem.
- *Brownian motion.* Lévy's construction of Wiener's measure. Scaling and symmetry properties. Martingales associated to Brownian motion. Strong Markov property, hitting times, reflection principle. Sample path properties, recurrence and transience. Brownian motion and the Dirichlet problem, Feynman-Kac formula. Skorokhod embedding, Donsker's invariance principle.
- *Poisson random measures.* Construction and basic properties, integrals with respect to a Poisson random measure.
- *Lévy processes.* Infinitely divisible laws, Lévy-Khinchin theorem, Lévy-Itô construction of Lévy processes.

Desirable previous knowledge

It will be assumed that students have some familiarity with the measure-theoretic formulation of probability – at the level of the Part II(B) course Probability and Measure, or Part A of Williams' book.

Level: General

Appropriate books

1. R. Durrett, *Probability: Theory and Examples.* Wadsworth (1991).
2. O. Kallenberg, *Foundations of Modern Probability.* Springer (1997).
3. D. Williams, *Probability with Martingales.* Cambridge University Press (1991).
4. L.C.G. Rogers and D. Williams, *Diffusions, Markov processes, and Martingales,* Vol. I (2nd edition). Wiley (1994) [Chapters I & II].
5. D.W. Stroock, *Probability Theory - An analytic view.* C.U.P. (1993) [Chapters I-V].

Quantum Information Theory (L24)

N. Datta

This course aims to provide a detailed mathematical treatment of various aspects of Quantum Information Theory.

The course will start with a short introduction to some of the basic concepts and tools of Classical Information Theory, which will prove useful in the study of Quantum Information Theory. Topics in this part of the course will include a brief dicussion of data compression, of transmission of data through noisy channels, Shannon's theorems, entropy and channel capacity.

The quantum part of the course will commence with a study of open systems and of a discussion of how they necessitate a generalization of the basic postulates of Quantum Mechanics. Topics will include quantum states, quantum operations, generalized measurements, POVMs and the Kraus Representation Theorem. Entanglement and some applications elucidating its usefulness as a resource in Quantum Information Theory will be discussed. The concept of decoherence and quantum error correction will be introduced and various examples of quantum error-correcting codes will be discussed in detail. This will be followed by a study of the von Neumann entropy, its properties and its interpretation as the data compression limit of a quantum information source. Schumacher's theorem will be discussed in detail. The definitions of ensemble average fidelity and entanglement fidelity will be introduced in this context. Various examples of quantum channels will be given and the different capacities of a quantum channel will be discussed. The Holevo bound on the accessible information and the Holevo-Schumacher-Westmoreland (HSW) Theorem will also be covered.

1. Summary of classical information theory: entropy, information rate, channel capacity, codes. [4]
2. Summary of basics of quantum theory: states, measurements, dynamics. [3]
3. Quantum Entanglement and its uses. [2]
4. Quantum error-correcting codes. [5]
5. Quantum channels and quantum entropy. [5]
6. Quantum coding theorems. [5]

Four sets of example sheets will be distributed and there will be four example classes to discuss the problems in these sheets.

Pre-requisite mathematics

Basic knowledge of the postulates of Quantum Mechanics will be assumed. However, an additional lecture could be arranged for students who do not have the necessary background. Elementary knowledge of Probability Theory, Vector Spaces and Group Theory will be useful. Students who have attended the Part III course on Coding and Information and/or a previous course in Quantum Mechanics will be at an advantage. It will be helpful for students to attend the Part III course on Introduction to Quantum Information Theory (lectured in the Michaelmas Term). The course on Quantum Information Science (lectured in the Lent Term) is also recommended.

Level: Additional

Literature

1. C.H. Bennett, P.W. Shor. *Quantum information theory.* Information Theory: 1948–1998. IEEE Trans. Inform Theory 44 (1998), no 6, 2724–2742.

2. C.H. Bennett et al. Entanglement-assisted capacity of a quantum channel and the reverse Shannon theorem. quant-ph/0106052, 2001.

3. J. Gruska, *Quantum Computing.* London: McGraw-Hill, 1999.

4. A.Yu. Kitaev. *Quantum computations: algorithms and error corrections.* Russian Math. Surveys 52, no 6 (1997), 1191–1249.

5. M.A. Nielsen, I.L.Chuang, *Quantum Computation and Quantum Information.* Cambridge, CUP, 2000.

6. M. Ohya and D. Petz. *Quantum Entropy and Its Use.* Berlin: Springer-Verlag, 1993.

7. J. Preskill. *Lecture notes on a course on Quantum Computation.* http://www.theory.caltech.edu/people/preskill.

8. P. Shor. The classical capacity achievable by a quantum channel assisted by limited entanglement. quant-ph/0402129, 2004.

9. P. Shor. Capacities of quantum channels and how to find them. quant-ph/0304102, 2003.

Stochastic Networks (M24)

F.P. Kelly

This course uses stochastic models to shed light on important issues in the design and control of communication networks. Randomness arises in communication systems at many levels: for example, the initiation and termination times of calls in a telephone network, or the statistical structure of the arrival streams of packets at routers in the Internet. How can routing, flow control and connection acceptance algorithms be designed to work well in uncertain and random environments?

The first part of the course will describe a variety of classical models that can be used to help understand the performance of large-scale communication networks. Queueing and loss networks will be studied, as well as random access schemes.

The second part of the course will study congestion control algorithms in the Internet. This is an area of some practical importance, with network operators, hardware and software vendors, and regulators actively seeking ways of delivering new services reliably and effectively. The interplay between end-systems and the network has attracted the attention of economists as well as mathematicians and engineers.

Pre-requisite Mathematics

Mathematics that will be assumed to be known before the start of the course: Part IB Optimization and Markov Chains. Familiarity with Part II Applied Probability would be useful, but is not assumed.

Literature

1. Kelly, F. Mathematical modelling of the Internet. In "Mathematics Unlimited - 2001 and Beyond" (Editors B. Engquist and W. Schmid). Springer-Verlag, Berlin, 2001. 685-702.

   http://www.statslab.cam.ac.uk/~frank/mmi.html

Stochastic Calculus and Applications (L24)

S. Großkinsky & J.R. Norris

Stochastic Calculus is an extension of classical calculus for functions of a single variable, which applies in particular to almost all functions arising as a path of Brownian motion, even though such paths are nowhere differentiable. The key result is Itô's formula which is a sort of chain rule. There are two important classes of continuous-time processes in $\mathbb{R}^d$: those which evolve by a discrete series of jumps and those which look locally like Brownian motion. There is a stochastic calculus associated to both classes of process which provides a powerful analytical tool in their study. The course will develop this approach and give examples of its application.

Those attending this course will normally also attend the Part III course Advanced Probability, which covers all the prerequisite material. A prior acquaintance with Brownian motion, continuous-time Markov chains and martingale theory is highly desirable, as given, for example, in Kallenberg's book, Chapters 6, 10, 11.

- *Stochastic calculus for continuous martingales*
  Martingales and local martingales. The Hilbert space $\mathcal{M}^2$ of L^2-bounded martingales. Finite variation processes: total variation, Lebesgue–Stieltjes integral. Any continuous local martingale of finite variation is constant. Adaptedness and previsibility. Stochastic integrals I: $H \in \mathcal{S}$, $M \in \mathcal{M}_2$. Quadratic variation in $\mathcal{M}^c_{\text{loc}}$. Stochastic integrals II: $H \in L^2(M)$, $M \in \mathcal{M}^c_2$, extension by localization, basic properties, approximation by Riemann sums. Covariation in $\mathcal{M}^c_{\text{loc}}$, Kunita–Watanabe identity. Semimartingales, Doob–Meyer decomposition. Itô formula. Stratonovich integrals. Differential calculus. Exponentials.

- *Stochastic calculus for jump processes*
  Stochastic integration with respect to an integer-valued random measure. Poisson random measures, construction of Lévy processes. Pure jump Markov processes in $\mathbf{R}^d$, Lévy kernel, Kurtz' theorem for the fluid limit.

- *Stochastic differential equations*
  Stochastic differential equations driven by Brownian motion. Existence and uniqueness for Lipschitz coefficients. Examples: Brownian exponential, Ornstein–Uhlenbeck process, noisy dynamical system, Bessel processes. Local existence and uniqueness for locally Lipschitz coefficients. Relation with second order elliptic and parabolic partial differential equations: Dirichlet problem and Cauchy problem. Feynman–Kač formula. Diffusion processes: L-diffusions, strong Markov property, construction via stochastic differential equations, identification of finite-dimensional distributions in terms of the heat kernel.

- *Applications*
  Lévy's characterization of Brownian motion; identification of Bessel processes with the radial part of Brownian motion, identification of the Ornstein–Uhlenbeck transition density. Continuous local martingales as time-changes of Brownian motion. Exponential martingale inequality. Girsanov's theorem, Cameron–Martin formula.

Level: Additional

Books

B. Oksendal, *Stochastic Differential Equations: an introduction with applications*, Springer, 1992.

O. Kallenberg, *Foundations of Modern Probability*, Springer (1997). Chapters 15, 16, 18, 21.

D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer, 1991.

L. C. G. Rogers and D. Williams, *Diffusions, Markov Processes and Martingales, Vol 2: Itô calculus*, Wiley, 1987.

D. W. Stroock, *Probability Theory: an analytic view*, C.U.P., 1994.

Spread of Epidemics and Rumours (L16)

M. Draief and L. Massoulié

Inaugurated by D. Bernoulli (1700-1782), the analysis of epidemics and their dissemination have been studied by various mathematicians. The benefits from epidemic modelling are three-fold: understanding mechanisms of spread of epidemics, predicting their future course and developing strategies to control them. Recently, epidemic algorithms (also known as gossip algorithms) have been proposed as means to disseminate information in large scale settings, such as the Internet, or "Peer-to-Peer" networks. Such algorithms operate by letting desired information spread in a distributed system as an epidemic would spread throughout a group of susceptible individuals. Their study has provided a renewed impetus in the study of epidemics.

This course gives an introduction to deterministic and stochastic models of epidemics. Applications to the spread of rumours and dissemination of information are covered. General results relating stochastic models to deterministic models specified by differential equations are covered (Kurtz's theorem), and applied to classical epidemic models. Basic phase transitions arising in the so-called Erdős-Rényi random graph model are treated, and their relation to the behaviour of epidemics in homogeneous populations is explained. Analysis of the small-world phenomenon on spatially structured graphs, augmented by random short-cuts, is provided. Recent aspects of the small-world phenomenon, and in particular, the so-called "navigability" of small-world models introduced by Kleinberg, are treated. Preferential attachment models for graph creation are analysed, and the "power-law" properties of resulting graphs is established. The historical model of Yule describing the sizes of families of species is also treated. Finally, the impact of topological properties of the space on which epidemics propagate on the thresholds for global epidemic outbreaks are covered.

The mathematical methods introduced in this course are the following. Coupling methods, Poisson approximation (the Stein-Chen method), concentration inequalities (Chernoff bounds, and Azuma-Hoeffding inequality), and branching processes.

Pre-requisite Mathematics

Familiarity with stochastic processes and martingales. Some material from the Advanced Probability course will be useful.

Level

Additional

Literature

1. N. Alon, J. Spencer, *The probabilistic method*, 2nd edition, Wiley, 2000.
2. H. Andersson and T. Britton, *Stochastic Epidemic Models and their Statistical Analysis*, Springer, 2000.
3. Athreya and Ney, *Branching processes*, Dover, 2004.
4. B. Bollobás, *Random Graphs*, Cambridge University Press, 2nd edition, 2001.
5. D.J. Daley and J. Gani, *Epidemic Modelling: an introduction*, Cambridge University Press, 2001.
6. R. Durrett, *Random graph dynamics*, Cambridge University Press, 2006.
7. R. Ethier and T. Kurtz, *Markov processes: characterization and convergence*, 1986.
8. S. Janson, T. Luczak, A. Ruciński, *Random Graphs*, Wiley Interscience, 2000.
9. T. Lindvall, *Lectures on the Coupling Method*, Wiley & Sons, New York, 1992.
10. D. Watts, *Small worlds*, Wiley, 1999.

Interacting Particle Systems (L24)

G.R. Grimmett

Imagine a family of particles inhabiting a container. They may interact, propagate, or die, and they do so at random at rates usually dependent on what is happening in their immediate neighbourhoods. This much-studied branch of statistical physics has a rigorous mathematical counterpart as a branch of probability theory, in which particles inhabit the points of a lattice and they evolve as a Markov process. It is an area of considerable contemporary importance.

The course begins with RANDOM WALK. We study the transience/recurrence of random walk on a general graph G, and we generalise Pólya's theorem using methods derived from Kirchhoff's 1847 theory of electrical networks. This leads to a discussion of UNIFORM SPANNING TREES via Wilson's algorithm and LOOP-ERASED RANDOM WALK.

Next comes the 'static' theory of interacting particles, namely PERCOLATION (including directed and first-passage percolation), GIBBS STATES, ISING/POTTS MODELS, and RANDOM-CLUSTER MODELS.

One of the targets is to add dynamics to situations where the usual physical models are of equilibrium type. Fundamental questions concern the interplay between the passage of time and the large/infinite number of particles involved. The basic processes are of 'spin-flip' type, in which the state of each particle evolves as time passes. Such processes may be used to model physical phenomena such as the evolution of ferromagnetism, for example, the STOCHASTIC ISING MODEL. Other specific examples include the CONTACT MODEL (a model for the spread of disease about a population), the VOTER MODEL (which describes the spread of consensus), and the EXCLUSION PROCESS (in which a fixed number of particles jump around the available region of space as a system of interacting random walks).

Use will be made of arguments from probability theory, particularly Markov processes, and the theory of disordered media, particularly percolation. The main concepts and results of these areas will be introduced when needed. There will be regular example sheets and classes.

Desirable Previous Knowledge

Elementary probability including probability spaces and Markov chains. Some material from the Advanced Probability course will be useful.

Introductory Reading

Take the flavour from the paper entitled 'Percolation', being item 21 on
`http://www.statslab.cam.ac.uk/~grg/preprints.html`

Reading to complement course material

1. Russell Lyons, Yuval Peres, *Probability on Trees and Networks*,
   `http://mypage.iu.edu/~rdlyons/prbtree/prbtree.html`
2. Geoffrey Grimmett, *Percolation*, Springer, 1999.
3. Tom Liggett, *Interacting Particle Systems*, Springer, 1985.
4. Tom Liggett, *Stochastic Interacting Systems: Contact, voter and exclusion processes*, Springer, 1999.

Stochastic Loewner Evolutions (L16)

J.R. Norris

Stochastic Loewner Evolution (SLE) was discovered by Oded Schramm in 1999. It provides a family of continuum models, depending on a parameter κ, various intances of which are believed, and in some cases known, to arise as limits for certain planar lattice-based models in statistical physics..

The course will focus on the continuum models alone. The basic properties of SLE will be explored for a number of key choices of κ. SLE is a generalization of Loewner's (non-stochastic) evolution of conformal maps, but now driven by a Brownian motion of diffusivity κ. So the fundamentals of conformal mapping will be needed, though most of this will be developed as required. A basic familiarity with Brownian motion and Itô calculus will be assumed.

The course material will be based on a selection from the accounts of Lawler and Werner listed below.

Literature

1. W. Werner, *Random planar curves and Schramm–Loewner evolutions*, math.PR/0303354, 2003.
2. G. F. Lawler, *Conformally Invariant Processes in the Plane*, AMS, 2005.

