THE HISTORY OF q-CALCULUS AND A NEW METHOD.

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Contents

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1. **Introduction.**

This Licentiate Thesis contains the following papers:

1. Ernst T., Silvestrov S. D.: Shift difference equations, symmetric polynomials and representations of the symmetric group, U. U. D. M. Report 1999:14, ISSN 1101-3591, Department of Mathematics, Uppsala University, 1999.

2. Ernst T.: A new notation for q-calculus and a new q-Taylor formula. U. U. D. M. Report 1999:25, ISSN 1101-3591, Department of Mathematics, Uppsala University, 1999.

3. Ernst T.: Generalized Vandermonde determinants. U. U. D. M. Report 2000:6, ISSN 1101-3591, Department of Mathematics, Uppsala University, 2000.

4. Ernst T.: A new Vandermonde-related determinant and its connection to difference equations. U. U. D. M. Report 2000:9, ISSN 1101-3591, Department of Mathematics, Uppsala University, 2000.

This is a history of q-calculus with a new notation and a new method for q-hypergeometric series. Of course it is not possible to cover all the details in this vast subject in only one book and the reader is invited to study the book by Gasper and Rahman [343] for additional information. Some parts of chapter 2 are based on the first chapters of this book. In section 2.25 a new q-Taylor formula is presented. Furthermore generalized Vandermonde determinants, symmetric functions and representation theory of the symmetric group are treated. These two parts will later be united when solving linear q-difference equations with constant coefficients. In the last decades q-calculus has developed into an interdisciplinary subject, which is briefly discussed in chapters 3 and 7. This is nowadays called q-disease.

> Despite the current increase of pessimism concerning the quantum group invasion (possibly the quantum group pest), all these applications should invite one to pursue the investigations of quantum groups [670].

In this context it would be natural to say something about the history of the connection between group representation theory and quantum mechanics, which was initiated by Eugene Wigner (1902-1995) between November 12 and November 26 1926 when Wigner's first papers on quantum mechanics reached the Zeitschrift der Physik, and both appeared in volume 40.

> It was John von Neumann who first proposed that group representation theory be used in quantum mechanics. Wigner was invited to Göttingen in 1927 as assistant to David Hilbert. Though the new quantum mechanics had been initiated only in 1925, already in 1926-27 the mathematician Hilbert in Göttingen gave lectures on quantum mechanics [935].

The Gruppenpest [842] (the pest of group theory) would last for three decades [935].

1.1. **Partitions, generalized Vandermonde determinants and representation theory.** The theory of *symmetric functions* appeared first in Newtons *Arithmetica Universalis.* A modern account is given by Macdonald (1995) [594] which we shall use as reference for matters of notation. The theory of symmetric functions is intimately connected to the symmetric group as was outlined in [594].

The basic theory of the symmetric group was developed by Young and Frobenius in the first two decades of the twentieth century. A *partition* [594] of $m \in \mathbb{N}$ is any finite sequence λ

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n),\ m \equiv \sum_{j=1}^{n} \lambda_j \equiv |\lambda| \tag{1}$$

of non-negative integers in decreasing order:

$$\lambda_1 \geq \lambda_2 \geq \ldots, \geq \lambda_n \geq 0$$

containing only finitely many non-zero terms such that the *weight* m of λ

$$\sum_{j=1}^{l(\lambda)} \lambda_j = m, \tag{2}$$

where $l(\lambda)$, the number of parts > 0 of λ, is called the *length* of λ. We shall find it convenient not to distinguish between two such sequences which differ only by a string of zeros at the end. The *Young frame* [802] of a partition λ may be formally defined as the set of points $(i, j) \in \mathbb{Z}^2$ such that $1 \leq j \leq \lambda_i$. The *conjugate* of a partition λ is the partition λ' whose Young frame is the transpose of the Young frame λ, i.e. the Young frame obtained by reflection in the main diagonal.

Let $x^\alpha = \prod_{j=1}^{n} x_j^{\alpha_j}$ be a monomoial, and consider the polynomial a_α obtained by antisymmetrizing x^α:

$$a_\alpha \equiv a_\alpha(x_1, \ldots, x_n) \equiv \sum_{w \in S_n} \epsilon(w) w(x^\alpha) = \begin{vmatrix} x_1^{\alpha_1} & x_2^{\alpha_1} & \vdots & x_n^{\alpha_1} \\ x_1^{\alpha_2} & x_2^{\alpha_2} & \vdots & x_n^{\alpha_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\alpha_n} & x_2^{\alpha_n} & \vdots & x_n^{\alpha_n} \end{vmatrix}, \tag{3}$$

where $\epsilon(w)$ is the sign of the permutation w. Given partitions $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ of m and $\delta = (n-1, n-2, \ldots, 1, 0)$ of $\binom{n}{2}$, the generalized Vandermonde determinant is defined by

$$a_{\lambda+\delta} \equiv \begin{vmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \vdots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \vdots & x_n^{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \vdots & x_n^{\lambda_n} \end{vmatrix}. \tag{4}$$

Some references on generalized Vandermonde determinants are [315], [785],[909].

To keep this paper as selfcontained as possible we need some definitions and theorems from representation theory. The following formulae [240] are needed in the proof of the second Frobenius character formula (21).

Theorem 1.1. *Let K be a field and G a group. Further let χ and ζ be the characters afforded by the irreducible KG-modules V and U. Then*

(1) *$V \ncong U$ implies*

$$\sum_{x\in G} \chi(x)\zeta(x^{-1}) = 0. \tag{5}$$

(2) *If K is a splitting field for G, then*

$$\sum_{x\in G} \chi(x)\chi(x^{-1}) = |G|. \tag{6}$$

(3) *Let K be a splitting field for G. Assume that G has exactly r conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_r$. Denote $r_i = |\mathcal{C}_i|$, choose $x_i \in \mathcal{C}_i$, and denote the irreducible characters of G by $\chi_1, \ldots, \chi_r$. Then*

$$\sum_{m=1}^{r} \chi_m(x_i)\chi_m(x_j^{-1}) = \frac{\delta_{ij}|G|}{r_j} = \delta_{ij}|c_{x_j}|, \tag{7}$$

where c_{x_j}=the centralizer of x_j in G= $\{x \in G | xx_j = x_jx\}$.

Theorem 1.2. *Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G over $\mathbb{C}$. Then*

$$\chi_i(x^{-1}) = \overline{\chi_i(x)}, \forall x \in G, \tag{8}$$

where the bar denotes complex conjugation.

A *Young tableau* Y is any placement of the numbers $1, 2, \ldots, m$ in the boxes of the Young frame belonging to the partition λ.

Remark 1. Proctor has defined Young tableau for the classical groups [722], [723].

Definition 1. For a given Young tableau Y, let the Young subgroup P_Y be the set of all permutations that permute the numbers in each row of Y among themselves. We denote the permutations in P_Y by letters such as p. Let W_Y be the set of all permutations in S_m that permute elements of each column of Y among themselves. We denote the permutations in W_Y by letters such as ω. Let A be the group algebra of the group $S_m : A \equiv A_{S_m}$. We consider the following elements of the algebra A :

$$f_Y = \sum_p p \tag{9}$$

$$\varphi_Y = \sum_\omega \pi_\omega \omega, \tag{10}$$

where π_ω is the sign of the permutation ω.

Definition 2. The *Young symmetrizer* h_Y corresponding to the Young tableau Y is defined by

$$h_Y = f_Y \varphi_Y. \tag{11}$$

The ideal

$$I_Y = A_{S_m} h_Y \tag{12}$$

is a minimal left ideal in A_{S_m}.

The restrictions $T_\lambda \cong T_Y$ of the left regular representation T to I_Y form a complete system of irreducible representations of the group S_m, as Y runs through all partitions. Now let m_Y denote the dimension of T_λ. The element

$$e_Y = \frac{m_Y}{m!} h_Y \tag{13}$$

is an idempotent in I_Y.

According to the *hook rule*,

$$m_Y = m! \frac{\prod_{p<r} (l_p - l_r)}{\prod_{j=1}^{n} \lambda_j!}, \tag{14}$$

where

$$l_k = \lambda_k + n - k. \tag{15}$$

Remark 2. The q-hook rule was defined in [332].

We know that every element of S_m is uniquely the product of disjoint cycles. The lengths of these cycles form a partition of m and, in this way, the conjugacy classes of S_m are indexed by the partitions of m [358]. The character χ_λ of the irreducible representation T_λ of the group S_m is written in terms of the symmetrizer h_Y as

$$\chi_\lambda(\mu) = \frac{m_Y}{m!} \sum_{a \in S_m} h_Y(a^{-1}g^{-1}a), \tag{16}$$

where g is any member of the conjugacy class μ of S_m.

Remark 3. For the general linear group $GL(n, F)$ over a field F with arbitrary characteristic, an element similar to (13) is called the *Schur module* [597].

1.2. **The Frobenius character formulae.** In our partition λ let

$$S_\lambda \equiv S_{\lambda_1} \times \cdots \times S_{\lambda_n} \tag{17}$$

be a Young subgroup. Consider the corresponding irreducible representation T_λ of the symmetric group S_m, which is induced from the trivial representation of the subgroup S_λ by the Frobenius character formula for induced representations. Let the conjugacy class

$$\mu = 1^{\mu_1} 2^{\mu_2} \ldots m^{\mu_m} \tag{18}$$

belong to S_m, let $n \geq m$ and put

$$\mathcal{S}_\mu(x_1, \ldots, x_n) \equiv \mathcal{S}_\mu \equiv \prod_{j=1}^{m} (\sum_{k=1}^{n} x_k^j)^{\mu_j}. \tag{19}$$

Theorem 1.3. The Frobenius character formula for S_m [802]*:*

$$\mathcal{S}_\mu a_\delta = \sum_\lambda \chi_\lambda(\mu) a_{\lambda+\delta}, \tag{20}$$

where the sum is taken over all partitions of m, adding the appropriate number of zeros and χ_λ is the character of the irreducible representation T_λ.

Sketch of proof Simon [802]:

(1) Let λ be a partition of m and let $\mathcal{F}$ be a Young frame of λ with rows $\lambda_1, \ldots, \lambda_n$, where we set some $\lambda_j = 0$ if need be. The notion $\mathcal{F}' \lhd \mathcal{F}$ for an m-frame $\mathcal{F}$ and an $m-1$-frame $\mathcal{F}'$ means that you can get $\mathcal{F}'$ from $\mathcal{F}$ by removing a square from $\mathcal{F}$. Each m-frame $\mathcal{F}$ has an associated irreducible representation which will be denoted $\mathcal{U}^{(\mathcal{F})}$.

(2) Using μ as a class label and λ as a frame label, we'll use induced representations to produce some (not necessarily irreducible) character Ψ_λ (the permutation character) [358].

(3) We'll compute a generating function for the Ψ_λ's, that is, suitable polynomials $\mathcal{S}_\mu(x_1, \dots, x_n)$ and $G_\lambda(x_1, \dots, x_n)$, so

$$\mathcal{S}_\mu(x_1, \dots, x_n) = \sum_\lambda \Psi_\lambda G_\lambda(x_1, \dots, x_n).$$

(4) With $a_\delta \equiv (-1)^{\binom{n}{2}} \prod_{1 \le j < i \le n} (x_i - x_j)$, we'll expand

$$a_\delta \mathcal{S}_\mu(x_1, \dots, x_n) \equiv F_\mu(x_1, \dots, x_n) = \sum_\lambda \tilde{\chi_\lambda}(\mu) a_{\lambda+\delta},$$

where

$$\tilde{\chi_\lambda} = \sum_\tau b_{\tau\lambda} \Psi_\tau$$

with $b_{\tau\lambda} \in \mathbb{Z}$.

(5) We'll prove

$$a_\delta G_\lambda(x_1, \dots, x_n) = \sum_\tau b_{\lambda\tau} a_{\tau+\delta}.$$

Thus, the $\tilde{\chi_\lambda}$ are integral combinations of characters in $\widehat{S_m}$, where $\widehat{S_m}$, the dual object, is the set of equivalence classes of irreducible representations, each class consisting of unitarily equivalent irreducible representations.

(6) We'll prove $\frac{1}{m!} \sum_\mu |\mu| |\tilde{\chi_\lambda}(\mu)|^2 = 1$, where $|\mu| = \dfrac{m!}{\prod_{j=1}^{m} \mu_j! \; j^{\mu_j}}$ is the number of elements in the conjugacy class μ.

(7) We'll prove $\tilde{\chi_\lambda}(\{e\}) > 0$. It follows that the $\tilde{\chi_\lambda}$ are irreducible characters.

(8) We'll prove a branching law:

$$\tilde{\chi_\lambda}_{S_{m-1}} = \sum_{\lambda^{(1)} | \mathcal{F}(\lambda^{(1)}) \triangleleft \mathcal{F}(\lambda)} \tilde{\chi_{\lambda^{(1)}}},$$

so that $\tilde{\chi_\lambda}$ is the character of $\mathcal{U}^{(\mathcal{F})}$ by induction.

This formula can be used to compute the characters of the symmetric group. Namely, $\chi_\lambda(\mu)$ is the coefficient of $x_1^{\lambda_1+n-1}\cdots x_n^{\lambda_n}$ in the polynomial $\mathcal{S}_\mu a_\delta$.

The following corollary gives a formula for the generalized Vandermonde determinant in terms of the irreducible characters of S_m.

Corollary 1.4. *(Frobenius)*

$$\frac{a_{\lambda+\delta}}{a_\delta}=\sum_{\mu\in S_m^*}\left(\chi_\lambda(\mu)c_\mu^{-1}\mathcal{S}_\mu\right), \tag{21}$$

where S_m^* denotes the set of all conjugacy classes of S_m, and the order of the centralizer of any permutation $\in\mu$ [358]

$$c_\mu=\prod_{j=1}^{m}\mu_j!\ j^{\mu_j}. \tag{22}$$

Proof. [801] Let $\lambda^{(1)}$, $\lambda^{(2)}$ and $\lambda^{(3)}$ denote partitions of m. We calculate the right-hand side of (21) and show that it is equal to the left-hand side.

$$\sum_\mu \chi_{\lambda^{(3)}}(\mu)c_\mu^{-1}\mathcal{S}_\mu=\frac{1}{a_\delta}\sum_\mu \chi_{\lambda^{(3)}}(\mu)c_\mu^{-1}\sum_{\lambda^{(1)}}\chi_{\lambda^{(1)}}(\mu)a_{\lambda^{(1)}+\delta}=$$

$$\frac{1}{a_\delta}\sum_{\lambda^{(1)}}\left(\sum_\mu \chi_{\lambda^{(1)}}(\mu)\chi_{\lambda^{(3)}}(\mu)c_\mu^{-1}\right)a_{\lambda^{(1)}+\delta}.$$

We must prove that

$$\sum_\mu \chi_{\lambda^{(1)}}(\mu)\chi_{\lambda^{(3)}}(\mu)c_\mu^{-1}=\begin{cases}0, & \text{if } \lambda^{(3)}\neq\lambda^{(1)};\\ 1, & \text{if } \lambda^{(3)}=\lambda^{(1)}.\end{cases} \tag{23}$$

From the proof of (20) we know that the $\chi_{\lambda^{(3)}}(\mu)$ are integers. Equations (7) and (8) now imply that we must prove that

$$\sum_\mu \chi_{\lambda^{(1)}}(\mu)\chi_{\lambda^{(3)}}(\mu)\frac{1}{\sum_{\lambda^{(2)}}\chi_{\lambda^{(2)}}(\mu)^2}=\begin{cases}0, & \text{if } \lambda^{(3)}\neq\lambda^{(1)};\\ 1, & \text{if } \lambda^{(3)}=\lambda^{(1)}.\end{cases} \tag{24}$$

First consider the case $\lambda^{(3)}=\lambda^{(1)}$. According to (6),

$$\sum_\mu \frac{|\mu|}{|S_m|}\chi_{\lambda^{(3)}}(\mu)^2=1,$$

and we find that

$$\frac{|\mu|}{|S_m|}=\frac{1}{\sum_{\lambda^{(2)}}\chi_{\lambda^{(2)}}(\mu)^2},$$

which is equation (7). Now let $\lambda^{(3)} \neq \lambda^{(1)}$. According to (5),

$$\sum_{\mu} |\mu| \chi_{\lambda^{(3)}}(\mu) \chi_{\lambda^{(1)}}(\mu) = 0.$$

Now we only have to use (7) again to complete the proof. □

Remark 4. The character χ_{1^m} is equal to the sign of the permutation which corresponds to the conjugacy class and is therefore called the signature [358].

The following equation is a consequence of the Frobenius character formula [802].

$$\chi_{\lambda'}(\mu) = (-1)^{sgn(\mu)} \chi_{\lambda}(\mu). \tag{25}$$

In particular, the dimensions of the representations associated with conjugate partitions are equal.

There are other methods to compute the irreducible characters $\chi_{\lambda}(\mu)$. One method uses the so-called lattice permutation [412], [750]. A graphical construction is given in [802].

Let M_{λ} be the permutation module affording Ψ_{λ}. M_{λ} has a natural basis $\{f_{\mathcal{P}} | \bar{\mathcal{P}} = \lambda\}$ permuted by S^m. Since χ_{λ} has multiplicity 1 in Ψ_{λ}, there is a unique submodule $X_{\lambda} \subseteq M_{\lambda}$ affording χ_{λ} which is called the Specht module. Let A_m denote the alternating group, and let S^{λ} denote the Specht module. If λ is a non-symmetric partition of m then the restrictions $S^{\lambda} \downarrow A_m$ and $S^{\lambda'} \downarrow A_m$ are irreducible and isomorphic to each other [98].

In [512] and [181] a Frobenius character formula for a Hecke algebra was presented. In [519], [887] a q-analogue of Frobenius character formula was presented. This was accomplished by calculating the irreducible representations of the Hecke algebra $H_n(q)$.

1.3. **Symmetric functions.** The symmetric group S_n acts on the ring $\mathbb{Z}[x_1, \ldots, x_n]$ of polynomials in n independent variabels $x_1, \ldots, x_n$ with rational integer coefficients by permuting the variables, and a polynomial is *symmetric* if it is invariant under this action. The symmetric polynomials form a graded subring

$$\Lambda_n \equiv \mathbb{Z}[x_1, \ldots, x_n]^{S_n} \equiv \bigoplus_{k \geq 0} \Lambda_n^k, \tag{26}$$

where Λ_n^k consists of the homogeneous symmetric polynomials in n variables of degree k, together with the zero polynomial.

The element Λ^k is obtained from Λ_n^k by letting the number of variables $\to \infty$ [594, p. 19]. The graded ring $\Lambda \equiv \bigoplus_{k \geq 0} \Lambda^k$ is called the *ring of symmetric functions* in countably many variables $x_1, x_2, \ldots$.

The k:th *elementary symmetric polynomial* $e_k(x_1,\ldots,x_n)$ in the indeterminates $x_1,\ldots,x_n$ is the sum of all possible distinct products, taken k at a time , of the elements $x_1,\ldots,x_n$ [843] i.e.

$$e_k(x_1,\ldots,x_n)=\begin{cases}\sum_{j_1<j_2\ldots<j_k\leq n} x_{j_1}\ldots x_{j_k}, & 1\leq k\leq n;\\ 1, & k=0\\ 0, & k<0 \text{ or } k>n.\end{cases} \tag{27}$$

Now follows a similar definition. The k:th *elementary symmetric function* e_k [594] is the sum of all products of k distinct variables x_j, i.e.

$$e_k=\begin{cases}\sum_{j_1<j_2\ldots<j_k} x_{j_1}\ldots x_{j_k}, & 1\leq k;\\ 1, & k=0\\ 0, & k<0.\end{cases} \tag{28}$$

We make the convention that always when the number of variabels is not specified we let e_k denote the elementary symmetric function etc. For each partition $\lambda=(\lambda_1,\lambda_2,\ldots)$ define

$$e_\lambda=e_{\lambda_1}e_{\lambda_2},\ldots. \tag{29}$$

For each integer $k\geq 0$ the k:th *complete symmetric polynomial* $h_k(x_1,\ldots,x_n)$ in the indeterminates $x_1,\ldots,x_n$ is the sum of all possible products, taken k at a time, chosen without restriction on repetition, of the elements $x_1,\ldots,x_n$ i.e.

$$h_k(x_1,\ldots,x_n)=\begin{cases}\sum_{1\leq j_1\leq j_2\ldots\leq j_k\leq n} x_{j_1}\ldots x_{j_k}, & 1\leq k;\\ 1, & k=0\\ 0, & k<0.\end{cases} \tag{30}$$

where the summation is extended over all vectors $(j_1,\ldots,j_k)$ of k integers satisfying $1\leq j_1\leq\cdots\leq j_k\leq n$.

Let λ be any partition of length $\leq n$. The polynomial

$$m_\lambda(x_1,\ldots,x_n)=\sum x^\alpha \tag{31}$$

summed over all distinct permutations α of $\lambda=(\lambda_1,\ldots,\lambda_n)$ is clearly symmetric. For each $k\geq 0$ the k:th *complete symmetric function* [594] h_k is the sum of all monomials of total degree k in the variables $x_1,x_2,\ldots$, so that

$$h_k=\sum_{|\lambda|=k} m_\lambda. \tag{32}$$

For each partition $\lambda=(\lambda_1,\lambda_2,\ldots)$ define

$$h_\lambda=h_{\lambda_1}h_{\lambda_2},\ldots. \tag{33}$$

The following equation [433] obtains $\forall n\geq 1$:

$$\sum_{k=0}^{n}(-1)^k e_k h_{n-k}=0 \tag{34}$$

Let ω be the involution and homomorphism of graded rings

$$\omega:\Lambda\mapsto\Lambda$$

defined by

$$\omega(e_k)=h_k,\ \forall k\geq 0.$$

The 'forgotten' symmetric functions are defined by

$$f_\lambda=\omega(m_\lambda), \tag{35}$$

they don't have any simple direct description.

The *Schur function* s_λ, defined by

$$s_\lambda=\frac{a_{\lambda+\delta}}{a_\delta}, \tag{36}$$

is a quotient of two homogeneous skew-symmetric polynomials and is thus a homogeneous symmetric polynomial [594]. The following equations obtain [433],[594]:

$$\begin{aligned} s_\lambda=&\begin{vmatrix} h_{\lambda_1} & \ldots & h_{\lambda_1+j-1} & \ldots & h_{\lambda_1+n-1}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ h_{\lambda_i+1-i} & \ldots & h_{\lambda_i+j-i} & \ldots & h_{\lambda_i+n-i}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ h_{\lambda_n-n+1} & \ldots & h_{\lambda_n-n+j} & \ldots & h_{\lambda_n} \end{vmatrix}=\\ =&\begin{vmatrix} e_{\lambda'_1} & \ldots & e_{\lambda'_1+j-1} & \ldots & e_{\lambda'_1+m-1}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ e_{\lambda'_i+1-i} & \ldots & e_{\lambda'_i+j-i} & \ldots & e_{\lambda'_i+m-i}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ e_{\lambda'_n-m+1} & \ldots & e_{\lambda'_n-m+j} & \ldots & e_{\lambda'_m} \end{vmatrix}, \end{aligned} \tag{37}$$

where $n\geq l(\lambda)$ and $m\geq l(\lambda')$.

Remark 5. The first equation is originally due to Jacobi 1841 [468] and is sometimes called the *Jacobi-Trudi identity*. The second identity was first proved by Aitken 1931 [14]. Aitken from Edinburgh had been greatly influenced by Young's work on the representation theory of the symmetric group [667].

There are five $\mathbb{Z}$-bases of Λ:

$$e_\lambda,\ h_\lambda,\ f_\lambda,\ m_\lambda \text{ and } s_\lambda.$$

Let R^m denote the $\mathbb{Z}$-module generated by the irreducible characters of S_m, and let

$$R = \bigoplus_{m\geq 0} R^m, \tag{38}$$

with the understanding that $S_0 = \{1\}$, so that $R^0 = \mathbb{Z}$. The $\mathbb{Z}$-module R has a ring structure and is called the *Littlewood-Richardson ring.*

R carries a scalar product: if $f, g \in R$, say

$$f = \sum_m f_m,\ g = \sum_m g_m,\ f_m, g_m \in R^m,$$

we define

$$< f, g >= \sum_{m\geq 0} < f_m, g_m >_{S_m} . \tag{39}$$

For each $k \geq 1$, the k:th *power sum* is defined by

$$p_k = \sum x_i^k, \tag{40}$$

and as before

$$p_\lambda = p_{\lambda_1} p_{\lambda_2}, \ldots . \tag{41}$$

If f_μ is the value of f at elements of cycle type μ, the *characteristic* of f is defined by

$$ch(f) = \sum_{|\mu|=m} c_\mu^{-1} f_\mu p_\mu. \tag{42}$$

The *characteristic map ch* is an isometric isomorphism of R onto Λ [594]. Under this isomorphism, the irreducible character χ_λ corresponds to the Schur function s_λ and the permutation character Ψ_λ corresponds to the complete symmetric function h_λ, where the permutation character Ψ_λ is the character on S_m induced by the identity character of

$$S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \ldots .$$

The following equation is of fundamental importance for the theory of the symmetric group:

$$\chi_\lambda = det(\Psi_{\lambda_i - i + j})_{1\leq i,j\leq m} \in R^m. \tag{43}$$

1.4. **The connection between analytic number theory and q-series.** It was Euler who started analytic number theory by inventing the *Euler-Maclaurin summation formula* 1732 [256] and 1736 (proof) [257], which expresses the average value of an arithmetical function using integrals and the greatest integer function [282], [41].

Then came the *Euler product* in 1737, which expresses the infinite sum of a multiplicative arithmetical function as the infinite product over the primes of an infinite sum of function values of prime powers. If the function is *completely multiplicative*, the product simplifies.

Euler developed the theory of partitions, also called *additive number theory* in the 1740s. The *partition function* $p(n)$ is the number of ways to write n as a sum of integers. There are also other partition functions, but they have some restriction on the parts, e.g. only odd parts are allowed. Euler found all the generating functions for these partitions. In 1748 Euler considered the infinite product $\prod_{k=0}^{\infty}(1-q^{k+1})^{-1}$ as a generating function for $p(n)$.

By using induction Euler proved the *pentagonal number theorem* in 1750

$$1+\sum_{m=1}^{\infty}(-1)^m(q^{\frac{m(3m-1)}{2}}+q^{\frac{m(3m+1)}{2}})=\prod_{m=1}^{\infty}(1-q^m), 0<|q|<1, \tag{44}$$

which was the first example of a q-series, and at the same time the first example of a theta-function.

Furthermore Euler discovered the first two q-exponential functions, a prelude to the q-binomial theorem and at the same time introduced an operator which would over hundred years later lead to the q-difference operator.

Yet another example of a q-series is the following result of Gauss, which was published in 1866, 11 years after his death 1855:

$$1+\sum_{m=1}^{\infty}q^{\binom{m+1}{2}}=\prod_{m=1}^{\infty}\frac{1-q^{2m}}{1-q^{2m-1}}, |q|<1. \tag{45}$$

1.5. **Some aspects of combinatorics.** We are now going to define a series of numbers and related polynomials. Some of their q-analogues will be presented in a subsequent chapter. It is not a coincidence that the next section is concerned with hypergeometric series. In fact it can be shown that the two approaches are equivalent as was first observed by Gauss.

Let the *Pochhammer symbol* $(a)_n$ be defined by

$$(a)_n = \prod_{m=0}^{n-1}(a+m),\ (a)_0 = 1. \tag{46}$$

Since *products* of Pochhammer symbols occur so often, we shall frequently use the more compact notation

$$(a_1, a_2, \ldots, a_m)_n \equiv \prod_{j=1}^{m}(a_j)_n. \tag{47}$$

We will take some of the following definitions from Riordan [748]. James Stirling (1692-1770) was a disciple of Newton [246, p. 17] who invented the *Stirling numbers* of the first and second kind $s(n,k)$, $S(n,k)$, which are defined as follows.

$$\begin{aligned} &s(0,0) = S(0,0) = 1 \\ &(t-n+1)_n = \sum_{k=0}^{n} s(n,k)t^k,\ n>0 \\ &t^n = \sum_{k=0}^{n} S(n,k)(t-k+1)_k,\ n>0. \end{aligned} \tag{48}$$

The following recurrence relations obtain:

$$\begin{aligned} &s(n+1,k) = s(n,k-1) - ns(n,k) \\ &S(n+1,k) = S(n,k-1) + kS(n,k). \end{aligned} \tag{49}$$

The *generalised Stirling numbers* and *generalised Stirling polynomials* are defined by [799], [352]

$$S^{(\alpha)}(n,k,\gamma) = (-1)^k/k! \sum_{j=0}^{k}(-1)^j(k/j)(\alpha+\gamma j)^n \tag{50}$$

and

$$T_n{}^{(\alpha)}(x,\gamma,-p) = \sum_{k=0}^{n} S^{(\alpha)}(n,k,\gamma)p^\gamma x^{\gamma k}. \tag{51}$$

The *Catalan numbers* $C_n = \frac{1}{n+1}\binom{2n}{n}$ [210] were first discovered in China about 1730 by J.Luo [566] and were studied by Euler in the middle of the 18th century. They became famous through the 1838 paper by Catalan.

The *Bernoulli numbers* B_n are defined by

$$\frac{z}{e^z-1}=1-\frac{z}{2}+\sum_{n=1}^{\infty}\frac{B_{2n}z^{2n}}{(2n)!}. \tag{52}$$

The *Bernoulli polynomials* are defined by

$$B_n(x)=(B+x)^n=\sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}, \tag{53}$$

where B^n must be replaced by B_n on expansion. The *generalized Bernoulli polynomials* $B_n^{(\alpha)}(x)$ are defined by [684, Ch. 6], [827]

$$(\frac{z}{e^z-1})^{\alpha}e^{xz}=\sum_{n=0}^{\infty}\frac{B_n^{(\alpha)}(x)z^n}{(n)!},\ |z|<2\pi,\ \alpha\in\mathbb{C}. \tag{54}$$

The *generalized Bernoulli numbers* are defined by $B_n(\alpha)=B_n^{(\alpha)}(0)$.

