Commentary

In Our Opinion

Collaboration and Respect

While the headlines scream that American math scores are shamefully low on a global scale, an important evolution is quietly taking place in the American mathematical community. Slowly, but with increasing momentum, American mathematicians and educators are joining in constructive dialogue to improve and strengthen mathematics education.

The growing dialogue between mathematicians and educators is not without its pains. Even within mathematics, people with different mathematical specializations have differing attitudes and viewpoints about mathematical knowledge and ways of using it. As Ronald D. Snee noted[1]:

"The presence of variation and uncertainty constitutes a fundamental difference between mathematics and statistics. Effective statisticians are comfortable with data that are variable and often suspect for many reasons. Many mathematicians are not comfortable with variation."

Similarly, mathematicians may not be comfortable with educational research or in teaching certain grade levels. The training and perspectives of mathematicians are quite different from the knowledge and experience of teachers and educational researchers. None of these types of expertise can replace the other, and the contributions of all are needed for meaningful gain. Thus our aim here is to stimulate discussion that will allow us to appreciate better each other's work and be more sensitized to our own mindsets and assumptions.

Those whose mathematical intuition is very reliable can sometimes overestimate the reliability of their pedagogical intuition. The logically most efficient presentation of a concept may not be the pedagogically most efficient. Working through a chain of deductive arguments to complete a proof may leave some with the impression that everything else is fairly straightforward. While the proof showcases the end result of mathematical thought, it does not convey the making of mathematics—the struggle that led to the result. Much of mathematics aims toward the greatest generality, whereas the teaching of mathematics often builds toward an abstract generality by a succession of concrete examples.

Mathematics research often centers on problems whose outcomes are well defined. This is quite different from research in mathematics education, where the results of thorny, open-ended issues are not always clear-cut. Both are different from classroom instruction, where the problem is to convey meaning, using such tools as timing and nuance.

Terminology in mathematics rests on precise, albeit arbitrary, definitions that retain their meanings in different mathematical contexts. In education, terminology often reflects the struggle to describe underlying notions ("understanding," "transferability") that are context-dependent and whose interpretations can change with time. Mathematicians used to working with clear terminology and precise definitions may become impatient with educational terminology. However, reflecting on what the terminology means and refers to can lead to significant insights about mathematics education.

Education research in this country grew out of statistical methodologies such as those used in agricultural research. Statistical studies can be useful for answering certain kinds of educational research questions that can be framed in quantitative terms, but they are not well suited to answering questions concerning the processes of learning mathematics. Other methodologies, such as ethnographic research or clinical, task-based interviews involving a small number of subjects are useful for probing these issues. For example, researchers have given us significant insights into how problem solvers use heuristics by carefully observing students in the process of doing mathematics.[2] Qualitative educational research that uses tight observational protocols and cross-supporting sources of evidence can provide a measure of objectivity. (Of course, poorly done qualitative research is just as useless as poorly done quantitative research.)

The most significant difference between mathematics and education research is that mathematics evolves and verifies largely by logical reasoning, whereas education research grows and confirms by observation. Thus we may use logical reasoning to rectify a mathematical misstatement, but only through careful, probing observations can we unearth a pervasive thought pattern of which this misstatement is an example.

There are many ways that mathematicians, teachers, and education researchers can work together. Mathematicians are our best source of information about what should be taught and with what emphases. Mathematicians can help students and teachers to appreciate better the nature of mathematics and how it is done. For example, they can provide opportunities such as mentorship programs for teachers to experience what mathematical research is like. And mathematicians can collaborate with education faculty to strengthen teacher programs.

To the mathematician's understanding of the subject the education researcher brings understanding of the learner. And the classroom teacher brings related practical experience to both. For example, the ability to do deductive reasoning may develop at different ages in different children. This does not mean that we must abandon mathematical maturity as a goal of education. Rather, it suggests that educational researchers, in collaboration with classroom teachers, help us assess what can be expected of students in different grade levels and at different stages of development. For meaningful gains in education, the results of these efforts and related research must be communicated throughout the mathematical community.

Surely mathematicians, educational researchers, and teachers can find other ways to work together. Working in isolation can lead to distortions and misperceptions. Working together, respectful of our diverse strengths and perspectives, we can obtain a richly textured picture of reality.