Topics in Stochastic Analysis (E8)

Non-Examinable (Graduate Level)

P.K. Friz

Brownian motion lies in the intersection of three important classes of stochastic processes: (i) Gaussian processes, (ii) martingales and (iii) Markov processes. We shall study certain aspects of such processes, in particular sample path regularity and existence of a stochastic area. We can then, depending on the interests of the audience, discuss some of the following topics: stochastic flows driven by the above processes, elements of anticipating stochastic calculus, large deviations, refined support theorems. Rough path theory is used as a tool when appropriate and its interplay with classical tools from probability theory is emphasized. The required background is covered by other Part III courses: Advanced Probability, Rough Paths and Stochastic Calculus.

Literature

1. Terry Lyons and Zhongmin Qian, *System Control and Rough Paths*, Oxford University Press, 2002.
2. Peter Friz and Nicolas Victoir, *Multidimensional Stochastic Processes as Rough Paths, Theory and Applications*, Cambridge University Press (forthcoming).

Operational Research, Statistics, and Mathematical Finance

Mathematics of Operational Research (M24)

R.R. Weber

This course is accessible to a candidate with mathematical maturity who has no previous experience of operational research; however it is expected that most candidates will already have had exposure to some of the topics listed below.

- Convexity, the supporting hyperplane theorem. Lagrangian sufficiency. Strong Lagrangian problems; sufficient conditions for convexity of the optimal value function. [3]
- Linear programming: the simplex algorithm, duality, shadow prices. [3]
- Complexity of algorithms: typical and worst-case behaviour. *NP-completeness.* Exponential complexity of the simplex algorithm. Polynomial time algorithms for linear programming. [3]
- Ford–Fulkerson algorithm; max-flow min-cut theorem. Minimal spanning trees, transportation algorithm, general circulation problems. Shortest and longest paths; critical paths; project cost-time functions. [5]
- Integer programming and tree searching. The branch and bound method. The travelling salesman problem. [3]
- Two-person zero-sum games. Cooperative and non-cooperative games. Nash equilibria. The core, nucleolus, Shapley value. Bargaining. Market games and oligopoly. Evolutionary games. Bidding and auctions. [7]

Level: General.

Books

1. L.C. Thomas, *Games, Theory and Application*, Wiley, Chichester (1984).
2. M.S. Bazaran, J.J. Harvis and H.D. Shara'i, *Linear Programming and Network Flows*, Wiley (1988).
3. D. Bertsimas and J.N. Tsitsiklis, *Introduction to Linear Optimization*, Athena Scientific (1997). Undoubtedly the best book on the market.

See the course web page: http://www.statslab.cam.ac.uk/rrw1/MOR.

Advanced Financial Models (M24)

D.P. Kennedy

This course is an introduction to the modelling of financial derivatives. It complements the material in Advanced Probability, Stochastic Calculus and Applications and Mathematical Models in Financial Management.

- *Discrete-time models.* Survey of minimum-variance approaches to hedging. Complete and incomplete markets; minimal martingale measure. Characterization of lack of arbitrage. Pricing assets in multi-period models. Optimal stopping; Snell envelope; relation to pricing American options. [8]

- *Brownian motion and stochastic calculus.* Introduction to Brownian motion; hitting times; martingales. Girsanov's Theorem and change of measure. Stochastic integrals; Itô's Lemma. Ornstein–Uhlenbeck process. [5]

- *Black–Scholes model.* European call option; Black–Scholes formula. Self-financing portfolios; general partial differential equation for pricing claims. Barrier and other exotic options. Computational issues. [5]

- *Interest-rate models.* One-dimensional models: Vasicek; Cox–Ingersoll–Ross. Whole yield-curve approaches: Heath–Jarrow–Morton; Gaussian random-field models; characterization of martingale measure; structure of covariance. Pricing interest-rate claims. [5]

Level: General

Books

1. M. Baxter and A. Rennie, *Financial Calculus*, Cambridge University Press (1996).

2. D. Duffie, *Dynamic Asset Pricing Theory*, 2nd ed., Princeton University Press (1996).

3. I. Karatzas, *Lectures on the Mathematics of Finance*, CRM Monograph Series, Vol.8, American Mathematical Society (1997).

4. D. Lamberton and B. Lapeyre, *Introduction to Stochastic Calculus Applied to Finance*, Chapman and Hall (1996).

5. M. Musiela and M. Rutkowski, *Martingale Methods in Financial Modelling*, Springer-Verlag (1997).

Optimal Investment (L16)

M. Tehranchi

The course will study a wide range of optimal investment/consumption problems that arise in theory and practice, and will discuss the usefulness of the conclusions. Most examples studied will be in a continuous-time setting. The following provisional list of topics indicates some of the intended content; not all the topics on this list will necessarily be covered, and topics may be covered that are not on this list.

- Self-financing portfolios and the wealth equation;
- the Merton problem and its solution in the CRRA case,using the Hamilton-Jacobi-Bellman approach;
- the Merton problem, general case, by martingale representation;
- the Merton problem, general case, using state-price density approach;
- (Davis-Varaiya) martingale principle of optimal control;
- dual methodology and the Pontryagin principle;
- equilibrium pricing;
- the equity premium puzzle;
- utility-indifference pricing;
- Lagrangian martingale characterisation of optimal solutions;
- imperfections: transaction costs, portfolio constraints, parameter uncertainty, infrequent rebalancing;
- varied objectives: ratcheting of consumption, habit formation, recursive utility;
- backward SDEs and optimal control;
- How good are any of these rules in practice?

Pre-requisites

A firm grasp of martingale theory, and a working knowledge of (Brownian) stochastic calculus will be required in the course.

Literature

1. Bloggs, F. *How to make money in publishing.* OUP.

Applied Statistics (M8+8, E4+4)

S. Pitts and B.D.M. Tom

This course will count as a 3 unit (24 lectures) course. There will be 8 lectures and 8 classes in the Michaelmas term followed by 4 lectures and 4 classes in the Easter term.

* Introduction to Linux, R and S-Plus on the Statistical Laboratory computing network. Use of LaTeX for report writing.

* Exploratory data analysis, graphical summaries. Introduction to non-parametric tests.

  The essentials of Generalized Linear Modelling.

* Linear regression, orthogonal polynomials, standard experimental designs: factorial experiments and interpretation of interactions.

* Regression diagnostics: residuals, leverages and other related plots. Collinearity. Box-Cox transformations.

* Discrete data analysis: binomial and Poisson regression. Multi-way contingency tables.

* Some special topics, e.g.
  (i) use of the Akaike Information Criterion for model-search
  (ii) quasi-likelihood and over-dispersion

* Modern regression, e.g. generalized additive regression.

* Some special topics, e.g. methods for survival data analysis and for time-series, disease progression models (msm), longitudinal data, density estimation, EM algorithm, cost effectiveness.

The above methods will be put into practice via S-Plus or R.

In the practical classes, emphasis is placed on the importance of the clear presentation of the analysis, so students are required to submit their written solutions to the lecturer.

Level: General

Appropriate books

1. W.N. Venables and B.D. Ripley, *Modern Applied Statistics and S-Plus*, 4th ed., Springer-Verlag, New York (2002).

Statistical Theory (M16)

R.J. Samworth

This is a course on parametric statistical theory that goes hand in hand with the Lent term course on nonparametric statistical theory. We begin with a discussion of concepts and principles of statistical inference and use these in the second chapter to develop inferential methods based on the likelihood function. These basic first-order techniques rely on large-sample asymptotic approximations and may be inappropriate in some situations. The third chapter, therefore, is devoted to modern, sophisticated refinements of the procedures which yield more reliable inference for small sample sizes.

Concepts and Principles: Likelihood and related quantities, sufficiency, linear models. Exponential families, transformation models, maximal invariants and equivariance. Discussion of approaches to inference, ancillarity, parameter orthogonality. [6]

First-order theory: Review of basic probability, modes of convergence, Slutsky's theorem, stochastic order notation, moments and cumulants, the delta method. Review of Wald, score, likelihood ratio statistics and signed root versions, distribution theory in no nuisance parameter case. Generalised linear models. Discussion of nuisance parameters, profile likelihood. [5]

Higher-order theory: Asymptotic expansions, Edgeworth expansions, saddlepoint approximations, Laplace's method, the p^*-formula, Bartlett correction, modified profile likelihood, Bayesian asymptotics. [5]

Pre-requisite Mathematics

Basic familiarity with statistical inference, including point estimation and hypothesis testing, will be assumed. Part IID Principles of Statistics is recommended as background. Measure theory is certainly not necessary, but would be a small bonus.

Literature

1. L. Pace and A. Salvan, *Principles of Statistical Inference*, World Scientific (1997).
2. T.A. Severini, *Likelihood Methods in Statistics*, Oxford University Press (2000).
3. A.C. Davison, *Statistical Models*, Cambridge University Press (2003).
4. G.A. Young and R.L. Smith, *Essentials of Statistical Inference*, Cambridge University Press (2005).

Actuarial Statistics (L16)

S.M. Pitts

This course provides an introduction to various topics in non-life insurance mathematics. These topics feature in the Institute and Faculty of Actuaries examinations, CT6 and ST3.

Topics covered in lectures include

1. Loss distributions
2. Reinsurance
3. Aggregate claims
4. Ruin theory
5. Credibility theory
6. No claims discount systems

Prerequisite mathematics

This course assumes

an introductory probability course (including moment generating functions, probability generating functions, conditional expectations and variances)

a statistics course (including maximum likelihood estimation, Bayesian methods)

that you know what a Poisson process is

that you have met discrete time finite statespace Markov chains

Backup to the lectures

Lectures will be supplemented with examples sheets and examples classes.

Literature

1. S. Asmussen *Ruin Probabilities.* World Scientific, 2000.
2. C.D. Daykin, T. Pentikäinen and E. Pesonen, *Practical Risk Theory for Actuaries and Insurers.* Chapman and Hall, 1993.
3. J. Grandell, *Aspects of Risk Theory.* Springer, 1991.
4. R.V. Hogg and S.A. Klugman, *Loss Distributions.* Wiley, 1984.
5. T. Rolski, H. Schmidli, V. Schmidt and J. Teugels, *Stochastic Processes for Insurance and Finance.* Wiley, 1999.

Biostatistics (L16)

This course consists of two components: Survival Data and Statistics in Medical Practice. Together these make up one 2 unit (16 lecture) course. You must take both components together for the examination. Survival Data has 10 lectures and 2 classes; Statistics in Medical Practice has 6 lectures and 1 class.

Survival Data (L10+2)

Part of

P. Treasure

Characteristics of survival data; censoring. Definition and properties of the survival function, hazard and integrated hazard. Examples.

Review of inference using likelihood. Estimation of survival function and hazard both parametrically and non-parametrically.

Explanatory variables: accelerated life and proportional hazards models. Special case of two groups. Model checking using residuals.

Case studies and recent developments.

Level: General

Principal book

1. D.R. Cox & D. Oakes, *Analysis of Survival Data,* London: Chapman & Hall (1984).

Other books

2. P. Armitage, J.N.S. Matthews & G. Berry, *Statistical Methods in Medical Research* (4th ed.), Oxford: Blackwell (2001) [Chapter on Survival Analysis for preliminary reading].
3. M.K.B. Parmar & D. Machin, *Survival Analysis: A Practical Approach* (1995), Chichester: Jon Wiley.

Statistics in Medical Practice (L6+1)

Part of

S. Bird & V. Farewell & D. Spiegelhalter

Each lecture will be a self-contained study of a recent problem in biostatistics. Topics may include AIDS forecasting, clinical trials, clinical databases, diagnostic testing, and institutional comparisons, illustrating the use of techniques such as Bayesian methods, longitudinal modelling, ranking, sequential analysis and shrinkage estimation.

Level: General

Appropriate books

There are no appropriate books but relevant handouts will be provided.

Applied Multivariate Analysis (L16)

S. Brooks

Aims

To introduce students to the main ideas of multivariate statistical analysis, that is, the analysis of sets of data where we have several measurements on each of a number of individuals.

Learning Objectives

By the end of the unit the successful student will:

1. Be fully at home with the notion, theoretical basis and techniques of principal components, cluster analysis, Hotelling's T^2 test, multivariate regression, MANOVA and discriminant analysis;
2. Know what sorts of data can and should be applied to different sorts of data; and
3. Have experience of applying these methods to a wide range of data sets.

Syllabus

Topics covered will include the following

- Multivariate data, graphical presentations.
- The multivariate normal distribution, conditional distributions.
- Classical likelihood inference for the parameters of the multivariate normal distribution: the mean and the covariance matrix.
- Multivariate regression and MANOVA. Graphical representations of dependencies among variables.
- Principal components analysis; pca and the scaling problem.
- Discriminant analysis.
- "Data-analytic" methods: hierarchical cluster analysis: the choice of a dissimilarity metric and the choice of a suitable clustering algorithm. The minimum spanning tree for a group of objects.
- Tree-based methods for partitioning large groups of objects.

Related Courses

Statistical Theory and Applied Statistics.

Literature

1. Krzanowski, W.J. (1990) *Principles of Multivariate Analysis: A User's Perspective.* Oxford University Press.
2. Mardia, K.V., J.T. Kent and J.M. Bibby (1979) *Multivariate Analysis.* Academic Press: London.
3. Ripley, B.D. (1996) *Pattern Recognition and Neural Networks.* Cambridge University Press.
4. Seber, G.A.F. (1984) *Multivariate Observations.* Wiley: New York.
5. Venables, W.N. and B.D. Ripley (1999) *Modern Applied Statistics with S-Plus.* Springer-Verlag: New York.
6. Webb, A. (1999) *Statistical Pattern Recognition.* Arnold.
7. Whittaker, J. (1990) *Graphical Models in Applied Multivariate Statistics.* Wiley: Chichester.

Time Series and Monte Carlo Inference

(L24)

The course consists of two components: Time Series and Monte Carlo Inference. Together these make up one 3 unit (24 lecture) course. You must take the two components together for the examination. Time Series has 8 lectures; Monte Carlo Inference has 16 lectures.

Time Series (L8)

Part of

S. Pitts

Time series analysis refers to problems in which observations are collected at regular time intervals and there are correlations among successive observations. Applications cover virtually all areas of Statistics but some of the most important include economic and financial time series, and many areas of environmental or ecological data.

This course will cover some of the most important methods for dealing with these problems. In the case of time series, these include the basic definitions of autocorrelations, etc. then time-domain fitting including auto-regressive and moving average processes, and spectral methods.

Backup to the lectures

Lectures will be supplemented with an examples sheet and an examples class.

Literature

1. C. Chatfield, *The Analysis of Time Series: Theory and Practice.* Chapman and Hall, 1975. Good general introduction, especially for those completely new to time series.
2. P.J. Brockwell and R.A. Davis, *Time Series: Theory and Methods.* Springer Series in Statistics, 1986.
3. P.J. Brockwell and R.A. Davis, *Introduction to Time Series and Forecasting.* Springer, 1996.
4. P.J. Diggle, *Time Series: A Biostatistical Introduction.* Oxford University Press, 1990.

Monte Carlo Inference (L16)

Part of

R.J. Samworth & R.B. Gramacy

Computational statistical methods (often referred to as "Monte Carlo methods") have had an enormous impact on statistical practice over the past 10-15 years in particular. This course reviews and discusses many of the ideas underlying these methods and illustrates how they may be applied in practice.

Lecture topics include

1. The uses and aims of simulation in statistical inference.
2. Pseudo-random numbers. Generation of random variables; principles, techniques and examples.
3. The bootstrap and jackknife. Bootstrap for estimation. Bootstrap confidence sets and hypothesis tests. The bootstrap and dependent data.
4. Markov chain Monte Carlo. Implementational issues, applications in biology and medicine.