The *Eulerian numbers* $H_n(\lambda)$ are defined by [157], [331], [259, p. 487].

$$\frac{1-\lambda}{e^z-\lambda}=\sum_{n=0}^{\infty}\frac{H_n(\lambda)z^n}{(n)!}. \tag{55}$$

These numbers must not be confused with the Euler or secant numbers defined in (81). The *Genocchi numbers* G_n are defined by

$$\frac{2x}{e^x+1}=x+\sum_{n=1}^{\infty}\frac{G_{2n}x^{2n}}{(2n)!}. \tag{56}$$

The Bell numbers $\mathcal{B}_n$ are defined by

$$\mathcal{B}_n=\sum_{k=0}^{n}S(n,k). \tag{57}$$

In 1877 Dobinski [231] proved the remarkable formula

$$\mathcal{B}_{n+1}=\frac{1}{e}\sum_{k=1}^{\infty}\frac{k^n}{(k-1)!}. \tag{58}$$

1.6. **A short history of hypergeometric series . . .** This section is about how some wellknown formulae from complex analysis, most of them hypergeometric series were discovered. In later sections we will prove their q-analogues. We start with a few definitions.

The *generalized hypergeometric series*, ${}_pF_r$, is given by

$$
\begin{aligned}
&{}_pF_r(a_1,\ldots,a_p;b_1,\ldots,b_r;z)\equiv\\
&\equiv\ {}_pF_r\left[\begin{array}{c}a_1,\ldots,a_p\\ b_1,\ldots,b_r\end{array};z\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{n!(b_1)_n\ldots(b_r)_n}z^n.
\end{aligned}
\tag{59}
$$

Denote the series (59) $\sum_{n=0}^{\infty}A_nz^n$. The quotient $\frac{A_{n+1}}{A_n}$ is a rational function $R(n)$ (Askey).

The following notation for the gamma function will sometimes be used [810]:

$$
\Gamma\left[\begin{array}{c}a_1,\ldots,a_p\\ b_1,\ldots,b_r\end{array}\right]\equiv\frac{\Gamma(a_1)\ldots\Gamma(a_p)}{\Gamma(b_1)\ldots\Gamma(b_r)}.
\tag{60}
$$

An ${}_{r+1}F_r$ series is called k- *balanced* if $b_1+\ldots+b_r=k+a_1+\ldots+a_{r+1}$ and $z=1$; and a 1-balanced series is called balanced or Saalschützian after L.Saalschütz (1835-1913). The hypergeometric series

$$
{}_{r+1}F_r(a_1,\ldots,a_{r+1};b_1,\ldots,b_r;z)
\tag{61}
$$

is called *well-poised* if its parameters satisfy the relations

$$
1+a_1=a_2+b_1=a_3+b_2=\ldots=a_{r+1}+b_r.
\tag{62}
$$

The hypergeometric series (61) is called *nearly-poised* [934] if its parameters satisfy the relation

$$
1+a_1=a_{j+1}+b_j
\tag{63}
$$

for all but one value of j in $1\leq j\leq r$. If the series (61) is well-poised and $a_2=1+\frac{1}{2}a_1$, then it is called a *very-well-poised* series.

The Oxford professor John Wallis (1616-1703) first used the word 'hypergeometric' (from the Greek) for the series

$$
\sum_{k=0}^{\infty}(a)_k
$$

in his work *Arithmetica Infinitorium* (1655).

The concept of differentiation (and integration) to a noninteger order appeared already in a letter from Leibniz to L'Hôpital dated September 30, 1695, and in another letter dated May 28, 1697, from Leibniz to Wallis [825]. For an extensive bibliography on occurences of fractional derivatives see [691].

The *binomial series*, first studied by Newton, is defined by

$$
(1-z)^{-\alpha}=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}z^n,\ |z|<1,\ \alpha\in\mathbb{C}.
\tag{64}
$$

Leonhard Euler (1707-1783) proved the following four equations: (see book by Gasper and Rahman [343], pp.10,19). The *beta integral*

(65)
$$\int_0^1 x^{s-1}(1-x)^{t-1}\,dx = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \equiv B(s,t),\ \ Re(s)>0,\ \ Re(t)>0;$$

the integral representation of the hypergeometric series ${}_2F_1(a,b;c;z)$:

$$(66)\quad {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}\,dx,$$
$$|arg(1-z)|<\pi,\ \ Re(c)>Re(b)>0;$$

Euler's transformation formula:

$$(67)\qquad {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}{}_2F_1(c-a,c-b;c;z).$$

and finally *Euler's dilogarithm* (Leibniz 1696, Euler 1768, Rogers 1906) [535],[520]:

$$(68)\qquad Li_2(z) \equiv \sum_{n=1}^{\infty}\frac{z^n}{n^2},\ \ |z|<1.$$

Remark 6. In fact Euler considered only the case $b=-n$ in (66) [810, p. 3]. The general equation (66) is now known as the *Pochhammer integral* [810, p. 20], [715].

After Wallis many other mathematicians studied similar series, one of them was Stirling who

> by a remarkable numerical analysis, confirmed a result which would now be written in the form [246, p. 18]
> $$\Gamma(\frac{1}{2}) = \sqrt{\pi}.$$

The only important new discoveries during the next one hundred and fifty years, except the above results of Euler, were the following three summation formulas, one of which was discovered in China 1303 ...

In 1772, the Frenchman, A.T. Vandermonde (1735-1796) [189], [669], [901] stated the *Chu-Vandermonde summation formula*

$$(69)\qquad {}_2F_1(-n,b;c;1) = \frac{(c-b)_n}{(c)_n},\ \ n=0,1,\ldots.$$

> Johann Friedrich Pfaff (1765-1825) was one of the leading mathematicians in Germany in the latter part of the eighteenth and early nineteenth centuries, and is perhaps best known for his work on differential equations. Beginning in the late 1790's, he was teacher and friend

> of C.F. Gauss at Helmstadt, but it is not known how much influence he exerted on the latter's mathematical work [246, p. 26].
>
> Pfaff [707, pp.46-48] introduces what were later called contiguous functions by Gauss [246, p. 28].

In 1797 Pfaff [707, pp.51-52] [768] stated a similar formula, which is now called the *Pfaff-Saalschütz* summation formula or the Saalschütz summation formula for a terminating balanced hypergeometric series:

$$ {}_3F_2(a,b,-n;c,1+a+b-c-n;1) = \frac{(c-a,c-b)_n}{(c,c-a-b)_n},\ n=0,1,\ldots. \tag{70}$$

The relevant work of Euler and Pfaff was mentioned by Jacobi and in a well known textbook by Heine [428, pp. 357-398] [246, p. 29]. Pfaff [707] also proved a transformation formula for a ${}_2\phi_1(a,b;c;q,z)$ hypergeometric series, which is now called the *Pfaff-Kummer transformation formula* [554]

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}{}_2F_1(a,c-b;c;\frac{z}{z-1}). \tag{71}$$

In the period 1770-1808 Göttingen had a strong combinatorics school under C.F. Hindenburg (1741-1808) [810, p. 2].

> Eminent figures in this eighteenth century period were Hindenburg, Euler, Lambert, Lagrange, Kramp, Eschenbach, and Toepfer [364, p. 409].

Then on January 20 1812 C.F. Gauss (1777-1855) [347] proved the famous *Gauss summation formula*

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ Re(c-a-b)>0, \tag{72}$$

a generalization of the Vandermonde summation formula. In the following year Gauss found that the series ${}_2F_1(a,b;c;z)$ satisfies the following second order differential equation

$$ z(1-z)\frac{d^2y}{dz^2} + \{c-(1+a+b)z\}\frac{dy}{dz} - aby = 0, \tag{73}$$

having *regular singular points* at 0, 1, ∞, and at no other points [108, p. 224]. Gauss also found contiguous relations for the hypergeometric series. Gauss viewed the function ${}_2F_1(a,b;c;z)$ as a function of four variables.

The following formula is usually called *Gauss' second summation theorem* [262]:

$$ {}_2F_1(a,b;\frac{1}{2}(1+a+b);\frac{1}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}b)}. \tag{74} $$

Gauss also proved the following equation (γ denotes Euler's constant)

$$ \Psi(x) \equiv \frac{\Gamma(x)'}{\Gamma(x)} = -\frac{1}{x} - \gamma + \sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{x+n}), \tag{75} $$

which can be used to give a simple proof of an advanced version of the Euler-Maclaurin summation formula. Nowadays this function is called the digamma function.

Furthermore Gauss continued Euler's work on the Γ function and together with Legendre (1752-1833) found the *Legendre duplication formula*, which says that

$$ \Gamma(2x)\Gamma(\frac{1}{2}) = 2^{2x-1}\Gamma(x)\Gamma(x+\frac{1}{2}). \tag{76} $$

The theory of hypergeometric series after Gauss had continued with the work of Clausen 1828 [211], who found a formula which expresses the square of a ${}_2F_1$ series as a ${}_3F_2$ series and with the long and important 1836 [554] paper of Ernst Kummer (1810-1893), which contains a formula for the sum of a well-poised ${}_2F_1$ series with argument -1.

$$ {}_2F_1(a,b;1+a-b;-1) = \frac{\Gamma(1+a-b)\Gamma(1+\frac{1}{2}a)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)}. \tag{77} $$

In the same paper Kummer showed that the differential equation (73) has in all 24 solutions in terms of similar Gauss functions. The Italian mathematician Francesco Faà di Bruno studied for Cauchy in Paris 1850-1859 [957]. Faà di Bruno found a beautiful formula for the n:th derivative of a composite function and introduced the Faà di Bruno polynomials in several variables. In 1866 Mehler [616] first stated [922] the famous Mehler formula for the Hermite polynomials $H_n(x)$. In modern notation Mehler's formula takes the following form:

$$ \sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{t^n}{n!} = (1-4t^2)^{-\frac{1}{2}} exp\left(\frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right) \tag{78} $$

The *tangent numbers* T_{2n+1} are integers defined by André 1879 [23], [26, p. 380]:

$$ tan\ x = \sum_{n=0}^{\infty} \frac{T_{2n+1}x^{2n+1}}{(2n+1)!}. \tag{79} $$

The following equation obtains [158, p. 413]:

$$T_{2n+1} = ((1+x^2)D)^{2n+1}x|_{x=0}. \tag{80}$$

The *Euler numbers* or *secant numbers* $\mathcal{S}_{2n}$ are integers defined by André 1879 [23], [158, p. 414]:

$$\frac{1}{cos\ x} = \sum_{n=0}^{\infty} \frac{\mathcal{S}_{2n} x^{2n}}{(2n)!}. \tag{81}$$

The following equation obtains [158, p. 414]:

$$\mathcal{S}_{2n} = (x + (1+x^2)D)^{2n}1|_{x=0}. \tag{82}$$

In 1880 *Sonine* [813, p. 41] generalized the Laguerre polynomials. In 1879 Thomae [866] found a transformation formula for a ${}_3F_2$ series with unit argument. In the same year Thomae [865] found all the relationships between Kummer's 24 solutions. At this time Johannes Thomae (1840-1921) worked in Halle. He would later contribute strongly to the development of mathematics at the university of Jena [830]. If the parameters a_i, b_j in (59) are chosen such that the series does not terminate and does not become undefined, it can be shown that the series converges for all z if $p \leq r$, converges for $|z| < 1$ if $p = r+1$, and diverges for all $z \neq 0$ if $p > r+1$. The differential equation of the generalized hypergeometric series (59) is

$$(z\frac{d}{dz}+a_1)\ldots(z\frac{d}{dz}+a_p) - \frac{d}{dz}(z\frac{d}{dz}+b_1-1)\ldots(z\frac{d}{dz}+b_r-1) = 0 \tag{83}$$

[629]. For example, the hypergeometric series

$$y(z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{n!(d)_n(e)_n} z^n \tag{84}$$

satisfies the third order differential equation

$$\begin{aligned} & z^2(z-1)\frac{d^3y}{dz^3} + z\left((a+b+c+3)z-(d+e+1)\right)\frac{d^2y}{dz^2} + \\ & + \left((bc+ac+ab+a+b+c+1)z - de\right)\frac{dy}{dz} + abcy = 0, \end{aligned} \tag{85}$$

as was shown by Thomae 1870 [863], by Goursat 1883 [382] and by Pochhammer 1888 [714]. Pochhammer (1841-1920) from Kiel was one of the greatest contributors to hypergeometric series during this time.

In the twentieth century the main battle-field had moved to England and America. Again the history of q-calculus took a great leap forward with the help of hypergeometric series. Several equations for products of hypergeometric series have been proved by Orr 1899 [702]. In 1903

Dixon [230] proved the following summation formula for the well-poised series

$$
\begin{aligned}
&{}_3F_2(a,b,c;1+a-b,1+a-c;1) = \\
&\Gamma\left[\begin{array}{c} 1+\frac{1}{2}a, 1+a-b, 1+a-c, 1+\frac{1}{2}a-b-c \\ 1+a, 1+\frac{1}{2}a-b, 1+\frac{1}{2}a-c, 1+a-b-c \end{array}\right],
\end{aligned} \tag{86}
$$

provided that the series is convergent, i.e. $Re(1+\frac{1}{2}a-b-c) > 0$. In 1907 *Dougall* [241] proved the following summation formula for the very-well-poised 2-balanced i.e.

$$1+2a+n = b+c+d+e$$

series

$$
\begin{aligned}
&{}_7F_6\left[\begin{array}{c} \alpha \\ \beta \end{array};1\right] = \\
&= \frac{(1+a, 1+a-b-c, 1+a-b-d, 1+a-c-d)_n}{(1+a-b, 1+a-c, 1+a-d, 1+a-b-c-d)_n},
\end{aligned} \tag{87}
$$

where

$$\alpha \equiv a, 1+\tfrac{1}{2}a, b, c, d, e, -n,$$

$$\beta \equiv \tfrac{1}{2}a, 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a+n.$$

> The basic idea of representing a function by a contour integral with gamma functions in the integrand seems to be due to S. Pincherle (1853-1936) who used contours of a type which stems from Riemann's work. This side of the subject was developed extensively by R.Mellin and E.Barnes [76] [810, p. 3].

Pincherle and Mellin proved a formula for the asymptotic behaviour of the factorial function in a vertical strip [583]. In 1908 Barnes (1874-1953) [77] found a contour integral representation of a ${}_2F_1$ hypergeometric series

(88)

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} (-z)^s ds,$$

where $|arg(-z)| < \pi$.

In 1923 Whipple [933] showed that by iterating Thomae's ${}_3F_2$ transformation formula, one obtains a set of 120 such series, and he tabulated the parameters of these 120 series [900].

In 1925 Watson [919] [810] proved the following generalization of Gauss' second summation theorem:

$$
\begin{aligned}
&{}_3F_2(a,b,c;\frac{1}{2}(1+a+b),2c;1) = \\
&\Gamma\left[\begin{array}{c} \frac{1}{2},\frac{1}{2}+c,\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b,\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b+c \\ \frac{1}{2}+\frac{1}{2}a,\frac{1}{2}+\frac{1}{2}b,\frac{1}{2}-\frac{1}{2}a+c,\frac{1}{2}-\frac{1}{2}b+c \end{array}\right],
\end{aligned} \tag{89}
$$

provided that the series is convergent, i.e. $Re(\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b+c)>0$.

The *Hille-Hardy formula*, Hille [434], Hardy [418] and Watson [921] for the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ looks as follows:

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{n!}{\Gamma(\alpha+n+1)}L_n^{(\alpha)}(x)L_n^{(\alpha)}(y)t^n = \\
&=(1-t)^{-1}(xyt)^{-(\frac{1}{2})\alpha}exp\left(-\frac{(x+y)t}{1-t}\right)I_\alpha\left(\frac{2\sqrt{xyt}}{1-t}\right),
\end{aligned} \tag{90}
$$

where $I_\alpha(z)$ denotes the modified Bessel function.

A formula for the product of two Hermite polynonials was proved in 1938 by Feldheim [280], by Watson [924] and in 1940 by Burchnall [135]. In 1945 Rainville [734] found contiguous relations for the generalized hypergeometric series (59).

Finally we prove Bailey's summation formula for a ${}_2F_1$ series with argument $\frac{1}{2}$ to give a hint of the technique which is used in such calculations.

$$
{}_2F_1(a,1-a;c;\frac{1}{2}) = \frac{\Gamma(\frac{1}{2}c)\Gamma(\frac{c+1}{2})}{\Gamma(\frac{c+a}{2})\Gamma(\frac{c+1-a}{2})}. \tag{91}
$$

Proof. [810]. Put $b=1-a$ and $z=\frac{1}{2}$ in (66), take $(1-x)^2$ as the new variable, use the Beta function identity to obtain the identity

$$
{}_2F_1(a,1-a;c;\frac{1}{2}) = 2^{a-1}\frac{\Gamma(c)\Gamma(\frac{c+a-1}{2})}{\Gamma(c+a-1)\Gamma(\frac{c+1-a}{2})}. \tag{92}
$$

Now use the Legendre duplication formula. □

The theory of *multiple Gaussian hypergeometric series* is treated in the book by H.M. Srivastava & Karlsson [826], where the authors have collected a huge material which was previously widely scattered in the literature. Some of the applications of multiple Gaussian hypergeometric series are [826, p. 47]

(1) Statistical distributions,
(2) Decision theory,
(3) Genetics,

(4) Mechanics of deformable media,
(5) Communications engineering,
(6) Theory of elasticity,
(7) Perturbation theory,
(8) Theory of heat conduction,
(9) Problems involving dual integral equations with trigonometric and Hankel kernels,
(10) Problems involving the Tissot-Pochhammer equation, Laplace's linear differential (or difference) equation of higher order, and Schrödinger's equation,
(11) Asymptotic theory of Hill's equation, generalized Lamé equation, and generalized Weber equation, and
(12) Theory of Lie algebras and Lie groups.

In 1926 Appell & Kampé de Fériet [42] introduced some 2-variable hypergeometric series. One example is the Appell function

$$F_2(a,b,b';c,c';x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{m!n!(c)_m(c')_n}x^m y^n. \tag{93}$$

In 1931 J. Horn (1867-1946) showed that there are essentially 34 distinct convergent 2-variable hypergeometric series which are of second order [439]. Burchnall & Chaundy 1940 [133] and 1941 [134] found some interesting expansions of Appell's double hypergeometric series. In 1942 Burchnall [136] found differential equations associated with the Appell hypergeometric functions.

Multiple hypergeometric series have recently been used in connection with 9-j coefficients [897], [898], [712], [899], [818]. As was shown in [609], the Appell functions can be used in $SU(2)$ Seiberg-Witten theory. We will come back later to more applications of Appell functions.

1.7. **A short history of elliptic functions.** This section is based on [485]. The formula for the arclength of the ellipse involves an integral of the form

$$E(z)=\int_0^z \frac{\sqrt[2]{(1-(kx)^2)}dx}{\sqrt[2]{(1-x^2)}}, \tag{94}$$

where $0<k<1$. Another *elliptic integral* is

$$F(z)=\int_0^z \frac{dx}{\sqrt[2]{(1-x^2)(1-(kx)^2)}}, \tag{95}$$

where $0<k<1$. $F(z)$ and $E(z)$ are called the Jacobi form for the elliptic integral of the first and second kind. Legendre first worked with these integrals and he was followed by Abel (1802-1829) and Jacobi (1804-1851) , who in the 1820s inspired by Gauss, discovered that *inverting* $F(z)$ gave the doubly periodic *elliptic function*

$$F^{-1}(\omega) = sn(\omega). \tag{96}$$

Just as the rational functions on the *Riemann sphere* Σ form a field denoted $\mathbb{C}(z)$, the meromorphic functions on the torus $\mathbb{C}/\Omega$ are the doubly periodic elliptic functions on $\mathbb{C}$, which form a field, denoted $E(\Omega)$, where Ω is a fixed *lattice*. Both $E(\Omega)$ and $E_1(\Omega)$, the field of even elliptic functions, are extension fields of $\mathbb{C}$. Every lattice has a *fundamental region*, e.g. the *Dirichlet region* $D(\Omega)$ is a fundamental region for Ω. Just as the rational functions have order equal to the maximum of the degrees in the numerator and in the denominator, the order of an elliptic function f is the sum of the orders of the congruence classes of poles of f. The order is denoted $ord(f) = N$. By using elementary complex function theory, it is possible to prove the following statements about elliptic functions: An analytic elliptic function must be constant. The sum of the residues of f within a fundamental region is zero. There are no elliptic functions of order $N = 1$. If f has order $N > 0$, then f takes each value $c \in \Sigma$ exactly N times.

About 1850 K. Weierstrass (1815-1897) introduced the Weierstrass *sigma-function*, $\sigma(z)$, which is z multiplied with the product over all lattice points of those *elementary factors* $\mathcal{E}_2$ which have simple zeros at the lattice points. The *Weierstrass zeta-function* $\zeta(z)$ is the logarithmic derivative of $\sigma(z)$. The Weierstrass $\wp$-function $\in E_1(\Omega)$ is $-\frac{d\zeta}{dz}$. The Weierstrass $\wp$-function satisfies a differential equation, whose coefficients are the *Eisenstein series* for Ω.

Let Ω be a lattice with basis $\{\omega_1, \omega_2\}$, and let

$$\omega_3 = \omega_1 + \omega_2$$

If P is a fundamental region with $0, \frac{1}{2}\omega_1, \frac{1}{2}\omega_2, \frac{1}{2}\omega_3$ in its interior, then $\frac{1}{2}\omega_1, \frac{1}{2}\omega_2, \frac{1}{2}\omega_3$ are the zeros of $\frac{d\wp}{dz}$ in P.

Define

$$e_j = \wp(\frac{1}{2}\omega_j)$$

for $j = 1, 2, 3$. These numbers are mutually distinct.

Although the functions $\zeta(z)$ and $\sigma(z)$ are not elliptic, they obey translation properties, which can be proved using the *Legendre relation*. An elliptic function f of order N, which is elliptic with respect to a lattice Ω induces a function $\widehat{f} : \mathbb{C}/\Omega \mapsto \Sigma$, which is an N-sheeted branched covering of Σ by $\mathbb{C}/\Omega$.

Hermite discovered that the general quintic equation can be solved in terms of elliptic functions.

Elliptic functions are used in many models of mathematical physics, e.g. the Calogero-Moser system and the Euler top [110].

Weierstrass' elliptic functions can be expressed in terms of the *double gamma function* as was outlined in [74].

Elliptic functions played an important part in Andrew Wiles' proof of Fermat's last theorem.

1.8. **The Jacobi theta functions.** This section is based on [740] and [706]. The *theta functions* were known in special cases to Euler and Gauss but were completely and systematically explored only by Jacobi 1829. He found the original Jacobi's four theta functions, one of which is

$$\theta_3(z) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz), \tag{97}$$

where $q = \exp(\pi it), t \in \mathcal{U}$. He was then able to express his elliptic functions in terms of his four theta functions.

These functions have series which converge very rapidly when q is small. With the aid of his famous transformation formula and his pupils in Königsberg Jacobi was able to compute his theta functions with rapidly converging series. This enabled him to compile highly accurate tables of his elliptic functions, and hence, of course, of the elliptic integral $F(z)$. Much more importantly, it led to a beautiful theory, which is still flourishing.

Another equivalent way to define the theta functions is via the *theta characteristic*, which is a two by one matrix of integers, written

$$\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}.$$

Next, given a complex number ζ and another complex number τ, satisfying $Im\tau > 0$, and characteristic, we define the *first order theta function* with argument ζ and *theta period* τ, by

$$\theta(\zeta,\tau) = \sum_{n=-\infty}^{+\infty} \exp(i\pi(\tau(n+\frac{\epsilon}{2})^2 + 2(n+\frac{\epsilon}{2})(\zeta+\frac{\epsilon'}{2}))).$$

This function is analytic with respect to both variables.

Up to sign this theta function is determined completely by the residue classes, mod 2, of ϵ and ϵ', this is expressed by the *reduction formula.* Now we define the *reduced characteristics* to be those characteristics in which ϵ and ϵ' are either 0 or 1, and the *reduced representative* of an arbitrary characteristic to be that reduced characteristic whose entries are the least nonnegative residues mod 2 of ϵ and ϵ' respectively. We define a characteristic, not necessarily reduced, to be even or odd according as $\epsilon\epsilon'$ is even or odd. The theta function is an even or odd function of ζ, according as the characteristic is even or odd.

The theta function considered as a function only of ζ is entire and quasi-double periodic, with periods 1 and τ.

The function $sn(u)$ can now be expressed as a quotient of theta functions. The *theta constant* is the function value for $\zeta = 0$. By using the concept of *half-period* it is possible to express the *modulus* of $sn(u)$ as a quotient of theta constants. An n^{th} order theta function may be expressed as the product of n first order theta functions. Its characteristic is given by the matrix sum of the n characteristics. There is a formula for the number of linear independent even and odd theta functions of n^{th} order with a certain characteristic.

The *addition theorem for* $sn(u)$ can be proved by using the *theta identities*. The first period of $sn(u)$ can be expressed in terms of a theta constant by using the product expansions for the theta functions.

The *theta group* is the group generated by $z \mapsto z+2$ and $z \mapsto -\frac{1}{z}$. It has a fundamental domain bounded by the vertical lines $z = 1$, $z = -1$ and the unit circle.

One hundred years ago *Thomae* was able to obtain a formula connecting hyperelliptic theta functions evaluated at half theta characteristics and the branch points of a hyperelliptic curve [666].

The theta functions are very important in connection with *representation theory* for *Lie groups*. In the twentieth century Siegel has developed theta functions of several complex variables in connection with quadratic forms. We can associate with a compact Riemann surface a collection of first order theta functions. An attempt to solve algebraic equations with the help of theta functions is made in [659].

1.9. **Meromorphic continuation and Riemann surfaces.** This section is based on [485]. Let (D, f) be a *function element*, $a \in D$ and γ is a path in Σ from a to some point b in Σ. Then a *meromorphic continuation of* (D, f) *along* γ is a finite sequence of direct meromorphic continuations $(D, f) \sim (D_1, f_1) \sim (D_2, f_2) \sim \cdots \sim (D_m, f_m)$ such that:

(1) each *region* D_i is an open disc in Σ with $a \in D_1 \subseteq D$;
(2) there is a subdivision $0 = s_0 < s_1 < \ldots < s_m = 1$ of I such that $\gamma([s_{i-1}, s_i]) \subseteq D_i$ for $i = 1, 2, \ldots, m$, $I = [0, 1]$.

Let $p(z)$ be a polynomial. The *Riemann surface* S of $\sqrt{p(z)}$ is a 2-sheeted covering of Σ, which can be extended to a 2-sheeted *branched* *covering map* $\Psi(z)$. A point of multiplicity $k>1$ is called a *branchpoint* of *order* $k-1$.

The *monodromy group* of a Riemann surface of any many-valued function is the group of permutations of the sheets induced by meromorphic continuation along closed paths in Σ. In the following we consider a few things similar to the theory of manifolds. An equivalence class of atlases on a surface S is called a *complex structure* on S, which is then called an *abstract Riemann surface.* If Ω is a lattice in $\mathbb{C}$ then $\mathbb{C}/\Omega$ is a Riemann surface. The set $\mathcal{M}$ of all *germs* $[f]_a$, for all $a\in\Sigma$ is called the *sheaf of germs of meromorphic functions*, and it is an abstract Riemann surface. Let $w(z)$ be a possibly many-valued *algebraic function* of the form $A(z,w)=0$ for some polynomial $A(z,w)$ with Riemann surface S_A. S_A is compact and if $A(z,w)$ is an irreducible polynomial then S_A is connected.

Every Riemann surface is *orientable.* This result together with the previous theorems and a wellknown theorem from algebraic topology implies that the Riemann surface S_A of an irreducible algebraic equation $A(z,w)=0$ of degree n in w, and with branch-points of orders $n_1,\ldots,n_r$, has a *genus* which is given by the *Riemann-Hurwitz formula*:

$$g=1-n+\frac{1}{2}\sum_{i=1}^{r}n_i \tag{98}$$

Thus the expression $\sum_{i=1}^{r}n_i$, called the *total order of branching* is an even integer, greater than or equal to $2(n-1)$.

The *universal covering surface* $\widehat{S}$ was invented by Schwarz who told Klein in 1882, and Klein wrote Poincaré soon after. In 1908 Koebe employed the method in a rigorous manner to prove the general *uniformisation theorem*, which states that every simply connected Riemann surface is conformally equivalent to just one of:

(1) the Riemann sphere Σ
(2) the complex plane $\mathbb{C}$
(3) the upper half plane $\mathcal{U}$

The tori $\mathbb{C}/\Omega$ and $\mathbb{C}/\Omega'$ are *conformally equivalent* if and only if Ω and Ω' are similar. A lattice Ω acts discontinuously on $\mathbb{C}$ as a group of translations.

Every connected Riemann surface S may be obtained from one of the three simply connected Riemann surfaces $\widehat{S}=\Sigma$, $\mathbb{C}$ or $\mathcal{U}$ by factoring out a discontinuous subgroup G of $Aut\widehat{S}$. This leads to the study of the *symmetric space* $\mathcal{U}$.

Some examples of applications of Riemann surfaces are

(1) String theory.

(2) Solutions of Maxwells equations in radar technology.
(3) Nonlinear differential equations of mathematical physics.
(4) Schottky problem.