—Warren Page
New York City Technical College (CUNY)
—Mark Saul
Associate Editor

[1]*Snee, Ronald D.,* Mathematics is Only One Tool That Statisticians Use, *The College Math. J.* ***19*** *(1988), 30–32.*

[2]*See, for example, James J. Kaput and Ed Dubinsky (eds.),* Research Issues in Undergraduate Mathematics Learning, *MAA Notes* ***33****, Math. Assoc. America, 1993.*

Letters to the Editor

Stamps on Cover

I was delighted to see the postage stamps on the cover of the March 1998 AMS *Notices*. Members of the AMS might be interested in knowing that there is an organization devoted to mathematical philately, the Mathematical Study Unit, which is an affiliate of the American Philatelic Society and the American Topical Assocation. In addition to a quarterly newsletter, *Philamath*, the unit publishes a checklist of mathematical postage stamps. There are approximately 150 members worldwide, including Robin Wilson (mentioned in the article) and the president of a major mathematical organization in this country. A deceased member was a Nobel laureate. A number of years ago there was an article on our organization in a SIAM publication, but nothing has appeared in an AMS or MAA journal. As president of the Mathematical Study Unit, I encourage interested people to contact our secretary-treasurer, Estelle Buccino, 5615 Glenwood Rd., Bethesda, MD 20817, for further information on our activities.

—Monty J. Strauss
Texas Tech University

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Author's Web Site Offers Precollege Math Lesson Suggestions

The 1989 NCTM document "Curriculum and Evaluation Standards for School Mathematics" at `http://www.enc.org/reform/journals/ENC2280/nf_280dtoc1.htm` does not provide methods to attain all the goals it sets. The 1995 British document "Tackling the Mathematics Problem" at `http://www.lms.ac.uk/policy/tackling/report.html` gives a UK perspective on causes, but still does not propose a full solution. A full solution to the mathematics education problem would have steps to explain each basic concept, since instruction like mathematical induction fails if one or more such steps are undefined or too high for most to climb.

Advice for instruction from primary school to college service courses is posted at my Web site at `http://www.cam.org/~aselby/`. Its main page now has 3,500 or so visits per week. The advice is written for precollege mathematics teachers trained in the discipline or seconded from another.

Collecting ideas easily described and repeated in the classroom advances the common knowledge as well as critical reading and writing skills in all disciplines. This eases one education problem. Others related to the contraction of schools and colleges still remain.

—Alan M. Selby
Montréal, Québec

(Received February 25, 1998)

E. J. McShane in 1938–42: A Personal Recollection

A fine retrospective on the career of E. J. (Jimmy) McShane (1904–1989) appeared in the year of his death: "A Tribute to E. J. McShane", by L. D. Berkowitz and Wendell H. Fleming, *SIAM Journal of Control and Optimization*, vol. 27, no. 5, pp. 909–915. Quoted there are some eloquent remarks by Victor Klee, a Ph.D. of McShane's from the year 1949, regarding his qualities of generosity, graciousness, and kindness. Coming across this recently in my files, it occurred to me that there might be some value in the kind of personal recollection recorded below.

McShane had come to the University of Virginia in 1935 to join G. T. Whyburn, from the University of Texas at Austin, as a full professor in the revitalization of what had become a moribund small department of mathematics. He had already attracted attention within the mathematical research community through a number of innovative postdoctoral papers and his superb English translation and enlargement of Richard Courant's monumental *Differential and Integral Calculus*, the result of the years 1932–33 spent with Courant in Göttingen.

When I arrived at the University of Virginia in the fall of 1938 with a fresh B.S. in mathematics and physics from then-tiny Stetson University in Florida, I was one of some twenty-odd mathematics graduate students. We students of those years quickly found that the department's two full professors had very different styles.

Whyburn, the department chairman, was stiff and formal. His topology seminar, conducted in his office-classroom in the manner of his famed Texas mentor, R. L. Moore, met three times a week. At the outset of each meeting he would state on the blackboard any new definitions required and the latest additions to his carefully ordered list of theorems to be proved. Then any student could volunteer to present *his own* proof of any previously stated theorem not yet proved in the seminar, accepting as known all the theorems that preceded that one on the list. Sometimes a difficult theorem would go unproved for weeks. Very occasionally Whyburn himself might have to present a correct proof. Apart from these seminar meetings and weekly department seminars, that was about all that we graduate students saw of Whyburn.

McShane was very different. By design he made his office-conference-room into an inviting center for graduate student study and activity, constantly buzzing with mathematical questions and challenges. In those days none of the instructors or teaching assistants had an office of his own, so everyone used McShane's.