Pre-requisite Mathematics

This course assumes that you have

- Attended an introductory Probability course;
- Attended a Statistics course (including hypothesis tests, estimation, confidence intervals, Bayesian methods);
- Some knowledge of Markov chains;
- a familiarity (or a willingness to learn how to use) the Splus package.

Literature

1 Efron, B. and R.J. Tibshirani, *An introduction to the Bootstrap.* Chapman and Hall: New York, (1993).

2 Gamerman, D., *Markov chain Monte Carlo.* Chapman and Hall: London, (1997).

3 Gentle, J.E., *Random number generation and Monte Carlo methods.* Springer: New York, (1998).

4 Lange, K,. *Numerical analysis for Statisticians.* Springer: New York, (1998).

5 Manly, B.F.J., *Randomization, Bootstrap and Monte Carlo methods in Biology.* Chapman and Hall: London, (1998).

6 Ripley, B.D., *Stochastic Simulation.* Wiley: London, (1987).

7 Robert, C.P. and G. Casella, *Monte Carlo Statistical methods.* Springer: New York, (1999).

Nonparametric Statistical Theory (L16)

R.J. Samworth

This course complements the Michaelmas term course on Statistical Theory that covers parametric models. We no longer assume that our data comes from a distribution belonging to a finite-dimensional class, and consider fundamental problems such as how to estimate a distribution function, a quantile, a density function or a regression function. I will also cover some extreme value theory, including analogues of the Central Limit Theorem for maxima and minima of a sample. There are many beautiful mathematical results in these areas that contribute greatly to our understanding of the statistical procedures, as well as several open problems.

Kernel density estimation: Optimality criteria, asymptotic approximations, asymptotically optimal bandwidth, canonical kernels, higher order kernels, bandwidth selection, multivariate density estimation. [4]

Nonparametric regression and classification: kernel nonparametric regression, asymptotic bias and variance approximations; linear discriminant analysis, nearest-neighbour methods, kernel discriminant analysis. [4]

Empirical processes and the bootstrap: Empirical distribution function, order statistics, sample quantiles and associated asymptotic distribution theory. The bootstrap. [4]

Extreme value theory: Domains of attraction, max-stability, extreme value types, extremal types theorem and related necessary and sufficient conditions. [4]

Pre-requisite Mathematics

Although very little of the material relies on the Michaelmas term Statistical Theory course, it will be very useful background. Measure theory is certainly not necessary, but would be a small bonus.

Literature

1 J. Galambos, *The Asymptotic Theory of Extreme Order Statistics*, Wiley (1978).

2 R.J. Serfling, *Approximation Theorems in Mathematical Statistics*, Wiley (1980).

3 A.W. van der Vaart, *Asymptotic Statistics*, Cambridge University Press (1998).

4 M.P. Wand and M.C. Jones, *Kernel smoothing*, Chapman and Hall (1995).

5 L. Wasserman, *All of Nonparametric Statistics*, Springer (2006).

Particle Physics, Quantum Fields and Strings

The courses on Symmetry and Particle Physics, Quantum Field Theory, Advanced Quantum Field Theory and The Standard Model are intended to provide a linked course covering High Energy Physics. The remaining courses extend these in various directions. Knowledge of Quantum Field Theory is essential for most of the other courses. The Standard Model course assumes knowledge of Symmetry and Particle Physics.

Desirable previous knowledge

Basic quantum mechanics, wave functions, amplitudes and probabilities. Quantisation in terms of commutation relations between coordinates q and corresponding momenta p. Schrödinger and Heisenberg pictures. Dirac bra and ket formalism.

Harmonic oscillator, its solution using creation and annihilation operators.

Angular momentum operators and their commutation relations. Determination of possible states $|jm\rangle$ from the basic algebra. Idea of spin as well as orbital angular momentum. Two body systems. Clebsch-Gordan coefficients for decomposition of products of angular momentum states.

Perturbation theory, degenerate case and to second order. Time dependent perturbations, 'Golden Rule' for decay rates. Cross sections, scattering amplitudes in quantum mechanics, partial wave decomposition.

Lagrangian formulation of dynamics. Normal modes. Familiarity with Lorentz transformations and use of 4-vectors in special relativity, 4-momentum p^μ for a particle and energy-momentum conservation in 4-vector form. Relativistic formulation of electrodynamics using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and Lagrangian density $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$.

Basic knowledge of δ-functions (including in 3 dimensions) and Fourier transforms. Basic properties of groups and the idea of a matrix representation. Permutation group.

The desirable previous knowledge needed to tackle the Particle Physics, Quantum and Statistical Mechanics courses is covered by the following Cambridge undergraduate courses. Students starting Part III from outside might like to peruse the syllabuses on the WWW at

`http://www.maths.cam.ac.uk/undergrad/sched ules/`

Year	*Courses*	
Second	*Essential*:	Quantum Mechanics, Special Relativity, Methods, Complex Methods.
	Helpful:	Electromagnetism, Principles of Dynamics.
Third	*Essential*:	Foundations of Quantum Mechanics, Applications of Quantum Mechanics.
	Very helpful:	Statistical Physics, Electrodynamics, Methods of Mathematical Physics.

If you have not taken the courses equivalent to those denoted 'essential', then you should review the relevant material over the vacation.

Quantum Field Theory (M24)

D Tong

Quantum Field Theory is the language in which all of modern physics is formulated. It represents the marriage of quantum mechanics with special relativity and provides the mathematical framework in which to describe the interactions of elementary particles.

This first Quantum Field Theory course introduces the basic types of fields which play an important role in high energy physics: scalar, spinor (Dirac), and vector (gauge) fields. The relativistic invariance and symmetry properties of these fields are discussed using the Lagrangian language and Noether's theorem.

The quantisation of the basic non-interacting free fields is developed in terms of operators which create and annihilate particles and anti-particles and the associated Fock space of quantum physical states is explained.

Interactions are introduced using perturbative techniques and the role of Feynman diagrams is explained. This is first illustrated for theories with a purely scalar field interaction, and then for a Yukawa coupling between scalar fields and fermions. Finally Quantum Electrodynamics, the theory of interacting photons, electrons and positrons, is introduced and elementary scattering processed are computed.

Necessary Previous Knowledge:

Lagrangian approach to Classical Dynamics Advanced Quantum Mechanics.

Introductory Reading:

L. Alvarez-Gaume and M. Vazquez-Mozo, "Introductory Lectures on Quantum Field Theory", hep-th/0510040. available from the hep-th archive [http://www.arxiv.org/abs/hep-th/0510040].

Books:

L.H. Ryder, Quantum Field Theory, Cambridge University Press (1996).

M.E. Peskin and D.V. Schroeder, An Introduction to Quantum Field Theory, Addison-Wesley (1996).

S. Weinberg, The Quantum Theory of Fields Vol I, Cambridge University Press (1995)

A. Zee, Quantum Field Theory in a Nutshell, Princeton University Press, (2003)

Symmetry and Particle Physics (M24)

J. B. Gutowski

The course provides introductory material on properties of Lie groups and Lie algebras with particular emphasis on the groups $SU(2)$, $SU(3)$ and their applications to particle physics.

The observed apparently elementary particles are reviewed and divided into hadrons and leptons. Hadrons are characterized by having strong interactions and are now regarded as composite particles made up of quarks. Hadrons and leptons also undergo electromagnetic and weak interactions, which are responsible for many decays. Various quantum numbers which differentiate particles and their interactions are described.

Basic properties of Lie groups and Lie algebras are introduced. The geometric structures underlying the theory of Lie groups and algebras are presented, and basic properties of representations are explained. Results for angular momentum which correspond to the group $SU(2)$ are reviewed, and properties of $SU(3)$ are examined in detail. Hadrons are classified in terms of representations of symmetry groups such as isospin and $SU(3)$, which motivates considering their construction in terms of quarks and gives rise to observed states.

The role of Lie groups in understanding spacetime symmetry is examined. The theory of Clifford algebras and spinors is developed. Properties of the Lorentz group and Poincaré group are examined.

The application of Lie groups to abelian and non-abelian gauge symmetry is reviewed. Local symmetry requires gauge fields, and local gauge theories are now the basis for particle physics. Weak and electromagnetic interactions are described by the well established Weinberg-Salam model, and strong interactions are described by QCD, which has gauge group $SU(3)$.

At the end of the course, the systematic classification of simple Lie algebras is reviewed. Ideas of roots and weights and their properties are explained, and their important role in the classification of representations is presented.

Reading to complement course material

1. D. H. Perkins, *Introduction to High energy Physics,* 4th ed., CUP (2000).

2. B. R. Martin and G. Shaw, *Particle Physics,* 2nd ed., Wiley (1998).

3. M. Nakahara, *Geometry, Topology and Physics,* 2nd ed., IOP Publishing (2003).

4. H. Georgi, *Lie Algebras in Particle Physics,* Perseus Books (1999).

5. J. Fuchs and C. Schweigert, *Symmetries, Lie Algebras and Representations,* 2nd ed., CUP (2003).

6. H. F. Jones, *Groups, Representations and Physics,* 2nd ed., IOP Publishing (1998).

7. I. Buchbinder and S. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity or a Walk through Superspace,* IOP Publishing (revised edition 1998).

8. T-P. Cheng and L-F. Li, *Gauge Theory of Elementary Particle Physics,* Oxford (1984).

TitleAdvanced Quantum Field Theory (L24) H. Osborn

Quantum field theory (QFT) is the crucial theoretical framework for describing elementary particles and their interactions (excluding gravity) and also it is essential in the understanding of string theory and is used in many other areas of physics. The standard model, which describes the basic interactions of particle physics is a quantum field theory involving gauge fields. Gauge field theories have local gauge symmetries corresponding to a gauge group, which is in general a non-abelian Lie Group. Quantising gauge field theories requires additional theoretical tools beyond those developed in the introductory quantum field theory course.

A variety of new concepts and methods are first introduced in the simpler context of scalar field theory. The functional integral approach provides a formal non-perturbative definition of any QFT. It reproduces the usual Feynman rules but also allows a natural treatment of gauge field theories. The course also discusses more systematically the treatment of the divergences which arise in perturbative calculations. The need for regularisation in QFT is explained, and the utility of dimensional regularisation in particular for calculations is emphasised. It is shown how renormalisation introduces an arbitrary mass scale and renormalisation group equations which reflect this arbitrariness are obtained. Various physical implications are then discussed.

The rest of the course is concerned specifically with gauge theories. The peculiar difficulties of quantising gauge fields are considered, before showing how these can be overcome using the functional approach in conjunction with ghost fields and BRST symmetry. A renormalisation group analysis reveals that the coupling constant of a quantum gauge theory can become effectively small at high energies. This is the phenomenon of asymptotic freedom, which is crucial for the understanding of QCD. It is then possible to make perturbative calculations which may be compared with experiment. Further properties of gauge theories are then discussed, including the possibility that a classical symmetry may be broken by quantum effects, and how these can be calculated. Such anomalies have important implications for the way in which gauge particles and fermions interact (as in the standard model).

Books

L.H. Ryder, *Quantum Field Theory*, 2nd ed., Cambridge University Press (1996).

M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley (1996).

S. Weinberg, *Quantum Theory of Fields, Vols. 1 & 2*, Cambridge University Press (1996).

Standard Model (L24)

B. Allanach

The Standard Model refers to the generally accepted quantum field theory which currently provides our most fundamental understanding of all observed elementary particle physics. It is the quantum theory of the non-abelian gauge (or Yang-Mills) field theory for gauge group $SU(3) \times SU(2) \times U(1)$. The course aims to demonstrate how this model is realised in nature and to point out some of its limitations as a fundamental theory. It is intended to complement the more general Advanced QFT course.

This course begins by defining the Standard Model in terms of its local (gauge) and global symmetries and its elementary particle content. The parity P, charge conjugation C and time-reversal T transformation properties of the theory are investigated. Some of the symmetries are not manifest in nature, and we must first develop the idea of spontaneous symmetry breaking. The Higgs mechanism provides a concrete realisation, providing masses for W and Z bosons as well as the fermions.

Features of weak interactions in spontaneously broken $SU(2) \times U(1)$ gauge theory (the Weinberg-Salam model) are discussed. It is shown how CP violation becomes possible when there are three generations of particles. The incorporation of massive neutrinos into the model is briefly discussed. Various scattering and decay processes can be calculated in the electroweak sector using perturbation theory because of the smallness of the couplings, and a number of examples are considered. Neutrino masses, an important window into physics beyond the Standard Model, are discussed.

The strong interactions are based upon the gauge theory with (unbroken) gauge group $SU(3)$, called quantum chromodynamics (QCD). As a non-abelian theory, its coupling constant has the feature that it decreases at higher energies. This result can be exploited to calculate high-energy processes using perturbation theory. As an example we consider electron-positron annihilation to final state hadrons at high energies. We also examine inelastic scattering of leptons on hadrons; to a zeroth order-approximation, this process can be described in terms of free quarks in the hadron, providing strong evidence for their existence.

The course ends with a brief discussion of grand unified theories, charge quantization and the cancellation of the triangle anomaly in the Standard Model.

Examples sheets and examples classes will complement the course.

Desirable Previous Knowledge

It is necessary to have attended the Quantum Field Theory and the Symmetries in Particle Physics courses, or be familiar with the material covered in them. It is advantageous to attend the Advanced Quantum Field Theory course at the same time as attending this course, or read on renormalisation and non-abelian gauge fixing.

Reading to complement course material

1. M.E. Peskin and D.V. Schroeder, An Introduction to Quantum Field Theory, Addison-Wesley (1996).
2. T-P. Cheng and L-F. Li, Gauge Theory of Elementary Particle Physics, Oxford University Press (1984)
3. J.F. Donoghue, E. Golowich and B.R. Holstein, Dynamics of the Standard Model, Cambridge University Press (1994)
4. I.J.R. Aitchison and A.J.G. Hey, Gauge Theories in Particle Physics, IoP Publishing (1989).

String Theory (L24)

M.B. Green

String Theory currently provides the most promising framework for constructing a unified theory of all the known fundamental forces of physics. The basic idea in String Theory is that the fundamental objects are one-dimensional line segments ("open strings") or closed loops ("closed strings"), rather than point particles, as in conventional quantum field theory.

Quantum theories of interacting relativistic extended strings must satisfy consistency constraints that are much stronger than for conventional quantum field theories. For example, string theories necessarily contain gravitational interactions, thus offering the prospect of achieving the elusive goal of a consistent quantum theory of gravity. Another consistency requirement is that the dimension of space-time should be either 10 or 26 (depending on the details of the theory considered). The dimensions in excess of the four familiar from everyday experience are usually supposed to be curled up tight to form a compact space. The momenta associated with these extra dimensions provide conserved quantum numbers, like electric charge, which is manifested as an abelian internal symmetry. But if the radii of the extra dimensions take on special values, related to the characteristic size of the string, larger non-abelian internal symmetries appear. It is possible that all the symmetries and forces observed in Nature arise in such ways.

String theory is a large and evolving subject and this course will only provide an introduction, covering: the quantization of a relativistic open and closed string; physical states and the Virasoro algebra; vertex operators and scattering amplitudes; toroidal compactification of 'extra' dimensions; an overview of the superstring. Those attending this course will find that they need a knowledge of the material of the Quantum Field Theory course and, as it progresses, the Advanced Quantum Field Theory course.