1.10. **Riemann theta functions, Schottky problem and hierarchies of integrable equations.** Consider again a Riemann surface S with genus g. Choose a symplectic basis $(\alpha_1, \ldots, \alpha_g; \beta_1, \ldots, \beta_g)$ of the homology group $H_1(S)$ [89]. The *Riemann period matrix* is defined by

$$\Omega_{ij} = \oint_{\beta_i} \omega_j, \tag{99}$$

where the $\omega_k's$ denote the h holomorphic differentials with the standard normalization

$$\oint_{\alpha_i} \omega_j = \delta_{ij}. \tag{100}$$

For a symmetric $g \times g$ matrix B with negative definite ReB (Riemann period matrix matrix) one associates the *Riemann theta function*

$$\Theta(z) = \sum_{m \in \mathbb{Z}^g} exp(\frac{1}{2} < Bm, m > + < m, z >).$$

In 1888 Schottky [789], [89] posed the *Schottky problem*: When is B the Riemann period matrix [214], [393] of a Riemann theta function? For $g = 4$ the problem was solved by Schottky 1903 [790]. For $g > 4$ the problem was solved by Novikov and Hirota with the help of the *KP-hierarchy.*

Most nonlinear dynamical systems are non-integrable [834]. In classical mechanics we find among these non-integrable systems those with chaotic behaviour. Now it is desirable to have a simple approach for deciding whether a dynamical system is integrable or not. In classical mechanics the so-called *Painlevé property* serves to distinguish between integrable or non-integrable systems. A necessary condition for an ordinary differential equation to have the Painlevé property is that there be a Laurent expansion which represents the general solution in a deleted neighbourhood of a pole. Recently, *Weiss* et.al. [927] have introduced the Painlevé property for partial differential equations. They applied the method to the *soliton* equations (KdV,KP) and found, in a remarkably straightforward manner, the well-known *Bäcklund transformation* [64].

Some nonlinear integrable differential equations of mathematical

physics are (the last three equations are systems of differential equations, which have a Lax pair.)

(1) sine-Gordon (SG) equation.
(2) Nonlinear Schrödinger (NLS) equation.
(3) KdV hierarchy.
(4) mKdV hierarchy.
(5) KP hierarchy.
(6) Ernst equation.
(7) Toda system.
(8) Whitham hierarchy.
(9) Supersymmetric KP(SKP) hierarchy [600].
(10) WDVV hierarchy.
(11) Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy.
(12) Jaulent-Miodek (JM) hierarchy.
(13) Boussinesq hierarchy [92], [606].
(14) xyz Heisenberg equation.
(15) Liouville equation [611], [578].
(16) Azbel-Hofstadter problem, Hofstadter problem, Harper equation, almost Mathieu equation.
(17) Discrete Painlevé equation [383].
(18) Nahm's equations.
(19) Burger's equation.
(20) Non-linear σ model [340], [441], [585].
(21) Harry Dym equation [726].
(22) Generalized Hénon-Heiles system [83] [248] [202] [742].
(23) Kaup equation [708].
(24) Zhiber-Mikhailov-Shabat (ZMS) equation [947].
(25) Zakharov-Shabat (ZS) hierarchy [896].
(26) Kupershmidt's equation [580].
(27) Sawada-Kotera equation [105], [944].
(28) BKP hierarchy [795],[582], [188].
(29) Calogero-Moser (CM) system.
(30) Quantum Calogero-Sutherland system [849], [850], [903].
(31) Maxwell-Bloch equation [359], [725].
(32) Ablowitz-Ladik hierarchy (ALH) [908].
(33) Davey-Stewartson (DS) equation [318], [555], [954].
(34) Kundu-Eckhaus equation [555].
(35) Landau-Lifshitz equation hierarchy [120], [121].
(36) $O(3)$ Chiral field equation hierarchy [120], [121].
(37) Tzitzeica equation [885], [787].
(38) Hirota-Satsuma equations [928].
(39) Hirota's bilinear difference equation [775].
(40) Bullough-Dodd equations [930].
(41) Caudrey-Dodd-Gibbon equations [930], [969].

(42) Ruijsenaars-Schneider (RS) model [762], [763], [764], [765], [766].
(43) Veselov-Novikov hierarchy [873].
(44) Wess-Zumino-Novikov-Witten (WZNW) model [654], [334].
(45) Ashkin-Teller model [15].
(46) Fateev-Zamolodchikov model [15].
(47) Potts model [15], [705].
(48) Heisenberg-Ising model [853], Ising model [128], [861], [553].
(49) Gelfand-Dickey hierarchies [34].
(50) Lattice Gelfand-Dickey hierarchies [674], [675].
(51) Yajima-Oikawa hierarchy [431].
(52) Melnikov hierarchy [431].
(53) Hubbard model [914].
(54) Thirring model [871].
(55) Calogero-Degasperis-Fokas equation [436].
(56) Interaction-round-a-face (IRF) model [486], [487].
(57) Kac-van Moerbeke (KM) system [281], [355].
(58) Lotka-Volterra system [281].
(59) Regge-Wheeler equation [63].
(60) Darboux-Zakharov-Manakov (DZM) system [220], [111].
(61) Hitchin systems [435], [508], [665], [854].
(62) Gaudin model [345], [854], [794].
(63) Haldane-Shastry spin chain [82].
(64) Garnier system [338], [728].
(65) Holt system [878].
(66) Kowalewski top [409], [746], [713], [440], [876], [874], [875].
(67) Goryachev-Chaplygin top [876], [109].
(68) Euler top [110].

It seems that all these equations are equivalent to linear systems of differential equations (for KdV *Lax-pair*). These linear systems contain a *spectral parameter*, which lives on a Riemann surface [548], [657]. An excellent introduction to this is given in [839].

> One of the most promising solution techniques for nonlinear differential equations rests on methods of algebraic geometry and leads to the so-called *finite-gap solutions* that can be expressed elegantly in terms of theta functions. Such methods were first used to construct periodic and quasiperiodic solutions to nonlinear evolution equations like the KdV and the sine-Gordon equation [524].

Usually these equations admit a whole hierarchy of *Poisson brackets* [47, p. 135].

The *KdV hierarchy* (3) was derived in 1895 by Korteweg and de Vries [536] from the *Navier-Stokes equation* of fluid dynamics as a special limit to give a model of nonlinear wave motions of shallow water observed in a canal. They showed that the KdV equation admits a solitary wave solution, or a one-soliton solution. The KdV hierarchy is connected with symmetric functions, Riemann surfaces, differential geometry, Hamiltonian structures, *Poisson brackets* and *Schouten brackets* [226], [100], [299], [594]. We will now try to give a 'short' introduction to the KdV hierarchy taken from [226, pp. 1-5] .

> The author ventures to offer one more monograph to the reader's attention despite a rather great amount of books in this field which recently appeared. The first of them was: Novikov 1980 [956] followed by many others. We do so because we believe that nowadays no one can claim to have written a book which can be regarded as a standard manual in this science as a whole. Neither do we. For a long time books had not been written but the flood of papers was overwhelming: many hundreds, maybe thousands of them. All this followed one single work by Gardner, Green, Kruskal and Miura 1967 [337] about the Korteweg-de Vries equation (KdV):
>
> $$\frac{\partial u}{\partial t} = 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial^3 x} \tag{101}$$
>
> which, before that work, seemed to be merely an unassuming equation of mathematical physics describing waves in shallow water. It is hard to understand how to guess regarding equation (101) that this equation is related to the *Schrödinger equation*, and that a remarkable algebra stands behind it. It is also surprising that the first idea about the exceptional properties of this equation was generated by computer calculations. This story was captivatingly narrated by one of the authors Kruskal M [550]. The relation of non-linear equations to linear ones was explained best by Lax [570]. More general hierarchies are obtained for operators of arbitrary orders and are called generalized KdV hierarchies.

The KP equation (5), which has the Weierstrass elliptic function as solution, is a three dimensional generalization of the KdV hierarchy (3), which was introduced in 1970 by Kadomtsev-Petviashvili [488], [657], [44]. After 1980, the KP theory has developed at an enormous rate and today encompasses a great number of mathematical and physical subjects. In 1997 [267] the Faà di Bruno polynomials were used in connection with cohomology in KP theory.

> It is generally believed that all known hierarchies of $(1+1)$-dimensional equations integrable by means of the inverse scattering method can be represented as certain reductions of a universal KP hierarchy of $(2+1)$-dimensional equations and/or of its extensions to modified and multicomponent cases. This hypothesis originates from a unifying *Sato theory* that describes the KP hierarchy in terms of a *pseudodifferential operator*[704].

The following can be said about the connection between *Bäcklund transformations* and the Painlevé property [929] Weiss:

> When a differential equation possesses the Painlevé property it is possible (for specific equations) to define a Bäcklund transformation. From the Bäcklund transformation, it is then possible to derive the Lax Pair, modified equations and *Miura transformations* associated with the completely integrable system under consideration. In this paper completely integrable systems are considered for which Bäcklund transformations may not be directly defined. These systems are of two classes. The first class consists of equations of Toda Lattice type e.g. sine-Gordon, Bullough-Dodd (40) equations. We find that these equations can be realized as the minus-one equation of sequences of integrable systems. Although the Bäcklund transformation may or may not exist for the minus-one equation, it is shown, for specific sequences, that the Bäcklund transformation does exist for the positive equations of the sequence. This, in turn, allows the derivation of Lax pairs and the recursion operation for the entire sequence. The second class of equations consists of sequences of *Harry Dym type* (21). These equations have *branch point singularities*, and, thus, do not directly possess the Painlevé property. Yet, by a process similar to the uniformization of algebraic curves, their solutions may be parametrically represented by meromorphic functions. For specific systems, this is shown to provide a natural extension of the Painlevé property.

The following models are used in particle physics. *String theory* is a special case of the *non-linear σ model* (20), and the *WZNW model*

(44) is a special case of the non-linear σ model except for an extra term. The WZNW model describes low-energy *hadrons*, and *nucleons* are described by solitons. Also π-meson fields are considered. In [13] Aitchison gives an elementary introduction to how the *Wess-Zumino terms* acts like a monopole in the space of scalar fields of the non-linear σ model. The $\mathrm{SL}(2,\mathbf{R})/\mathrm{U}(1)$ WZNW model can also be used in the theory of black holes. In [340] a Lax pair for the $(1+1)$-dimensional non-linear σ model was found.

In [953], the Painlevé property was applied to the classical Boussinesq hierarchy (13) and the Jaulent-Miodek hierarchy (12) to obtain Miura type transformations. In 1989 [559] the question of the integrability of the quantum Korteweg-de Vries (q-KdV) and quantum Boussinesq (q-B) equations and the relation of their conservation laws to the integrals of certain perturbed conformal field theories was discussed.

> W-algebras [226] play a prominent role in two dimensional physics. They first appeared in the context of integrable models (although under a different name) as Poisson structures associated with generalized KdV hierarchies [605].

The q-deformation of nonlinear integrable differential equations started in 1996 when Edward Frenkel, introduced a q-deformation of the KdV hierarchy and related soliton equations [323], see also [410], [514]. In the same paper a q-deformed N:th affine Toda equation and a *q-Miura transformation* was presented. Also in 1996 a q-Toda equation was presented in [880]. In 1998 Iliev [448], [449] solved the q-deformation of the KP hierarchy. Compare [57]. Also in 1998 Adler, Horozov and van Moerbeke [8] found a connection between the solutions of the q-KP and the 1-Toda lattice equations. The q-Calogero-Moser equation was presented in [450], [451]. In [17], [877] a q-deformed spherical top was presented.

The famous Burger's equation (19) reduces to the heat equation by the Cole-Hopf transformation [335]. In [837], [835], [859] a Lax pair was constructed for a space-dependent Burger's equation.

The Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) hierarchy (10)

> is a remarkable system of partial differential equations determining deformations of $2-$dimensional topological field theories, which was introduced in 1990 by Witten E. [940] and Dijkgraaf R., Verlinde E., Verlinde H. [228]. The WDVV equations consist of three groups of differential equations called the associativity equations,

the normalisation equations, and the quasihomogeneity equations [668].

The AKNS hierarchy (11) is an important bosonic, integrable hierarchy which, among other things, includes the KdV equation, the mKdV equation, the nonlinear Schrödinger equation (2), sine-Gordon equation (1) [43]. In [952] a connection between the AKNS hierarchy and the JM hierarchy was found. The vector constrained KP hierarchy [431] contains many interesting integrable systems like the AKNS, Yajima-Oikawa (51) and Melnikov hierarchies (52).

The Potts model (47) is used in statistical mechanics, phase transitions and in probability theory [398]. The ising model (48), which is connected to Potts model, is also used in ferromagnetism [128] and in two-dimensional gravity [553].

In [848]-[850] Sutherland obtained exact results for a quantum many-body problem for either fermions or bosons interacting in one dimension by a two-body potential $V(r) = \frac{g}{r^2}$ with periodic boundary conditions. The Calogero-Moser system (29) is Newton's equations for a remarkable class of solvable models describing interacting particles which was discovered about thirty years ago [704].

There is strong evidence that the CM model plays an important role in understanding the universal behaviour in *quantum chaos* [676]. The CM system may be viewed as being related to the A_{N-1} root system, and occurs in three variants: rational, hyperbolic/trigonometric, and elliptic. The Ruijsenaars-Schneider model (42) is a relativistic generalization of the CM system appearing also in three variants (rational, hyperbolic/trigonometric, and elliptic).

The nonlinear Schrödinger (NLS) equation (2) [236],[502]

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi + V\psi + F(\psi,\overline{\psi})\psi, \tag{102}$$

where $F(\psi,\overline{\psi})$ is a real-valued functional, is a Schrödinger equation admitting diffusion currents, and whose probability density satisfies a Fokker-Planck equation. An excellent account of the semiclassical limit of the defocusing NLS hierarchy is given in [796]. The Whitham equations (8) describe the semiclassical limit of the defocusing nonlinear Schrödinger equation [869]. In [362] the second derivatives of the prepotential with respect to Whitham time variables in the Seiberg-Witten theory are expressed in terms of Riemann theta-functions.

The stationary axisymmetric Einstein equations for the vacuum case in general relativity is equivalent to a single nonlinear differential equation for a complex potential, the so-called Ernst equation (6) [250]. The solutions found differ from similar solutions of other equations in several aspects; e.g. they are in general not periodic or quasiperiodic. The main difference is that this class is much richer than previously obtained ones [524].

The xyz Heisenberg equation (14) introduced in 1928 [429] as a quantum mechanical model of ferromagnetism

> was actively investigated by many scholars beginning with Hans Bethe. In the present-day literature on mathematical physics it is known by the jargon name the xyz model, which is used in the situation of general position $J_x \neq J_y \neq J_z$. The particular cases $J_x = J_y \neq J_z$ and $J_x = J_y = J_z = J$ are called the xxz and the xxx model, respectively.
>
> Heisenberg's model turned out to be very fruitful in the theory of magnetism and there is an extensive physical literature devoted to it. In recent years it has attracted the attention of mathematical physicists, since it turned out that the problem of finding the eigenvalues and eigenvectors for the Heisenberg Hamiltonean H can be solved exactly, in a certain sense, and requires beautiful mathematical constructions. The first step on the way to a solution of this problem was taken by Bethe in 1931 [99], when he examined the completely isotropic case of the xxx model and found the eigenvalues and eigenvectors of its Hamiltonean. Bethe's exact solution counts as one of the fundamental results in the theory of spin models, and the method proposed by him, the famous Bethe Ansatz, has been applied successfully to other many-particle models in one-dimensional mathematical physics.
>
> In 1972 Rodney Baxter in his remarkable papers [86]-[87] (the results were announced by him in 1971 in [84]-[85]) gave a solution for the xyz model. He discovered a link between the quantum xyz model and a problem of two-dimensional classical physics, the so-called eight-vertex model [855].

The xyz Heisenberg equation is equivalent to a linear system, whose spectral parameter lives on a torus [445]. In [961] the Lax pair was found from the Yang-Baxter equation and was applied to the xyz Heisenberg equation.

> One of the classical problems of the 19-century geometers was the study of the connection between differential geometry of submanifolds and nonlinear (integrable) PDE's. For instance, Liouville found the general solution of the equation (known now as the Liouville equation (15)) which describes minimal surfaces in E^3 [578]. Bianchi solved the general Goursat problem for the sine-Gordon equation (1) [103, p. 137], which encodes the whole geometry of the *pseudospherical surfaces.* Moreover the method of construction of a new pseudospherical surface from a given one, proposed by Bianchi [102], gives rise to the Bäcklund transformation for the SG equation [65].
>
> The connection between geometry and integrable PDE's became even deeper when Hasimoto [420] found the transformation between the equations governing the curvature and torsion of a nonstretching thin vortex filament moving in an incompressible fluid and the NLS equation. Several authors, including Lamb, Lakshmanan, Sasaki, Chern and Tenenblat related the ZS (25) spectral problem and the associated AKNS hierarchy (11) to the motion of curves in E^3 or to the pseudospherical surfaces and certain foliations on them [238].

In [834] the method of Weiss et.al. [927] was applied to the Liouville equation (15).

The *Azbel-Hofstadter problem* (16), [437], [266],[741] [729], [139], [444],[547] [1], [53], [709], [936], which in the Landau gauge is equivalent to a one-dimensional quasiperiodic difference equation, also known as the Harper equation, belongs to the class of integrable models of quantum field theory. Despite being just a one-particle problem [1], it has been solved by the Bethe Ansatz [936].

Nahm's equations (18) arise in Nahm's construction of monopole solutions in Yang-Mills theory [836], [584].

> The Tzitzeica equation (37) was derived in a classical geometric context as long ago as 1910 by Tzitzeica [885]. Indeed, even a linear representation and a Bäcklund transformation are cited therein. Seventy years later, the Tzitzeica equation was rediscovered and subsequently analysed by several authors [235], [474], [621], [786] in the setting of modern soliton theory. In terms of other physical applications, Gaffet [336] has shown that, for a

> particular class of gas laws, a (1+1)-dimensional anisentropic gas dynamics system may be reduced to the Tzitzeica equation [787].

In 1986 Weiss [929] and in 1996 Schief [787] showed that the Tzitzeica equation passes the Painlevé test. Schief [787] showed that the Tzitzeica equation possesses a Bäcklund transformation. In 1999 Conte & Musette & Grundland [216] found a Bäcklund transformation for the Tzitzeica equation and a Bäcklund transformation for the Hirota-Satsuma equation (38) at the same time.

In [537] a Hilbert space approach was applied to the study of symmetries and first integrals for analytic systems of ordinary differential equations. In [538] the Hilbert space approach was extended to the case of nonlinear partial differential equations.

The theory of integrable and partially integrable nonlinear evolution equations are presented in a didactic and concise way in [662]. Some examples of partially integrable PDEs are Kuramoto-Sivashinsky (KS) equation, Kolmogorov-Petrovskij-Piskunov (KPP) equation [588], [95], [510], [569]; Ginzburg-Landau equations.

1.11. **Multiple gamma functions, multiple q-gamma functions and $|q| = 1$.** The beginning of this section is based on Ueno K. & Nishizawa M. 1997 [889], [890].

> Multiple gamma functions were introduced by Barnes as an infinite product regularized by the multiple Hurwitz zeta function [72],[73],[74],[75]. Hardy [416],[417] studied this function from the viewpoint of the theory of elliptic functions. Kurokawa [560] showed that multiple gamma functions play an essential role to express gamma factors of the Selberg zeta function of compact Riemann surfaces and the determinants of the Laplacians on some Riemannian manifolds. In 1984 Moak [648] found a *q-analogue of Stirling's formula* for Jackson's q-gamma function. The multiple q-gamma function was introduced by Nishizawa in [680]. Inspired by Moak [648], Ueno & Nishizawa [888] derived a new representation for Jackson's q-gamma function. By the use of the Euler-Maclaurin summation formula, an asymptotic expansion (the higher Stirling formula) and an infinite product representation (the Weierstrass canonical product formula) for the Vignèras multiple gamma functions by considering the classical limit of the multiple

> q-gamma function was presented by Ueno & Nishizawa in [889], [890].

We next quote Nishizawa 1998 [682].

> q-Analysis with $|q| = 1$ is important for the studies on the XXZ-model etc. In [681], Ueno & Nishizawa presented a method of q-analysis with $|q| = 1$. By using Kurokawa's double sine function in place of the q-shifted factorial, they proved that the Euler type integral makes sense and that it gives a solution of a q-difference analogue of Gauss' hypergeometric equation under some conditions on parameters. In [682], this method is applied to a q-difference analogue of Lauricella's D-type hypergeometric equation.

2. **Introduction to q-calculus.**

2.1. **The q-difference operator.** In the rest of this paper we choose the principal branch of the logarithm:

$$-\pi < Im(log(q)) \leq \pi,$$

where Im denotes the imaginary part of a complex number.

The variables $a, b, c, a_1, a_2, \ldots, b_1, b_2, \ldots \in \mathbb{C}$ denote parameters in hypergeometric series or q-hypergeometric series. The variables i, j, k, l, m, n, p, r will denote natural numbers except for certain cases where it will be clear from the context that i will denote the imaginary unit.

The power function is defined by $q^a = e^{alog(q)}$.
Furthermore the q-analogues of the natural numbers,of the factorial function and of the semifactorial function [206, p. 43], [461, p. 314] are defined by:

$$\{n\}_q = \sum_{k=1}^{n} q^{k-1},\ \{0\}_q = 0,\ q \in \mathbb{C}, \tag{103}$$

$$\{n\}_q! = \prod_{k=1}^{n} \{k\}_q,\ \{0\}_q! = 1,\ q \in \mathbb{C}. \tag{104}$$

$$\{2n-1\}_q!! = \prod_{k=1}^{n} \{2k-1\}_q,\ q \in \mathbb{C}, \tag{105}$$

$$\{2n\}_q!! = \prod_{k=1}^{n} \{2k\}_q,\ q \in \mathbb{C}. \tag{106}$$

In 1908 Jackson [458] reintroduced [916],[357] the Euler-Jackson q-difference operator

$$(D_q\varphi)(x) = \frac{\varphi(x) - \varphi(qx)}{(1-q)x},\ q \in \mathbb{C}\backslash\{1\}. \tag{107}$$

If we want to indicate the variable which the q-difference operator is applied to, we denote the operator $(D_{q,x}\varphi)(x,y)$. The limit as q approaches 1 is the derivative

$$\lim_{q\to 1} (D_q\varphi)(x) = \frac{d\varphi}{dx}, \tag{108}$$

if φ is differentiable at x.

Example 1.

$$D_q(x^\alpha) = \frac{x^\alpha - (qx)^\alpha}{(1-q)x} = \frac{x^\alpha(1-q^\alpha)}{x(1-q)} = \{\alpha\}_q x^{\alpha-1},\ \alpha \in \mathbb{C}. \tag{109}$$

The formulae for the q-difference of a sum, a product and a quotient of functions are [261]:

$$D_q(u(x) + v(x)) = D_q u(x) + D_q v(x). \tag{110}$$

$$D_q(u(x)v(x)) = D_q u(x)v(x) + u(qx)D_q v(x). \tag{111}$$

$$D_q\left(\frac{u(x)}{v(x)}\right) = \frac{v(x)D_q u(x) - u(x)D_q v(x)}{v(qx)v(x)},\ v(qx)v(x) \neq 0. \tag{112}$$

Applying the Taylor formula to the right hand side of (107) we obtain the following expression for the q-difference operator:

$$D_q(f(x)) = \sum_{k=0}^{\infty} \frac{(q-1)^k}{(k+1)!} x^k f^{(k+1)}(x), \tag{113}$$

provided that f is analytic.

The following lemma was proved by Hjalmar Rosengren. Compare [407], where the special case $k = 1$ of this equation was stated.

Lemma 2.1. *The chain rule for the q-difference operator. Let $g(x)$ be the function $g(x) : x \mapsto cx^k$. Then*

$$D_q(f \circ g)(x) = \left(D_{q^k}(f)\right)(g(x))D_q(g)(x). \tag{114}$$

Proof.

$$LHS = \frac{f(cx^k) - f(c(xq)^k)}{x(1-q)}$$

$$RHS = \{k\}_q\ cx^{k-1}\frac{f(cx^k) - f((cx^k)q^k)}{cx^k(1-q^k)} = LHS.$$

□

Remark 7. In 1982 Ira Gessel [353] proposed a q-analogue of the chain rule based on a q-analogue of functional composition for Eulerian generating functions.

2.2. **Heine's letter to Dirichlet 1846.** In 1846 Dr.E.Heine, a 'Privatdocent' from Bonn wrote the following letter to professor Dirichlet, which was published in the Crelle journal the same year [425].

> Sehr viele Reihen, darunter auch solche, auf welche die elliptischen Functionen führen, sind in der allgemeinen Reihe
>
> $$1+\sum_{k=1}^{\infty}\frac{\prod_{m=0}^{k-1}(q^{a+m}-1)\prod_{m=0}^{k-1}(q^{b+m}-1)}{\prod_{m=0}^{k-1}(q^{m}-1)\prod_{m=0}^{k-1}(q^{c+m}-1)}z^k$$
>
> enthalten, die ich zur Abkürzung mit
>
> $${}_2\phi_1(a,b;c;q,z)$$
>
> bezeichne, gerade so wie es bei der hypergeometrischen Reihe zu geschehen pflegt, in welche unser ϕ für $q=1$ übergeht. Es scheint mir nicht uninteressant, die ϕ ganz ähnlich zu behandeln, wie **Gauss** die ${}_2F_1$ in den 'Disquisitiones generales' untersucht hat. Ich will hier nur flüchtige **Andeutungen** zu einer solchen Übertragung geben. Es entspricht jedem F im &5 der Disquisitiones eine Reihe ϕ. Einige Formeln entsprechen zwei oder mehr verschiedene ϕ, nämlich denen, in welchen a oder b unendlich werden. So hat die Reihe für e^t zwei Analoge.

The two series that Heine is talking about are the two q-analogues of the exponential function found by Euler.

2.3. **Basic definitions for q-hypergeometric series with the classical notation (Watson).** The q-hypergeometric series considered by Heine 1846 [425] is a generalization of the hypergeometric series, given by the formula

$$ {}_2\phi_1(a,b;c;q,z)=\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}z^n, \tag{115}$$

where the q-shifted factorial is defined by

$$(116)\qquad (a;q)_n = \begin{cases} 1, & n = 0; \\ \prod_{m=0}^{n-1}(1-aq^m), & n = 1,2,\ldots, \end{cases}$$

and it is assumed that (see [343, p. 3]) $c \neq q^{-m},\ m = 0,1,\ldots.$

The series (115) converges absolutely for $|z| < 1$ when $0 < |q| < 1$ and it converges (at least termwise) to Gauss' series as $q \to 1$.

As Heine pointed out: Der Fall $q > 1$ lässt sich leicht auf den Fall zurückführen, wo $0 < |q| < 1$; man kann sich dazu die Formel

$$(117)\qquad {}_2\phi_1(q^a, q^b; q^c; q, z) = {}_2\phi_1(q^a, q^b; q^c; \frac{1}{q}, q^{a+b-c-1}z)$$

bedienen, die sich unmittelbar aus der Definition der ϕ ergibt.

2.4. **A new notation and a new method for q-hypergeometric series.** Dear reader, please take a look at the following pages. q-Calculus has in the last twenty years served as a bridge between mathematics and physics. The majority of scientists in the world who use q-calculus today are physicists.

> The field has expanded explosively, due to the fact that applications of basic hypergeometric series to the diverse subjects of combinatorics, quantum theory, number theory, statistical mechanics, are constantly being uncovered. The subject of q-hypergeometric series is the frog-prince of mathematics, regal inside, warty outside [939].

Furthermore it is said that

> progress in q-calculus is heavily dependent on the use of a proper notation [506].

This is a modest attempt to present a new notation for q-calculus and in particular for q-hypergeometric series, which is compatible with the old notation. Also a new method, which follows from this notation is presented. This notation will be similar to Gauss' notation for hypergeometric series and in the spirit of Heine [425], Pringsheim [719] and Smith [811]. Jackson also used a similar notation in some of his last papers [465], [466].

Remark 8. A similar notation was proposed by Rajeswari V. & Srinivasa Rao K. in 1991 [736] and in 1993 [815, p. 72] in connection with the q-analogues of the 3-j and 6-j coefficients. See also [816] and [817].

Van der Jeugt, Srinivasa Rao and Pitre have recently published several interesting articles on both hypergeometric series and multiple hypergeometric series [897], [898], [899], [818].

In 1999 Van der Jeugt & Srinivasa Rao [900] showed that certain two-term transformation formulas between q-hypergeometric series easily can be described by means of invariance groups.

The q-hypergeometric series was developed by Heine 1846 [425] as a generalization of the hypergeometric series:

$$ {}_2\phi_1(a,b;c|q,z)=\sum_{n=0}^{\infty}\frac{\langle a;q\rangle_n\langle b;q\rangle_n}{\langle 1;q\rangle_n\langle c;q\rangle_n}z^n, \tag{118}$$

with my new notation for the q-shifted factorial

$$\langle a;q\rangle_n=\begin{cases}1, & n=0;\\ \prod_{m=0}^{n-1}(1-q^{a+m}) & n=1,2,\ldots.\end{cases} \tag{119}$$

Remark 9. The relation between the new and the old notation is

$$\langle a;q\rangle_n=(q^a;q)_n. \tag{120}$$

The following modification of the q-shifted factorial will be useful:

$$\widetilde{\langle a;q\rangle}_n=\prod_{m=0}^{n-1}(1+q^{a+m}), \tag{121}$$

where the tilde denotes an involution which changes a minus sign to a plus sign in all the n factors of $\langle a;q\rangle_n$. The tilde can also enter as a parameter in a q-hypergeometric series as is shown in the proof of (398).

Remark 10. In a few cases the parameter a in (119) will be infinite $(0<|q|<1)$. Three examples are (143), (395), the proof of (397) ; they correspond to multiplication by 1.

Recall that in the classical notation $\widetilde{\langle a;q\rangle}_n$ is denoted by $(-q^a;q)_n$.