Typically, about midmorning he would blow in, sit down, put his feet up on his desk, light his pipe, and ask about the mathematical question of the moment. Whatever it was and however vainly we might have been puzzling over it, he would puff on his pipe for a few moments and then go to the blackboard and make an incisive contribution, sometimes solving the problem outright. (Years later when his doctors warned him that smoking was beginning to seriously threaten his health, he simply stopped, humorously lamenting how much he missed it.)

In 1940, with entry of the U.S. into World War II imminent, McShane—unathletic and peaceable to a degree—was waggishly self-deprecatory about his volunteering his mathematical expertise to the Army, observing that with no hair, bad eyes, false teeth, and flat feet he knew he would hardly be of use as a soldier. But having witnessed at first hand Hitler's coming to power in Germany, he was entirely serious about doing what he best could to contribute to the defeat of Hitler's Germany. He began spending his summers and half of most academic-year weeks at the Army's Aberdeen Proving Ground, doing immediately useful research in exterior ballistics.

To complete that part of the story: McShane took a leave of absence from the University of Virginia for the period 1943–45 to continue full-time his work at Aberdeen Proving Ground in a section headed at that time by astronomer Edwin Hubble, now memorialized by the Hubble Space Telescope. By 1944 he had succeeded in bringing to join him in that work J. L. Kelley, a brilliant 1940 University of Virginia Ph.D. in topology. In 1953, well after World War II was over, there appeared *Exterior Ballistics*, McShane's joint book with Kelley and Aberdeen's Frank Reno.

—Truman Botts
Arlington, VA

(Received February 28, 1998)

Teach Calculus with Big O

I am pleased to see so much serious attention being given to improvements in the way calculus has traditionally been taught, but I'm surprised that nobody has been discussing the kinds of changes that I personally believe would be most valuable. If I were responsible for teaching calculus to college undergraduates and advanced high school students today and if I had the opportunity to deviate from the existing textbooks, I would certainly make major changes by emphasizing several notational improvements that advanced mathematicians have been using for more than a hundred years.

The most important of these changes would be to introduce the O notation and related ideas at an early stage. This notation, first used by Bachmann in 1894 and later popularized by Landau, has the great virtue that it makes calculations simpler, so it simplifies many parts of the subject, yet it is highly intuitive and easily learned. The key idea is to be able to deal with quantities that are only partly specified and to use them in the midst of formulas.

I would begin my ideal calculus course by introducing a simpler "A notation", which means "absolutely at most". For example, $A(2)$ stands for a quantity whose absolute value is less than or equal to 2. This notation has a natural connection with decimal numbers: Saying that π is approximately 3.14 is equivalent to saying that $\pi = 3.14 + A(.005)$. Students will easily discover how to calculate with A:

$$\begin{aligned}
&10^{A(2)} = A(100);\\
&(3.14 + A(.005))\,(1 + A(0.01))\\
&\quad = 3.14 + A(.005) + A(0.0314)\\
&\qquad + A(.00005)\\
&\quad = 3.14 + A(0.03645)\\
&\quad = 3.14 + A(.04)\,.
\end{aligned}$$

I would of course explain that the equality sign is not symmetric with respect to such notations; we have $3 = A(5)$ and $4 = A(5)$ but not $3 = 4$, nor can we say that $A(5) = 4$. We can, however, say that $A(0) = 0$.

The A notation applies to variable quantities as well as to constant ones. For example,

$$\begin{aligned}
\sin x &= A(1);\\
x &= A(x);\\
A(x) &= xA(1);\\
A(x) \pm A(y) &= A(x + y)\\
&\qquad \text{if } x \ge 0 \text{ and } y \ge 0;\\
(1 + A(t))^2 &= 1 + 3A(t)\\
&\qquad \text{if } t = A(1).
\end{aligned}$$

Once students have caught on to the idea of A notation, they are ready for O notation, which is even less specific. In its simplest form, $O(x)$ stands for something that is $CA(x)$ for some constant C, but we don't say what C is. We also define side conditions on the variables that appear in the formulas. For example, if n is a positive integer, we can say that any cubic polynomial in n is $O(n^3)$.

I would define the derivative by first defining what might be called a "strong derivative": The function f has a strong derivative $f'(x)$ at point x if

$$f(x + \epsilon) = f(x) + f'(x)\epsilon + O(\epsilon^2)$$

whenever ϵ is sufficiently small. The vast majority of all functions that arise in practical work have strong derivatives, so I believe this definition best captures the intuition I want students to have about derivatives.