Header Books:

B. Zweibach, *First Course in String Theory* (C.U.P. 2004).

M. Green, J.H. Schwarz and E. Witten *Superstring Theory: Volume 1 Introduction* (C.U.P., 1987);

M. Green, J.H. Schwarz and E. Witten *Superstring Theory: Volume 2 Loop Amplitudes, Anomalies & Phenomenology* (C.U.P., 1987);

J. Polchinski *String Theory: Volume I An Introduction to the Bosonic String* (C.U.P., 1998);

J. Polchinski *String Theory: Volume II Superstring Theory and Beyond* (C.U.P., 1998).

Header Sources on the Internet:

E. D'Hoker *Lectures on String Theory*, given at the Institute for Advanced Study, Princeton (1997)

H. Ooguri and Z. Yin *TASI Lectures on Perturbative String Theories* (1996) `hep- th/9612254` available from the `hep-th` archive [`http://xxx.soton.ac.uk/archive/hep- th`].

R. Dijkgraaf *String Theory*, Lectures given at the University of Amsterdam (2000) [`http://www.science.uva.nl/research/itf/education.html`].

Supersymmetry and Extra Dimensions (L24)

F. Quevedo

Supersymmetry (SUSY) is a spacetime symmetry that relates bosons (force carriers) and fermions (constituents of matter). It is an essential ingredient in most attempts to extend and improve the present standard model of particle physics and seems to be crucial in the attempts to reconcile quantum mechanics and gravitation. The possible existence of extra dimesions also enhances the spacetime symmetries of the standard model in a direction different but complementary with supersymmetry. Both are basic ingredients of string theory. This course will provide an introduction to the main ideas and techniques in supersymmetric and extra dimensional theories. Including a brief presentation of the minimal supersymmetric standard model (MSSM), Kaluza-Klein theories and the brane-world scenario and their possible physical implications.

Desirable Previous Knowledge

The course will assume knowledge of the Quantum Field Theory, General Relativity and Symmetry and Particle Physics courses given in Michaelmans Term and it is designed to be followed in conjuntion with the Advanced Quantum Field Theory course in Lent Term. It also fits well with the courses on the Standard Model and String Theory.

Introductory Reading

1. D. Bailin and A. Love, *Supersymmetric Gauge Field Theory and String Theory.* IOP Publishing, Bristol (1994).

2. I.L. Buchbinder and S.M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity or a walk Through Superspace.* IOP Publishing, Bristol and Philadelphia (1995) (revised edition 1998).

Reading to complement course material

1. S. Weinberg, *Quantum Theory of Fields* Vol III, Cambridge University Press (2000).
2. A. Bilal, *Introduction to Supersymmetry*, hep-th/0101055.
3. M. J. Duff, B. E. Nilsson and C. N. Pope, *Kaluza-Klein Supergravity,* Phys. Rept. **130** (1986) 1.
4. D. Bailin and A. Love, *Kaluza-Klein Theories,* Rept. Prog. Phys. **50** (1987) 1087.
5. V. A. Rubakov, *Large and infinite extra dimensions: An introduction,* Phys. Usp. **44** (2001) 871 [Usp. Fiz. Nauk **171** (2001) 913], hep-ph/0104152.

Solitons and Instantons (E16)

M. Dunajski

In many quantum field theories, including gauge theories, it is necessary to consider solutions of the nonlinear field equations that are topologically distinct from the vacuum. Stable, localized particle-like solutions of this kind are called topological solitons. Solutions that are also localised in Euclidean time are called instantons. In certain cases, the second order field equations can be reduced to first order Bogomolny equations, which makes it easier to analyse the soliton solutions. The resulting equations are often integrable, and explicit solutions can be found using Lax pairs and basic twistor methods.

The examples of scalar kinks, magnetic monopoles in Yang-Mills-Higgs theory and anti-self-dual instantons in pure Yang-Mills theory will be discussed. Some basic ideas from topology and global complex analysis will be introduced as needed.

Prerequisites: Some knowledge of differential geometry and classical gauge theory would be an advantage.

Recommended Books:

L.J. Mason and N.M.J. Woodhouse (1996), *Integrability, Self-Duality and Twistor Theory*, OUP. Chapters 3, 4, 10.

R.S. Ward amd R. Wells (1990), *Twistor Geometry and Field Theory*, CUP, Chapters 5 and 8.

R. Rajaraman (1982), *Solitons and instantons*, An introduction to solitons and instantons in quantum field theory. North-Holland Publishing Co., Chapters 2, 3, 4.

N.S. Manton and P.M. Sutcliffe (2004), *Topological Solitons*, CUP, Chapters 3,4,5,6,8,10.

Supergravity (E16)

A C Davis

Supergravity (E16) Supergravity extends the supersymmetry course in the Lent term but the inclusion of gravity. This means that supersymmetry becomes a local symmetry and could provide a low energy description of string theory. The course will give a basic introduction to supergravity including the main ideas and techniques. Simple models will be discussed and recent developments reviewed.

The course will assume knowledge of Quantum Field Theory, General Relativity, Advanced Quantum Field Theory and Supersymmetry.

Recommended Books

R. Bailin and A. Love, Supersymmetric Gauge Field Theory and String Theory

J. Wess and J. Bagger Supersymmetry and Supergravity

P.C. West, An Introduction to supersymmetry and supergravity.

Quantum Information

Desirable previous knowledge

Introduction to Quantum Computation (M16)

A.K. Ekert

This course is an introduction to current developments in quantum computation and quantum information theory. Topics include: quantum logic gates, quantum networks, quantum entanglement, decoherence, selected quantum algorithms, elements of computational complexity, quantum error correction, fault tolerant quantum computation, and physical implementations of quantum computation.

Desirable Previous Knowledge

The course is largely self-contained but elementary knowledge of quantum mechanics, at the level of The Feynman Lectures on Physics vol. III, is assumed. Familiarity with basic concepts of information theory and computational complexity is useful, however, the relevant material is introduced so you don't need to worry if you haven't seen it before. Some knowledge of rudimentary group theory while not essential, would also be helpful.

Introductory Reading

1. Deutsch, D. and Ekert, A. Quantum computation. Physics World , Vol. 11 No.3, pp.47-52 (March 1998).
2. Deutsch, D. The Fabric of Reality. Allen Lane, The Penguin Press.
3. Milburn, G. Schrödinger's Machines. W.H. Freeman & Company.
4. Brown, J. Minds, machines, and the multiverse. Simon & Schuster.
5. The Centre for Quantum Computation (http://www.qubit.org)has several WWW pages and links devoted to quantum computation and cryptography.

Reading to complement course material

1. Peres, A. Quantum Theory: Concepts and Methods. Kluwer.
2. Nielsen, M. and Chuang, I. Quantum Computation and Quantum Information. CUP.
3. Feynman, R.P. Feynman Lectures on Computation. Addison-Wesley.
4. Bouwmeester, D. et al. The physics of quantum information. Springer.
5. Papadimitriou, C.H. Computational Complexity. Addison-Wesley.
6. Welsh, D. Codes and Cryptography. Clarendon Press.

Quantum Information, Entanglement and Nonlocality (L16)

Adrian Kent

This course develops from scratch the theories of quantum information, entanglement and quantum nonlocality and explores their interrelations. These topics lie at the heart of modern theoretical physics, and anyone focussing on either theoretical physics or quantum information science options in Part 3 should seriously consider taking the course.

The course is taught in a style somewhere between that of a traditional lecture and a graduate seminar, with questions encouraged and some opportunities for discussion. Some of the course material is covered by going through some recent research papers, which are distributed in advance. Those taking the course in previous years have generally found this course style works very well.

Topics to be covered, insofar as time permits, are:

Pure and mixed quantum states. Quantum entanglement: definition of entanglement for pure and mixed states. Measures of entanglement, the concentration, purification and manipulation of entanglement. The relation between quantum theory and special relativity. Quantum nonlocality: the Bell's theorem, the CHSH inequalities, the Braunstein-Caves inequalities, experimental tests of nonlocality. Efficient simulation of quantum correlations by classical communication and shared randomness. Quantum information: no-go theorems for quantum information processing, elementary applications of quantum information.

Desirable Previous Knowledge

The course assumes familiarity with the basic principles of quantum mechanics, but is otherwise self-contained.

Introductory Reading

No textbook or review article adequately covers all the course material at the appropriate level. Printed lecture notes are distributed along with a small number of research papers. Some relevant introductory material is in chapters one and (particularly) two of

1. Nielsen, M. and Chuang, I. Quantum Computation and Quantum Information. Cambridge University Press.

Reading to complement course material

Research papers likely to be discussed during the course are:

1. A. Kent, Entangled Mixed States and Local Purification, Phys.Rev.Lett. 81 (1998) 2839-2841, quant-ph/9805088.
2. D. Rohrlich and S. Popescu, Nonlocality as an axiom for quantum theory, quant-ph/9508009.
3. B. F. Toner and D. Bacon, The Communication Cost of Simulating Bell Correlations, Phys. Rev. Lett. 91, 187904 (2003), quant-ph/0304076.

(The references quant-ph/9805088 etc mean that the papers can be downloaded from the quant-ph archive at www.arxiv.org.)

Control of Quantum Systems (L16)

Sonia Schirmer

The course will introduce basic concepts in control theory such as controllability and control design in the context of quantum systems. It will be application-driven and consider the unique challenges of controlling quantum systems – such as the difficulty of implementing feedback due to the nature of quantum measurements, the importance of coherent interaction between the system and the controller, the problem of decoherence, etc.

The applications considered will range from some traditional ones in quantum chemistry such as selective excitation of molecular states to the challenge of controlling solid-state devices, e.g. to implement quantum state preparation and quantum operations necessary for quantum computing.

Tentative list of topics

1. Objectives of Control
   - optimization of observables in quantum chemistry
   - quantum state engineering (quantum chemistry, quantum computing, etc.)
   - engineering of quantum processes in quantum computing, etc.
   - decoherence control
   - engineering of DFS or IFS subsystems
2. Prerequisites for Control – Brief overview of System Identification Techniques
   - Tomography
   - Hamiltonian Identification, etc.
3. Control Strategies
   - open-loop control strategies
   - closed-loop feedback control
   - learning and model-based learning (AI approaches)
4. Controllability of Quantum Systems
   - pure-state controllability
   - mixed-state controllability
   - simultaneous controllability
   - dissipative systems, etc.
5. Control Field Design
   - Geometric control approaches (NMR and beyond)
   - Optimal control field design
   - Robustness issues
6. Applications
   - Control of gated semiconductor quantum dot architectures
   - Optically controlled quantum dots and issues of optimal field design

There will be example sheets and at least two discussion sessions led by the lecturer. Projects/essay topics will also be available.

Desirable Previous Knowledge

Quantum Mechanics, Linear Algebra, Calculus. Basic knowledge of Lie algebras and Lie groups.

Introductory Reading

No introductory reading is required but to get a general flavour of the applications (if not the mathematics involved), you may wish to read the following article:

1. H. Rabitz, R. de Vivie-Riedle, M. Motzkus, K. Kompa, Whither the future of controlling quantum phenomena?, Science **288**, p. 824 (2000)

Reading to complement course material

Detailed lecture notes and further reading material will be provided for all topics covered in the course. Recommended background reading:

1. Optical Control of Molecular Dynamics by S. A. Rice and M. Zhao (Wiley 2000), ISBN 0-471-35423-6

2. The theory of open quantum systems by H. P. Breuer and F. Petruccione (OUP 2002), ISBN 0-19-852063-8

Relativity and Gravitation

Desirable previous knowledge

Suffix notation, vector and tensor analysis. Variational principle and Lagrangian formulation of dynamics. Familiarity with Lorentz transformations and use of 4-vectors in special relativity, 4-momentum p^μ for a particle and energy-momentum conservation in 4-vector form. Relativistic formulation of electrodynamics using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and Lagrangian density $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$.

Knowledge of basic mathematical methods, including Fourier transforms, normal modes, and δ-function (including 3-dimensions). Basic quantum mechanics, wave functions, amplitudes and probabilities. Familiarity with aspects of statistical physics and thermodynamics, including notions of thermal equilibrium, entropy, black body radiation, and Fermi-Dirac, Bose-Einstein and Boltzmann distributions.

The desirable previous knowledge needed to tackle the Relativity and Gravitation courses is covered by the following Cambridge undergraduate courses. Students starting Part III from outside might like to peruse the syllabuses on the WWW at

`http://www.maths.cam.ac.uk/undergrad/schedules/`

Year	*Courses*	
Second	*Essential*:	Methods, Special Relativity, Principles of Dynamics, Quantum Mechanics.
	Helpful:	Electromagnetism, Geometry, Complex Methods.
Third	*Very helpful*:	General Relativity, Statistical Physics, Electrodynamics, Methods of Mathematical Physics.

If you have not taken the courses equivalent to those denoted 'essential', then you should review the relevant material over the vacation. For those who have not previously attended a course on General Relativity, some familiarity with the content of Chapters 5-10 of R. D'Inverno, *Introducing Einstein's Relativity* (OUP, 1992) would be very helpful (but not essential) for the Part III GR course.

General Relativity (M24)

J.M. Stewart

Contents

Local differential geometry - Differentiable manifolds, tangent vectors, tangent spaces, tensor algebra, tensor fields and commutators, maps of manifolds, intergral curves and Lie derivatives, linear connections, geodesics, torsion and curvature, the pseudo-Riemannian metric.

Gravitation is geometry - Newtonian spacetimes, special relativity, the general theory of relativity.

Linearized theory - perturbations, gauge invariance, gravitational waves.

Variational principles - Integration on manifolds, variational principles for gravity.

Some more advanced topics - to be announced.

Previous Knowledge Required

It is not necessary to have studied general relativity or differential geometry in order to take this course, because the material will be developed from first principles. Familiarity with Newtonian gravity and special relativity is essential. Some knowledge of other areas of mathematical physics, eg, the continuity and Euler equations from fluid dynamics, is desirable.

On the mathematical side, considerable familiarity with finite-dimensional vector spaces, the calculus of functions $f : R^m \rightarrow R^n$ and the chain rule will be needed.

Books

One of the problems is that of notation. Books marked with a **D** use a different notation to that of the course. Rindler, d'Inverno, Stephani, Hughston & Tod and Burke are slightly below the level of the course, but are well written and so form valuable introductory material. The first half of the course is based on Stewart. The other books in their different ways cover some of the foundations and applications.

- W. Rindler, *Essential Relativity* (Springer, 1977).
- R. d'Inverno, *Introducing Einstein's Relativity* (OUP, 1992).
- H. Stephani (**D**) *General Relativity* (CUP, 1982).
- L.P. Hughston & K.P. Tod, *An Introductin to General Relativity* (CUP, 1990).
- W.L. Burke (**D**) *Spacetime, Geometry, Cosmology* (University Science Books, 1980).
- J.M. Stewart, *Advanced General Relativity* (CUP, 1991).
- L. Landau & E. Lifshitz, *Classical Theory of Fields* (Pergamon, 1971).
- C.W. Misner, K.S. Thorne & J.W. Wheeler (**D**), *Gravitation* (Freeman, 1973).
- S. Weinberg (**D**), *Gravitation and Cosmology* (Wiley, 1972).
- R.M. Wald (**D**) *General Relativity* (Chicago UP, 1984).

Cosmology (M24)

N.G. Turok

Over the past decade, cosmology has made substantial progress. The quality of data has improved dramatically, allowing accurate measurements of the basic parameters of the universe - the expansion rate, the densities of various components and the primordial density variation amplitude. The basic Big Bang picture continues to hold good, explaining Hubble's expansion law, the formation of galaxies and large-scale structure, the cosmic background radiation, the origin of the light elements and much besides.