Definition 3. Generalizing Heine's series, we shall define a q-hypergeometric series by [343, p. 4], [401, p. 345]:

$$\begin{aligned}&{}_p\phi_r(a_1,\ldots,a_p;b_1,\ldots,b_r|q,z)\equiv {}_p\phi_r\left[\begin{array}{c}a_1,\ldots,a_p\\ b_1,\ldots,b_r\end{array}|q,z\right]=\\ &=\sum_{n=0}^{\infty}\frac{\langle a_1;q\rangle_n\ldots\langle a_p;q\rangle_n}{\langle 1;q\rangle_n\langle b_1;q\rangle_n\ldots\langle b_r;q\rangle_n}\left[(-1)^nq^{\binom{n}{2}}\right]^{1+r-p}z^n,\end{aligned} \tag{122}$$

where $q \neq 0$ when $p > r+1$. Furthermore $a_1, \ldots, a_p,\ b_1, \ldots, b_r \in \mathbb{C}$,

$$b_j \neq 0,\ j = 1, \ldots, r,\ b_j \neq -m,\ j = 1, \ldots, r, m \in \mathbb{N},$$

$b_j \neq \frac{2m\pi i}{logq},\ j = 1, \ldots, r, m \in \mathbb{N}$ [811].

The series

$$ {}_{r+1}\phi_r(a_1, \ldots, a_{r+1}; b_1, \ldots, b_r|q, z) \tag{123}$$

is called *k-balanced* if $b_1 + \ldots + b_r = k + a_1 + \ldots + a_{r+1}$ and $z = q$; and a 1-balanced series is called balanced (or Saalschützian). Analogous to the hypergeometric case, we shall call the q-hypergeometric series (123)*well-poised* if its parameters satisfy the relations

$$1 + a_1 = a_2 + b_1 = a_3 + b_2 = \ldots = a_{r+1} + b_r. \tag{124}$$

The q-hypergeometric series (123) is called *nearly-poised* [71] if its parameters satisfy the relation

$$1 + a_1 = a_{j+1} + b_j \tag{125}$$

for all but one value of j in $1 \leq j \leq r$.

The q-hypergeometric series (123) is called *almost poised* [127] if its parameters satisfy the relation

$$\delta_j + a_1 = a_{j+1} + b_j,\ 1 \leq j \leq r, \tag{126}$$

where δ_j is $0, 1$ or 2.

If the series (123) is well-poised and if, in addition

$$a_2 = 1 + \tfrac{1}{2}a_1,\ a_3 = \widetilde{1 + \tfrac{1}{2}a_1}, \tag{127}$$

then it is called a very-well-poised series.

We have changed the notation for the q-hypergeometric series (118) slightly according to the new notation which is introduced in this paper. The terms to the left of | in (118) and in (122) are thought to be exponents, and the terms to the right of | in (118) and in (122) are thought to be ordinary numbers.

There are several advantages with this new notation:

(1) The theory of hypergeometric series and the theory of q-hypergeometric series will be united.
(2) We work on a logarithmic scale; i.e. we only have to add and subtract exponents in the calculations. Compare with the 'index calculus' from [41].
(3) The conditions for k-balanced hypergeometric series and for k-balanced q-hypergeometric series are the same.

(4) The conditions for well-poised and nearly-poised hypergeometric series and for well-poised and nearly-poised q-hypergeometric series are the same. Furthermore the conditions for almost poised q-hypergeometric series are expressed similarly.
(5) The conditions for very-well-poised hypergeometric series and for very-well-poised q-hypergeometric series are similar. In fact, the extra condition for a very-well-poised hypergeometric series is $a_2 = 1 + \frac{1}{2}a_1$, and the extra conditions for a very-well-poised q-hypergeometric series are $a_2 = 1 + \frac{1}{2}a_1$ and $a_3 = \widetilde{1 + \frac{1}{2}a_1}$.
(6) We don't have to distinguish between the notation for integers and non-integers in the q-case anymore.
(7) The q-analogues are easier to recognize.

Remark 11. A q-analogue of a ${}_{p-1}\phi_{p-2}$ 2-balanced hypergeometric series can be a balanced ${}_{p}\phi_{p-1}$ q-hypergeometric series. One example is Jackson's formula (426). This is because of (195), which expresses a q-shifted factorial times its tilde-version as a q-shifted factorial with base q^2.

To justify the following three definitions of infinite products we remind the reader of the following wellknown theorem from complex analysis Rudin [761, p. 300]:

Theorem 2.2. *Let Ω be a region in the complex plane. Suppose $f_n \in H(\Omega)$ for $n = 1, 2, 3, \ldots$, no f_n is identically 0 in any component of Ω, and*

$$\sum_{n=1}^{\infty} |1 - f_n(z)| \tag{128}$$

converges uniformly on compact subsets of Ω. Then the product

$$f(z) = \prod_{n=1}^{\infty} f_n(z) \tag{129}$$

converges uniformly on compact subsets of Ω. Hence $f \in H(\Omega)$.

Now we define

$$\langle a; q\rangle_\infty = \prod_{m=0}^{\infty} (1 - q^{a+m}),\ 0 < |q| < 1. \tag{130}$$

$$\widetilde{\langle a; q\rangle}_\infty = \prod_{m=0}^{\infty} (1 + q^{a+m}),\ 0 < |q| < 1. \tag{131}$$

$$(a;q)_\infty = \prod_{m=0}^{\infty}(1-aq^m),\ \ 0<|q|<1. \tag{132}$$

We shall henceforth assume that $0<|q|<1$ whenever $\langle a;q\rangle_\infty$ or $(a;q)_\infty$ appears in a formula, since the infinite product in (130) diverges when

$$q^a \neq 0,\ \ |q| \geq 1.$$

Since products of q-shifted factorials occur so often, to simplify them we shall frequently use the more compact notations

$$\langle a_1,\ldots,a_m;q\rangle_n \equiv \prod_{j=1}^{m}\langle a_j;q\rangle_n. \tag{133}$$

$$\langle a_1,\ldots,a_m;q\rangle_\infty \equiv \prod_{j=1}^{m}\langle a_j;q\rangle_\infty. \tag{134}$$

Let k denote a positive integer. Then we define

$$\langle a;q^k\rangle_n = \begin{cases} 1, & n=0; \\ \prod_{m=0}^{n-1}(1-q^{a+mk}), & n=1,2,\ldots. \end{cases} \tag{135}$$

$$\langle a;q^k\rangle_\infty = \prod_{m=0}^{\infty}(1-q^{a+mk}),\ \ 0<|q|<1. \tag{136}$$

$$(a;q^k)_n = \begin{cases} 1, & n=0; \\ \prod_{m=0}^{n-1}(1-aq^{mk}), & n=1,2,\ldots. \end{cases} \tag{137}$$

$$(a;q^k)_\infty = \prod_{m=0}^{\infty}(1-aq^{mk}),\ \ 0<|q|<1. \tag{138}$$

$$\langle a;q^{-k}\rangle_\infty = \prod_{m=0}^{\infty}(1-q^{-a-mk}),\ \ 1<|q|. \tag{139}$$

$$(a;q^{-k})_\infty = \prod_{m=0}^{\infty}(1-aq^{-mk}),\ \ 1<|q|. \tag{140}$$

For negative subscripts, the shifted factorial and the q-shifted factorials are defined by

$$(a)_{-n} \equiv \frac{1}{(a-1)(a-2)\cdots(a-n)} \equiv \frac{1}{(a-n)_n} = \frac{(-1)^n}{(1-a)_n}. \tag{141}$$

$$\begin{aligned}
&\langle a;q\rangle_{-n} \equiv \\
&\equiv \frac{1}{(1-q^{a-n})(1-q^{1+a-n})\cdots(1-q^{a-1})} \equiv \frac{1}{\langle a-n;q\rangle_n} = \\
&= \frac{q^{0+1+2+\ldots+(n-1)}}{(1-q^{a-1})(q-q^{a-1})\cdots(q^{n-1}-q^{a-1})} = \\
&= \frac{q^{n(1-a)+0+1+2+\ldots+(n-1)}}{(q^{1-a}-1)(q^{2-a}-1)\cdots(q^{n-a}-1)} = \\
&\frac{(-1)^n q^{n(1-a)+\binom{n}{2}}}{(1-q^{1-a})(1-q^{2-a})\cdots(1-q^{n-a})} = \frac{(-1)^n q^{n(1-a)+\binom{n}{2}}}{\langle 1-a;q\rangle_n}.
\end{aligned} \tag{142}$$

Denote the series (122) $\sum_{n=0}^{\infty} A_n z^n$. The quotient $\frac{A_{n+1}}{A_n}$ is a rational function $R(q^n)$. $R(q^n)$ contains the factor $(1-q^{1+n})$ in the denominator. After simplification, the maximal degree of the numerator and of the denominator is called the order of the generalized Heine series. We can say the following about the convergence of the q-series ${}_p\phi_r$ Hahn 1949 [401]: The series of type $(p, p-1)$ converge for $|z| < 1$ when $0 < |q| < 1$; because the coefficient of z^n is bounded and tends to 1. The series of type (p, r) with $p \leq r$ are entire functions, and the series converge faster, the bigger the difference $r - p$. The series of type (p, r) with $p > r - 1$ diverge $\forall z \neq 0$ if no $\{a_j\}_{j=1}^{p}$ is equal to a negative integer, when the series reduces to a polynomial.

The function (122) is one-valued and regular, and its only singularity is a pole of order 1 at $z = (\frac{1}{q})^n$, $n = 0, 1, \ldots$ Pringsheim 1910 [719], Smith 1911 [811].

The analytic continuation of a series of type $(p, p-1)$ can be made with the help of its q-difference equation [401] or by its asymptotic expansion see Watson 1910 [918]; in this way we obtain function values in concentric circles of increasing radius.

In 1998 [686] the analytic continuation of a q-hypergeometric series was obtained by an operator algebra method.

Heine had shown that his series satisfies at least one second order linear q-difference equation and in [864] his pupil Thomae found all the linear relations between the solutions of Heine's q-difference equation.

2.5. **The q-binomial theorem.** Euler found the following two q-analogues of the exponential function:

(143)

$$e_q(z) \equiv {}_1\phi_0(\infty; -|q, z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{\langle 1; q\rangle_n} = \frac{1}{(z;q)_\infty},\ |z|<1,\ 0<|q|<1.$$

(144)

$$e_{\frac{1}{q}}(z) \equiv {}_0\phi_0(-; -|q, -z) \equiv \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{\langle 1; q\rangle_n} z^n = (-z;q)_\infty,\ 0<|q|<1,$$

where ${}_0\phi_0$ is defined by (122). The second function is an entire function just as the usual exponential function. Our next goal is to prove a q-analogue of (64). The history of the q-binomial theorem goes back to an 1820 book of Schweins who gives this formula and refers to an earlier work of Rothe [760] from 1811 [629]. Gauss published similar formulae in 1811. In 1829 Jacobi [467] referred to the book of Schweins and in 1843 Cauchy [172] was the first person to prove the q-binomial theorem:

Theorem 2.3.
(145)

$${}_1\phi_0(a; -|q, z) \equiv \sum_{n=0}^{\infty} \frac{\langle a; q\rangle_n}{\langle 1; q\rangle_n} z^n = \frac{(zq^a;q)_\infty}{(z;q)_\infty},\ |z|<1,\ 0<|q|<1.$$

The proof is due to Heine 1878 [427, p. 97ff]. We will prove the following slightly more general theorem, which will be used in the proof of the Jacobi triple product identity (216). The proof is a modification of the argument on page 8 in the book by Gasper and Rahman [343]. Heine's proof is obtained by putting $k = 1$.

Theorem 2.4. *Let k denote an arbitrary positive integer. Then*

$$\sum_{n=0}^{\infty} \frac{\langle a; q^k\rangle_n}{\langle k; q^k\rangle_n} z^n = \frac{(zq^a;q^k)_\infty}{(z;q^k)_\infty}, |z|<1, 0<|q|<1. \tag{146}$$

Proof. Put

$$h_a(z) = \sum_{n=0}^{\infty} \frac{\langle a; q^k\rangle_n}{\langle k; q^k\rangle_n} z^n, |z|<1, 0<|q|<1. \tag{147}$$

Compute the difference

$$h_a(z) - h_{a+k}(z) = \sum_{n=1}^{\infty} \frac{\langle a; q^k\rangle_n - \langle a+k; q^k\rangle_n}{\langle k; q^k\rangle_n} z^n =$$

$$= \sum_{n=1}^{\infty} \frac{\langle a+k; q^k \rangle_{n-1}}{\langle k; q^k \rangle_n} (1 - q^a - (1 - q^{a+kn})) z^n = q^a \sum_{n=1}^{\infty} \frac{\langle a+k; q^k \rangle_{n-1}}{\langle k; q^k \rangle_n} \times$$

$$\times (q^{kn} - 1) z^n = -zq^a \sum_{n=1}^{\infty} \frac{\langle a+k; q^k \rangle_{n-1}}{\langle k; q^k \rangle_{n-1}} z^{n-1} = -zq^a h_{a+k}(z).$$

This implies the following recurrence for $h_a(z)$:

$$h_a(z) = (1 - zq^a) h_{a+k}(z). \tag{148}$$

Next compute the difference

$$h_a(z) - h_a(q^k z) = \sum_{n=1}^{\infty} \frac{\langle a; q^k \rangle_n}{\langle k; q^k \rangle_n} (z^n - q^{kn} z^n) = \sum_{n=1}^{\infty} \frac{\langle a; q^k \rangle_n}{\langle k; q^k \rangle_{n-1}} z^n =$$

$$= z(1 - q^a) \sum_{n=1}^{\infty} \frac{\langle a+k; q^k \rangle_{n-1}}{\langle k; q^k \rangle_{n-1}} z^{n-1} = z(1 - q^a) h_{a+k}(z).$$

This implies the following equation for $h_a(z)$:

$$h_a(z) = \frac{1 - zq^a}{1 - z} h_a(q^k z). \tag{149}$$

The proof is completed by iterating this n times and by letting n go to infinity. □

Now equation (143) follows from (145) by letting $a \to \infty$, and (144) follows from (145) by substituting $z \to zq^{-a}$ and letting $a \to -\infty$. This procedure is the q-analogue [255] of the formula

$$\lim_{a \to \infty} (1 + \frac{z}{a})^a = e^z.$$

Remark 12. In the following proofs we will often use the following special case of the q-binomial theorem, which is obtained by substituting $z \mapsto q^b$ in (146):

$$\sum_{n=0}^{\infty} \frac{\langle a; q^k \rangle_n}{\langle k; q^k \rangle_n} q^{bn} = \frac{\langle a+b; q^k \rangle_\infty}{\langle b; q^k \rangle_\infty}, 0 < |q| < 1. \tag{150}$$

The 'dual' of the above equation is

$$\sum_{n=0}^{\infty} \frac{\langle a; q^k \rangle_n}{\langle k; q^k \rangle_n} (-q^b)^n = \frac{\langle \widetilde{a+b}; q^k \rangle_\infty}{\langle \tilde{b}; q^k \rangle_\infty}, 0 < |q| < 1. \tag{151}$$

The following special case of the q-binomial theorem will sometimes be used:

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{\langle 1; q \rangle_n} z^n = \frac{(za; q)_\infty}{(z; q)_\infty}, \ |z| < 1, \ 0 < |q| < 1. \tag{152}$$

The following corollaries of (146) are extensions of Euler's formulae (143) and (144):

$$\sum_{n=0}^{\infty} \frac{z^n}{\langle k; q^k \rangle_n} = \frac{1}{(z; q^k)_\infty}, \ |z| < 1, \ 0 < |q| < 1, \tag{153}$$

$$\sum_{n=0}^{\infty} \frac{q^{k\binom{n}{2}}}{\langle k; q^k \rangle_n} z^n = (-z; q^k)_\infty, \ 0 < |q| < 1, \ z \in \mathbb{C}. \tag{154}$$

With the help of the q-binomial theorem (145) we can now explain what a q-analogue really is. In the process of q-deformation a branching singularity ($q = 1$) disappears and is replaced by an infinite sequence of poles and zeros. According to Johannes Thomae 1869 [862, p. 262], Smith 1911 [811, p. 51], the solution of the difference equation

$$(1 - z)U(z) = (1 - q^a z)U(qz) \tag{155}$$

is the function

$$p(a, q, z) = \lim_{n\to\infty} (1 - q^n z)^{-a} \prod_{m=0}^{n-1} \frac{1 - zq^{a+m}}{1 - zq^m}, \tag{156}$$

which makes sense both for $0 < |q| < 1$ and $|q| > 1$. The first statement is obvious by (145). The last statement is proved as follows.

$$p(a, q, z) = \lim_{n\to\infty} (-z)^{-a} (1 - \frac{1}{zq^n})^{-a} \prod_{m=0}^{n-1} \frac{1 - \frac{1}{zq^{a+m}}}{1 - \frac{1}{zq^m}} \tag{157}$$

or

$$p(a, q, z) = (-z)^{-a} p(a, \frac{1}{q}, \frac{1}{z}), \tag{158}$$

so that this follows from the previous case. For $q = 1$, the infinite product is equal to one and the result is the multiply valued function$(1 - z)^{-a}$.

We infer that $p(a, q, z)$ is analytic in q and z, and for $0 < |q| < 1$, $p(a, q, z) \equiv {}_1\phi_0(a; -|q, z)$. This means that both functions and their analytic continuations are identical everywhere.

2.6. **Some aspects on the development of q-calculus in the period 1893-1950.** In his work 1893-1895, Rogers inspired by Heine's book 'Handbuch der Kugelfunktionen', introduced a set of orthogonal polynomials which can be expressed in terms of q-hypergeometric series and which have the Gegenbauer polynomials as limits when $q \to 1$. They are orthogonal with respect to an absolutely continuous measure on $(-1, 1)$. As a result of this Rogers proved the two Rogers-Ramanujan identities, which express an infinite sum as a quotient of infinite products.

At this stage the main battle-field had moved to England and America. Some of the greatest q-analysts of this time were Ramanujan, Watson, Jackson, Bailey, Daum, Pringsheim, Smith, Ward, MacMahon, Carmichael, Mason, Ryde, Birkhoff, Adams, Starcher, and Trjitzinsky. The first 7 of these worked mainly on q-hypergeometric series except Jackson who did much more and except Ramanujan who also worked on number theory and theta functions. G.N.Watson (1886-1965) became famous when his giant book 'A treatise on the theory of Bessel functions' was published in 1952. The last 7 of the abovementioned persons worked on the solutions of so-called q-difference equations. Starting from 1904 the English reverend Jackson published a number of mathematical articles devoted entirely to q-calculus which would last until 1951. Jackson had studied elliptic functions, special functions and the work of Heine, Thomae and Rogers. Jackson started to find q-analogues of trigonometric functions, Bessel functions, Legendre polynomials and the gamma function. He explained the connection between the q-gamma function and elliptic functions.

In 1910 Watson [918] proved a q-analogue of Barnes contour integral expression for a hypergeometric series.

Almost everybody had now forgotten Rogers' brilliant proof of the Rogers-Ramanujan identities in 1894, when Ramanujan (1887-1920) wrote to professor Hardy from India in 1909 and told him about a number of certain identities which he could not prove. Some of them were just corollaries of the Rogers-Ramanujan identities. The following citation is from Andrews excellent book [31]:

> In 1913 Ramanujan astounded mathematicians by presenting them with a long list of remarkable propositions that he claimed to have discovered. On examination some of them proved old; many could be proved (and a few disproved) by standard methods; but some remained doubtful. Among these were two simple but beautiful propositions, of which one alone need be stated. It was

$$\sum_{j=0}^{\infty}\frac{q^{j^2}}{\langle 1;q\rangle_j}=\frac{\langle 2,3,5;q^5\rangle_\infty}{\langle 1;q\rangle_\infty}. \tag{159}$$

> Ramanujan, even when his powers were developed by education at Cambridge, (Hardy managed to get Ramanujan to come to Cambridge in 1914) could not prove

> them; nor could MacMahon, the leading English authority on the subject, who devoted to them a whole chapter of his treatise *Combinatory Analysis.* Later Ramanujan, searching for something else in old *Proceedings of the London Mathematical Society*, found that Rogers had stated and proved them in 1894.

Ramanujan returned to India in 1919 and continued to work with his new love, the mock theta functions. His work was continued by Watson [923].

P.A.MacMahon used q-series as combinatorial generating functions and began a work that would lead to applications in representation theory of the symmetric group, root systems of Lie algebras etc.[32]. In 1929 Watson [920] proved a transformation formula for a terminating very-well-poised ${}_8\phi_7$ series, which turned out to be equivalent to Jacksons' q-analogue of Dougalls summation formula and at the same time gave the first comprehensible proof of the Rogers-Ramanujan identities.

A q-analogue of Kummer's formula (77), was proved independently by Bailey 1941 [68] and Daum 1942 [223]. This was one of the first examples of a summation formula for a q-hypergeometric series with argument -1. W.N.Bailey (1893-1961) had been greatly influenced by Ramanujan as an undergraduate at Cambridge 1914 and became professor of mathematics at Bedford College in the 1940s with speciality q-calculus.

2.7. **The q-integral.** Archimedes calculated the integral of $f(x) = x^2$ as the sum of a finite geometric series. In the 1650s, the famous Frenchman Pierre de Fermat (1608-1665), Pascal and others [33, p. 485] found a way to generalize Archimedes' results. Fermat introduced the q-integral of the function $f(x) = x^\alpha,\ \alpha \in \mathbb{Q}$ [33, p. 485] on the interval $[0,1]$ [48]. This was done by introducing the *Fermat measure*, which puts mass $a(1-q)q^n$ at $x = aq^n$ [48].

Thomae was a pupil of Heine who in 1869 introduced the so-called q-integral [862], [863]

$$\int_0^1 f(t)\, d_q(t) = (1-q)\sum_{n=0}^{\infty} f(q^n)q^n,\ 0 < q < 1. \tag{160}$$

In fact Thomae proved that the Heine transformation formula for the ${}_2\phi_1(a,b;c|q,z)$ was in fact a q-analogue of the Euler beta integral, which can also be expressed as a quotient of gamma functions. In 1910 Jackson defined the general q-integral [343], [460],[862]:

$$\int_a^b f(t)\,d_q(t) = \int_0^b f(t)\,d_q(t) - \int_0^a f(t)\,d_q(t), \tag{161}$$

where

$$\int_0^a f(t)\,d_q(t) = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n,\ 0<|q|<1. \tag{162}$$

There is no unique canonical choice for the q-integral from 0 to ∞ [534]. Following Jackson we will put

$$\int_0^{\infty} f(t)\,d_q(t) = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n,\ 0<|q|<1, \tag{163}$$

provided the sum converges absolutely [534]. This choice of values for q was also used in [351]. The bilateral q-integral is defined by

$$\int_{-\infty}^{\infty} f(t)\,d_q(t) = (1-q)\sum_{n=-\infty}^{\infty} [f(q^n)+f(-q^n)]q^n,\ 0<|q|<1. \tag{164}$$

If f is continuous on $[0,x]$, then

$$\begin{aligned} &D_q\left(\int_0^x f(t)\,d_q(t)\right) = D_q\left(x(1-q)\sum_{n=0}^{\infty} f(xq^n)q^n\right) = \\ &\frac{x(1-q)\sum_{n=0}^{\infty} f(xq^n)q^n - x(1-q)\sum_{n=0}^{\infty} f(xq^{n+1})q^{n+1}}{x(1-q)} = f(x). \end{aligned} \tag{165}$$

The following formulae for integration by parts will be useful [261], [460]:

Theorem 2.5. *If u and v are continuous on $[a,b]$, then*

$$\int_a^b u(t)D_q v(t)\,d_q(t) = [u(t)v(t)]_a^b - \int_a^b v(qt)D_q u(t)\,d_q(t). \tag{166}$$

$$\int_a^b u(qt)D_q v(t)\,d_q(t) = [u(t)v(t)]_a^b - \int_a^b v(t)D_q u(t)\,d_q(t). \tag{167}$$

Remark 13. In 1993 Matsuo [613] defined the q-integral on $[0,s\infty]$ by

$$\int_0^{s\infty} f(t)\,d_q(t) = s(1-q)\sum_{n=-\infty}^{\infty} f(sq^n)q^n,\ 0<|q|<1,\ s\in\mathbb{C}\backslash\{0\}, \tag{168}$$

provided the sum converges absolutely.

2.8. **Elementary q-functions.** If $|q| > 1$, or
$0 < |q| < 1$ and $|z| < |1-q|^{-1}$, the q-exponential function $E_q(z)$ was defined by Jackson [456] 1904, and by Exton[261]

$$E_q(z) = \sum_{k=0}^{\infty} \frac{1}{\{k\}_q!} z^k. \tag{169}$$

The q-difference equation for $E_q(z)$ is

$$D_q E_q(az) = aE_q(az). \tag{170}$$

If $|q| > 1$, $E_q(z)$ is an entire function, which by the Weierstrass factorization theorem [12], [761] can be expressed as the infinite product

$$E_q(z) = e^{g(z)} \prod_{n=0}^{\infty} \mathcal{E}_{p_n}(\frac{z}{z_n}), \tag{171}$$

where $\{z_n\}_{n=0}^{\infty}$ are the zeros of $E_q(z)$, listed according to their multiplicities, $\{p_n\}_{n=0}^{\infty}$ is a sequence of nonnegative integers, g is an entire function and $\mathcal{E}_p$ are the Weierstrass elementary factors

$$\mathcal{E}_p(z) = (1-z)exp(\sum_{n=1}^{p} \frac{z^n}{n}). \tag{172}$$

A variation of the following formula was stated in [916], and another variation was stated in [912, p. 12].

Theorem 2.6. [261, p. 127]

$$E_q(z) = \prod_{n=0}^{\infty} \left(1 + (q-1)\frac{z}{q^{n+1}}\right), \ |q| > 1. \tag{173}$$

Proof. We see that at least equation (170) is satisfied and $E_q(0) = 1$ as in (169). We must prove by induction that

$$\frac{1}{\{n\}_q!} = \sum_{k_1=0}^{\infty} \sum_{k_2=k_1+1}^{\infty} \cdots \sum_{k_n=k_{n-1}+1}^{\infty} \prod_{l=1}^{n} \left(\frac{q-1}{q^{k_l+1}}\right). \tag{174}$$

The induction hypothesis is true for $n = 1$, because

$$\sum_{k_1=0}^{\infty} \frac{q-1}{q^{k_1+1}} = 1. \tag{175}$$

Now assume that the induction hypothesis is true for for $n = m - 1$. Then it is also true for $n = m$, because

$$\sum_{k_1=0}^{\infty} \frac{q-1}{q^{k_1+1} q^{(m-1)(k_1+1)}} = \frac{q-1}{q^m - 1} = \frac{1}{\{m\}_q}. \tag{176}$$

$\square$

By (173) $E_q(z)$ is a slowly increasing function of order 0. By the Weierstrass factorization theorem, the zeros of $E_q(z)$ are $\{-\frac{q^{k+1}}{q-1}\}_{k=0}^{\infty}$ and we obtain the following equality

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(k+1)}}{(q-1)^n\{n\}_q!}=0,\quad k=0,1\ldots,\ |q|>1. \tag{177}$$

There is another q-exponential function which is entire when $0<|q|<1$ and which converges when $|z|<|1-q|^{-1}$ if $|q|>1$. To obtain it we must invert the base in (169), i.e. $q\to\frac{1}{q}$. This is a common theme in q-calculus.

$$E_{\frac{1}{q}}(z)=\sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}}{\{k\}_q!}z^k. \tag{178}$$

A solitary factor of $q^{\binom{k}{2}}$ has appeared in the general term of the series. Such terms whose exponents are quadratic in the index of summation of the series concerned are often characteristic of q-functions [261]. In this case the term results from the summation of an arithmetic series. In many ways, $E_q(z)$ and $E_{\frac{1}{q}}(z)$ are mutually complementary [261]. The q-difference equation corresponding to (170) is

$$D_qE_{\frac{1}{q}}(az)=aE_{\frac{1}{q}}(qaz), \tag{179}$$

which also reduces to the differential equation of the exponential function when q tends to unity as expected. Compare formulae (143) and (144), which give two other q-analogues of the exponential function. We immediately obtain

$$E_{\frac{1}{q}}(z)=\prod_{n=0}^{\infty}\left(1+(1-q)zq^n\right),\ 0<|q|<1. \tag{180}$$

The zeros of the above function are $\{-\frac{1}{q^k(1-q)}\}_{k=0}^{\infty}$ and we obtain the following equality

$$\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}}{\{n\}_q!}(-\frac{1}{(1-q)q^k})^n=0,\ k=0,1\ldots,\ 0<|q|<1. \tag{181}$$

This equation is equivalent to (144).

The power function z^α is the branch of the power function, which takes the value q^α for $z=q$.

A k-function is a q-periodic function, i.e. a solution of the functional equation

$$k(qz) = k(z) \text{ or } D_q k(z) = 0. \tag{182}$$

These functions play in the theory of q-difference equations the rôle of the arbitrary constants of the differential equations [399], [166].

In 1871 Thomae [864, p. 106] gave several interesting examples of k-functions e.g. $z^{2\pi i log q}$, and its positive or negative integer powers.

One of the most important k-functions is [402, p. 262]

$$z^{a-b}\frac{(q^a z;q)_\infty (q(q^a z)^{-1};q)_\infty}{(q^b z;q)_\infty (q(q^b z)^{-1};q)_\infty}. \tag{183}$$

In [634] Mimachi uses a k-function which is a quotient of theta functions.

Another interesting function is the one-parameter family of q-exponential functions

$$E_q^{(\alpha)}(z) = \sum_{k=0}^{\infty} \frac{q^{\alpha\frac{k^2}{2}}}{\langle 1;q\rangle_k} z^k \tag{184}$$

which has been considered in [311],[51] .