I'm sure it would be a pleasure for both students and teacher if calculus were taught in this way. The extra time needed to introduce O notation is amply repaid by the simplifications that occur later. In fact, there probably will be time to introduce the "o notation", which is equivalent to the taking of limits, and to give the general definition of a not necessarily strong derivative:

$$f(x + \epsilon) = f(x) + f'(x)\epsilon + o(\epsilon)\,.$$

But I would not mind leaving a full exploration of such things to a more advanced course, when it will easily be picked up by anyone who has learned the basics with O alone.

Students will be motivated to use O notation for two important reasons. First, it significantly simplifies calculations because it allows us to be sloppy—but in a satisfactorily controlled way. Second, it appears in the power series calculations of symbolic algebra systems like Maple and Mathematica, which today's students will surely be using.

For more than twenty years I have dreamed of writing a calculus text entitled *O Calculus*, in which the subject would be taught along the lines sketched above. Perhaps my ideas are preposterous, but I'm hoping that this letter will catch the attention of people who are much more capable than I of writing calculus texts for the new millenium. And I hope that some of these now-classical ideas will prove to be at least half as fruitful for students of the next generation as they have been for me.

Further details appear at `http://www-cs-faculty.stanford.edu/~knuth/calc/`.

—*Donald E. Knuth*
Stanford University

(Received March 18, 1998)

Conference to Honor Memory of Pontryagin

Readers may have noticed that a conference is scheduled in Moscow August 31–September 5, 1998, to honor the memory of the late Russian mathematician L. S. Pontryagin. What may be less well known are the reasons why this conference has been the subject of considerable controversy.

In late January I was invited to speak at the conference. Soon after, I noted the name of I. R. Shafarevich as a member of the Organizing Committee. I found that troubling, because he is the well-known author of the extreme right-wing and anti-Semitic polemic "Russophobia", a former editor of the anti-Semitic daily *Deyn*, and at the present time one of the most outspoken political activists in the extreme nationalistic and anti-Semitic Pamyat Party. On making inquiries, I learned that S. Novikov had resigned from the Organizing and Program Committees in November. In a letter to the organizers of the conference he explained his reasons. Professor Novikov has given me permission to quote from his letter:

> I respect Pontryagin and was ready to give his memory proper respect; however I already warned you that this occasion may be used for making an anti-semitic shabash. I certainly refuse to participate jointly with Shafarevich, who has nothing in common with Pontryagin scientifically.

Upon further inquiry I received a letter from a member of the Program Committee in my area, who wrote that "good mathematicians who agreed to come take only about 1/3 of places, and many weak persons with suspicious past and strange science have applied."

I am writing to alert other would-be participants to the actual state of affairs. The decision as to whether to boycott this meeting or to participate in it is not a simple matter. One does not wish to lend one's support, even indirectly, to abhorrent ideas, but on the other hand one does not wish to turn one's back on those among our colleagues who are trying hard to keep the focus of this meeting on mathematics. After some consideration I have decided not to participate. I urge those who do participate to make every effort to disassociate themselves publicly from political events having nothing to do with the mathematics of Pontryagin. Those who decide not to participate because of the prominence of Shafarevich in the Organizing Committee are urged to send a letter to the organizers giving their reasons.

—*Joan S. Birman*
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(Received March 23, 1998)

Remarks about Birman's Letter

Professor J. Birman, in her "Letter to the Editor" concerning the Pontryagin memorial conference, gives the following quotation from my private letter: "…good mathematicians who agreed to come take only about 1/3 of places, and many weak persons with suspicious past and strange science have applied."

I find it necessary to make two contextual remarks, which I consider as extremely important for adequate reading of this sentence.

1. In this part of the letter I analyzed only the situation with applications to the topology section of the conference; in the other three sections it was much better. Indeed, a preceding part of the letter was as follows: "I am afraid that if nothing changes then our section can be the weakest among all four sections….Because good…etc."

2. This letter was written in February, about two months before the deadline for applications. Since then something has changed: many good topologists and geometers have agreed to come; thus my fears that many places would remain for weak mathematicians (who, as usual, also have applied) were not realized.

Independently of these remarks, I use this opportunity to confirm my deep respect for Professor J. Birman and also my very negative estimation of the national theories of Professor I. R. Shafarevich.

—*V. A. Vassiliev*
Independent Moscow University and
Steklov Mathematical Institute

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