This course will review the Big Bang model and its major quantitative successes. It will also discuss the challenges now faced at the cutting edge. Students should be familiar with special relativity and basic statistical physics. If students have not already attended a general relativity course they should plan to follow the one given in Part III: it would be also be useful to have looked at an elementary text at the level of Schutz's book (see below) in advance of the course.

A. The Expanding Universe

Homogeneity and isotropy: the Friedmann-Robertson-Walker metric and Friedmann's equation. Kinematics of particles in an expanding universe. Solutions of Friedmann's equation for different cosmologies. Gravitational instability. The horizon and flatness puzzles.

B. Thermal History of the Universe

Review of relevant statistical physics including equilibrium particle distributions. The cosmic background radiation. Decoupling and 'freeze-out' of particle species. Creation of light elements in the early universe; predicted and measured abundances.

C. Topics in Modern Cosmology

Outstanding puzzles: dark matter, dark energy and the origin of large scale structure. Cosmic inflation and the quantum origin of density variations. Predictions for the cosmic background anisotropy (simple treatment). Alternative proposals and future tests.

Course texts

Kolb, E.W. & M.S. Turner, *The Early Universe*, Addison-Wesley (1990).

Peebles, P.J.E., *Principles of Physical Cosmology*, Princeton (1993).

Useful references

Padmanabhan, T. *Structure Formation in the Universe*, Cambridge (1993).

Peacock, A., et. al (eds.), *The Physics of the Early Universes*, Scottish Universities (1989).

Peebles, P.J.E., *The Large Scale Structure of the Universe*, Princeton (1980).

Schutz, B.F., *A First Course in General Relativity*, Cambridge (1985).

Weinberg, S.W., *Gravitation and Cosmology*, Wiley (1972) (chapters 14-16).

Black Holes (L24)

M.J. Perry

This course aims to explore the fundamental physics of black holes. Black holes are regions of spacetime that are removed from causal contact with the rest of the universe. These results in much fascinating physics which explores what spacetime is, and confronts some fundamental issues in quantum theory. Topics that will be covered include:

- Gravitational Collapse. Why black holes necessarily form under certain circumstances
- The Event Horizon. General properties.
- Exact Black Hole Metrics. An exploration of the Schwarzschild, Reissner-Nordstrom and Kerr metrics.
- The Laws of Black Hole Mechanics. The analogy with laws of thermodynamics.
- The positive energy theorem and the stability of gravitation.
- The Hawking effect. Black hole evaporation and the information paradox.
- The fundamental explanation of black hole entropy from string theory (qualitative only).

Introductory courses on General Relativity and Quantum Field theory are essential.

Reading

Useful texts containing material relevant to the course:

- S.W. Hawking and G.F.R. Ellis, *The Large-Scale Structure of Spacetime*, CUP, 1973.
- R.M. Wald, *General Relativity*, Chicago, 1984.
- I.D. Novikov & V.P. Frolov, *Black Hole Physics*, Kluwer, 1998 (revised edition).
- N. Birrell & P.C.W. Davies, *Quantum Fields in Curved Space*, CUP, 1982.
- S.W. Hawking, *Hawking on the Big Bang and Black Holes*, World Scientific, 1992.
- G.W. Gibbons & S.W. Hawking (Eds.) *Euclidean Quantum Gravity*, World Scientific, 1992.
- B. Carter, 'Black hole equilibrium states', in *Black Holes; Les Houches 1972*, Eds. C. de Witt & B.S. de Witt, Gordon and Breach, 1973.
- A. Strominger, *Les Houches Lectures on Black Holes*, hep-th/9501071.

Advanced Cosmology (L16)

E.P.S. Shellard

This course will take forward at much greater depth the topics in modern cosmology covered at the end of the Michaelmas Term course. The prediction from fundamental theory for the primordial perturbation spectrum remains the key area of confrontation with cosmological observations, both from large-scale structure and the cosmic microwave sky. This course will develop the mathematical tools and physical understanding necessary for research in this very active area. If time permits we will also consider applications for specific models of particular current interest.

Cosmological Perturbation Theory

- The 3+1 formalism and the Einstein equations
- Linearised Einstein equations for an expanding universe
- Quantum fluctuations in inflationary models
- The density perturbation and the synchronous gauge
- Cold dark matter perturbations and transfer functions
- Non-relativistic matter perturbations
- Adiabatic vs. isocurvature fluctuations

Cosmic Microwave Sky

- Relativistic kinetic theory
- Collisionless Boltzmann equation
- Photon scattering and diffusion
- The CMB power spectrum on large angular scales
- Small angular scales and the Doppler peaks

Topical issues

- Nongaussianity from cosmic strings and inflation
- Signatures of brane inflation and extra dimensions
- Future prospects

Course texts

Bardeen, J.M., Cosmological Perturbations From Quantum Fluctuations To Large Scale Structure, DOE/ER/40423-01-C8 Lectures given at 2nd Guo Shou-jing Summer School on Particle Physics and Cosmology, Nanjing, China, Jul 1988. (Available on request).

Dodelson, S., *Modern Cosmology*, Elsevier 2003.

Efstathiou, G., in Physics of the Early Universe, Proc. 36th Scottish Summer School, eds. J. Peacock, et al., p. 361, Adam Hilger (1990).

Liddle, A. & Lyth, D., Cosmological Inflation and Large Scale Structure, Cambridge (2000).

Peebles, P.J.E., Principles of Physical Cosmology, Princeton (1993).

Useful references

Kolb, E.W.& M.S. Turner, The Early Universe, Addison-Wesley (1990).

Linde, A., Particle Physics and Inflationary Cosmology, Harwood (1990).

Ma, C., & Bertschinger, E., *Cosmological Perturbation Theory in Synchronous and Conformal Newtonian Gagues*, *Astrophysical Journal* **455**, 7 (1995) [astro-ph/9506072].

Misner, C.W., Thorne, K.S., and Wheeler, J.A., Gravitation, Freeman, 1973.

Mukhanov, V., Feldman, H., & Brandenberger, R., Phys. Reports 205, 203 (1992).

Padmanabhan, T., Structure Formation in the Universe, Cambridge (1993).

Peacock, J.A., Cosmological Physics, Cambridge (1999).

Peebles, P.J.E., Principles of Physical Cosmology, Princeton (1993).

Weinberg, S., Gravitation and Cosmology Wiley (1973).

Astrophysics

Desirable previous knowledge

Different courses in this area require different background knowledge, but none requires a great deal. You should be comfortable with basic ideas of Newtonian mechanics, hydrodynamics and electromagnetism. Refer to each course description for specifics. Courses are constructed to be mainly self-contained, though it is advantageous to combine them.

Astrophysical Fluid Dynamics (M24)

G. I. Ogilvie

Fluid dynamics describes a very wide range of astrophysical phenomena, such as the formation and internal dynamics of stars and giant planets, the workings of jets and accretion discs around stars and black holes, and the dynamics of the expanding Universe. Effects that can be important in astrophysical fluids include compressibility, self-gravitation and the dynamical influence of the magnetic field that is 'frozen in' to a highly conducting plasma.

The basic models introduced and applied in this course are Newtonian gas dynamics and magnetohydrodynamics (MHD) for an ideal compressible fluid. The aim of the course is to provide familiarity with the basic phenomena and techniques that are of general relevance to astrophysics. Wherever possible the emphasis will be on simple examples, physical interpretation and application of the results in astrophysical contexts.

Provisional synopsis

- Equations of ideal gas dynamics and MHD, including compressibility, thermodynamic relations and self-gravitation. Microphysical basis and validity of a fluid description.
- Physical interpretation of MHD, with examples of basic phenomena.
- Conservation laws, symmetries and hyperbolic structure. Stress tensor and virial theorem.
- Linear waves in homogeneous media.
- Nonlinear waves, shocks and other discontinuities.
- Spherically symmetric steady flows: stellar winds and accretion.
- Axisymmetric rotating magnetized flows: astrophysical jets.
- Waves and instabilities in stratified rotating astrophysical bodies.

This course is suitable for both astrophysicists and fluid dynamicists. An elementary knowledge of fluid dynamics, thermodynamics and electromagnetism will be assumed. Related courses include 'Accretion Discs' (J. E. Pringle), 'Magnetohydrodynamics and Turbulence' (A. A. Schekochihin), 'Stellar and Planetary Magnetic Fields' (M. R. E. Proctor) and 'Structure and Evolution of Stars' (J. C. B. Papaloizou).

Course-related literature

1. Choudhuri, A. R. (1998). *The Physics of Fluids and Plasmas.* Cambridge University Press. (An accessible introduction to many of the topics covered in this course.)
2. Landau, L. D., & Lifshitz, E. M. (1987). *Fluid Mechanics*, 2nd ed. Pergamon Press.
3. Shu, F. H. (1992). *The Physics of Astrophysics*, vol. 2: *Gas Dynamics.* University Science Books.

Structure and Evolution of Stars (M24)

F C B Papaloizou

The structure of a star can be mathematically described by certain differential equations which can be derived from the principles of hydrodynamics, electromagnetic theory, thermodynamics, quantum mechanics, atomic and nuclear physics. Some familiarity with these theories will be assumed.

The basic equations of a spherical star will be derived in detail and the mode of energy transport, the equation of state, and the physics of the opacity sources and the nuclear reactions will be discussed.

Approximate solutions of the equations for stellar structure will be given. Attention will be given to the virial theorem, polytropic gas spheres and homology principles. The procedure for numerical solution of the equations will be mentioned briefly.

The evolution of a star will be discussed with reference to its main-sequence evolution, the exhaustion of various nuclear fuels and the end points of evolution such as white dwarfs, neutron stars and black holes.

Applications of the theory to young stars with protoplanetary discs and interacting binary stars will be considered.

Throughout the course, reference will be made to the observational properties of stars and these will be discussed at appropriate times with particular reference to the Hertzsprung–Russell diagram, the mass-luminosity law and spectroscopic information.

There will be four examples sheets each of which will be discussed during an examples.

Introductory Reading

1. Shu, F. *The Physical Universe*, W.H. Freeman University Science Books. 1991.
2. Phillips, A. *The Physics of Stars*, Wiley, 1999.

Reading to complement course material

1. Prialnik, D. *An Introduction to the Theory of Stellar Structure and Stellar Evolution*, CUP, 2000.
2. J. E. Pringle & R. A. Wade (1985) *"Interacting Binary Stars"*, CUP

Physical Cosmology (L24)

Max Pettini

Recent years have seen remarkable progress in observational cosmology, with major improvements in our knowledge of the nature of the universe and of how the structures within it have evolved. The currently popular model is a flat universe with about 1/3 of the critical density in the form of matter (in turn made up of baryons and cold dark matter in roughly 1:5 proportions), and 2/3 in the form of 'dark energy' (which may be Einstein's cosmological constant). The course will concentrate on the major probes of light and mass in the universe, and on how observations and mathematical modelling together lead to this broad picture.

The main topics which will be covered are:

* Introduction to the observed universe: review of recent determinations of cosmological parameters.

* Standard Friedmann-Robertson-Walker cosmology: Friedmann equations; world models; redshifts and distances in cosmology; particle horizon.

* Distant supernovae: cosmological constant; dark energy; concordance models.

* Baryons at high redshift: quasar absorption lines; the Lyman alpha forest – a window on the intergalactic medium; protogalaxies.

* Big bang nucleosynthesis: confronting theoretical predictions with astrophysical measurements.

* Large scale structure I: growth of density fluctuations; the spectrum of fluctuations; galaxy clustering; redshift distortions and bias; the matter power spectrum.

* Large scale structure II: evolution of density perturbations in linear theory; spherical collapse and virialization.

* Gravitational lensing as a probe of dark matter: strong lensing – constraining dark matter on galaxy and galaxy cluster scales; weak lensing and cosmic shear – cosmological parameter determination; lensing of the cosmic microwave background.

Desirable Previous Knowledge

The course is fully self-contained, but students who have previously attended introductory courses in General Relativity and in Cosmology will have an easier start.

Reading to complement course material

Many excellent books on Cosmology are available. The following complement the course well.

1. Coles, P. & Lucchin, F., *Cosmology - The Origin and Evolution of Cosmic Structure* (second edition), 2002, Wiley.
2. Peacock, J. A., *Cosmological Physics*, 1999, Cambridge University Press.
3. Rauch, M. *The Lyman Alpha Forest in the Spectra of QSOs*, 1998, Annual Review of Astronomy & Astrophysics, 36, 267.
4. Wolfe, A. M., Gawiser, E., & Prochaska, J. X. *Damped Lyman Alpha Systems*, 2005, Annual Review of Astronomy & Astrophysics, 43, 861.
5. *Gravitational Lensing: Strong, Weak, and Micr*o, Proceedings of the 33rd Saas-Fee Advanced Course. Kochanek, C.S., Schneider, P., Wambsganss, J. (Springer-Verlag:Berlin) available from www.astro.uni-bonn.de/∼peter/SaasFee.html

Stellar and Planetary Magnetic Fields (L24)

M.R.E. Proctor

It been known since the dynamo was invented by Siemens that Faraday induction can sustain currents in a circuit (and hence magnetic fields) against the effects of Ohmic resistance. This dynamo process is now almost universally recognised as the mechanism that maintains the large scale magnetic fields in astrophysical bodies such as the Earth, Sun and some planets. In addition it has been more recently recognised that dynamo action is likely to occur in all sufficiently vigorous turbulent flows of conducting fluids. The course will cover both fundamentals of the theory and applications to the solar dynamo, geodynamo, magnetic fields in galaxies etc. Some knowledge of standard fluid dynamics and electromagnetism would be an advantage.

1. **Fundamentals.** Induction equation and Lorentz Force. Definition of kinematic dynamo.

Astrophysical Dynamics (L24)

N W Evans

Astrophysics provides many examples of complex and fascinating dynamical systems. This course covers the mathematical tools to describe the dynamical features of Galaxies and Solar systems. The behaviour of these systems is controlled by Newton's laws of motion and Newton's laws of gravity.

The Solar system is dominated by the Sun and Jupiter. It is dynamically very old, and its great age allows many subtle dynamical mechanisms to manifest themselves. Galaxies are dynamically very young, a typical star like the Sun having orbited only thirty or so times around the Galaxy. The motions of stars in Galaxies are described using classical statistical mechanics, since the number of stars in a Galaxy is so great. The study of large assemblies of stars interacting via long-range forces provides many unusual examples of collective phenomena, such as bars and spiral structure. The interplay between astrophysical dynamics and modern cosmology is also important. For example, much of the evidence for the dark matter problem is dynamical in origin.

A provisional synopsis is:

Violent relaxation, the virial equilibrium of galaxies.

Potential theory of Galactic disks, haloes and bulges.

The orbits of stars in galaxies, spherical axisymmetric and triaxial potentials.

Jeans theorem and steady-state models of galaxies. Non-classical integrals.

The equations of stellar hydrodynamics and their applications.

Theories of spiral structure and stellar bars.

Dark matter in galaxies.

The Solar system, commensurabilities, secular resonances, dissipative trapping, spin-orbit coupling.

An elementary knowledge of Lagrangian and Hamiltonian dynamics will be assumed, as will some familiarity with the contents of a course on Mathematical Methods. No prior knowledge of classical astronomy will be assumed.