2.9. Some useful identities.

Theorem 2.7. *The following formulae hold* , $0 < |q| < 1$:

$$\langle a;q\rangle_n = \frac{\langle a;q\rangle_\infty}{\langle a+n;q\rangle_\infty}. \tag{185}$$

$$(aq^{-n};q)_n \equiv \frac{1}{(a;q)_{-n}} = \frac{(aq^{-n};q)_\infty}{(a;q)_\infty}. \tag{186}$$

$$\frac{(a;q^2)_\infty}{(aq^{2n};q^2)_\infty} = (a;q^2)_n. \tag{187}$$

Proof. These identities follow immediately from the definition (119). □

Theorem 2.8. *The following formulae hold whenever* $q \neq 0$ *and* $q \neq$ *root of* 1 :

$$\langle -a+1-n;q\rangle_n = \langle a;q\rangle_n (-1)^n q^{-\binom{n}{2}-na}. \tag{188}$$

$$\langle a;q\rangle_{n-k} = \frac{\langle a;q\rangle_n}{\langle -a+1-n;q\rangle_k}(-1)^k q^{\binom{k}{2}+k(1-a-n)}. \tag{189}$$

$$\langle a+n;q\rangle_k=\frac{\langle a;q\rangle_k\langle a+k;q\rangle_n}{\langle a;q\rangle_n}. \tag{190}$$

$$\langle a+k;q\rangle_{n-k}=\frac{\langle a;q\rangle_n}{\langle a;q\rangle_k}. \tag{191}$$

$$\langle a+2k;q\rangle_{n-k}=\frac{\langle a;q\rangle_n\langle a+n;q\rangle_k}{\langle a;q\rangle_{2k}}. \tag{192}$$

$$\langle -n;q\rangle_k=\frac{\langle 1;q\rangle_n}{\langle 1;q\rangle_{n-k}}(-1)^k q^{\binom{k}{2}-nk}. \tag{193}$$

$$\langle a-n;q\rangle_k=\frac{\langle a;q\rangle_k\langle 1-a;q\rangle_n}{\langle -a+1-k;q\rangle_n q^{nk}}. \tag{194}$$

Proof. Formula (188) is proved as follows:

$$LHS=(1-q^{-a+1-n})(1-q^{-a+2-n})\cdots(1-q^{-a})$$

$$\begin{aligned}RHS&=(1-q^{a})(1-q^{a+1})\cdots(1-q^{a+n-1})(-1)^n q^{-na-(0+1+2+\ldots+(n-1))}=\\&=(1-q^{-a})(q-q^{-a})\cdots(q^{n-1}-q^{-a})q^{-(0+1+2+\ldots+(n-1))}=\\&=(1-q^{-a})(1-q^{-a-1})\cdots(1-q^{-a-n+1})=LHS.\end{aligned}$$

Let us prove (189). There are two cases to consider. Case 1: $n>k$.

$$LHS=(1-q^{a})(1-q^{a+1})\cdots(1-q^{a+n-k-1})$$

$$\begin{aligned}RHS&=\frac{(1-q^{a})(1-q^{a+1})\cdots(1-q^{a+n-1})(-1)^k q^{k(1-a)+0+1+2+\ldots+(k-1)}}{(1-q^{-a+1-n})(1-q^{-a+2-n})\cdots(1-q^{-a+k-n})q^{nk}}=\\&=\frac{(1-q^{a})(1-q^{a+1})\cdots(1-q^{a+n-1})}{(1-q^{a+n-1})(1-q^{a+n-2})\cdots(1-q^{a+n-k})}=LHS.\end{aligned}$$

Case 2: $n<k$.

$$LHS=((1-q^{a+n-k})(1-q^{a+n-k+1})\cdots(1-q^{a-1}))^{-1}=RHS.$$

The identities (190) - (192) follow easily from the definition (119) and the formula (193) is proved as follows:

$$LHS=(1-q^{-n})(1-q^{-n+1})\cdots(1-q^{-n+k-1})$$

$$\begin{aligned}RHS&=\frac{(1-q)(1-q^2)\cdots(1-q^n)(-1)^k q^{0+1+2+\ldots+(k-1)}}{(1-q)(1-q^2)\cdots(1-q^{n-k})q^{nk}}=\\&=\frac{(1-q^{n-k+1})(1-q^{n-k+2})\cdots(1-q^n)(-1)^k q^{0+1+2+\ldots+(k-1)}}{q^{nk}}=\\&=(q^{-k+1}-q^{-n})(q^{-k+2}-q^{-n})\cdots(1-q^{-n})q^{0+1+2+\ldots+(k-1)}=\\&=(1-q^{k-n-1})(1-q^{k-n-2})\cdots(1-q^{-n})=LHS.\end{aligned}$$

Finally, the following computation proves (194):

$$LHS = (1 - q^{a-n})(1 - q^{a-n+1}) \cdots (1 - q^{a-n+k-1})$$

$$RHS = \frac{(1-q^a)(1-q^{a+1})\cdots(1-q^{a+k-1})(1-q^{1-a})(1-q^{2-a})\cdots(1-q^{n-a})}{(1-q^{-a+1-k})(1-q^{-a+2-k})\cdots(1-q^{n-a-k})q^{nk}} =$$

$$= \frac{(1-q^a)(1-q^{a+1})\cdots(1-q^{a+k-1})(q^{1-a}-1)(q^{2-a}-1)\cdots(q^{n-a}-1)}{(q^{-a+1}-q^k)(q^{-a+2}-q^k)\cdots(q^{n-a}-q^k)} =$$

$$= \frac{(1-q^a)(1-q^{a+1})\cdots(1-q^{a+k-1})(1-q^{a-1})(q-q^{a-1})\cdots(q^{n-1}-q^{a-1})}{(1-q^{a+k-1})(q-q^{a+k-1})\cdots(q^{n-1}-q^{a+k-1})} =$$

$$= \frac{(1-q^a)(1-q^{a+1})\cdots(1-q^{a+k-1})(1-q^{a-1})(1-q^{a-2})\cdots(1-q^{a-n})}{(1-q^{a+k-1})(1-q^{a+k-2})\cdots(1-q^{a+k-n})} = LHS.$$

□

Theorem 2.9. *The following formulae hold for any $q \in \mathbb{C}$:*

$$\langle 2a; q^2\rangle_n = \langle a; q\rangle_n \widetilde{\langle a; q\rangle}_n. \tag{195}$$

$$\langle a; q\rangle_{2n} = \langle a; q^2\rangle_n \langle a+1; q^2\rangle_n. \tag{196}$$

$$\langle a; q^k\rangle_{n+l} = \langle a; q^k\rangle_n \langle a+kn; q^k\rangle_l. \tag{197}$$

$$\langle a; q\rangle_{n+k} = \langle a; q\rangle_n \langle a+n; q\rangle_k. \tag{198}$$

Proof. Equation (195) can be proved as follows:

$$LHS = (1 - q^{2a})(1 - q^{2a+2}) \cdots (1 - q^{2a+2n-2})$$

$$RHS = (1-q^a)(1-q^{a+1})\cdots(1-q^{a+n-1})(1+q^a)(1+q^{a+1})\cdots(1+q^{a+n-1}) = LHS.$$

The equality (196) is proved similarly:

$$LHS = (1 - q^a)(1 - q^{a+1}) \cdots (1 - q^{a+2n-1})$$

$$RHS = (1-q^a)(1-q^{a+2})\cdots(1-q^{a+2n-2})(1-q^{a+1})(1-q^{a+3})\cdots(1-q^{a+2n-1}) = LHS.$$

□

The following limits will be useful in the proof of the Rogers-Ramanujan identities. Remember that $0 < |q| < 1$. In all cases the limits of the variables are independent of each other.

$$\lim_{d,e \to -\infty} \frac{\langle d, e; q\rangle_j}{\langle d+e-n-a; q\rangle_j} = (-1)^j q^{\binom{j}{2} + j(n+a)}. \tag{199}$$

Proof.

$$\lim_{d,e\to-\infty}\frac{\langle d,e;q\rangle_j}{\langle d+e-n-a;q\rangle_j}=$$

$$=\lim_{d,e\to-\infty}\frac{(1-q^d)\cdots(1-q^{d+j-1})(1-q^e)\cdots(1-q^{e+j-1})}{(1-q^{d+e-n-a})(1-q^{d+e-n-a+1})\cdots(1-q^{d+e-n-a+j-1})}=$$

$$=(-1)^j q^{\binom{j}{2}+j(n+a)}.$$

□

$$\lim_{b,c\to-\infty}\frac{\langle a+1-b-c;q\rangle_j}{\langle a+1-b,a+1-c;q\rangle_j}=1. \tag{200}$$

Proof.

$$\lim_{b,c\to-\infty}\frac{\langle a+1-b-c;q\rangle_j}{\langle a+1-b,a+1-c;q\rangle_j}=$$

$$=\lim_{b,c\to-\infty}\frac{(1-q^{a+1-b-c})(1-q^{a+2-b-c})\cdots(1-q^{a-b-c+j})}{(1-q^{a+1-b})\cdots(1-q^{a+j-b})(1-q^{a+1-c})\cdots(1-q^{a+j-c})}=1.$$

□

$$\lim_{b\to-\infty}\frac{\langle b;q\rangle_j}{\langle a+1-b;q\rangle_j}q^{-bj}=(-1)^j q^{\binom{j}{2}}. \tag{201}$$

Proof.

$$\lim_{b\to-\infty}\frac{\langle b;q\rangle_j}{\langle a+1-b;q\rangle_j}q^{-bj}=$$

$$=\lim_{b\to-\infty}\frac{(1-q^b)(1-q^{b+1})\cdots(1-q^{b+j-1})q^{-bj}}{(1-q^{a+1-b})(1-q^{a+2-b})\cdots(1-q^{a-b+j})}=(-1)^j q^{\binom{j}{2}}.$$

□

$$\lim_{n\to+\infty}\langle a+1;q\rangle_n\langle -n;q\rangle_j q^{nj}=(-1)^j q^{\binom{j}{2}}\langle a+1;q\rangle_\infty. \tag{202}$$

Proof.

$$\lim_{n\to+\infty}\langle a+1;q\rangle_n\langle -n;q\rangle_j q^{nj}=$$

$$=\lim_{n\to+\infty}(1-q^{a+1})(1-q^{a+2})\cdots(1-q^{a+n})\times$$

$$\times(1-q^{-n})(1-q^{-n+1})\cdots(1-q^{-n+j-1})q^{nj}=$$

$$=\lim_{n\to+\infty}(1-q^{a+1})(1-q^{a+2})\cdots(1-q^{a+n})(q^n-1)(q^n-q)\cdots(q^n-q^{j-1})=$$

$$=(-1)^j q^{\binom{j}{2}}\langle a+1;q\rangle_\infty.$$

□

$$\lim_{n\to+\infty}\frac{\langle -n;q\rangle_j q^{jn}}{\langle a+n+1;q\rangle_j}=(-1)^j q^{\binom{j}{2}}. \tag{203}$$

Proof.

$$\lim_{n\to+\infty}\frac{\langle -n;q\rangle_j q^{jn}}{\langle a+n+1;q\rangle_j}=\lim_{n\to+\infty}\frac{(1-q^{-n})(1-q^{-n+1})\cdots(1-q^{-n+j-1})q^{jn}}{(1-q^{a+n+1})(1-q^{a+n+2})\cdots(1-q^{a+n+j})}=$$

$$=\lim_{n\to+\infty}(-1)^j\frac{(1-q^n)(q-q^n)\cdots(q^{j-1}-q^n)}{(1-q^{a+n+1})(1-q^{a+n+2})\cdots(1-q^{a+n+j})}=(-1)^j q^{\binom{j}{2}}.$$

□

The following two formulae serve as definitions for $\langle a;q\rangle_\alpha$ and $(a;q)_\alpha$, $\alpha\in\mathbb{C}$. This is an analytic continuation of (185). Compare [401, p. 342].

Definition 4.

$$\langle a;q\rangle_\alpha=\frac{\langle a;q\rangle_\infty}{\langle a+\alpha;q\rangle_\infty},\ a\neq -m-\alpha, m=0,1,\ldots. \tag{204}$$

$$(a;q)_\alpha=\frac{(a;q)_\infty}{(aq^\alpha;q)_\infty},\ a\neq q^{-m-\alpha}, m=0,1,\ldots. \tag{205}$$

Theorem 2.10. *The following equation holds for* $\alpha\in\mathbb{C}$. *Compare* [401, p. 342].

$$D_q(x;q)_\alpha=-\{\alpha\}_q(qx;q)_{\alpha-1},\quad \alpha\in\mathbb{C}. \tag{206}$$

Proof. We will prove that $D_q(-x;q)_\alpha = \{\alpha\}_q(-qx;q)_{\alpha-1}$. Then the result follows by (114).

$$
\begin{aligned}
LHS &\overset{\text{by}(205)}{=} D_q\frac{(-x;q)_\infty}{(-xq^\alpha;q)_\infty} = D_q\frac{\prod_{m=0}^{\infty}(1+xq^m)}{\prod_{m=0}^{\infty}(1+xq^{m+\alpha})} = \\
&= \frac{1}{(q-1)x}\left(\frac{\prod_{m=0}^{\infty}(1+xq^{m+1})}{\prod_{m=0}^{\infty}(1+xq^{m+\alpha+1})} - \frac{\prod_{m=0}^{\infty}(1+xq^m)}{\prod_{m=0}^{\infty}(1+xq^{m+\alpha})}\right) = \\
&= \frac{1}{(q-1)x}\frac{\prod_{m=0}^{\infty}(1+xq^{m+1})}{\prod_{m=0}^{\infty}(1+xq^{m+\alpha})}(1+xq^\alpha-(1+x)) = \\
&= \frac{q^\alpha-1}{q-1}\frac{\prod_{m=0}^{\infty}(1+xq^{m+1})}{\prod_{m=0}^{\infty}(1+xq^{m+\alpha})} \overset{\text{by}(205)}{=} RHS.
\end{aligned}
\tag{207}
$$

$\square$

Theorem 2.11. *The following formulae hold for* $0<|q|<1$ *and* $\alpha,\beta,\gamma\in\mathbb{C}$*:*

$$\langle\gamma;q\rangle_{\alpha+\beta} = \langle\gamma;q\rangle_\alpha\langle\gamma+\alpha;q\rangle_\beta. \tag{208}$$

$$\frac{\langle\gamma+\alpha;q\rangle_\beta}{\langle\gamma;q\rangle_\beta} = \frac{\langle\gamma+\beta;q\rangle_\alpha}{\langle\gamma;q\rangle_\alpha},\ \gamma\neq -m, m=0,1,\ldots. \tag{209}$$

$$\langle\gamma+\beta;q\rangle_{\alpha-\beta} = \frac{\langle\gamma;q\rangle_\alpha}{\langle\gamma;q\rangle_\beta},\ \gamma\neq -m, m=0,1,\ldots. \tag{210}$$

$$\frac{\langle\gamma;q\rangle_\alpha}{\langle\gamma;q\rangle_{2\beta}} = \frac{\langle\gamma+2\beta;q\rangle_{\alpha-\beta}}{\langle\gamma+\alpha;q\rangle_\beta},\ \gamma\neq -m, m=0,1,\ldots,\ \gamma+\alpha\neq -m, m=0,1,\ldots. \tag{211}$$

Proof. Equation (208) is proved as follows:

$$LHS = \frac{\langle\gamma; q\rangle_\infty}{\langle\gamma+\alpha+\beta; q\rangle_\infty}$$

$$RHS = \frac{\langle\gamma; q\rangle_\infty}{\langle\gamma+\alpha; q\rangle_\infty}\frac{\langle\gamma+\alpha; q\rangle_\infty}{\langle\gamma+\alpha+\beta; q\rangle_\infty} = LHS$$

Equation (209) is proved as follows:

$$LHS = \frac{\langle\gamma+\alpha; q\rangle_\infty}{\langle\gamma+\alpha+\beta; q\rangle_\infty}\frac{\langle\gamma+\beta; q\rangle_\infty}{\langle\gamma; q\rangle_\infty}$$

$$RHS = \frac{\langle\gamma+\beta; q\rangle_\infty}{\langle\gamma+\alpha+\beta; q\rangle_\infty}\frac{\langle\gamma+\alpha; q\rangle_\infty}{\langle\gamma; q\rangle_\infty} = LHS$$

Equation (210) is proved in a similar way:

$$LHS = \frac{\langle\gamma+\beta; q\rangle_\infty}{\langle\gamma+\alpha; q\rangle_\infty}$$

$$RHS = \frac{\langle\gamma; q\rangle_\infty}{\langle\gamma+\alpha; q\rangle_\infty}\frac{\langle\gamma+\beta; q\rangle_\infty}{\langle\gamma; q\rangle_\infty} = LHS$$

Finally equation (211) is proved as follows:

$$LHS = \frac{\langle\gamma; q\rangle_\infty}{\langle\gamma+\alpha; q\rangle_\infty}\frac{\langle\gamma+2\beta; q\rangle_\infty}{\langle\gamma; q\rangle_\infty}$$

$$RHS = \frac{\langle\gamma+2\beta; q\rangle_\infty}{\langle\gamma+\alpha+\beta; q\rangle_\infty}\frac{\langle\gamma+\alpha+\beta; q\rangle_\infty}{\langle\gamma+\alpha; q\rangle_\infty} = LHS$$

□

Theorem 2.12. *The following formulae hold* , $0 < |q| < 1$ *and* $\alpha, \beta, \gamma \in \mathbb{C}$.

$$(\gamma; q)_{\alpha+\beta} = (\gamma; q)_\alpha(\gamma q^\alpha; q)_\beta. \tag{212}$$

$$\frac{(\gamma q^\alpha; q)_\beta}{(\gamma; q)_\beta} = \frac{(\gamma q^\beta; q)_\alpha}{(\gamma; q)_\alpha},\ \gamma \neq q^{-m}, m = 0, 1, \ldots. \tag{213}$$

$$(\gamma q^\beta; q)_{\alpha-\beta} = \frac{(\gamma; q)_\alpha}{(\gamma; q)_\beta},\ \gamma \neq q^{-m}, m = 0, 1, \ldots. \tag{214}$$

(215)
$$\frac{(\gamma; q)_\alpha}{(\gamma; q)_{2\beta}} = \frac{(\gamma q^{2\beta}; q)_{\alpha-\beta}}{(\gamma q^\alpha; q)_\beta},\ \gamma \neq q^{-m}, m = 0, 1, \ldots,\ \gamma \neq q^{-m-\alpha}, m = 0, 1, \ldots.$$

Remark 14. In equations (185), (190) through (192), (196) through (198), (204) and (208) through (211) we can put a tilde over all the factors.

Our next aim is to prove the famous Jacobi triple product identity [467], [912], [41], [343] from 1829 with the help of Euler's equations (153) and (154). The following formula for the theta function holds:

Theorem 2.13.

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = (q^2, -qz, -qz^{-1}; q^2)_\infty, \tag{216}$$

where $z \in \mathbb{C}\backslash\{0\},\ 0 < |q| < 1$.

This proof is due to Vilenkin and Klimyk [912]:

Proof. If $|z| > |q|$ and $0 < |q| < 1$ then taking into account the fact that $\langle 2+2n; q^2\rangle_\infty = 0$ for negative n, we obtain

$$\begin{aligned}
&(-qz;q^2)_\infty \overset{\text{by}(154)}{=} \sum_{n=0}^{\infty} \frac{(zq)^n q^{n(n-1)}}{\langle 2;q^2\rangle_n} \overset{\text{by}(187)}{=} \\
&= \frac{1}{\langle 2;q^2\rangle_\infty} \sum_{n=-\infty}^{\infty} z^n q^{n^2} \langle 2+2n;q^2\rangle_\infty \overset{\text{by}(154)}{=} \\
&= \frac{1}{\langle 2;q^2\rangle_\infty} \sum_{n=-\infty}^{\infty} z^n q^{n^2} \sum_{m=0}^{\infty} \frac{(-1)^m q^{2m+2mn+m(m-1)}}{\langle 2;q^2\rangle_m} = \\
&= \frac{1}{\langle 2;q^2\rangle_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m z^{-m} q^m}{\langle 2;q^2\rangle_m} \sum_{n=-\infty}^{\infty} z^{m+n} q^{(m+n)^2} = \\
&= \frac{1}{\langle 2;q^2\rangle_\infty} \sum_{m=0}^{\infty} \frac{(-\frac{q}{z})^m}{\langle 2;q^2\rangle_m} \sum_{n=-\infty}^{\infty} z^n q^{n^2} = \\
&\overset{\text{by}(153)}{=} \frac{1}{\langle 2;q^2\rangle_\infty (-\frac{q}{z};q^2)_\infty} \sum_{n=-\infty}^{\infty} z^n q^{n^2}.
\end{aligned} \tag{217}$$

If $|z| > |q|$ and $0 < |q| < 1$, then the series converges absolutely. Transferring $\langle 2;q^2\rangle_\infty(-\frac{q}{z};q^2)_\infty$ to the left hand side and continuing analytically in z completes the proof of (216). □

Remark 15. Another proof of the Jacobi triple product identity can be found in [41] and finally a very difficult proof of the Jacobi triple product identity from the Weyl-Kac character formula for affine Lie algebras is given in [332].

2.10. The Gauss q-binomial coefficients and the Leibniz q-theorem.

Definition 5. Let the Gauss q-binomial coefficients [348] be defined by

$$\binom{n}{k}_q = \frac{\langle 1;q\rangle_n}{\langle 1;q\rangle_k \langle 1;q\rangle_{n-k}}, \tag{218}$$

for $k = 0, 1, \ldots, n$, and by

$$\binom{\alpha}{\beta}_q = \frac{\langle \beta+1, \alpha-\beta+1;q\rangle_\infty}{\langle 1, \alpha+1;q\rangle_\infty} = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\beta+1)\Gamma_q(\alpha-\beta+1)} \tag{219}$$

for complex α and β when $0 < |q| < 1$.

The q-binomial coefficient $\binom{n}{k}_q$ is a polynomial of degree $k(n-k)$ in q with integer coefficients, whose sum equals $\binom{n}{k}$ [97].

Remark 16. If we define

$$\binom{-n}{k}_q = \frac{\{-n\}_q\{-n-1\}_q \ldots \{-n-k+1\}_q}{\{k\}_q!},\ n = 1, 2, \ldots, \tag{220}$$

under suitable conditions equation (251) holds for all integers [203], [703].

Let $q = p^s$ be a prime power. The Galois field is denoted $GF(q)$. Let V_n denote a vector space of dimension n over $GF(q)$. The number of subspaces of V_n is

$$G_n = \sum_{k=0}^{n} \binom{n}{k}_q, \tag{221}$$

which is called the Galois number [357, p. 77].

Some important relations for the Gauss q-binomial coefficients are collected in [369, p. 91].

Definition 6. Let the operators T, M and the functions $(x-a)^{(n)}$, $(\widetilde{x-a})^{(n)}$ be defined by

$$[T_x f](x) \equiv [Tf](x) = f(qx). \tag{222}$$

$$[M_x f](x) \equiv [Mf](x) = xf(x). \tag{223}$$

$$(x-a)^{(n)} = \begin{cases} 1, & n = 0; \\ \prod_{m=0}^{n-1}(x - aq^m), & n = 1, 2, \ldots. \end{cases} \tag{224}$$

$$
(225) \qquad \widetilde{(x-a)}^{(n)} = \begin{cases} 1, & n=0; \\ \prod_{m=0}^{n-1}(xq^m - a), & n = 1, 2, \ldots. \end{cases}
$$

Remark 17. The first function, which was preferred by Jackson has recently been used by Bowman [122], and the second one is preferred by Cigler. The relation between them is

$$
(226) \qquad (xq^{n-1}-a)^{(n)} = q^{\binom{n}{2}}\widetilde{(x-a)}^{(n)}.
$$

The following result will be useful in the proof of the Leibniz q-theorem:

Theorem 2.14. The q-Pascal identity.

$$
(227) \quad \binom{\alpha+1}{k}_q = \binom{\alpha}{k}_q q^k + \binom{\alpha}{k-1}_q = \binom{\alpha}{k}_q + \binom{\alpha}{k-1}_q q^{\alpha+1-k},
$$

where $0 < |q| < 1$.

Proof. The first identity is proved by the following calculation:

$$
LHS = \frac{\langle k+1, \alpha-k+2; q\rangle_\infty}{\langle 1, \alpha+2; q\rangle_\infty}.
$$

$$
RHS = \frac{\langle k+1, \alpha-k+1; q\rangle_\infty}{\langle 1, \alpha+1; q\rangle_\infty} q^k + \frac{\langle k, \alpha-k+2; q\rangle_\infty}{\langle 1, \alpha+1; q\rangle_\infty} \stackrel{\text{by}(185)}{=}
$$

$$
= \frac{\langle k+1, \alpha-k+2; q\rangle_\infty}{\langle 1, \alpha+2; q\rangle_\infty} \frac{q^k(1-q^{\alpha+1-k}) + 1 - q^k}{1-q^{\alpha+1}} = LHS.
$$

The second identity is proved in a similar way:

$$
LHS = \frac{\langle k+1, \alpha-k+2; q\rangle_\infty}{\langle 1, \alpha+2; q\rangle_\infty}
$$

$$
RHS = \frac{\langle k+1, \alpha-k+1; q\rangle_\infty}{\langle 1, \alpha+1; q\rangle_\infty} + \frac{\langle k, \alpha-k+2; q\rangle_\infty}{\langle 1, \alpha+1; q\rangle_\infty} q^{\alpha+1-k} \stackrel{\text{by}(185)}{=}
$$

$$
= \frac{\langle k+1, \alpha-k+2; q\rangle_\infty}{\langle 1, \alpha+2; q\rangle_\infty} \frac{1-q^{\alpha+1-k} + q^{\alpha+1-k}(1-q^k)}{1-q^{\alpha+1}} = LHS.
$$

□

Remark 18. When $\alpha \in \mathbb{N}$, the result holds for all q.

The following theorem was obtained by Euler 1748 [258], [671], [917] and by Gauss 1876 [346].

Theorem 2.15.

$$\sum_{n=0}^{m}(-1)^n \binom{m}{n}_q q^{\binom{n}{2}} u^n = (u;q)_m. \tag{228}$$

Proof. The formula is true for $m = 1$. Assume that the formula is proved for $m-1$. Then it is also true for m, because

(229)
$$\sum_{n=0}^{m}(-1)^n \binom{m}{n}_q q^{\binom{n}{2}} u^n \overset{\text{by}(227)}{=} \sum_{n=0}^{m-1}(-1)^n \binom{m-1}{n}_q q^{\binom{n}{2}} u^n +$$
$$+\sum_{n=0}^{m-1}(-1)^n \binom{m-1}{n-1}_q q^{\binom{n}{2}+m-n} u^n + (-1)^m q^{\binom{m}{2}} u^m = (-1)^m q^{\binom{m}{2}} u^m +$$
$$+ (u;q)_{m-1} + \sum_{n=0}^{m-2}(-1)^{n+1} \binom{m-1}{n}_q q^{\binom{n}{2}+m-1} u^{n+1} = (-1)^m q^{\binom{m}{2}} u^m +$$
$$+ (u;q)_{m-1}(1-uq^{m-1}) + (-1)^{m-1} q^{\binom{m-1}{2}+m-1} u^m = (u;q)_m.$$

$\square$

Remark 19. In 1843 Cauchy [173] observed that (228) implies and is in fact a finite version of Jacobi's triple product identity.

Theorem 2.16. The Leibniz q-theorem [399, 2.5], [647, p. 33]. *Let $f(x)$ and $g(x)$ be n times q-differentiable functions. Then $fg(x)$ is also n times q-differentiable and*

$$D_q^n(fg)(x) = \sum_{k=0}^{n} \binom{n}{k}_q D_q^k(f)(xq^{n-k}) D_q^{n-k}(g)(x). \tag{230}$$

Proof. For $n = 1$ the formula above becomes (111). Assume that the formula is proved for $n = m$. Then it is also true for $n = m+1$, because

(231)
$$\begin{aligned}
&D_q^{m+1}(fg)(x) = D_q(D_q^m(fg)(x)) = \\
&= D_q \sum_{k=0}^{m} \binom{m}{k}_q D_q^k(f)(xq^{m-k}) D_q^{m-k}(g)(x) = \\
&\overset{\text{by}(114)}{=} \sum_{k=0}^{m} \binom{m}{k}_q \big(q^{m-k} D_q^{k+1}(f)(xq^{m-k}) D_q^{m-k}(g)(x) + \\
&+ D_q^k(f)(xq^{m+1-k}) D_q^{m+1-k}(g)(x)\big) = \\
&= \sum_{k=0}^{m} \binom{m}{k}_q D_q^k(f)(xq^{m+1-k}) D_q^{m+1-k}(g)(x) + \\
&+ \sum_{k=1}^{m+1} \binom{m}{k-1}_q q^{m+1-k} D_q^k(f)(xq^{m+1-k}) D_q^{m+1-k}(g)(x) = \\
&= f(xq^{m+1}) D_q^{m+1}(g)(x) + \sum_{k=1}^{m} \left(\binom{m}{k}_q + q^{m+1-k} \binom{m}{k-1}_q \right) \times \\
&\times D_q{}^k(f)(xq^{m+1-k}) D_q^{m+1-k}(g)(x) + D_q^{m+1}(f)(x) g(x) = \\
&\overset{\text{by}(227)}{=} \sum_{k=0}^{m+1} \binom{m+1}{k}_q D_q^k(f)(xq^{m+1-k}) D_q^{m+1-k}(g)(x).
\end{aligned}$$

□

Another interesting formula is the following one (compare [458, p. 255]) [402, p. 260], [767], [647, p. 33]:

(232)
$$(D_q^n f)(x) = (q-1)^{-n} x^{-n} q^{-\binom{n}{2}} \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} f(q^{n-k}x).$$

Proof. For $n=1$ the formula above becomes (107). Assume that the formula is proved for $n=m$. Then it is also true for $n=m+1$, because

(233)

$$\begin{aligned}&\left(D_q^{m+1}f\right)(x)=\\&=D_q\left((q-1)^{-m}x^{-m}q^{-\binom{m}{2}}\sum_{k=0}^{m}\binom{m}{k}_q(-1)^kq^{\binom{k}{2}}f(q^{m-k}x)\right)\overset{\text{by}(111)}{=}\\&=(q-1)^{-m}q^{-\binom{m}{2}}\left(-\frac{q^m-1}{q-1}q^{-m}x^{-m-1}\sum_{k=0}^{m}\binom{m}{k}_q(-1)^kq^{\binom{k}{2}}f(q^{m-k}x)+\right.\\&\left.q^{-m}x^{-m}\sum_{k=0}^{m}-\binom{m}{k}_q(-1)^k\frac{1}{(q-1)x}q^{\binom{k}{2}}\left(f(q^{m-k}x)-f(q^{m+1-k}x)\right)\right)\overset{\text{by}(227)}{=}\\&=(q-1)^{-m-1}x^{-m-1}q^{-\binom{m}{2}}\sum_{k=0}^{m+1}\binom{m+1}{k}_q(-1)^kq^{\binom{k}{2}}f(q^{m+1-k}x).\end{aligned}$$

□

Corollary 2.17. *If $q^p=1$ and p is prime, then*

$$D_q^p(f)=0. \tag{234}$$

Proof. For $p=2$ the corollary is true by inspection. For any other p the terms $f(x)$ and $f(q^px)$ appear with different sign in (232) and cancel each other. All other terms in (232) contain the factor $\{p\}_q$, which is zero by hypothesis. □

Remark 20. If $f(x)$ is analytic and $q^n=1,\ n\in\mathbb{N}$, then $D_q^n(f)=0$ [430].