Introductory Reading

Elmegreen D.M., "Galaxies and Galactic Structure", Prentice Hall, 1997 Sparke L. & Gallagher J., "Galaxies in the Universe: An Introduction", CUP, 2000 Ryabov Y., "An Elementary Survey of Celestial Mechanics", Dover, 2006

Reading to Complement the Course Material

Binney J.& Merrifield M., "Galactic Astronomy", Princeton, 1998 Binney J., & Tremaine S., "Galactic Dynamics", Princeton, 1987 Heggie D., & Hut P., "The Gravitational Million-Body Problem", CUP, 2003 Murray C.D. & Dermott S., "Solar System Dynamics", CUP, 1999 Pagel B.E.J, "Nucleosynthesis and Chemical Evolution of Galaxies", CUP, 1997 Roy A., "Orbital Motion", Adam Hilger, 1988

Accretion Discs (L16)

J E Pringle

For a particle orbiting in a spherical potential, the orbit with lowest energy for fixed angular momentum is a circular one. When gas is launched into orbit in a spherical potential it tends to lose energy (by colliding with itself, shock heating and radiating) much faster than it loses angular momentum (which it has to do by some mechanical means). The resulting flow is therefore typically one of a thin disc of centrifugally supported material moving on circular orbits.

Examples of such discs can be found:

Around planets, such as Saturn's rings.

In star forming regions, where the material forming a star has typically to shed angular momentum and leaves behind a disc in which planets may form.

In close binary star systems, especially those containing a compact object such as a neutron star or black hole, where the larger star transfers matter with excess angular momentum to the smaller by means of a disc.

In spiral galaxies, where the gas rotates in a thin plane and forms stars in the spiral arms.

In active galactic nuclei and quasars where material is thought to fall towards, and form a disc around, the central massive black hole.

This course deals with the general properties of such discs.

Useful courses:

Astrophysical Fluid Dynamics

Structure and Evolution of Stars

Preliminary reading material:

Pringle, J. E., 1981. Accretion Discs. ARA&A, 19, 137 (available from http://adsabs.harvard.edu/abstract_service.html).

Frank, J., King A., Raine, D., 2002. Accretion Power in Astrophysics (3rd edition). Cambridge University Press.

Applied and Computational Analysis

Desirable previous knowledge

Numerical Solution of Differential Equations (M24)

A. Iserles

Desirable Previous Knowledge

Good preparation for this course assumes relatively little in numerical mathematics per se, except for basic understanding of elementary computational techniques in linear algebra and approximation theory. Prior knowledge of numerical methods for differential equations will not be assumed and is not necessarily an advantage. Experience with programming and application of computational techniques will obviously assist comprehension but is neither assumed nor expected.

Fluency in linear algebra and decent understanding of mathematical analysis are a necessary prerequisite. Thus, linear spaces (inner products, norms, basic theory of function spaces, differential operators), complex analysis (analytic functions, complex integrals, the Cauchy formula), Fourier series, basic facts about dynamical systems and, needless to say, elements from the theory of differential equations.

There are several undergraduate textbooks on numerical analysis. The following describe material at a reasonable level of sophistication. Often they present material well in excess of the requirements for the course in computational differential equations, yet their contents (even the bits that have nothing to do with the course) will help you to acquire valuable background in numerical techniques:

S. Conte & C. de Boor, *Elementary Numerical Analysis,* McGraw–Hill, New York, 1980.

G.H. Golub & C.F. van Loan, *Matrix Computations,* 3rd edition. Johns Hopkins Press 1996.

A. Iserles, *A First Course in the Numerical Analysis of Differential Equations,* Cambridge University Press, Cambridge, 1996.

M.J.D. Powell, *Approximation Theory and Methods,* Cambridge University Press, Cambridge, 1981.

G.D. Smith, *Numerical Solution of Partial Differential Equations: Finite Difference Methods,* 3rd edition **!!!**, Oxford University Press, Oxford, 1985.

G. Strang, *Linear Algebra and Its Applications,* Academic Press, New York, 1976.

Outline

The goal of this lecture course is to present and analyse modern numerical methods for differential equations. The exposition is based on few basic ideas from approximation theory, complex analysis, theory of differential equations and linear algebra, leading in a natural way to a wide range of numerical methods and computational strategies. The emphasis is on algorithms that can be applied to a wide range of application areas and on their mathematical analysis, rather than on specific applications.

The course consists of three parts: methods for *ordinary differential equations* (with an emphasis on initial-value problems and a thorough treatment of stiff equations), numerical schemes for *partial differential equations* (both boundary and initial-boundary value problems, featuring finite differences and finite element methods) and, time allowing, *numerical algebra of sparse systems* (inclusive of fast Poisson solvers, sparse Gaussian elimination and iterative methods). We start from the very basics, analysing approximation of differential operators in a finite-dimensional framework, and proceed to the design of state-of-the-art numerical algorithms. Time allowing, we will consider numerical phenomena that are of interest in fluid and gas dynamics and in nonlinear dynamical systems.

Literature

1. O. Axelsson, *Iterative Solution Methods,* Cambridge University Press, Cambridge, 1996.
2. E. Hairer, S. P. Nørsett and G. Wanner, *Solving Ordinary Differential Equations I: Nonstiff Problems*, Springer–Verlag, Berlin, 2nd ed. 1993.
3. E. Hairer, and G. Wanner *Solving Ordinary Differential Equations II: Stiff and Differential Algebraic Problems*, Springer-Verlag, Berlin, 2nd ed. 1996.
4. A. Iserles, *A First Course in the Numerical Analysis of Differential Equations,* Cambridge University Press, Cambridge, 1996.
5. A. R. Mitchell and R. Wait, *The Finite Element Method in Partial Differential Equations*, Wiley, London, 1977 (out of print).
6. G. Strang and G. Fix, *An Analysis of the Finite Element Method*, Prentice–Hall, Englewood Cliffs, 1973 (out of print).

Computer-aided Geometric Design (M16)

M.A. Sabin

This year the course shows a significant difference from that given the last few years. Instead of aiming to reach and cover the methods currently standard in manufacturing industry and animation for representing the shapes of curves and surfaces, the syllabus will instead focus on methods which are likely to become standard in the next few years.

These do not involve closed-forms but, although smooth, are inherently fractal in their nature.

The prior mathematics needed is only a school-level familiarity with coordinates, real functions and calculus, but links will be made with numerical analysis and algebraic and differential geometry within the course.

Although the course will focus on the mathematics, not the computing exercises, which should be completed to get the most from the course will involve programming, and so some programming experience, together with access to computing facilities is highly desirable. The programming language is not constrained.

The major parts of the course are:-

- Linear Geometry (Points, Lines, Planes and their representations)
- Curves (piecewise parametric representations)
- Surfaces (piecewise parametric representations)
- Curves and surfaces defined by recursive refinement (classification, construction, analysis)

By the end of the course the student should be able to determine the smoothness and approximation order of the limit curve or surface of any given refinement scheme, and also be able to design appropriate representations and algorithms for implementing that refinement for surfaces of general topology.

Most of the 16 slots will take the form of lectures, but at suitable places within the course there will be sessions used to review the exercises.

Desirable previous knowledge

Computational Neuroscience (L16)

Stephen Eglen

This course will show how mathematical and computational techniques can be applied to investigate various problems in neuroscience. In particular, one key focus of the course will be how networks of neurons can learn to produce particular behaviours.

Lecture content: Introduction to the nervous system: how neurons encode and decode information. Hodgkin-Huxley models of action potential propagation. Introduction to network-level models. Associative networks for long-term storage. Supervised learning methods. Reinforcement learning methods. Unsupervised learning methods. Application of techniques to understanding visual system development.

Lab sessions will be held to demonstrate techniques discussed in lectures. Part III students will be expected to complete these computer-based assignments..

Desirable Previous Knowledge

Basic background in dynamical systems is helpful, but the level of mathematics will not be advanced; the focus of the course will be more upon application of computational techniques to neuroscience.

Introductory Reading

1. Computational approaches to brain function. Special supplement to the journal *Nature Neuroscience*, available online at http://www.nature.com/neuro/journal/v3/n11s/index.html

Reading to complement course material

1. P. Dayan and L. F. Abbott. Theoretical Neuroscience. MIT Press.
2. H. R. Wilson. Spikes decisions and actions: dynamical foundations of neuroscience. OUP.

Systems Biology (E16)

S Tavare and J Paulsson

Continuum Mechanics

Desirable previous knowledge

For all the fluid dynamics courses, previous attendance at an introductory course in fluid dynamics will be assumed. In practice familiarity with the continuum assumption and the material derivative will be assumed, as will basic ideas concerning incompressible, inviscid fluids mechanics (e.g. Bernoulli's Theorem, vorticity, potential flow). Some knowledge of basic viscous flow, such as Stokes flow, lubrication theory and elementary boundary-layer theory, is highly desirable.

For solid mechanics courses no previous knowledge of solid mechanics is required, but prior knowledge of some continuum mechanics (e.g. an introductory course in fluid dynamics) will be assumed.

For both fluid dynamics and solid mechanics courses previous attendance at a course on wave theory covering concepts such as wave energy and group velocity, is highly desirable.

In summary, knowledge of Chapters 1-8 of 'Elementary Fluid Dynamics' (D.J. Acheson, Oxford), plus Chapter 3 of 'Waves in Fluids' (J. Lighthill, Cambridge)(which deals with dispersive waves) would give a student an excellent grounding.

Familiarity with basic vector calculus (including Cartesian tensors), differential equations, complex variable techniques (e.g. Fourier Transforms) and techniques for solution of elementary PDEs, such as Laplace's equation, Poisson's equation, the diffusion equation and the simple wave equation, will be assumed. Knowledge of elementary asymptotic techniques would be helpful.

A Cambridge student taking continuum courses in Part III would be expected to have attended the following undergraduate courses.

Year	*Courses*
First	Differential Equations, Dynamics, Calculus & Methods.
Second	Methods, Complex Methods, Fluid Dynamics.
Third	Fluid Dynamics, Waves in Fluid and Solid Media, Methods of Mathematical Physics.

Students starting Part III from outside Cambridge might like to peruse the syllabuses for the above courses, which may be found on WWW with URL:

`http://www.maths.cam.ac.uk/undergrad/schedules/`

Magnetohydrodynamics and Turbulence (M16)

Alexander Schekochihin

It is sometimes said that turbulence is the last great unsolved problem of classical physics. MHD turbulence, or turbulence of a magnetised conducting fluid, exists in many physical systems: liquid-metal experiments, fusion devices, the Earth's interior and virtually all astrophysical plasmas from stars to galaxies and galaxy clusters. Many observed properties of astrophysical bodies (and, in some cases, their very existence) cannot be explained without recourse to some model of turbulence and turbulent transport in the constituent plasma: thus, one could view theory of MHD turbulence as a theory of the fundamental properties of luminous matter that makes up large-scale astrophysical objects. MHD turbulence is an area of very active current research, motivated by rapid and simultaneous progress in astrophysical observations (especially of the solar photosphere, interstellar and intracluster medium) and high-resolution numerical simulations.

The aim of this course is first to provide a basic introduction both to the laws of fluid motion in conducting media (magnetohydrodynamics) and to the fundamental theory of turbulence, and then to bring together these two subjects in presenting the modern state of the MHD turbulence theory. The approximate list of topics to be covered is as follows

- What is turbulence?

- Kolmogorov's 1941 dimensional theory of turbulence.
- Passive advection (mixing) of a scalar field.
- Measures of turbulence: correlation functions and spectra.
- MHD equations.
- Lagrangian MHD, flux freezing, conservation laws.
- MHD waves.
- Dimensional theories of MHD turbulence in the presence of a mean magnetic field (Alfvén-wave phenomenologies).
- Generation of small-scale magnetic fluctuations by turbulence (small-scale dynamo).
- Introduction to turbulence in the solar wind, interstellar medium and glaxy clusters.
- Introduction to kinetic theory of plasma turbulence.

Desirable Previous Knowledge

This course is suitable both for astrophysicists and fluid dynamicists. It will not require any previous knowledge of either astrophysics or MHD. Basic familiarity with the equations of fluid dynamics and of electricity and magnetism will be helpful.

Reading

1. L. D. Landau and E. M. Lifschitz, *Course of Theoretical Physics,* vol. 6: *Fluid Mechanics* (Butterworth-Heinemann, 1995), §§33–34.
2. P. A. Davidson, *Turbulence — An Introduction for Scientists and Engineers* (OUP, 2004).
3. P. A. Davidson, *An Introduction to Magnetohydrodynamics* (CUP, 2001).
4. P. A. Sturrock, *Plasma Physics* (CUP, 1994), §§12–17.
5. R. M. Kulsrud, *Plasma Physics for Astrophysics* (Princeton University Press, 2005).
6. A. A. Schekochihin and S. C. Cowley, *Turbulence and Magnetic Fields in Astrophysical Plasmas*, a chapter in the book *Magnetohydrodynamics: Historical Evolution and Trends*, S. Molokov, R. Moreau, and H. K. Moffatt, Eds. (Springer, 2006) — available on the web from http://www.damtp.cam.ac.uk/user/as6

Nonlinear Continuum Mechanics (M24)

J. R. Willis

The purpose of this course is to provide the basic theory that applies to all continua, whether solid or fluid or somewhere between, and which therefore underpins more specialised applications. Although the course could in principle be taken by someone with no previous exposure to the mechanics of any continuum, it is anticipated that students generally will be familiar with basic notions (such as stress, strain, strain-rate) through prior study of some course concerning the mechanics of a continuous medium, such as a course in elasticity, or viscous flow etc. It is intended that, by the end of the course, students will have the knowledge necessary for the in-depth study of fields including nonlinear elasticity, plasticity and rheology, as well as having a broader perspective from which to view fields such as linear elasticity or classical fluid mechanics.

Syllabus

Kinematics: Eulerian and Lagrangian descriptions. Deformation tensor, polar decomposition theorem, measures of strain, strain-rate tensor. *Balance laws*: Balance of linear momentum, moment of momentum.

Cauchy stress, nominal stress. *Conjugate measures of stress and strain*: Rate of working, work-conjugate stress and strain pairs, brief discussion of stress-rates. *Constitutive relations*: Simple materials. Fluids versus solids. Principle of frame indifference. Elastic solids, viscous fluids. Principle of "fading memory" (brief outline only). *Elementary continuum thermodynamics*: balance of energy, entropy inequality. Implications for constitutive relations. Treatment of constraints; incompressibility. Internal variables, dissipation potential. *Isothermal nonlinear elasticity*: Free energy function; examples including neo-Hookean material, Mooney-Rivlin material. Simple solutions, including inflation of a spherical shell. *Rheological models*: Examples such as the Reiner-Rivlin fluid, Bingham fluid. Solutions involving simple shear, Couette flow. *Theories of plasticity (isothermal treatment)*: Classical plasticity; yield surface, flow potential, normality, hardening; localisation of flow. Elementary solutions employing slip-line theory. Finite-deformation plasticity; multiplicative decomposition of deformation, plastic deformation tensor as an internal variable. Single-crystal plasticity. Theories incorporating rate-dependence.

Preliminary reading

None is essential but potential students may wish to familiarise themselves by looking over the "basic" sections of any book on elasticity or fluid mechanics. The lecture notes of C. Truesdell make stimulating reading.

Recommended books

No book will be followed precisely but the course will run approximately parallel to the book by S.C. Hunter, excluding Chapters 10 to 15, and with more detail on plasticity than in Chapter 16. The basic continuum mechanics is also covered well in the book by P. Chadwick.