Remark 21. An analogous result for a graded q-differential algebra equipped with an endomorphism d together with a corresponding q-Leibniz rule was presented in [244].

The formula (232) can be inverted [402, p. 260].

$$f(q^nx)=\sum_{k=0}^{n}(q-1)^kx^kq^{\binom{k}{2}}\binom{n}{k}_q D_q^k(f(x)). \tag{235}$$

The following two theorems for the Gauss q-binomial coefficients were proved in [187]:

Theorem 2.18. *If q is a primitive $(n+1)$ root of unity then*

$$\left|\binom{n}{k}_q\right|=1,\ k=0,1,\ldots,n. \tag{236}$$

Theorem 2.19. *If q is a primitive $(2m+2)$ root of unity then*

$$\left| \binom{2m}{m}_q \right| \geq 1. \tag{237}$$

Another beautiful identity due to Gauss [346] is stated in [357]:

$$\sum_{k=0}^{2n} (-1)^n \binom{2n}{k}_q = \langle 1; q^2 \rangle_n. \tag{238}$$

The q-multinomial coefficients [483, p. 305] can be defined by

$$\binom{n}{k_1, k_2, \ldots, k_m}_q = \frac{\langle 1; q \rangle_n}{\langle 1; q \rangle_{k_1} \langle 1; q \rangle_{k_2} \ldots \langle 1; q \rangle_{k_m}}, \tag{239}$$

where $k_1+k_2+\ldots+k_m = n$. In the same paper [483] Johnson presented q-multinomial extensions of equations (251), (227) and (230). Via a slight modification of Gessel's chain rule [353] Johnson proved two q-Faà di Bruno formulas for the n:th q-difference of a composite function. The second of these formulas involves the q-Bell polynomials.

2.11. **Cigler's operational methods for q-identities.** This section is based on [203]. We present a number of identities for linear operators on the vector space P of all polynomials in the variabel x over the field $\mathbb{R}$. We start with a few simple examples.

Example 2.

$$D_q x - q x D_q = I. \tag{240}$$

$$D_q x - x D_q = T. \tag{241}$$

Proof. Just apply both sides to an arbitrary polynomial. □

These two equations are equivalent to

$$\{n+1\}_q - q\{n\}_q = 1. \tag{242}$$

$$\{n+1\}_q - \{n\}_q = q^n. \tag{243}$$

Furthermore

$$D_q x^k - q^k x^k D_q = \{k\}_q x^{k-1}. \tag{244}$$

$$D_q x^k - x^k D_q = \{k\}_q x^{k-1} T. \tag{245}$$

Proof. Just apply both sides to an arbitrary polynomial. □

These two equations are equivalent to

$$\{n+k\}_q - q^k\{n\}_q = \{k\}. \tag{246}$$

$$\{n+k\}_q - \{n\}_q = \{k\}q^n. \tag{247}$$

The following two dual identities are equivalent to the formulae (227) for the q-binomial coefficients.

$$D_q^k x - q^k x D_q^k = \{k\}_q D_q^{k-1}. \tag{248}$$

$$D_q^k x - x D_q^k = T\{k\}_q D_q^{k-1}. \tag{249}$$

We now prove a variation of the q-binomial theorem, which is applicable to the quantum plane [507].

Theorem 2.20. *Let A and B be linear operators on P with*

$$BA = qAB. \tag{250}$$

Then

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k}_q A^k B^{n-k},\ n = 0, 1, 2, \ldots. \tag{251}$$

Proof. The theorem is obviously true for $n = 1$. Assume that it is true for $n = m-1$. Then it is also true for $n = m$, because

(252)

$$\begin{aligned}
&(A+B)^m = (A+B)\sum_{k=0}^{m-1} \binom{m-1}{k}_q A^k B^{m-1-k} = by(250) = \\
&= A^m + B^m + \sum_{k=1}^{m-1} \binom{m-1}{k-1}_q A^k B^{m-k} + \sum_{k=1}^{m-1} \binom{m-1}{k}_q A^k B^{m-k} q^k = \\
&= by(227) = \sum_{k=0}^{m} \binom{m}{k}_q A^k B^{m-k}.
\end{aligned}$$

□

Remark 22. This equation was first stated in [792]. Note the similarity with Leibniz' q-theorem.

Example 3. Choose $A = MT$ and $B = aT$, where a is a constant, then the requirements are fulfilled. Because of the relations

$$(MT)^n 1 = q^{\binom{n}{2}} x^n \text{ and } (MT - aT)^n 1 = (\widetilde{x-a})^{(n)}, \tag{253}$$

we thus obtain the following wellknown result. An equation equivalent to the following formulas was obtained by Euler 1748 [258], [671], [917] and by Gauss [346]. This is a finite form of (144) [703]. See also Exton's book [261], [142, p. 128] and [507] or compare with (228):

$$(\widetilde{x-a})^{(n)} = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}_q q^{\binom{k}{2}} x^k a^{n-k}. \tag{254}$$

Hence by (226) or by symmetry a similar equation follows:

$$(x-a)^{(n)} = \sum_{k=0}^{n}\binom{n}{k}_q q^{\binom{k}{2}} x^{n-k}(-a)^k. \tag{255}$$

Theorem 2.21. *The following equation relates the n:th q-difference operator at zero to the n:th derivative at zero for an analytic function* $f(z)$ [407]

$$D_q^n f(0) = \frac{\{n\}_q!}{n!} f^{(n)}(0). \tag{256}$$

Proof.

$$\begin{aligned}
D_q^n f(0) &\overset{\text{by}(232)}{=} \lim_{z\to 0}\frac{\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k q^{\binom{k}{2}} f(q^{n-k}z)}{(q-1)^n z^n q^{\binom{n}{2}}} = \\
&= \lim_{z\to 0}\frac{\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k q^{\binom{k}{2}} q^{n(n-k)} f^{(n)}(q^{n-k}z)}{(q-1)^n n! q^{\binom{n}{2}}} = \\
&\overset{\text{by}(254)}{=} \frac{f^{(n)}(0)(-1)^n(\widetilde{1-q^n})^{(n)}}{(q-1)^n n! q^{\binom{n}{2}}} = \\
&= \frac{f^{(n)}(0)(-1)^n(1-q^n)(q-q^n)\dots(q^{n-1}-q^n)}{(q-1)^n n! q^{0+1+\dots+(n-1)}} = \\
&= \frac{f^{(n)}(0)(q^n-1)(q^{n-1}-1)\dots(q-1)}{(q-1)^n n!} = \frac{\{n\}_q!}{n!} f^{(n)}(0).
\end{aligned} \tag{257}$$

In the proof the limit $\lim_{z\to 0}$ in the numerator was zero because of (254) and we could use L'Hôpital's rule n times. □

Remark 23. Another proof of this theorem was given in [481, p. 119]

2.12. q-Analogues of the trigonometric and hyperbolic functions.

Two q-analogues of the trigonometric functions are defined by

$$Sin_q(x) = \frac{1}{2i}(E_q(ix) - E_q(-ix)), \tag{258}$$

and

$$Cos_q(x) = \frac{1}{2}(E_q(ix) + E_q(-ix)), \tag{259}$$

with q-difference

$$D_q Cos_q(ax) = -aSin_q(ax) \tag{260}$$

$$D_q Sin_q(bx) = bCos_q(bx). \tag{261}$$

The following equation is easily proved.

$$Cos_q(x)Cos_{\frac{1}{q}}(x) + Sin_q(x)Sin_{\frac{1}{q}}(x) = 1. \tag{262}$$

In order to prove the following equations we introduce a new notation [401, p. 362]. Let $f(x) = \sum_{n=-\infty}^{+\infty} A_n x^n$ be a power series in x. Then put

$$f[x \pm y]_q = \sum_{n=-\infty}^{+\infty} A_n x^n (\mp \frac{y}{x}; q)_n. \tag{263}$$

The following two addition theorems obtain [456, p. 32]:

$$Cos_q(x)Cos_{\frac{1}{q}}(y) \pm Sin_q(x)Sin_{\frac{1}{q}}(y) = Cos_q[x \mp y]_q. \tag{264}$$

$$Sin_q(x)Cos_{\frac{1}{q}}(y) \pm Cos_q(x)Sin_{\frac{1}{q}}(y) = Sin_q[x \pm y]_q. \tag{265}$$

Proof. Expand the factors on the RHS and use (228). □

The following equations are proved in the same way:

$$E_q(x)E_{\frac{1}{q}}(y) = E_q[x + y]_q. \tag{266}$$

$$e_q(x)e_{\frac{1}{q}}(y) = e_q[x + y]_q. \tag{267}$$

From the q-binomial theorem (143), (152) we obtain

$$\frac{e_q(y)}{e_q(x)} = e_q[y - x]_q. \tag{268}$$

The following equations obtain:

$$D_q e_q(x) = \frac{e_q(x)}{1-q}, \tag{269}$$

$$D_q e_{\frac{1}{q}}(x) = \frac{e_{\frac{1}{q}}(qx)}{1-q}. \tag{270}$$

Instead Euler proved the equations

$$\triangle^+ e_q(x) = e_q(x). \tag{271}$$

$$\triangle^{+} e_{\frac{1}{q}}(x) = e_{\frac{1}{q}}(qx), \tag{272}$$

where $\triangle^{+}$ is defined by (298). We can now define four other q-analogues of the trigonometric functions [401].

$$sin_q(x) = \frac{1}{2i}(e_q(ix) - e_q(-ix)) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{\langle 1;q\rangle_{2n+1}},\ |x|<1. \tag{273}$$

$$cos_q(x) = \frac{1}{2}(e_q(ix) + e_q(-ix)) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{\langle 1;q\rangle_{2n}},\ |x|<1. \tag{274}$$

$$sin_{\frac{1}{q}}(x) = \frac{1}{2i}(e_{\frac{1}{q}}(ix) - e_{\frac{1}{q}}(-ix)) = \sum_{n=0}^{\infty}(-1)^n q^{n(2n+1)} \frac{x^{2n+1}}{\langle 1;q\rangle_{2n+1}}. \tag{275}$$

$$cos_{\frac{1}{q}}(x) = \frac{1}{2}(e_{\frac{1}{q}}(ix) + e_{\frac{1}{q}}(-ix)) = \sum_{n=0}^{\infty}(-1)^n q^{n(2n-1)} \frac{x^{2n}}{\langle 1;q\rangle_{2n}}, \tag{276}$$

where $x \in \mathbb{C}$ in the last two equations.

The following two addition theorems obtain [456, p. 32]:

$$cos_q(x)cos_{\frac{1}{q}}(y) \pm sin_q(x)sin_{\frac{1}{q}}(y) = cos_q[x \mp y]_q. \tag{277}$$

$$sin_q(x)cos_{\frac{1}{q}}(y) \pm cos_q(x)sin_{\frac{1}{q}}(y) = sin_q[x \pm y]_q. \tag{278}$$

The functions $sin_{\frac{1}{q}}(x)$ and $cos_{\frac{1}{q}}(x)$ solve the q-difference equation

$$(q-1)^2 D_q^2 f(x) + qf(q^2x) = 0, \tag{279}$$

and the functions $sin_q(x)$ and $cos_q(x)$ solve the q-difference equation

$$(q-1)^2 D_q^2 f(x) + f(x) = 0. \tag{280}$$

The *q-tangent numbers* $T_{2n+1}(q)$ are defined by [26, p. 380]:

$$\frac{sin_q(x)}{cos_q(x)} \equiv tan_q(x) = \sum_{n=0}^{\infty} \frac{T_{2n+1}(q)x^{2n+1}}{\langle 1;q\rangle_{2n+1}}. \tag{281}$$

It was proved by Andrews and Gessel [26, p. 380] that the polynomial $T_{2n+1}(q)$ is divisible by $\widetilde{\langle 1;q\rangle}_n$.

The *q-Euler numbers* or *q-secant numbers* $\mathcal{S}_{2n}(q)$ are defined by [27, p. 283]:

$$\frac{1}{cos_q(x)} = \sum_{n=0}^{\infty} \frac{\mathcal{S}_{2n}(q)x^{2n}}{\langle 1;q\rangle_{2n}}. \tag{282}$$

The congruence

$$\mathcal{S}_{2n} \equiv 1 \bmod 4 \tag{283}$$

was proved by Sylvester (1814-1897) [673, p. 260], [27, p. 283].

It was proved by Andrews and Foata [27, p. 283] that

$$\mathcal{S}_{2n}(q) \equiv q^{2n(n-1)} \bmod (q+1)^2. \tag{284}$$

We can now define eight q-analogues of the hyperbolic functions [949]. In the first two equations, $|q| > 1$, or $0 < |q| < 1$ and $|x| < |1-q|^{-1}$.

$$Sinh_q(x) = \frac{1}{2}(E_q(x) - E_q(-x)) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{\{2n+1\}_q!}. \tag{285}$$

$$Cosh_q(x) = \frac{1}{2}(E_q(x) + E_q(-x)) = \sum_{n=0}^{\infty} \frac{x^{2n}}{\{2n\}_q!}. \tag{286}$$

$$Sinh_{\frac{1}{q}}(x) = \frac{1}{2}(E_{\frac{1}{q}}(x) - E_{\frac{1}{q}}(-x)) = \sum_{n=0}^{\infty} q^{n(2n+1)} \frac{x^{2n+1}}{\{2n+1\}_q!}. \tag{287}$$

$$Cosh_{\frac{1}{q}}(x) = \frac{1}{2}(E_{\frac{1}{q}}(x) + E_{\frac{1}{q}}(-x)) = \sum_{n=0}^{\infty} q^{n(2n-1)} \frac{x^{2n}}{\{2n\}_q!}. \tag{288}$$

$$sinh_q(x) = \frac{1}{2}(e_q(x) - e_q(-x)) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{\langle 1;q\rangle_{2n+1}},\ |x| < 1. \tag{289}$$

$$cosh_q(x) = \frac{1}{2}(e_q(x) + e_q(-x)) = \sum_{n=0}^{\infty} \frac{x^{2n}}{\langle 1;q\rangle_{2n}},\ |x| < 1. \tag{290}$$

$$sinh_{\frac{1}{q}}(x) = \frac{1}{2}(e_{\frac{1}{q}}(x) - e_{\frac{1}{q}}(-x)) = \sum_{n=0}^{\infty} q^{n(2n+1)} \frac{x^{2n+1}}{\langle 1;q\rangle_{2n+1}}. \tag{291}$$

$$cosh_{\frac{1}{q}}(x) = \frac{1}{2}(e_{\frac{1}{q}}(x) + e_{\frac{1}{q}}(-x)) = \sum_{n=0}^{\infty} q^{n(2n-1)} \frac{x^{2n}}{\langle 1;q\rangle_{2n}}, \tag{292}$$

where $x \in \mathbb{C}$ in the last two equations.

The following four addition theorems obtain:

$$cosh_q(x)cosh_{\frac{1}{q}}(y) \pm sinh_q(x)sinh_{\frac{1}{q}}(y) = cosh_q[x \pm y]_q. \tag{293}$$

$$sinh_q(x)cosh_{\frac{1}{q}}(y) \pm cosh_q(x)sinh_{\frac{1}{q}}(y) = sinh_q[x \pm y]_q. \tag{294}$$

$$Cosh_q(x)Cosh_{\frac{1}{q}}(y) \pm Sinh_q(x)Sinh_{\frac{1}{q}}(y) = Cosh_q[x \pm y]_q. \tag{295}$$

$$Sinh_q(x)Cosh_{\frac{1}{q}}(y) \pm Cosh_q(x)Sinh_{\frac{1}{q}}(y) = Sinh_q[x \pm y]_q. \tag{296}$$

2.13. **Some variants of the q-difference operator, of the q-analogue, fractional q-differentiation and functions of matrix argument.** In such an interdisciplinary subject as q-calculus, many different definitions have been used, and in this section we try to collect some of them. In general, physicists tend to use symmetric operators.

A *symmetric q-difference operator* is defined by

$$D_{q_1,q_2} f(x) = \frac{f(q_1 x) - f(q_2 x)}{(q_1 - q_2)x}, \tag{297}$$

where $q_1 = q_2^{-1}$. Although the difference operators D_q and D_{q_1,q_2} convey the same idea, it turns out that D_{q_1,q_2} is the proper choice in constructing the Fourier transform between configuration and momentum space [296, p. 1797].

Euler used the operator

$$\triangle^+ \varphi(x) = \frac{\varphi(x) - \varphi(qx)}{x}. \tag{298}$$

Sometimes Jackson [461] used the operator

$$[\wp]\varphi(x) = \frac{\varphi(x) - \varphi(qx)}{1-q},\ q \in \mathbb{C}\backslash\{1\}. \tag{299}$$

Jackson [461, p. 305] expressed (232) as

$$x^n D_q^n = q^{-\binom{n}{2}} \prod_{m=0}^{n-1} [\wp - m], \tag{300}$$

where the m is counted as the q-analogue of m and $[\wp]$ is defined by (299). This formula can be proved by the q-Pascal identity or by (255). A related operator has been used by Horikawa [438]. Hahn called the $\{a\}_q$ basische Zahlen. We give two examples of *symmetric q-analogues*:

$$[a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}, \tag{301}$$

[438].

$$[a] = \frac{q^{\frac{a}{2}} - q^{-\frac{a}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \tag{302}$$

this is used in knot-theory in connection with R-matrices, [660].

In 1994 [200] Chung K.S. & Chung W.S. & Nam S.T. & Kang H.J. rediscovered 2 q-operations (q-addition and q-subtraction) which lead to new q-binomial formulas and consequently to a new form of the q-derivative.

> Some applications are given, such as a new Leibniz rule for the q-derivative of a product of functions and the q-Taylor expansion of a function. Most remarkably, the authors are able to define a q-log and a q-exponential which fulfill known properties of the ordinary log and exponential and hence might be more suitable for practical applications. The q-addition is defined by (compare [917, p. 256][18, p. 240])

$$(a \oplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k}, \ n = 0, 1, 2, \ldots, a \neq b. \tag{303}$$

q-Addition has the following properties:

$$\begin{aligned} &a \oplus_q b = b \oplus_q a \\ &a \oplus_q 0 = 0 \oplus_q a = a \\ &ka \oplus_q kb = k(a \oplus_q b). \end{aligned} \tag{304}$$

The q-subtraction is defined by

$$\begin{aligned} &a \ominus_q b = a \oplus_q -b \\ &a \ominus_q b = -(b \ominus_q a). \end{aligned} \tag{305}$$

Already in 1936 Morgan Ward (1901-1963) [917, p. 256] proved the following equations for q-subtraction:

$$(x \ominus_q y)^{2n+1} = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{k}_q x^k y^k (x^{2n+1-2k} - y^{2n+1-2k}). \tag{306}$$

(307)

$$(x \ominus_q y)^{2n} = (-1)^n \binom{2n}{n}_q x^n y^n + \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k}_q x^k y^k (x^{2n-2k} + y^{2n-2k}).$$

Furthermore [917, p. 262]:

$$E_q(x) E_q(-x) = \sum_{n=0}^{\infty} \frac{(1 \ominus_q 1)^{2n}}{\{2n\}_q!} x^{2n}. \tag{308}$$

$$e_q(x) e_q(-x) = \sum_{n=0}^{\infty} \frac{(1 \ominus_q 1)^{2n}}{\langle 1; q \rangle_{2n}} x^{2n}. \tag{309}$$

Ward [917, p. 258] also showed that q-addition can be a function value as follows.

If $F(x)$ denotes the formal power series

$$F(x) = \sum_{n=0}^{\infty} c_n x^n, \tag{310}$$

we define $F(x \oplus_q y)$ to mean the series

$$\sum_{n=0}^{\infty} c_n (x \oplus_q y)^n \equiv \sum_{n=0}^{\infty} \sum_{k=0}^{n} c_n \binom{n}{k}_q x^{n-k} y^k. \tag{311}$$

In like manner

$$\begin{aligned} F(x_1 \oplus_q x_2 \oplus_q \ldots \oplus_q x_k) = \sum_{n=0}^{\infty} c_n (x_1 \oplus_q x_2 \oplus_q \ldots \oplus_q x_k)^n \\ = \sum_{n=0}^{\infty} c_n P_{kn}(x). \end{aligned} \tag{312}$$

We immediately obtain the following rules for the product of two q-exponential functions

$$\begin{aligned} E_q(x)E_q(y) = E_q(x \oplus_q y) \\ e_q(x)e_q(y) = e_q(x \oplus_q y) \end{aligned} \tag{313}$$

Compare with the following two expression for the quotient of two q-exponential functions [916], [203, p. 91]

$$\frac{E_q(y)}{E_q(x)} = \sum_{k=0}^{\infty} \frac{(y-x)^{(k)}}{\{k\}_q!}, \tag{314}$$

[821, p. 71]

$$\frac{e_q(y)}{e_q(x)} = \sum_{k=0}^{\infty} \frac{(y-x)^{(k)}}{\langle 1; q \rangle_k}, \tag{315}$$

compare (268).

In order to present Ward's q-analogue of De Moivre's formula (316) and (317) we need a new notation. Let

$$\bar{n}_q \equiv 1 \oplus_q 1 \oplus_q \ldots \oplus_q 1, \; n \in \mathbb{N}.$$

Then

$$Cos_q(\bar{n}_q x) + iSin_q(\bar{n}_q x) = (Cos_q(x) + iSin_q(x))^n. \tag{316}$$

$$cos_q(\bar{n}_q x) + isin_q(\bar{n}_q x) = (cos_q(x) + isin_q(x))^n. \tag{317}$$

Furthermore Chung K.S. & Chung W.S. & Nam S.T. & Kang H.J. [200, p. 2023] defined a new q-derivative, which has the same value as the

q-difference operator D_q for all analytic functions. This q-derivative is defined by

$$D_x f(x) = \lim_{\delta x \to 0} \frac{f(x \oplus_q \delta x) - f(x)}{\delta x}. \tag{318}$$

This q-derivative D_x satisfies the following rules:

$$D_x(x \oplus_q a)^n = \{n\}_q (x \oplus_q a)^{n-1}. \tag{319}$$

$$D_x E_q(x) = E_q(x). \tag{320}$$

$$D_x e_q(x) = \frac{e_q(x)}{1-q}. \tag{321}$$

The function $Log_q(x)$ is [200, p. 2025] the inverse function to $E_q(x)$ and satisfies the following logarithm laws with addition replaced by q-addition:

$$\begin{aligned} Log_q(ab) &= Log_q(a) \oplus_q Log_q(b) \\ Log_q(\frac{a}{b}) &= Log_q(a) \ominus_q Log_q(b) \\ Log_q(a^n) &= nLog_q(a) \end{aligned} \tag{322}$$

The function $log_q(x)$ is the inverse function to $e_q(x)$ and satisfies the following logarithm laws with addition replaced by q-addition:

$$\begin{aligned} log_q(ab) &= log_q(a) \oplus_q log_q(b) \\ log_q(\frac{a}{b}) &= log_q(a) \ominus_q log_q(b) \\ log_q(a^n) &= nlog_q(a) \end{aligned} \tag{323}$$

Another power function is defined by $a_q^r = E_q(rlog_q(a))$. This power function satisfies the following laws:

$$\begin{aligned} a_q^x a_q^y &= a_q^{x \oplus_q y} \\ \frac{a_q^x}{a_q^y} &= a_q^{x \ominus_q y} \\ (ab)_q^x &= a_q^x b_q^x \\ (\frac{a}{b})_q^x &= \frac{a_q^x}{b_q^x} \end{aligned} \tag{324}$$

We obtain the following addition theorems for the q-analogues of the trigonometric functions.

$$Cos_q(x)Cos_q(y) + Sin_q(x)Sin_q(y) = Cos_q(x \ominus_q y). \tag{325}$$

$$Cos_q(x)Cos_q(y) - Sin_q(x)Sin_q(y) = Cos_q(x \oplus_q y). \tag{326}$$

$$Sin_q(x)Cos_q(y) + Sin_q(y)Cos_q(x) = Sin_q(x \oplus_q y). \tag{327}$$

$$Sin_q(x)Cos_q(y) - Sin_q(y)Cos_q(x) = Sin_q(x \ominus_q y). \tag{328}$$

$$cos_q(x)cos_q(y) + sin_q(x)sin_q(y) = cos_q(x \ominus_q y). \tag{329}$$

$$cos_q(x)cos_q(y) - sin_q(x)sin_q(y) = cos_q(x \oplus_q y). \tag{330}$$

$$sin_q(x)cos_q(y) + sin_q(y)cos_q(x) = sin_q(x \oplus_q y). \tag{331}$$

$$sin_q(x)cos_q(y) - sin_q(y)cos_q(x) = sin_q(x \ominus_q y). \tag{332}$$

We obtain the following addition theorems for the q-analogues of the hyperbolic functions.

$$Cosh_q(x)Cosh_q(y) + Sinh_q(x)Sinh_q(y) = Cosh_q(x \oplus_q y). \tag{333}$$

$$Cosh_q(x)Cosh_q(y) - Sinh_q(x)Sinh_q(y) = Cosh_q(x \ominus_q y). \tag{334}$$

$$Sinh_q(x)Cosh_q(y) + Sinh_q(y)Cosh_q(x) = Sinh_q(x \oplus_q y). \tag{335}$$

$$Sinh_q(x)Cosh_q(y) - Sinh_q(y)Cosh_q(x) = Sinh_q(x \ominus_q y). \tag{336}$$

$$cosh_q(x)cosh_q(y) + sinh_q(x)sinh_q(y) = cosh_q(x \oplus_q y). \tag{337}$$

$$cosh_q(x)cosh_q(y) - sinh_q(x)sinh_q(y) = cosh_q(x \ominus_q y). \tag{338}$$

$$sinh_q(x)cosh_q(y) + sinh_q(y)cosh_q(x) = sinh_q(x \oplus_q y). \tag{339}$$

$$sinh_q(x)cosh_q(y) - sinh_q(y)cosh_q(x) = sinh_q(x \ominus_q y). \tag{340}$$

The q-coaddition is defined by [18, p. 240])

$$(a \oplus^q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q q^{k(k-n)} a^k b^{n-k},\ n = 0, 1, 2, \ldots, a \neq b. \tag{341}$$

q-coaddition has the following properties:

$$\begin{aligned} &a \oplus^q b = b \oplus^q a \\ &a \oplus^q 0 = 0 \oplus^q a = a \\ &ka \oplus^q kb = k(a \oplus^q b). \end{aligned} \tag{342}$$

The q-cosubtraction is defined by

$$\begin{aligned} &a \ominus^q b = a \oplus^q -b \\ &a \ominus^q b = -(b \ominus^q a). \end{aligned} \tag{343}$$

The following equations for q-cosubtraction obtain, compare [917, p. 256]
(344)

$$(x\ominus^q y)^{2n+1}=\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}_q q^{k(k-2n-1)}x^k y^k(x^{2n+1-2k}-y^{2n+1-2k}).$$

(345)

$$\begin{aligned}(x\ominus^q y)^{2n}&=(-1)^n\binom{2n}{n}_q q^{-n^2}x^n y^n+\\&+\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}_q q^{k(k-2n)}x^k y^k(x^{2n-2k}+y^{2n-2k}).\end{aligned}$$

Furthermore

$$E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(-x)=\sum_{n=0}^{\infty}\frac{(1\ominus^q 1)^{2n}}{\{2n\}_q!}q^{\binom{2n}{2}}x^{2n}. \tag{346}$$

$$e_{\frac{1}{q}}(x)e_{\frac{1}{q}}(-x)=\sum_{n=0}^{\infty}\frac{(1\ominus^q 1)^{2n}}{\langle 1;q\rangle_{2n}}q^{\binom{2n}{2}}x^{2n}. \tag{347}$$

We assume that formulas similar to equations (311) and (312) for q-coaddition exist. We immediately obtain the following rules for the product of two q-exponential functions

$$\begin{aligned}E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)&=E_{\frac{1}{q}}(x\oplus^q y)\\ e_{\frac{1}{q}}(x)e_{\frac{1}{q}}(y)&=e_{\frac{1}{q}}(x\oplus^q y)\end{aligned} \tag{348}$$

Compare with the following two expression for the quotient of two q-exponential functions [916], [203, p. 91]

$$\frac{E_{\frac{1}{q}}(y)}{E_{\frac{1}{q}}(x)}=\sum_{k=0}^{\infty}\frac{\widetilde{(y-x)}^{(k)}}{\{k\}_q!}, \tag{349}$$

[821, p. 71]

$$\frac{e_{\frac{1}{q}}(y)}{e_{\frac{1}{q}}(x)}=\sum_{k=0}^{\infty}\frac{\widetilde{(y-x)}^{(k)}}{\langle 1;q\rangle_k}. \tag{350}$$

In order to present the two following q-analogues of De Moivre's formula we need a new notation. Let

$$\bar{n}^q\equiv 1\oplus^q 1\oplus^q\ldots\oplus^q 1,\ n\in\mathbb{N}.$$

Then

$$Cos_{\frac{1}{q}}(\bar{n}^q x)+iSin_{\frac{1}{q}}(\bar{n}^q x)=(Cos_{\frac{1}{q}}(x)+iSin_{\frac{1}{q}}(x))^n. \tag{351}$$

$$cos_{\frac{1}{q}}(\bar{n}^q x) + isin_{\frac{1}{q}}(\bar{n}^q x) = (cos_{\frac{1}{q}}(x) + isin_{\frac{1}{q}}(x))^n. \tag{352}$$