Bibliography

General continuum mechanics:
P. Chadwick, *Continuum Mechanics*, Allen and Unwin (1976).
S.C. Hunter, *Mechanics of Continuous Media*, Ellis Horwood (1976).
I-S, Liu, *Continuum Mechanics*, Springer (2002).
C. Truesdell, *The Elements of Continuum Mechanics*, Springer (reprinted 1985).

Nonlinear Elasticity:
R.W. Ogden, *Nonlinear Elastic Deformations*, Dover (1997).
G.A. Holzapfel, *Nonlinear Solid Mechanics*, Wiley (2001).

Rheology:
R.B. Bird, R.C. Armstrong and O. Hassager, *Dynamics of Polymeric Liquids* Vol. 1, Wiley (1987).

Plasticity:
V. Lubarda, *Elastoplasticity Theory*, CRC (2002).

Fundamentals of Atmosphere–Ocean Dynamics (M24)

M. E. McIntyre

Much of our planetary environment is fluid, from the upper atmosphere to the depths of the oceans. It is to fluid motions that we in Europe owe our temperate climate, and our food supplies both from the land and from the ocean. It is to fluid motions that we, and other forms of life, owe a tolerable local environment, despite the continual production of waste substances. It is fluid motions that control the global amount and distribution of stratospheric ozone and of many other greenhouse gases, and the timescale, of the order of a century, on which man-made chlorofluorocarbons are removed from the atmosphere. Climate changes, on timescales of years to centuries or longer, depend on the fluid-dynamical behaviour of the entire atmosphere–ocean system. Much of this behaviour involves the effects of density or entropy stratification under gravitational attraction, and Coriolis accelerations associated with the Earth's rotation.

Achieving a better understanding of our fluid environment is one of the great intellectual challenges facing mankind today. These lectures set the theoretical background to that challenge by presenting a modern view of the basic gravitational and Coriolis effects upon fluid motion. The processes and concepts to be studied are relevant also to understanding the behaviour of many other naturally occurring large bodies of fluid, such as stellar interiors — indeed, the differential rotation of the Sun's interior is now being made sense of, for the first time, in terms of the processes and concepts studied here. These include certain fundamental wave propagation and instability mechanisms, the ability of waves to transport momentum and angular momentum over large distances (overturning the old 'turbulence' paradigms), the nature of 'quasi-geostrophic' or 'balanced' motion, and the properties of key invariants such as the quantity known as potential vorticity.

Desirable previous knowledge:

Nothing beyond the basic fluid dynamics and elementary vector calculus detailed at the start of the Continuum Mechanics section of this booklet.

Recommended books

Gill, A. E., 1982: Atmosphere-Ocean Dynamics. Academic Press, 662pp.

Additional reading

Andrews, D. G., Holton, J. R., and Leovy, C. B., 1987: Middle Atmosphere Dynamics. Academic Press, 489pp.

Hoskins, B. J., McIntyre, M. E., and Robertson, A. W. 1985: On the use and significance of isentropic potential-vorticity maps. Q. J. Roy. Meteorol. Soc., **111**, 877-946. Also **113**, 402-404.

James, I. N., 1994: Introduction to circulating atmospheres. Cambridge, University Press, 422pp.

Pedlosky, J., 1979: Geophysical Fluid Dynamics. Second edition, Springer-Verlag, 624pp.

Slow Viscous Flow (M24)

J.R. Lister

In many flows of natural interest or technological importance, the inertia of the fluid is negligible. This may be due to the small scale of the motion, as in the swimming of micro-organisms and the settling of fine sediments, or due to the high viscosity of the fluid, as in the processing of glass and the convection of the Earth's mantle.

The course will begin by presenting the fundamental principles governing flows of negligible inertia. A number of elegant results and representations of general solutions will be derived for such flows. The motion of rigid particles in a viscous fluid will then be discussed. Many important phenomena arise from the deformation of free boundaries between immiscible liquids under applied or surface-tension forcing. The flows generated by variations in surface tension due to a temperature gradient or contamination by surfactants will be analysed in the context of the translation and deformation of drops and bubbles and in the context of thin films. The small cross-stream lengthscale of thin films renders their inertia negligible and allows them to be analysed by lubrication or extensional-flow approximations. Problems such as the fall of a thread of honey from a spoon and the subsequent spread of the pool of honey will be analysed in this way. Inertia is also negligible in flows through porous media such as the extraction of oil from sandstone reservoirs, movement of groundwater through soil or the migration of melt through a partially molten mush. Some basic flows in porous media will be analysed.

The course aims to examine a broad range of slow viscous flows and the mathematical methods used to analyse them. The course is thus generally suitable for students of fluid mechanics, and provides background for applied research in geological, biological or rheological fluid mechanics.

Desirable Previous Knowledge

As described above in the introduction to courses in Continuum Mechanics. Familiarity with basic vector calculus including Cartesian tensors and the summation convention is particularly useful for the first half ot the course.

Introductory Reading

1. D.J. Acheson. *Elementary Fluid Dynamics.* OUP (1990). Chapter 7

2. G.K. Batchelor. *An Introduction to Fluid Dynamics.* CUP (1970). pp.216–255.

3. L.G. Leal. *Laminar flow and convective transport processes.* Butterworth (1992). Chapters 4 & 5.

Reading to complement course material

1. J. Happel & H. Brenner. *Low Reynolds Number Hydrodynamics.* Kluwer (1965).

2. S. Kim & J. Karrila. *Microhydrodynamics: Principles and Selected Applications.* (1993)

3. C. Pozrikidis. *Boundary Integral and Singularity Methods for Linearized Viscous Flow.* CUP (1992).

4. O.M. Phillips. *Flow and Reactions in Permeable Rocks.* CUP (1991).

Perturbation and Stability Methods (M24)

Dr J.M. Rallison & Dr S.J. Cowley

This first part of this course will deal with the asymptotic solution to problems in applied mathematics in general when some parameter or coordinate in the problem assumes large or small values. Many problems of physical interest are covered by such asymptotic limits. The methods developed have significance, not only in revealing the underlying structure of the solution, but in many cases providing accurate predictions when the parameter or coordinate has only moderately large or small values.

A number of the most useful mathematical tools for research will be covered, and a range of physical applications will be provided. Specifically, the course will start with a brief review of classical asymptotic methods for the evaluation of integrals, but most of the lectures will be devoted to singular perturbation problems (including the methods of multiple scales and matched asymptotic expansions, and so-called 'exponential asymptotics'), for which straightforward asymptotic methods fail in one of a number of characteristic ways.

The second part of the course covers applications of perturbation methods to the study of fluid flows. So-called 'hydrodynamic stability' is a very broad discipline, and in this course we will concentrate on the stability of nearly parallel-flows (as for example arise in boundary-layer flows).

More details of the material are as follows, with approximate numbers of lectures in brackets:

- *Methods for Approximating Integrals.* This section will start with a brief review of asymptotic series. This will be followed by various methods for approximating integrals including the 'divide & conquer' strategy, Laplace's method, stationary phase and steepest descents. This will be followed by a discussion of Stokes lines and an introduction to 'asymptotics beyond all orders' in which exponentially small corrections are extracted from the tails of asymptotic series. [6]

- *Multiple Scales.* This method is generally used to study problems in which small effects accumulate over large times or distances to produce significant changes (the 'WKBJLG' method can be viewed as a special case). It is a systematic method, capable of extension in many ways, and includes such ideas as those of 'averaging' and 'time scale distortion' in a natural way. A number of applications will be studied including ray tracing and turning points (e.g. sound or light propagation in an inhomogeneous medium, including investigation of the rescaling required near 'hot spots', or'caustics'). [5]

- *The Summation of Series.* Cesàro, Euler and Borel sums, Padé approximants, continued fractions, Shanks' transformations, Richardson extrapolation, Domb-Sykes plots. [1]

- *Matched Asymptotic Expansions.* This method is applicable, broadly speaking, to problems in which regions of rapid variation occur, and where there is a drastic change in the structure of the problem when the limiting operation is performed. Boundary-layer theory in fluid mechanics was the subject in which the method was first developed, but it has since been greatly extended and applied to many fields. At the end of this section further examples will be given of asymptotics beyond all orders. [6]

- *Stability Theory.* This section will review both eigenvalue and 'non-eigenvalue' aspects of stability theory as applied to fluid flows, concentrating on nearly-parallel flows. Aspects that will be covered include the concepts of 'causality' and the Briggs-Bers technique, the continuous spectrum, and the transitory algebraic growth that can follow from the fact that the operators in hydrodynamic stability theory are often not self-adjoint. [6]

In addition to the lectures, a series of examples sheets will be provided. The lecturers will run examples classes in parallel to the course.

Desirable Previous Knowledge

Although many of the techniques and ideas originate from fluid mechanics and classical wave theory, no specific knowledge of these fields will be assumed. The only pre-requisites are familiarity with techniques from the theory of complex variables, such as residue calculus and Fourier transforms, and an ability to solve simple differential equations and partial differential equations and evaluate simple integrals.

Introductory Reading

1. E.J. Hinch. *Perturbation Methods*, Cambridge University Press (1991).

2. M.D. Van Dyke. *Perturbation Methods in Fluid Mechanics*, Parabolic Press, Stanford (1975).

Reading to complement course material

1. C.M. Bender and S. Orszag. *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill (1978). *Beware: Bender and Orszag call Stokes lines anti-Stokes lines, and vice versa.*

2. John P. Boyd. *The Devil's invention: asymptotic, superasymptotic and hyperasymptotic series* Acta Applicandae, **56**, 1-98 (1999), and also available at

   `http://www-personal.engin.umich.edu/~jpboyd/boydactaapplicreview.pdf`

3. M.V. Berry. *Waves near Stokes lines*, Proc. R. Soc. Lond. A, **427**, 265–280 (1990).

4. P.G. Drazin and W. H. Reid. *Hydrodynamic Stability*, Cambridge University Press (1981 and 2004).

5. J. Kevorkian and J.D. Cole. *Perturbation Methods in Applied Mathematics*, Springer (1981).

6. Peter Schmid and Dan S. Henningson. *Stability and Transition in Shear Flows*, Springer-Verlag (2001).

Computational Methods in Fluid Dynamics (M16)

Non-Examinable (Part III Level)

Prof. E.J. Hinch

The aim of this course is to provide an overview of some of the computational methods used to solve the partial differential equations that arise in fluid dynamics and related fields. The idea is to provide a feel for the computational methods rather than study them in depth (cf. the complementary aim of the course on the Numerical Analysis of Differential Equations). Although the course is non-examinable, project-type essays will be set on some of the material.

The course will start with a four-lecture introduction to the numerical solution of the Navier-Stokes equations at moderate Reynolds number; the issues and difficulties will be highlighted.

Next some general issues will be covered in greater detail.

- Discretisations: finite difference, finite element and spectral.
- Time-Stepping: explicit, implicit, multi-step, splitting, symplectic.
- Solution of Linear Systems: dense solvers, structured matrices, iterative methods: multigrid, conjugate gradient, GMRES and alternatives, preconditioning, sparse direct methods, eigensolvers, pseudo-spectra.

The remaining lectures will focus on specific issues selected from the following.

- Demonstration of the commercial software FLUENT.
- Implementation issues: code design, testing, data prefetch, cache issues, use of black-box routines, modular code design, language compliance.
- Methods for hyperbolic systems of equations such as the compressible Euler equations.
- Representation of surfaces: splines for curves, diffuse interface method, indicator functions in Volume of Fluid methods, level sets.
- Boundary Integral/Element Method.
- Fast Multipole Method.
- Parameter continuation.
- Lattice-Boltzmann and similar methods.

Pre-requisite Mathematics

Attendance at an introductory course in Numerical Analysis that has covered (at an elementary level) the solution of ordinary differential equations and linear systems will be assumed. Some familiarity with the Navier-Stokes equations and basic fluid phenomena will be helpful (as covered by a first course in Fluid Dynamics).

Literature

1. Boyd, J.P. (2000) *Chebyshev and Fourier Spectral Methods* Dover.
2. Acton, F.S. (1990) *Numerical Methods That Work* Mathematical Association of America.
3. LeVeque, R.J. (1992) *Numerical Methods for Conservation Laws* Birkhauser-Verlag.
4. Saad, Y. (1996) *Iterative Methods for Sparse Linear Systems* PWS.
5. Barrett, R. et al. (1994) *Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods* SIAM.
6. Iserles, A. (1996) *A First Course in the Numerical Analysis of Differential Equations* CUP.

Classical Wave Scattering (L16)

Orsola Rath-Spivack

The application of wave theory to scattering problems by acoustic and electromagnetic waves is of considerable relevance to many applications of practical significance. These include, for example, underwater sound propagation, radar remote sensing, the effect of noise in built-up areas, and a variety of medical diagnostic applications.

This course aims to provide the basic theory of wave scattering and an overview of the mathematical methods and approximations used to tackle these problems.

After a few lectures devoted to the fundamental theory and the governing equations (Helmholtz equation, boundary conditions), solutions for some canonical cases will be derived (e.g. flat infinite surface, wedge). The main approximation methods for the scattering problem will be given (Kirchoff and perturbation methods, parabolic equation, etc.). Scattering by randomly rough surfaces and finite objects will then be described in some detail, and some applications in underwater acoustics and remote sensing will be studied. Finally, a few lectures will be devoted to the inverse scattering problem.

Desirable Previous Knowledge

No previous knowledge is required in addition to that specified in the introduction to courses in Continuum Mechanics, but some familiarity with the wave equation and a basic knowledge of electromagnetism are helpful.

Introductory Reading

For example:

1. Dowling, AP and Ffowcs Williams, JE. Sound and Sources of Sound. Ellis Horwood.
2. Landau, LD and Lifschitz, EM. Fluid Mechanics. Butterworth-Heinemann. [Chapter 8]
3. R. Feynman, R. Leighton and M. Sands The Feynman Lectures on Physics, Vol 2. Addison and Wesley.
4. Landau, LD and Lifschitz, EM. Classical Theory of Fields. Butterworth-Heinemann. [Chapters 4 & 6]

(or similar introductions to the wave equation in acoustics, and to Maxwell's equations for the electromagnetic field.)

Reading to complement course material

1. Crighton, DG et al. Modern Methods in Analytical Acoustics. Springer.
2. Jackson, JD. Classical Electrodynamics. Wiley
3. Colton, D & Kress R. Inverse Acoustic and Electromagnetic Scattering Theory. Springer.

The Fluid Dynamics of Swimming Organisms (L16)

T J Pedley

The aim of this course is to demonstrate the application of familiar fluid mechanics to the scientific understanding of important real systems, in this case the interaction of active organisms with their aqueous environment. A neutrally buoyant body experiences no net external force, so in order to propel itself it must generate a mean thrust equal and opposite to the mean viscous drag. We shall study in detail the fluid dynamic mechanisms of thrust generation in the undulatory swimming of fish and cetaceans (at large Reynolds number: the first third of the course) and of long-tailed micro-organisms such as spermatozoa (at small Reynolds number: the second third). In each case, the body undulations will be taken as given, but in the case of fish we shall also investigate how the contraction of the swimming muscles must be coordinated in order to achieve those undulations. Other modes of locomotion such as jet propulsion (squid, octopus, scallops) and the coordinated waving of many tiny hairs (cilia) on the surface of certain protozoa may also be analysed.

Approximately the last third of the course will be devoted to the collective behaviour of large numbers of swimming micro-organisms, in particular the pattern-forming phenomenon of bioconvection, which is observed in laboratory suspensions of certain upward-swimming algae and bacteria. The aim is not only to be able to explain quantitatively a number of observed phenomena, but also, more generally to see how to approach the modelling of a large-scale randomly-distributed population on the basis of knowledge of individual behaviour.