We obtain the following addition theorems for the q-analogues of the trigonometric functions.

$$Cos_{\frac{1}{q}}(x)Cos_{\frac{1}{q}}(y) + Sin_{\frac{1}{q}}(x)Sin_{\frac{1}{q}}(y) = Cos_{\frac{1}{q}}(x \ominus^q y). \tag{353}$$

$$Cos_{\frac{1}{q}}(x)Cos_{\frac{1}{q}}(y) - Sin_{\frac{1}{q}}(x)Sin_{\frac{1}{q}}(y) = Cos_{\frac{1}{q}}(x \oplus^q y). \tag{354}$$

$$Sin_{\frac{1}{q}}(x)Cos_{\frac{1}{q}}(y) + Sin_{\frac{1}{q}}(y)Cos_{\frac{1}{q}}(x) = Sin_{\frac{1}{q}}(x \oplus^q y). \tag{355}$$

$$Sin_{\frac{1}{q}}(x)Cos_{\frac{1}{q}}(y) - Sin_{\frac{1}{q}}(y)Cos_{\frac{1}{q}}(x) = Sin_{\frac{1}{q}}(x \ominus^q y). \tag{356}$$

$$cos_{\frac{1}{q}}(x)cos_{\frac{1}{q}}(y) + sin_{\frac{1}{q}}(x)sin_{\frac{1}{q}}(y) = cos_{\frac{1}{q}}(x \ominus^q y). \tag{357}$$

$$cos_{\frac{1}{q}}(x)cos_{\frac{1}{q}}(y) - sin_{\frac{1}{q}}(x)sin_{\frac{1}{q}}(y) = cos_{\frac{1}{q}}(x \oplus^q y). \tag{358}$$

$$sin_{\frac{1}{q}}(x)cos_{\frac{1}{q}}(y) + sin_{\frac{1}{q}}(y)cos_{\frac{1}{q}}(x) = sin_{\frac{1}{q}}(x \oplus^q y). \tag{359}$$

$$sin_{\frac{1}{q}}(x)cos_{\frac{1}{q}}(y) - sin_{\frac{1}{q}}(y)cos_{\frac{1}{q}}(x) = sin_{\frac{1}{q}}(x \ominus^q y). \tag{360}$$

We obtain the following addition theorems for the q-analogues of the hyperbolic functions.

$$Cosh_{\frac{1}{q}}(x)Cosh_{\frac{1}{q}}(y) + Sinh_{\frac{1}{q}}(x)Sinh_{\frac{1}{q}}(y) = Cosh_{\frac{1}{q}}(x \oplus^q y). \tag{361}$$

$$Cosh_{\frac{1}{q}}(x)Cosh_{\frac{1}{q}}(y) - Sinh_{\frac{1}{q}}(x)Sinh_{\frac{1}{q}}(y) = Cosh_{\frac{1}{q}}(x \ominus^q y). \tag{362}$$

$$Sinh_{\frac{1}{q}}(x)Cosh_{\frac{1}{q}}(y) + Sinh_{\frac{1}{q}}(y)Cosh_{\frac{1}{q}}(x) = Sinh_{\frac{1}{q}}(x \oplus^q y). \tag{363}$$

$$Sinh_{\frac{1}{q}}(x)Cosh_{\frac{1}{q}}(y) - Sinh_{\frac{1}{q}}(y)Cosh_{\frac{1}{q}}(x) = Sinh_{\frac{1}{q}}(x \ominus^q y). \tag{364}$$

$$cosh_{\frac{1}{q}}(x)cosh_{\frac{1}{q}}(y) + sinh_{\frac{1}{q}}(x)sinh_{\frac{1}{q}}(y) = cosh_{\frac{1}{q}}(x \oplus^q y). \tag{365}$$

$$cosh_{\frac{1}{q}}(x)cosh_{\frac{1}{q}}(y) - sinh_{\frac{1}{q}}(x)sinh_{\frac{1}{q}}(y) = cosh_{\frac{1}{q}}(x \ominus^q y). \tag{366}$$

$$sinh_{\frac{1}{q}}(x)cosh_{\frac{1}{q}}(y) + sinh_{\frac{1}{q}}(y)cosh_{\frac{1}{q}}(x) = sinh_{\frac{1}{q}}(x \oplus^q y). \tag{367}$$

$$sinh_{\frac{1}{q}}(x)cosh_{\frac{1}{q}}(y) - sinh_{\frac{1}{q}}(y)cosh_{\frac{1}{q}}(x) = sinh_{\frac{1}{q}}(x \ominus^q y). \tag{368}$$

One definition of fractional derivative is the following generalization of Cauchy's integral formula to arbitrary v [322, p. 11]:

$$D^v f(z) = \frac{\Gamma(v+1)}{2\pi i} \int_c^{x+} (t-z)^{-v-1} f(t)\, dt, \tag{369}$$

where we use the principal branch of the logarithm and $c < x$. In 1990 [783] Saxena and Gupta

> unify and extend the results on q-integral operators existing in the literature and define two multidimensional q-integral operators associated with a basic analogue of the Srivastava-Daoust generalized Lauricella hypergeometric function of several complex variables introduced by H.M.Srivastava [212] 1983

and in 1991 [784] Saxena and Kumar

> evaluate certain integrals associated with the basic hypergeometric functions of two variables by the application of certain operators of *fractional q-differentiation*, which generalizes the results of Upadhyay [893].

In [949], a matrix q-hypergeometric series was explored. In [633] a Ramanujan ${}_1\Psi_1$ summation theorem for a Laurent series extension of Macdonald's Schur function multiple basic hypergeometric series of matrix argument was proved. This result contains as special, limiting cases our Schur function extension of the q-binomial theorem and the Jacobi triple product identity. Also a Heine transformation and q-Gauss summation theorem for Schur functions was proved.

2.14. **Heine's transformation formula for the ${}_2\phi_1$ and the q-gamma function.** Heine proceeded to prove a transformation formula for the ${}_2\phi_1$ which turned out to be the q-analogue of (66) [426],[427]:

Theorem 2.22.

$$ {}_2\phi_1(a,b;c|q,q^z) = \frac{\langle b, a+z; q\rangle_\infty}{\langle c, z; q\rangle_\infty}\, {}_2\phi_1(c-b, z; a+z|q, q^b), \tag{370}$$

where $|q^z| < 1$ *and* $|q^b| < 1$.

Proof.

$$
\begin{aligned}
{}_2\phi_1(a,b;c|q,q^z) &= \sum_{n=0}^{\infty}\frac{\langle a;q\rangle_n\langle b;q\rangle_n}{\langle 1;q\rangle_n\langle c;q\rangle_n}q^{zn} \overset{\text{by}(185)}{=} \\
&= \frac{\langle b;q\rangle_\infty}{\langle c;q\rangle_\infty}\sum_{n=0}^{\infty}\frac{\langle a;q\rangle_n\langle c+n;q\rangle_\infty}{\langle 1;q\rangle_n\langle b+n;q\rangle_\infty}q^{zn} \overset{\text{by}(150)}{=} \\
&= \frac{\langle b;q\rangle_\infty}{\langle c;q\rangle_\infty}\sum_{n=0}^{\infty}\frac{\langle a;q\rangle_n}{\langle 1;q\rangle_n}q^{zn}\sum_{m=0}^{\infty}\frac{\langle c-b;q\rangle_m}{\langle 1;q\rangle_m}q^{m(b+n)} = \\
&= \frac{\langle b;q\rangle_\infty}{\langle c;q\rangle_\infty}\sum_{m=0}^{\infty}\frac{\langle c-b;q\rangle_m}{\langle 1;q\rangle_m}q^{mb}\sum_{n=0}^{\infty}\frac{\langle a;q\rangle_n}{\langle 1;q\rangle_n}q^{n(z+m)} \overset{\text{by}(150)}{=} \qquad (371)\\
&= \frac{\langle b;q\rangle_\infty}{\langle c;q\rangle_\infty}\sum_{m=0}^{\infty}\frac{\langle c-b;q\rangle_m}{\langle 1;q\rangle_m}q^{mb}\frac{\langle a+z+m;q\rangle_\infty}{\langle z+m;q\rangle_\infty} \overset{\text{by}(185)}{=} \\
&= \frac{\langle b;q\rangle_\infty}{\langle c;q\rangle_\infty}\sum_{m=0}^{\infty}\frac{\langle c-b,z;q\rangle_m}{\langle 1,a+z;q\rangle_m}\frac{\langle a+z;q\rangle_\infty}{\langle z;q\rangle_\infty}q^{mb} = \\
&= \frac{\langle b,a+z;q\rangle_\infty}{\langle c,z;q\rangle_\infty}\ {}_2\phi_1(c-b,z;a+z|q,q^b).
\end{aligned}
$$

□

The following simple rules will sometimes be used:

$$\widetilde{a}\pm b=\widetilde{a\pm b}. \tag{372}$$

$$\widetilde{a}\pm\widetilde{b}=a\pm b. \tag{373}$$

$$q^{\widetilde{a}}=q^a. \tag{374}$$

$${}_p\phi_r(a_1,\ldots,a_p;b_1,\ldots,b_r|q,\widetilde{t})={}_p\phi_r(a_1,\ldots,a_p;b_1,\ldots,b_r|q,-t). \tag{375}$$

One example of the application of (373) is (412) and two examples of the application of (374) are (408) and (410).

Motivation for equations (372) through (375): We generalize Heine's transformation formula for the ${}_2\phi_1$ in the following way:

Theorem 2.23.

$${}_2\phi_1(a,b;c|q,-q^z)=\frac{\langle b,\widetilde{a+z};q\rangle_\infty}{\langle c,\widetilde{z};q\rangle_\infty}\ {}_2\phi_1(c-b,\widetilde{z};\widetilde{a+z}|q,q^b), \tag{376}$$

where $|q^z|<1$ *and* $|q^b|<1$.

Proof.

$$
\begin{aligned}
{}_2\phi_1(a,b;c|q,-q^z) &= \sum_{n=0}^{\infty} \frac{\langle a;q\rangle_n \langle b;q\rangle_n}{\langle 1;q\rangle_n \langle c;q\rangle_n}(-q^z)^n \overset{\text{by}(185)}{=} \\
&= \frac{\langle b;q\rangle_\infty}{\langle c;q\rangle_\infty} \sum_{n=0}^{\infty} \frac{\langle a;q\rangle_n \langle c+n;q\rangle_\infty}{\langle 1;q\rangle_n \langle b+n;q\rangle_\infty}(-q^z)^n \overset{\text{by}(151)}{=} \\
&= \frac{\langle b;q\rangle_\infty}{\langle c;q\rangle_\infty} \sum_{n=0}^{\infty} \frac{\langle a;q\rangle_n}{\langle 1;q\rangle_n}(-q^z)^n \sum_{m=0}^{\infty} \frac{\langle c-b;q\rangle_m}{\langle 1;q\rangle_m} q^{m(b+n)} = \\
&= \frac{\langle b;q\rangle_\infty}{\langle c;q\rangle_\infty} \sum_{m=0}^{\infty} \frac{\langle c-b;q\rangle_m}{\langle 1;q\rangle_m} q^{mb} \sum_{n=0}^{\infty} \frac{\langle a;q\rangle_n}{\langle 1;q\rangle_n}(-q^{z+m})^n \overset{\text{by}(151)}{=} \qquad (377) \\
&= \frac{\langle b;q\rangle_\infty}{\langle c;q\rangle_\infty} \sum_{m=0}^{\infty} \frac{\langle c-b;q\rangle_m}{\langle 1;q\rangle_m} q^{mb} \frac{\langle a+\widetilde{z+m};q\rangle_\infty}{\langle \widetilde{z+m};q\rangle_\infty} \overset{\text{by}(185)}{=} \\
&= \frac{\langle b;q\rangle_\infty}{\langle c;q\rangle_\infty} \sum_{m=0}^{\infty} \frac{\langle c-b,\widetilde{z};q\rangle_m}{\langle 1,\widetilde{a+z};q\rangle_m} \frac{\langle \widetilde{a+z};q\rangle_\infty}{\langle \widetilde{z};q\rangle_\infty} q^{mb} = \\
&= \frac{\langle b,\widetilde{a+z};q\rangle_\infty}{\langle c,\widetilde{z};q\rangle_\infty}\ {}_2\phi_1(c-b,\widetilde{z};\widetilde{a+z}|q,q^b).
\end{aligned}
$$

□

Let the q-gamma function be defined by [862], [457], [545]

$$
\Gamma_q(x) = \begin{cases} \frac{\langle 1;q\rangle_\infty}{\langle x;q\rangle_\infty}(1-q)^{1-x}, & \text{if } 0<|q|<1 \\ \frac{\langle 1;q^{-1}\rangle_\infty}{\langle x;q^{-1}\rangle_\infty}(q-1)^{1-x} q^{\binom{x}{2}}, & \text{if } 1<q. \end{cases} \tag{378}
$$

Let k denote a positive integer. In the same spirit we define

$$
\Gamma_{q^k}(x) = \begin{cases} \frac{\langle k;q^k\rangle_\infty}{\langle kx;q^k\rangle_\infty}(1-q^k)^{1-x}, & \text{if } 0<|q|<1 \\ \frac{\langle k;q^{-k}\rangle_\infty}{\langle kx;q^{-k}\rangle_\infty}(q^k-1)^{1-x} q^{k\binom{x}{2}}, & \text{if } 1<q. \end{cases} \tag{379}
$$

With the help of equations (185), (378) and (161) the transformation formula (370) can be rewritten as a q-integral expression, first found by Thomae 1869 [862], which is the q-analogue of (66) and takes the following form in the new notation:

$$
{}_2\phi_1(a,b;c|q,q^z) = \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)} \int_0^1 t^{b-1} \frac{(qt;q)_{c-b-1}}{(q^z t;q)_a}\, d_q(t). \tag{380}
$$

We proceed to find the q-analogue (382) of Legendre's duplication formula (76) [343, p. 18]:

$$\begin{aligned}\frac{\Gamma_{q^2}(x)\Gamma_{q^2}(x+\frac{1}{2})}{\Gamma_{q^2}(\frac{1}{2})} &= \frac{\langle 1,2;q^2\rangle_\infty}{\langle 2x,2x+1;q^2\rangle_\infty}(1-q^2)^{1-2x} = \\ &\overset{\text{by}(196)}{=} \frac{\langle 1;q\rangle_\infty}{\langle 2x;q\rangle_\infty}(1-q^2)^{1-2x} = (1+q)^{1-2x}\Gamma_q(2x).\end{aligned} \tag{381}$$

Next multiply by $(1+q)^{2x-1}\Gamma_{q^2}(\frac{1}{2})$ to obtain the equality

$$\Gamma_q(2x)\Gamma_{q^2}(\frac{1}{2}) = (1+q)^{2x-1}\Gamma_{q^2}(x)\Gamma_{q^2}(x+\frac{1}{2}). \tag{382}$$

Our next goal is to prove a q-analogue of (67) (see Gasper and Rahman [343, p. 10], formula (1.4.3)). Also see Heine 1878 [427, p. 115]. In the new notation this formula takes the following form:

Theorem 2.24.
(383)
$$ {}_2\phi_1(a,b;c|q,q^z) = \frac{\langle a+b+z-c;q\rangle_\infty}{\langle z;q\rangle_\infty}\, {}_2\phi_1(c-a,c-b;c|q,q^{a+b+z-c}).$$

Proof. Just iterate (370) as follows:

$$\begin{aligned} {}_2\phi_1(a,b;c|q,q^z) &= \frac{\langle b,a+z;q\rangle_\infty}{\langle c,z;q\rangle_\infty}\, {}_2\phi_1(c-b,z;a+z|q,q^b) = \\ &= \frac{\langle c-b,b+z;q\rangle_\infty}{\langle c,z;q\rangle_\infty}\, {}_2\phi_1(a+b+z-c,b;b+z|q,q^{c-b}) = \\ &= \frac{\langle a+b+z-c;q\rangle_\infty}{\langle z;q\rangle_\infty}\, {}_2\phi_1(c-a,c-b;c|q,q^{a+b+z-c}).\end{aligned}$$

$\square$

2.15. **A q-beta analogue and a q-analogue of the dilogarithm and of the digamma function.** The q-beta function is defined by

$$B_q(x,y) \equiv \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. \tag{384}$$

Our next goal is to prove a q-analogue of (65). Let $Re(x) > 0$. Then (see Gasper and Rahman [343, p. 19], formula (1.11.7)).

$$B_q(x,y) = \int_0^1 t^{x-1}\frac{(tq;q)_\infty}{(tq^y;q)_\infty}\, d_q(t),\ \ y \neq 0,-1,-2,\ldots. \tag{385}$$

Proof.

$$B_q(x,y) = (1-q)\frac{\langle 1, x+y; q\rangle_\infty}{\langle x,y;q\rangle_\infty} \overset{\text{by}(150)}{=}$$

(386)
$$= (1-q)\frac{\langle 1;q\rangle_\infty}{\langle y;q\rangle_\infty}\sum_{n=0}^{\infty}\frac{\langle y;q\rangle_n}{\langle 1;q\rangle_n}q^{nx} \overset{\text{by}(185)}{=}$$

$$= (1-q)\sum_{n=0}^{\infty}\frac{\langle n+1;q\rangle_\infty}{\langle n+y;q\rangle_\infty}q^{nx} \overset{\text{by}(160)}{=} \int_0^1 t^{x-1}\frac{(tq;q)_\infty}{(tq^y;q)_\infty}\,d_q(t).$$

□

Kirillov A.N. [520], [535] has found the following q-analogue of the dilogarithm (68):

$$Li_2(z,q) \equiv \sum_{n=1}^{\infty}\frac{z^n}{n(1-q^n)},\ |z|<1,\ 0<q<1. \tag{387}$$

Remark 24. The dilogarithm is used in representation theory of Virasoro and Kac-Moody algebras, and conformal field theory.

A q-analogue of the power series for $-log(1-z)$ was given by Heine

$$z\ {}_2\phi_1(1,1;2|q,z). \tag{388}$$

In [545] the following q-analogue of the digamma function was found, whereby

> numerous summation formulas for hypergeometric and basic hypergeometric series were derived. Among these summation formulas are nonterminating extensions and q-extensions of identities recorded by Lavoie, Luke, Watson, and Srivastava.

(389)
$$\Psi_q(x) = \begin{cases} -log(1-q) + logq\sum_{n=0}^{\infty}\frac{q^{n+x}}{1-q^{n+x}} & \text{if } 0<|q|<1 \\ -log(q-1) + (x-\frac{1}{2})logq + logq\sum_{n=0}^{\infty}\frac{q^{-n-x}}{q^{-n-x}-1} & \text{if } 1<q. \end{cases}$$

2.16. **Heine's q-analogue of Gauss' summation formula.** Heine's q-analogue of Gauss' summation formula (see Gasper and Rahman [343, p. 10], formula (1.5.1)) takes the following form in the new notation:

Theorem 2.25.

$${}_2\phi_1(a,b;c|q,q^{c-a-b}) = \frac{\langle c-a, c-b;q\rangle_\infty}{\langle c, c-a-b;q\rangle_\infty},\ |q^{c-a-b}|<1. \tag{390}$$

Proof. Put $z = c-a-b$ in (370), assume that $|q^b| < 1$, $|q^{c-a-b}| < 1$ to obtain

(391)
$$\begin{aligned} &{}_2\phi_1(a,b;c|q,q^{c-a-b}) = \frac{\langle b, c-b; q\rangle_\infty}{\langle c, c-a-b; q\rangle_\infty}\, {}_1\phi_0(c-a-b;-|q,q^b) = \\ &\overset{\text{by}(150)}{=} \frac{\langle c-a, c-b; q\rangle_\infty}{\langle c, c-a-b; q\rangle_\infty}. \end{aligned}$$

By analytic continuation, we can drop the assumption $|q^b| < 1$ and require only $|q^{c-a-b}| < 1$ in (390). □

In the terminating case $a = -n$, (390) reduces to (see Gasper and Rahman [343, p. 11], formula (1.5.2)):

(392)
$${}_2\phi_1(-n,b;c|q,q^{c+n-b}) \overset{\text{by}(185)}{=} \frac{\langle c-b; q\rangle_n}{\langle c; q\rangle_n},\ n = 0,1,\ldots,$$

which is one q-analogue of (69).

We proceed to prove Jackson's 1910 [462] q-analogue of (71) (see Gasper and Rahman [343, p. 11], formula (1.5.4)), which takes the following form in the new notation:

(393)
$$\begin{aligned} &{}_2\phi_1(a,b;c|q,q^z) = \\ &= \frac{\langle a+z; q\rangle_\infty}{\langle z; q\rangle_\infty} \sum_{n=0}^{\infty} \frac{\langle a, c-b; q\rangle_n}{\langle 1, c, a+z; q\rangle_n} (-1)^n q^{n(b+z)+\binom{n}{2}} \equiv \\ &\equiv \frac{\langle a+z; q\rangle_\infty}{\langle z; q\rangle_\infty}\, {}_2\phi_2(a, c-b; c, a+z|q, q^{b+z}). \end{aligned}$$

Proof.

$$
\begin{aligned}
&{}_2\phi_1(a,b;c|q,q^z) \overset{\text{by}(392)}{=} \sum_{m=0}^{\infty} \frac{\langle a;q\rangle_m}{\langle 1;q\rangle_m} q^{mz} \times \\
&\times \sum_{n=0}^{m} \frac{\langle -m, c-b;q\rangle_n}{\langle 1,c;q\rangle_n} q^{(b+m)n} \overset{\text{by}(193)}{=} \\
&= \sum_{n=0}^{\infty}\sum_{m=n}^{\infty} \frac{\langle a;q\rangle_m \langle c-b;q\rangle_n}{\langle 1,c;q\rangle_n \langle 1;q\rangle_{m-n}} q^{mz+bn+\binom{n}{2}}(-1)^n = \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\langle a;q\rangle_{m+n} \langle c-b;q\rangle_n}{\langle 1,c;q\rangle_n \langle 1;q\rangle_{m}} q^{(m+n)z+bn+\binom{n}{2}}(-1)^n \overset{\text{by}(198)}{=} \\
&= \sum_{n=0}^{\infty} \frac{\langle a,c-b;q\rangle_n}{\langle 1,c;q\rangle_n} (-1)^n q^{(b+z)n+\binom{n}{2}} \sum_{m=0}^{\infty} \frac{\langle a+n;q\rangle_m}{\langle 1;q\rangle_m} q^{mz} \overset{\text{by}(150)}{=} \\
&= \sum_{n=0}^{\infty} \frac{\langle a+z+n;q\rangle_\infty}{\langle z;q\rangle_\infty} \frac{\langle a,c-b;q\rangle_n}{\langle 1,c;q\rangle_n} (-1)^n q^{n(b+z)+\binom{n}{2}} \overset{\text{by}(185)}{=} \\
&= \frac{\langle a+z;q\rangle_\infty}{\langle z;q\rangle_\infty} \sum_{n=0}^{\infty} \frac{\langle a,c-b;q\rangle_n}{\langle 1,c,a+z;q\rangle_n} (-1)^n q^{n(b+z)+\binom{n}{2}}.
\end{aligned}
\tag{394}
$$

□

If $a=-n$, then the series on the right hand side of (393) can be reversed to yield Sears' 1951 transformation formula [793] (see Gasper and Rahman [343, p. 11], formula (1.5.6)). In the new notation this formula takes the following form:

$$
\begin{aligned}
&{}_2\phi_1(-n,b;c|q,q^z) = \frac{\langle c-b;q\rangle_n}{\langle c;q\rangle_n} q^{(b+z-1)n} \times \\
&\times \;{}_3\phi_2(-n,1-z,-c+1-n;b-c+1-n,\infty|q,q), \\
&0<|q|<1.
\end{aligned}
\tag{395}
$$

Proof.

$$
\begin{aligned}
&{}_2\phi_1(-n,b;c|q,q^z) \overset{\text{by}(393)}{=} \\
&= \frac{\langle -n+z;q\rangle_\infty}{\langle z;q\rangle_\infty} \sum_{l=0}^{n} \frac{\langle -n,c-b;q\rangle_l}{\langle 1,c,-n+z;q\rangle_l} (-1)^l q^{l(b+z)+\binom{l}{2}} = \\
&= \langle -n+z;q\rangle_n \sum_{k=0}^{n} \frac{\langle -n,c-b;q\rangle_{n-k}}{\langle 1,c,-n+z;q\rangle_{n-k}} (-1)^{n-k} q^{(n-k)(b+z)+\binom{n-k}{2}} =
\end{aligned}
$$

$$\overset{\text{by}4\times(189),(193)}{=} \frac{\langle c-b,-n;q\rangle_n}{\langle c,1;q\rangle_n}(-1)^n q^{(b+z)n+\binom{n}{2}}\times$$

$$\times\ {}_3\phi_2(-n,1-z,-c+1-n;b-c+1-n,\infty|q,q) \overset{\text{by}(188)}{=}$$

$$= \frac{\langle c-b;q\rangle_n}{\langle c;q\rangle_n} q^{(b+z-1)n}\ {}_3\phi_2(-n,1-z,-c+1-n;b-c+1-n,\infty|q,q).$$

□

2.17. **A q-analogue of Saalschütz's summation formula.** We proceed to prove Jackson's 1910 [462] q-analogue of (70) (see Gasper and Rahman [343, p. 13], formula (1.7.2)). In the new notation this formula takes the following form:

Theorem 2.26.
(396)

$${}_3\phi_2(a,b,-n;c,1+a+b-c-n|q,q) = \frac{\langle c-a,c-b;q\rangle_n}{\langle c,c-a-b;q\rangle_n} \quad n=0,1\ldots.$$

Proof. By the q-binomial theorem (150),

$$\frac{\langle z+a+b-c;q\rangle_\infty}{\langle z;q\rangle_\infty} = \sum_{n=0}^{\infty} \frac{\langle a+b-c;q\rangle_n}{\langle 1;q\rangle_n} q^{zn}$$

equation (383) gives

$$\sum_{n=0}^{\infty} \frac{\langle a,b;q\rangle_n}{\langle 1,c;q\rangle_n} q^{zn} = \sum_{k=0}^{\infty} \frac{\langle a+b-c;q\rangle_k}{\langle 1;q\rangle_k} \sum_{j=0}^{\infty} \frac{\langle c-a,c-b;q\rangle_j}{\langle c,1;q\rangle_j} q^{j(a+b-c)} q^{z(k+j)},$$

and hence, equating the coefficients of q^{zn} gives

$$\frac{\langle a,b;q\rangle_n}{\langle 1,c;q\rangle_n} = \sum_{j=0}^{n} \frac{\langle a+b-c;q\rangle_{n-j}}{\langle 1;q\rangle_{n-j}} \frac{\langle c-a,c-b;q\rangle_j}{\langle 1,c;q\rangle_j} q^{j(a+b-c)}.$$

By (189) this is equivalent to

$$\frac{\langle a,b;q\rangle_n}{\langle a+b-c,c;q\rangle_n} = \sum_{j=0}^{n} \frac{\langle c-a,c-b,-n;q\rangle_j}{\langle 1,c,1-a-b+c-n;q\rangle_j} q^j.$$

Replacing a,b by $c-a,c-b$, respectively completes the proof of (396).

□

Remember that $0<|q|<1$. Now let $a\to\infty$ in (396) and observe that

$$\lim_{a\to+\infty} \frac{\langle c-a;q\rangle_n}{\langle c-a-b;q\rangle_n} = q^{bn}.$$

The formula (1.5.3) in Gasper and Rahman [343, p. 11], takes the following form in the new notation :

$$
{}_2\phi_1(-n,b;c|q,q)=\frac{\langle c-b;q\rangle_n}{\langle c;q\rangle_n}q^{bn},\ n=0,1,\ldots, \tag{397}
$$

which is another q-analogue of (69).