A knowledge of basic fluid mechanics will be assumed, so the courses Fluid Dynamics (in Part 1B of the Mathematical Tripos) and Fluid Dynamics II (Part II D), or their equivalents elsewhere, are prerequisites.

Suggested Reading

Childress, S. Mechanics of swimming and flying, Cambridge University Press, 1981.

Lighthill, J. Mathematical biofluiddynamics, S.I.A.M., 1975

Pedley, T.J. & Kessler, J.O. Hydrodynamic phenomena in suspensions of swimming micro-organisms. Ann. Rev. Fluid Mech. 24: 313-358, 1992.

Wu, T.-Y. Swimming of a waving plate. J. Fluid Mech. 10: 321-344 1961 (plus three papers in vol. 46, 1971).

Polar Oceanography (L16)

P Wadhams

PART A. THE POLAR OCEANS AND THEIR ICE COVER (7)

1. The Arctic and Antarctic Oceans Geography Water masses Surface and deep currents Fresh water fluxes Cabelling in Antarctic Circulation under Antarctic ice shelves 2. The physics of sea ice and ice formation What happens when sea water cools Growth of ice crystals Brine cells and brine rejection Salinity structure Summer melt processes First- and multi-year ice 3. Ice growth and decay Thermodynamic model Equilibrium thickness Sensitivity of thickness to changes in forcing Sensitivity to albedo 4. Ice dynamics and the ice thickness distribution Ice motion - driving forces Free drift solution Ice interaction 5. The ice thickness distribution Ridge and lead formation Geometry of pressure ridges The probability density of ice thickness and its evolution Mathematical form of ridges and leads 6. The marginal ice zone Ice floes Eddies Waves in ice 7. Icebergs Sources Distribution in Arctic and Antarctic Physical properties Dynamics Decay and breakup Role in the oceans and in sediment transport Iceberg scouring

PART B. THE PHYSICS OF GLOBAL WARMING (3)

8. History and physics of the greenhouse effect History - from Fourier to Revelle Basic physics of a planet without an atmosphere The effect of atmospheric absorption Specific active gases - water vapour, CO_2, methane 9. Current and future emissions Changes in emissions The role of ozone 10. Climate models and their predictions Temperature - with polar amplification Rainfall Sea levels Extreme events

PART C. IMPACTS OF GLOBAL WARMING ON THE POLAR OCEANS (5)

11. Thinning and retreat of sea ice Satellite data on retreat Parkinson - retreat in sectors, Arctic and Antarctic What is found in Antarctic Thinning - the submarine and other evidence Model predictions of a seasonal Arctic ice cover 12. Convection and the thermohaline circulation The great conveyor belt and the two Arctic sinking zones Greenland Sea convection - chimneys The salt flux model Disappearance of the Odden Has convection slowed or stopped? 13. Ice and global sea levels Sea level = steric + eustatic The steric change from ocean warming Glacial retreat and advance The eustatic change - the anomaly of sea ice melt Experimental evidence 14. Expecting the unexpected - hydrates, fisheries, marine life, offshore oil, harbours Methane hydrates and permafrost decay Changes in Barents Sea fisheries Impact on polar bears and other high-profile animals Offshore oil exploration Japanese harbours Northern Sea Route Arctic haze and its removal 15. Conclusions 1 - how did it start and where will it end? Ice ages and their causes Earlier ice-free periods Is Man the only cause of current changes? What will happen in the longer term? The danger of a runaway greenhouse effect - a warning from Venus 16. Conclusions 2 Latest results from observations and models Potential feedbacks on feedbacks.

ASSIGNMENTS:-

I expect to give the students an essay of about, 2500 words.

Here is a typical choice:-

ICE FORMATION AND GROWTH 1.Write a critical account of the two methods of ice formation and growth in the ocean, based on published papers provided but going beyond these papers to review all aspects of our knowledge of sea ice formation.

THINNING AND RETREAT OF SEA ICE 2. What do we know about the thinning and retreat of sea ice? Write a critical account of its magnitude and the different ideas about its causes.

CHANGES IN THE NORTHERN SEAS 3. Describe how the warming of the Arctic is affecting ocean structure and processes, and what the implications of these effects are for future climate.

BACKGROUND KNOWLEDGE AND READING:

Degree in physics or maths. Background reading of general introductory oceanographic books. Book of the course is "Ice in the Ocean" by P Wadhams (Taylor and Francis, 2000) Another very useful book is "Global Warming - the Complete Briefing" by Sir John Houghton, 3rd Edn (CUP). More specialised books will be mentioned at beginning of course.

Environmental Fluid Dynamics (L16)

Non-Examinable (Part III Level)

S B Dalziel

This course studies the fluid dynamics of environmental processes which are unaffected by the Earth's rotation. (The effects of rotation are covered in the course 'Fundamentals of Atmosphere-Ocean Dynamics', in the Michaelmas Term.) These processes may range in scale from the flow through a doorway to the accidental release of a pollutant, which may spread many kilometres. The buoyancy effects associated with density differences are central to the flows that will be discussed. These density differences may be due to differences in temperature, concentration of a solute, composition, or the presence of a second phase. For some flows the density differences are large and variations in inertia important. For others, the density differences are small and the Boussinesq approximation may be applied.

In the absence of external forcing, a variable density fluid evolves towards a stable stratification. This course concentrates on systematic flows driven by or inhibited by density differences. We shall examine steady hydraulic flows and unsteady gravity currents that result from an imposed horizontal density gradient, including those due to gradients in particle concentration and those within or over porous media. Localised sources of density variation can create plumes (steady or time-dependent), and we shall investigate these flows in a variety of situations, with non-zero initial momentum, stratification or motion in the ambient, and in confined spaces such as buildings. We shall also study the effect of turbulence on density stratifications, and how the stratification alters the characteristics of the turbulence.

A combination of classical examples and current research will be used to illustrate these topics. Laboratory demonstrations of some examples will be included in the non-examinable course 'Demonstrations in Fluid Mechanics' (also in Lent term).

Desirable previous knowledge

Undergraduate fluid dynamics.

Reading to complement course material

1. J.S. Turner, Buoyancy Effects in Fluids, Cambridge University Press (1979).

2. J.E. Simpson, Gravity Currents in the Environment and the Laboratory (2nd edition), Cambridge University Press (1997).

3. F.M. Henderson, Open Channel Flow, Macmillan (1966).

4. B. Gebhart et al., Buoyancy-induced Flows and Transport, Hemisphere Publishing Corp. (1988).

Quantum Fluids (L16)

Natalia Berloff

Vortices are often associated with fluid dynamics: in this course we shall concentrate on quantised vortices. These are vortices in nonlinear fields (superfluid flow fields being but one example) that owe their existence and perseverance to the topology of the order-parameter field describing a medium with broken symmetry. The basic Ginzburg-Landau model serves to describe the vortex core structure in systems as diverse as chemical patterns, liquid crystals, atomic condensates, superconductors, and relativistic strings.

Most of the course will be devoted to superfluid motion that, in its simplest form, can be formulated as a conservative dynamical system. At the most basic level, the theory is projected on the classical inviscid compressible fluid dynamics. But it will be shown that acoustic dispersion creates an effective dissipation mechanism, replacing friction, and the latter makes a comeback when more realistic models are considered.

The topics covered are:

Introduction to order parameter space, broken symmetry and foundations of topological theory of defects. Nomenclature of vortex core studies.

Nonlinear Schrödinger Equation and its properties.

Gross-Pitaevskii equation (GPE) and classical hydrodynamics. Vortices in uniform Bose-Einstein condensates (BECs). Solitary waves: vortex rings and rarefaction pulses and their stability. Vortex nucleation.

Superfluid He4 and He3. Landau two-fluid model and HVBK model. Motion of vortex lines. Dissipative and nonlocal GPE. Superfluid turbulence.

Vortices in non-uniform Bose-Einstein condensates. Rotating condensates.

BECs in optical lattices. Multi-component condensates. Bose-Fermi mixtures.

Cosmology in the lab (COSLAB).

Desirable Previous Knowledge

Pre-requisites for the course include previous attendance at first courses in fluid and quantum mechanics. Familiarity with solution methods for partial differential equations will be assumed.

Introductory Reading

1. P.H. Roberts and N.G. Berloff, "Nonlinear Schrodinger equation as a model of superfluid helium," "Quantized Vortex Dynamics and Superfluid Turbulence" edited by C.F. Barenghi, R.J. Donnelly and W.F. Vinen, Lecture Notes in Physics, volume 571, Springer-Verlag, 2001. (Available also at http://www.damtp.cam.ac.uk/user/ngb23/my_Publications.html)

Reading to complement course material

1. L.M. Pismen "Vortices in nonlinear fields: from liquid crystals to superfluids; from non-equilibrium patterns to cosmic strings", International series of monographs in physics 100, Clarendon Press Oxford, 1999.
2. R.J. Donnelly: Quantized Vortices in Helium II, Cambridge University Press, Cambridge, 1991.
3. C. J. Pethick and H.Smith "Bose-Einstein Condensation in Dilute Gases", Cambridge University Press, 2002.
4. A.L. Fetter and A.A. Svidzinsky "Vortices in a trapped dilute Bose-Einstein condensate", J. Phys.: Condens. Matter 13, R135-R194 (2001).

Demonstrations in Fluid Mechanics (L8; non-examinable)

Non-Examinable (Part III Level)

S B Dalziel

While the equations governing most fluid flows are well known, they are often extremely difficult to solve. To make progress it is therefore necessary to make various assumptions about the nature of the flow, and then to use these assumptions to derive a simpler, more tractable, set of equations. For this process to be meaningful, it is critical that the relevant physics of the flow is retained in the simplified equations. Deriving such equations requires a combination of mathematical analysis and physical insight. Laboratory experiments play an integral role in providing physical insight into the flow and in providing both qualitative and quantitative data against which theoretical and numerical models may be tested.

The purpose of this demonstration course is to help develop an intuitive "feeling" for fluid flows through laboratory experimental investigation. This intuition can be used to construct simplified mathematical models, and this course will demonstrate how experiments may best be used to increase our understanding of flows, ranging from purely fundamental problems, to the most applied situations.

The demonstrations will include a range of flows currently being studied in a range of research projects in addition to classical experiments illustrating some of the flows studied in lectures. The demonstrations are likely to include:

- methods of flow visualisation
- Reynolds' experiments on the transition from laminar to turbulent flow in a pipe
- instability of jets, shear layers and boundary layers
- solitary, progressive and standing surface waves on water
- gravity waves
- thermal convection
- salt fingers and double-diffusive convection
- gravity currents, plumes and thermals
- ventilation flows
- vortex rings and tornado-like vortices
- granular media flows
- multiphase flows
- sedimentation and resuspension
- rotationally dominated flows
- shock waves in a supersonic wind tunnel
- flows in open channels
- non-Newtonian and low Reynolds' number flows

It should be noted that students attending this course are not required to undertake laboratory work on their own account.

Desirable Previous Knowledge

Undergraduate Fluid Dynamics.

Reading to complement course material

(a) M. Van Dyke. An Album of Fluid Motion. Parabolic Press.

(b) G. M. Homsy, H. Aref, K. S. Breuer, S. Hochgreb, J. R. Koseff, B. R. Munson, K. G. Powell, C. R. Robertson, S. T. Thoroddsen. Multimedia Fluid Mechanics (Multilingual Version CD-ROM). CUP.

(c) M. Samimy, K. Breuer, P. Steen, & L. G. Leal. A Gallery of Fluid Motion. CUP.

Desirable previous knowledge

Symmetric Dynamical Systems (L24)

J.H.P. Dawes

This course will introduce ideas and methods from nonlinear dynamics which are widely and routinely used to understand models of a wide range of physical systems, for example fluid flows, population dynamics, chemical reactions and coupled oscillators. The 'dynamical systems viewpoint' is to concentrate on features of the dynamics that are independent of the coordinate system, for example the long-term behaviour that the system 'settles down to'.

The first half of the course will be concerned with the qualitative behaviour of solutions to nonlinear ordinary differential equations, with an emphasis on structural changes in response to variations in parameters (bifurcation theory). There will be a brief discussion of the generation of complicated dynamics.

The second half of the course will extend these ideas to the case of symmetric systems. Symmetry naturally arises either from physical constraints, or from modelling assumptions. The existence of symmetry in the differential equations is, however, not 'generic'. This change in genericity comes with additional structure that provides ways of understanding the notion of a 'typical' bifurcation in the presence of symmetry, and typical dynamical behaviours. Basic ideas from group theory and representation theory will be developed as needed to describe the action of the group on the phase space of the dynamical system. Local bifurcations will be discussed in some detail, after which a variety of directions are possible including applications to pattern formation in two and three dimensions, symmetric chaos, heteroclinic cycling and coupled cell systems.

There will be four examples sheets and problem classes.

Desirable previous knowledge

Previous exposure to nonlinear dynamics would be extremely advantageous; familiarity with the subject at the level of the Cambridge Part II undergraduate course *Dynamical Systems* would be ideal. Only very elementary ideas from group theory will be assumed.

Introductory reading

(a) P. Drazin, *Nonlinear Systems.* Cambridge University Press, 1992.

(b) P. Glendinning, *Stability, Instability and Chaos.* Cambridge University Press, 1994.

(c) M. Hirsch & S. Smale, *Differential Equations, Dynamical Systems and Linear Algebra.* Academic Press, New York, 1974.

(d) J.D. Crawford, Introduction to bifurcation theory. *Rev. Mod. Phys.* **63**, 1991.

Reading to complement course material

(a) J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.* Springer, (2nd edition) 1986.

(b) Y.A. Kuznetsov, *Elements of Applied Bifurcation Theory.* Second edition, Springer, 1998.

(c) S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos.* Springer 1991.

(d) M. Golubitsky, I.N. Stewart & D.G. Schaeffer, *Singularities and Groups in Bifurcation Theory. Volume II.* Springer, Applied Mathematical Sciences Series **69**, 1988.

(e) M. Golubitsky and I.N. Stewart, *The Symmetry Perspective.* Progress in Mathematics, Volume 200. Birkhäuser, 2002.

(f) R.B. Hoyle, *Pattern Formation: An Introduction to Methods.* CUP 2006.

Imaging, boundary Value Problem and Integrability (L24)

A.S. Fokas

The goal of the course is to present new methods which can be used to i) generalise and ii) nonlinearise the classical Fourier transform.

Regarding i) : The Fourier transform can be used for the solution of inverse problems such as the one appearing in computerised tomography. An important new application of a *generalised* Fourier transform is the inversion of the so called attenuated Radon transform. The significance of this transform for the functional imaging of the brain will be elucidated.

Regarding ii): A well known application of the Fourier transform is the solution of initial value problems for linear evolution PDEs. An example of a *nonlinearised* Fourier transform is the Riemann-Hilbert formalism used for the solution of the initial value problem of the nonlinear Schroedinger equation.

Related to the topic of Fourier Transforms is the existence of certain integral representations for the solution of initial boundary value problems. A new method for making such representations effective will also be dicussed. Applications include evolution PDEs with spatial derivatives of arbitrary order as well as the basic elliptic PDEs.

Desirable Previous Knowledge

The only prerequisite is some familiarity with complex variables, e.g. from the course IB Complex Methods.

Introductory Reading

M.J. Ablowitz and A.S.Fokas, Introduction to Complex Variables and Applications, Cambridge University Press, Second Edition, 2002