2.18. **The Bailey-Daum summation formula.** We proceed to prove the following q-analogue of Kummer's formula (77), which was proved independently by Bailey 1941 [68] and Daum 1942 [223] (see Gasper and Rahman [343, p. 14], formula (1.8.1)). In the new notation this formula takes the following form:

$$
{}_2\phi_1(a,b;1+a-b|q,-q^{1-b})=\frac{\langle\widetilde{1};q\rangle_\infty\langle a+1,2(1-b)+a;q^2\rangle_\infty}{\langle 1+a-b,\widetilde{1-b};q\rangle_\infty}. \tag{398}
$$

Proof.

$$
\begin{aligned}
&{}_2\phi_1(a,b;1+a-b|q,-q^{1-b}) \stackrel{\text{by}(370)}{=}\\
&=\frac{\langle a,\widetilde{1};q\rangle_\infty}{\langle 1+a-b,\widetilde{1-b};q\rangle_\infty}\,{}_2\phi_1(1-b,\widetilde{1-b};\widetilde{1}|q,q^a)=\\
&\frac{\langle a,\widetilde{1};q\rangle_\infty}{\langle 1+a-b,\widetilde{1-b};q\rangle_\infty}\sum_{n=0}^{\infty}\frac{\langle 1-b,\widetilde{1-b};q\rangle_n}{\langle 1,\widetilde{1};q\rangle_n}q^{an}=\\
&\stackrel{\text{by}(195)}{=}\frac{\langle a,\widetilde{1};q\rangle_\infty}{\langle 1+a-b,\widetilde{1-b};q\rangle_\infty}\sum_{n=0}^{\infty}\frac{\langle 2(1-b);q^2\rangle_n}{\langle 2;q^2\rangle_n}q^{an}=\\
&\stackrel{\text{by}(150)}{=}\frac{\langle a,\widetilde{1};q\rangle_\infty}{\langle 1+a-b,\widetilde{1-b};q\rangle_\infty}\frac{\langle 2(1-b)+a;q^2\rangle_\infty}{\langle a;q^2\rangle_\infty}\stackrel{\text{by}(196)}{=}\\
&=\frac{\langle\widetilde{1};q\rangle_\infty\langle a+1,2(1-b)+a;q^2\rangle_\infty}{\langle 1+a-b,\widetilde{1-b};q\rangle_\infty}.
\end{aligned} \tag{399}
$$

□

2.19. **A general expansion formula.** First a lemma

Lemma 2.27.

$$
\begin{aligned}
&(-1)^j q^{-ji}q^{-\binom{j}{2}}\frac{\langle a+j,-n;q\rangle_j\langle -n+j,a+2j;q\rangle_i}{\langle 1,a+j;q\rangle_i}=\\
&\frac{\langle a+i+j,-i-j;q\rangle_j\langle -n;q\rangle_{i+j}q^j}{\langle 1;q\rangle_{i+j}}.
\end{aligned} \tag{400}
$$

Proof. This follows from (198), (190) and (193)

$$\langle -n;q\rangle_{i+j}=\langle -n;q\rangle_j\langle -n+j;q\rangle_i. \tag{401}$$

$$\langle a+2j;q\rangle_i\langle a+j;q\rangle_j=\langle a+j;q\rangle_i\langle a+i+j;q\rangle_j. \tag{402}$$

$$\langle -i-j;q\rangle_j\langle 1;q\rangle_i=\langle 1;q\rangle_{i+j}(-1)^j q^{-j(i+j)}q^{\binom{j}{2}}. \tag{403}$$

□

In the q-Saalschütz formula (396) replace

$$n,a,b,c \;\text{ by } k,1+a-b-c,a+k,1+a-b,$$

respectively and use (188) twice to get (see Gasper and Rahman [343, p. 32], formula (2.2.1)). In the new notation this formula takes the following form:

$$\begin{aligned}&{}_3\phi_2(-k,a+k,1+a-b-c;1+a-b,1+a-c|q,q)=\\&\frac{\langle c,1-b-k;q\rangle_k}{\langle 1+a-b,c-a-k;q\rangle_k}=\frac{\langle b,c;q\rangle_k}{\langle 1+a-b,1+a-c;q\rangle_k}q^{(1+a-b-c)k},\end{aligned} \tag{404}$$

so that

$$\begin{aligned}&\sum_{k=0}^{n}\frac{\langle b,c,-n;q\rangle_k}{\langle 1,1+a-b,1+a-c;q\rangle_k}A_k \overset{\text{by}(404)}{=}\\&=\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{\langle -k,a+k,1+a-b-c;q\rangle_j\langle -n;q\rangle_k}{\langle 1,1+a-b,1+a-c;q\rangle_j\langle 1;q\rangle_k}q^{j+k(b+c-1-a)}A_k=\\&=\sum_{j=0}^{n}\sum_{i=0}^{n-j}\frac{\langle -i-j,a+i+j,1+a-b-c;q\rangle_j\langle -n;q\rangle_{i+j}}{\langle 1,1+a-b,1+a-c;q\rangle_j\langle 1;q\rangle_{i+j}}\times\\&\times q^{j+(i+j)(b+c-1-a)}A_{i+j}=\\&\overset{\text{by}(400)}{=}\sum_{j=0}^{n}\frac{\langle 1+a-b-c,a+j,-n;q\rangle_j}{\langle 1,1+a-b,1+a-c;q\rangle_j}(-1)^j q^{-\binom{j}{2}}\\&\sum_{i=0}^{n-j}\frac{<j-n,a+2j;q\rangle_i}{\langle 1,a+j;q\rangle_i}q^{-ij+(i+j)(b+c-1-a)}A_{i+j},\end{aligned} \tag{405}$$

where $\{A_k\}$ is an arbitrary sequence. This is equivalent to Bailey's 1949 [70] lemma. Choosing

$$A_k = \frac{\langle a, a_1, \ldots, a_l; q\rangle_k}{\langle b_1, \ldots, b_{l+1}; q\rangle_k} q^{zk} \tag{406}$$

and using (198) 4 times we obtain the following expansion formula (see Gasper and Rahman [343, p. 33], formula (2.2.4)). In the new notation this formula takes the following form:

$$\begin{aligned}
&{}_{l+4}\phi_{l+3}\left[\begin{matrix} a, b, c, a_1, \ldots, a_l, -n \\ 1+a-b, 1+a-c, b_1, \ldots, b_{l+1}\end{matrix} |q, q^z\right] = \\
&= \sum_{k=0}^{n} \frac{\langle b, c, -n, a, a_1, \ldots, a_l; q\rangle_k}{\langle 1, 1+a-b, 1+a-c, b_1, \ldots, b_{l+1}; q\rangle_k} q^{zk} \\
&= \sum_{j=0}^{n} \frac{\langle -n, a_1, \ldots, a_l, 1+a-b-c; q\rangle_j \langle a; q\rangle_{2j}}{\langle 1, 1+a-b, 1+a-c, b_1, \ldots, b_{l+1}; q\rangle_j} \times \\
&\times (-1)^j q^{j(b+c-1-a+z)-\binom{j}{2}} \times \\
&\times {}_{l+2}\,\phi_{l+1}\left[\begin{matrix} a+2j, a_1+j, \ldots, a_l+j, j-n \\ b_1+j, \ldots, b_{l+1}+j\end{matrix} |q, q^{b+c+z-1-a-j}\right],
\end{aligned} \tag{407}$$

which is a q-analogue of Bailey's formula from 1935 [67], 4.3(1).

Formula (407) enables one to reduce a ${}_{l+4}\phi_{l+3}$ series to a sum of ${}_{l+2}\phi_{l+1}$ series (see Gasper and Rahman [343, p. 33]).

2.20. **A summation formula for a terminating very-well-poised ${}_4\phi_3$ series.**

Setting $b = 1+\frac{1}{2}a,\ c = \widetilde{1+\frac{1}{2}a},\ a_k = b_k,\ k = 1, 2, \ldots, l$ and $b_{l+1} = a+n+1$, formula (407) implies that the formula (2.3.1) in Gasper and Rahman [343, p. 33] takes the following form in the new notation :

$$\begin{aligned}
&{}_4\phi_3(a, 1+\tfrac{1}{2}a, \widetilde{1+\tfrac{1}{2}a}, -n; \tfrac{1}{2}a, \widetilde{\tfrac{1}{2}a}, 1+a+n|q, q^z) \overset{\text{by}(374)}{=} \\
&= \sum_{j=0}^{n} \frac{\langle \widetilde{-1}, -n; q\rangle_j \langle a; q\rangle_{2j}}{\langle 1, \frac{1}{2}a, \widetilde{\frac{1}{2}a}, 1+a+n; q\rangle_j} \times (-1)^j q^{(1+z)j} q^{-\binom{j}{2}} \\
&\times {}_2\phi_1(a+2j, j-n; 1+a+n+j|q, -q^{z+1-j}).
\end{aligned} \tag{408}$$

If $z = n$, then the ${}_2\phi_1$-series (408) can be summed by means of the Bailey-Daum summation formula (398) (see Gasper and Rahman [343,

p. 34], formula (2.3.2)). In the new notation this formula takes the following form:

$$
\begin{aligned}
&{}_2\phi_1(a+2j, j-n; 1+a+n+j|q, -q^{n+1-j}) = \\
&= \frac{\langle \widetilde{1}; q\rangle_\infty \langle 1+a+2j, 2(1+n)+a; q^2\rangle_\infty}{\langle 1+a+j+n, \widetilde{1+n}-j; q\rangle_\infty}.
\end{aligned} \tag{409}
$$

We obtain

$$
\begin{aligned}
&{}_4\phi_3(a,1+\tfrac{1}{2}a,\widetilde{1+\tfrac{1}{2}a},-n;\tfrac{1}{2}a,\widetilde{\tfrac{1}{2}a},1+a+n|q,q^n) \overset{\text{by}(409)}{=}\\
&=\sum_{j=0}^{n}\frac{\langle\widetilde{-1},-n;q\rangle_j\langle a;q\rangle_{2j}}{\langle 1,\frac{1}{2}a,\widetilde{\frac{1}{2}a},1+a+n;q\rangle_j}(-1)^j q^{(1+n)j}q^{-\binom{j}{2}}\times\\
&\times\frac{\langle\widetilde{1};q\rangle_\infty\langle 1+a+2j,2(1+n)+a;q^2\rangle_\infty}{\langle 1+a+j+n,\widetilde{1+n-j};q\rangle_\infty}\overset{\text{by}(195),(196)}{=}\\
&=\sum_{j=0}^{n}\frac{\langle\widetilde{-1},-n;q\rangle_j\langle 1+a;q^2\rangle_j}{\langle 1,1+a+n;q\rangle_j}(-1)^j q^{(1+n)j}q^{-\binom{j}{2}}\times\\
&\times\frac{\langle\widetilde{1};q\rangle_\infty\langle 1+a+2j,2(1+n)+a;q^2\rangle_\infty}{\langle 1+a+j+n,\widetilde{1+n-j};q\rangle_\infty}\overset{\text{by}2\times(185)}{=}\\
&=\sum_{j=0}^{n}\frac{\langle\widetilde{-1},-n;q\rangle_j\langle 1+a;q^2\rangle_j\langle\widetilde{1};q\rangle_{n-j}}{\langle 1;q\rangle_j}(-1)^j q^{(1+n)j}q^{-\binom{j}{2}}\times\\
&\times\frac{\langle 1+a+2j,2(1+n)+a;q^2\rangle_\infty}{\langle 1+a+n;q\rangle_\infty}\overset{\text{by}(189),(374)}{=}\\
&=\sum_{j=0}^{n}\frac{\langle\widetilde{-1},-n;q\rangle_j\langle 1+a;q^2\rangle_j\langle\widetilde{1};q\rangle_{n}}{\langle 1,\widetilde{-n};q\rangle_j}q^j\times\\
&\times\frac{\langle 1+a+2j,2(1+n)+a;q^2\rangle_\infty}{\langle 1+a+n;q\rangle_\infty}\overset{\text{by}(187)}{=}\\
&=\sum_{j=0}^{n}\frac{\langle\widetilde{-1},-n;q\rangle_j\langle\widetilde{1};q\rangle_{n}}{\langle 1,\widetilde{-n};q\rangle_j}q^j\frac{\langle 1+a,2(1+n)+a;q^2\rangle_\infty}{\langle 1+a+n;q\rangle_\infty}\overset{\text{by}(411)}{=}\\
&=\sum_{j=0}^{n}\frac{\langle\widetilde{-1},-n;q\rangle_j\langle\widetilde{1};q\rangle_{n}}{\langle 1,\widetilde{-n};q\rangle_j}q^j\times\\
&\times\frac{\langle 1+a,2+a;q^2\rangle_\infty\langle 1+a;q\rangle_n}{\langle 2+a;q^2\rangle_n\langle 1+a;q\rangle_\infty}\overset{\text{by}(195),(196)}{=}\\
&=\frac{\langle\widetilde{1},1+a;q\rangle_n}{\langle 1+\frac{1}{2}a,\widetilde{1+\frac{1}{2}a};q\rangle_n}\,{}_2\phi_1(-n,\widetilde{-1};\widetilde{-n}|q,q),
\end{aligned}
\tag{410}
$$

where by (185) and (187)

$$
\frac{\langle 2(1+n)+a;q^2\rangle_\infty}{\langle 1+a+n;q\rangle_\infty}=\frac{\langle 2+a;q^2\rangle_\infty\langle 1+a;q\rangle_n}{\langle 2+a;q^2\rangle_n\langle 1+a;q\rangle_\infty}.
\tag{411}
$$

Both sides of (410) are equal to 1 when $n=0$. By (373), (374), (397), (see Gasper and Rahman [343, p. 34]) the ${}_2\phi_1$ series in the last row of (410) has the sum

$$\frac{\langle 1-n;q\rangle_n}{\langle \widetilde{-n};q\rangle_n}(q^{-1})^n, \tag{412}$$

when $n=0,1,\ldots$. Since $\langle 1-n;q\rangle_n=0$ unless $n=0$, it follows that the formula (2.3.4) (see Gasper and Rahman [343, p. 34]) takes the following form in the new notation:

$$ {}_4\phi_3(a,1+\tfrac{1}{2}a,\widetilde{1+\tfrac{1}{2}a},-n;\tfrac{1}{2}a,\widetilde{\tfrac{1}{2}a},1+a+n|q,q^n)=\delta_{n,0}, \tag{413}$$

where $\delta_{n,0}$ is the Kronecker delta function. This summation formula will be used in the next session to obtain the sum of a ${}_6\phi_5$ series.

2.21. **A summation formula for a terminating very-well-poised ${}_6\phi_5$ series.**

Let us now put

$$a_1=1+\tfrac{1}{2}a,\ a_2=\widetilde{1+\tfrac{1}{2}a},\ b_1=\tfrac{1}{2}a,\ b_2=\widetilde{\tfrac{1}{2}a},\ b_{l+1}=1+a+n,$$

and $a_k=b_k$, $k=3,4,\ldots,l$. Formula (407) now implies that the formula (2.4.1) in Gasper and Rahman [343, p. 34] takes the following form in the new notation :

(414)

$$\begin{aligned}
&{}_6\phi_5\left[\begin{matrix} a,b,c,1+\frac{1}{2}a,\widetilde{1+\frac{1}{2}a},-n \\ 1+a-b,1+a-c,\frac{1}{2}a,\widetilde{\frac{1}{2}a},1+a+n\end{matrix}|q,q^z\right]= \\
&=\sum_{j=0}^{n}\frac{\langle 1+a-b-c,1+\frac{1}{2}a,\widetilde{1+\frac{1}{2}a},-n;q\rangle_j\langle a;q\rangle_{2j}}{\langle 1,1+a-b,1+a-c,\frac{1}{2}a,\widetilde{\frac{1}{2}a},1+a+n;q\rangle_j}q^{j(b+c-1-a+z)-\binom{j}{2}}\times \\
&\times(-1)^j\ {}_4\phi_3\left[\begin{matrix} a+2j,1+\frac{1}{2}a+j,\widetilde{1+\frac{1}{2}a}+j,j-n \\ \frac{1}{2}a+j,\widetilde{\frac{1}{2}a}+j,1+a+n+j\end{matrix}|q,q^{b+c+z-1-a-j}\right].
\end{aligned}$$

If $z=1+a+n-b-c$, then we can sum the above ${}_4\phi_3$ series by means of (413) and obtain the summation formula (see Gasper and Rahman [343, p. 34], formula (2.4.2)). In the new notation this formula takes the following form:

$$
\begin{aligned}
&{}_6\phi_5\left[\begin{array}{c} a,b,c,1+\frac{1}{2}a,\widetilde{1+\frac{1}{2}a},-n \\ 1+a-b,1+a-c,\frac{1}{2}a,\widetilde{\frac{1}{2}a},1+a+n \end{array}|q,q^{1+a+n-b-c}\right]= \\
&=\frac{\langle 1+a-b-c,1+\frac{1}{2}a,\widetilde{1+\frac{1}{2}a},-n;q\rangle_n\langle a;q\rangle_{2n}}{\langle 1,1+a-b,1+a-c,\frac{1}{2}a,\widetilde{\frac{1}{2}a},1+a+n;q\rangle_n}(-1)^n q^{\binom{n+1}{2}}= \\
&\overset{\text{by}(191),(193)}{=}(k=n)\overset{\text{by}(196),(195)}{=}\frac{\langle 1+a,1+a-b-c;q\rangle_n}{\langle 1+a-b,1+a-c;q\rangle_n}.
\end{aligned}
\tag{415}
$$

2.22. **Watson's transformation formula for a terminating very-well-poised ${}_8\phi_7$ series.** We will now use (415) to prove Watson's 1929 [920] transformation formula for a terminating very-well-poised ${}_8\phi_7$ series (see Gasper and Rahman [343, p. 35], formula (2.5.1)). Denoting

$$\alpha\equiv a,b,c,d,1+\tfrac{1}{2}a,\widetilde{1+\tfrac{1}{2}a},e,-n \tag{416}$$

$$\beta\equiv 1+a-b,1+a-c,1+a-d,1+a-e,\tfrac{1}{2}a,\widetilde{\tfrac{1}{2}a},1+a+n \tag{417}$$

$$\gamma\equiv a+2j,d+j,e+j,1+\tfrac{1}{2}a+j,\widetilde{1+\tfrac{1}{2}a+j},j-n \tag{418}$$

and

$$\delta\equiv 1+a-d+j,1+a-e+j,\tfrac{1}{2}a+j,\widetilde{\tfrac{1}{2}a+j},1+a+n+j \tag{419}$$

we find that this formula takes the following form in the new notation:

Theorem 2.28.

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{array}{c}\alpha\\ \beta\end{array}|q,q^{2a+2+n-b-c-d-e}\right]=\frac{\langle 1+a,1+a-d-e;q\rangle_n}{\langle 1+a-d,1+a-e;q\rangle_n}\times \\
&\times\ {}_4\phi_3\left[\begin{array}{c} d,e,1+a-b-c,-n \\ 1+a-b,1+a-c,d+e-n-a\end{array}|q,q\right].
\end{aligned}
\tag{420}
$$

Proof. First observe that by (188), (198), (191) and (192)

(421)
$$\begin{aligned}
&\langle 1+a+n-d-e-j;q\rangle_j^{-1} = \langle -a-n+d+e;q\rangle_j^{-1}(-1)^j q^{(d+e-n-a)j+\binom{j}{2}}\\
&\langle 1+a-d+j;q\rangle_{n-j} = \frac{\langle 1+a-d;q\rangle_n}{\langle 1+a-d;q\rangle_j}\\
&\langle 1+a-d-e;q\rangle_{n-j} = \frac{\langle 1+a-d-e;q\rangle_n}{\langle 1+a-d-e+n-j;q\rangle_j}\\
&\langle 1+a+2j;q\rangle_{n-j} = \frac{\langle 1+a;q\rangle_n\langle 1+a+n;q\rangle_j}{\langle 1+a;q\rangle_{2j}}.
\end{aligned}$$

In (407) put

$$\begin{aligned}
&l=4, a_1=d, a_2=1+\tfrac{1}{2}a,\ a_3=\widetilde{1+\tfrac{1}{2}a}, a_4=e,\ b_1=1+a-d,\\
&b_2=\tfrac{1}{2}a,\ b_3=\widetilde{\tfrac{1}{2}a}, b_4=1+a-e,\\
&b_5=1+a+n, z=2a+2+n-b-c-d-e,
\end{aligned} \tag{422}$$

and in (415) put

$$a=a+2j,\ b=e+j,\ c=d+j,\ n=n-j. \tag{423}$$

We obtain

(424)

$$
{}_8\phi_7\left[\begin{matrix}\alpha\\ \beta\end{matrix}|q,q^{2a+2+n-b-c-d-e}\right]\stackrel{\text{by}(407)}{=}
$$

$$
=\sum_{j=0}^{n}\frac{\langle -n,d,e,1+\frac{1}{2}a,\widetilde{1+\frac{1}{2}a},1+a-b-c;q\rangle_j\langle a;q\rangle_{2j}(-1)^j q^{j(1+a+n-d-e)-\binom{j}{2}}}{\langle 1,1+a-b,1+a-c,1+a-d,1+a-e,\frac{1}{2}a,\widetilde{\frac{1}{2}a},1+a+n;q\rangle_j}\times
$$

$$
\times_6\,\phi_5\left[\begin{matrix}\gamma\\ \delta\end{matrix}|q,q^{1+a+n-j-d-e}\right]\stackrel{\text{by}(415)}{=}
$$

$$
=\sum_{j=0}^{n}\frac{\langle -n,d,e,1+\frac{1}{2}a,\widetilde{1+\frac{1}{2}a},1+a-b-c;q\rangle_j\langle a;q\rangle_{2j}(-1)^j q^{j(1+a+n-d-e)-\binom{j}{2}}}{\langle 1,1+a-b,1+a-c,1+a-d,1+a-e,\frac{1}{2}a,\widetilde{\frac{1}{2}a},1+a+n;q\rangle_j}\times
$$

$$
\times\frac{\langle 1+a+2j,1+a-d-e;q\rangle_{n-j}}{\langle a+j+1-e,a+j+1-d;q\rangle_{n-j}}\stackrel{\text{by}(196),by(195)}{=}
$$

$$
=\sum_{j=0}^{n}\frac{\langle -n,d,e,1+a-b-c;q\rangle_j\langle 1+a;q\rangle_{2j}(-1)^j q^{j(1+a+n-d-e)-\binom{j}{2}}}{\langle 1,1+a-b,1+a-c,1+a-d,1+a-e,1+a+n;q\rangle_j}\times
$$

$$
\times\frac{\langle 1+a+2j,1+a-d-e;q\rangle_{n-j}}{\langle a+j+1-e,a+j+1-d;q\rangle_{n-j}}\stackrel{\text{by}(421)}{=}
$$

$$
=\frac{\langle 1+a,1+a-d-e;q\rangle_n}{\langle 1+a-d,1+a-e;q\rangle_n}\,{}_4\phi_3\left[\begin{matrix}d,e,1+a-b-c,-n\\ 1+a-b,1+a-c,d+e-n-a\end{matrix}|q,q\right].
$$

∎

2.23. **Jackson's sum of a terminating very-well-poised balanced ${}_8\phi_7$ series.** The following theorem for a terminating very-well-poised balanced ${}_8\phi_7$ series was proved by Jackson in 1921 [464] (see Gasper and Rahman [343, p. 35], formula (2.6.2)). In the new notation this theorem takes the following form:

Theorem 2.29. *Let the ${}_8\phi_7$ series in (420) be balanced i.e. the six parameters a,b,c,d,e and n satisfy the relation*

$$
2a+n+1=b+c+d+e. \tag{425}
$$

Then

(426)

$$
{}_8\phi_7\left[\begin{matrix}a,b,c,d,1+\frac{1}{2}a,\widetilde{1+\frac{1}{2}a},e,-n\\ 1+a-b,1+a-c,1+a-d,1+a-e,\frac{1}{2}a,\widetilde{\frac{1}{2}a},1+a+n\end{matrix}|q,q\right]=
$$

$$
=\frac{\langle 1+a,1+a-b-c,1+a-b-d,1+a-c-d,;q\rangle_n}{\langle 1+a-b,1+a-c,1+a-d,1+a-b-c-d;q\rangle_n},
$$

when $n=0,1,2,\ldots$.

Proof. This follows directly from (420), since the ${}_4\phi_3$ series on the right of (420) becomes a balanced ${}_3\phi_2$ series when (425) holds and therefore can be summed by the q-Saalschütz's summation formula (396) (see Gasper and Rahman [343, p. 36]). A permutation of the variables completes the proof. □

Note that (426) is a q-analogue of (87).

2.24. **Watson's proof of the Rogers-Ramanujan identities.** Watson [920] used his transformation formula (420) to give a simple proof of the famous Rogers-Ramanujan identities ([419]):

$$\sum_{j=0}^{\infty}\frac{q^{j^2}}{\langle 1;q\rangle_j}=\frac{\langle 2,3,5;q^5\rangle_\infty}{\langle 1;q\rangle_\infty}, \tag{427}$$

$$\sum_{j=0}^{\infty}\frac{q^{j(j+1)}}{\langle 1;q\rangle_j}=\frac{\langle 1,4,5;q^5\rangle_\infty}{\langle 1;q\rangle_\infty}, \tag{428}$$

where $|q|<1$.

Proof. (See Gasper and Rahman [343, pp. 36-37]). In the new notation this proof takes the following form: Put

$$\alpha\equiv a,b,c,d,1+\tfrac{1}{2}a,\widetilde{1+\tfrac{1}{2}a},e,-n$$

and

$$\beta\equiv 1+a-b,1+a-c,1+a-d,1+a-e,\tfrac{1}{2}a,\widetilde{\tfrac{1}{2}a},1+a+n.$$

Let $b, c, d, e \to -\infty$ in (420) to obtain

(429)

$$\lim_{b,c,d,e\to-\infty} {}_8\phi_7 \left[\begin{matrix} \alpha \\ \beta \end{matrix} |q, q^{2a+2+n-b-c-d-e} \right] \overset{\text{by}(195)}{=} \lim_{b,c,d,e\to-\infty} \sum_{j=0}^{n}$$

$$\frac{\langle a,b,c,d,e,-n;q\rangle_j \langle a+2;q^2\rangle_j q^{j(2a+2+n-b-c-d-e)}}{\langle 1,1+a-b,1+a-c,1+a-d,1+a-e,1+a+n;q\rangle_j \langle a;q^2\rangle_j} =$$

$$\overset{\text{by}(201)}{=} \sum_{j=0}^{n} \frac{\langle a,-n;q\rangle_j \langle a+2;q^2\rangle_j q^{j(2a+2j+n)}}{\langle 1,1+a+n;q\rangle_j \langle a;q^2\rangle_j} =$$

$$= \sum_{j=0}^{n} \frac{\langle a,-n;q\rangle_j (1-q^{a+2j}) q^{j(2a+2j+n)}}{\langle 1,1+a+n;q\rangle_j (1-q^a)} =$$

$$= \lim_{b,c,d,e\to-\infty} \frac{\langle 1+a,1+a-d-e;q\rangle_n}{\langle 1+a-d,1+a-e;q\rangle_n}$$

$$\sum_{j=0}^{n} \frac{\langle d,e,1+a-b-c,-n;q\rangle_j}{\langle 1,1+a-b,1+a-c,d+e-n-a;q\rangle_j} q^j \overset{\text{by}(199),\text{by}(200)}{=}$$

$$= \langle 1+a;q\rangle_n \sum_{j=0}^{n} \frac{\langle -n;q\rangle_j (-1)^j q^{(a+n+\frac{j+1}{2})j}}{\langle 1;q\rangle_j}.$$

Thus we have

$$\sum_{j=0}^{n} \frac{\langle a,-n;q\rangle_j (1-q^{a+2j}) q^{j(2a+2j+n)}}{\langle 1,1+a+n;q\rangle_j (1-q^a)} = \tag{430}$$

$$= \langle 1+a;q\rangle_n \sum_{j=0}^{n} \frac{\langle -n;q\rangle_j (-1)^j q^{(a+n+\frac{j+1}{2})j}}{\langle 1;q\rangle_j}.$$

The next step is to let $n \to \infty$. If

$$A_j(n) = \frac{\langle -n;q\rangle_j q^{jn}}{\langle 1+a+n;q\rangle_j},$$

then by (203)

$$\lim_{n\to+\infty} A_j(n) = (-1)^j q^{\binom{j}{2}}, \tag{431}$$

for any fixed value of j. For all values of n, $|A_j(n)| < L$, a constant. Hence, in the series on the left of (430) the modulus of each term is

$$< L \left| \frac{\langle a;q\rangle_j (1-q^{a+2j}) q^{j(2a+2j)}}{\langle 1;q\rangle_j (1-q^a)} \right| = C \left| q^{2aj+2j^2} \right|,$$

which is the term of a convergent series. If

$$B_j(n) = \langle 1+a;q\rangle_n \langle -n;q\rangle_j q^{nj},$$

then by (202)

$$\lim_{n\to+\infty} B_j(n) = (-1)^j q^{\binom{j}{2}} \langle 1+a;q\rangle_\infty, \tag{432}$$

for any fixed value of j. For all values of n, $|B_j(n)| < K$, a constant. Hence, in the series on the right of (430) the modulus of each term is

$$< K \left| \frac{q^{(a+\frac{j+1}{2})j}}{\langle 1;q\rangle_j} \right| = D \left| q^{(a+\frac{j+1}{2})j} \right|,$$

which is the term of a convergent series. Now by (202), (203) and (430)

(433)

$$1+\sum_{j=1}^{\infty} \frac{\langle 1+a;q\rangle_{j-1}(1-q^{a+2j})}{\langle 1;q\rangle_j} q^{2aj} q^{\frac{j(5j-1)}{2}} (-1)^j = \langle 1+a;q\rangle_\infty \sum_{j=0}^{\infty} \frac{q^{j(a+j)}}{\langle 1;q\rangle_j}.$$

Now put $a=0$ in (433):

(434)

$$1+\sum_{j=1}^{\infty} \frac{\langle 1;q\rangle_{j-1}(1-q^{2j})}{\langle 1;q\rangle_j} q^{\frac{j(5j-1)}{2}} (-1)^j = 1+\sum_{j=1}^{\infty} (1+q^j) q^{\frac{j(5j-1)}{2}} (-1)^j =$$

$$= 1+\sum_{j=1}^{\infty} (-1)^j \left(q^{\frac{j(5j-1)}{2}} + q^{\frac{j(5j+1)}{2}}\right) = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j}{2}+\frac{5j^2}{2}} \overset{\text{by}(216)}{=}$$

$$= \langle 2,3,5;q^5\rangle_\infty = \langle 1;q\rangle_\infty \sum_{j=0}^{\infty} \frac{q^{j^2}}{\langle 1;q\rangle_j}.$$

This proves(427).

Next put $a=1$ in (433):

$$1+\sum_{j=1}^{\infty} \frac{\langle 2;q\rangle_{j-1}(1-q^{1+2j})}{\langle 1;q\rangle_j} q^{\frac{j(5j+3)}{2}} (-1)^j = \langle 2;q\rangle_\infty \sum_{j=0}^{\infty} \frac{q^{j(1+j)}}{\langle 1;q\rangle_j}.$$

Multiply by $(1-q)$

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{q^{j(1+j)}}{\langle 1;q\rangle_j} &= \frac{1}{\langle 1;q\rangle_\infty} \left(1-q+\sum_{j=1}^{\infty} (1-q^{1+2j}) q^{\frac{j(5j+3)}{2}} (-1)^j\right) = \\ &= \frac{1}{\langle 1;q\rangle_\infty} \sum_{-\infty}^{\infty} q^{\frac{j(5j+3)}{2}} (-1)^j \overset{\text{by}(216)}{=} \frac{\langle 1,4,5;q^5\rangle_\infty}{\langle 1;q\rangle_\infty}. \end{aligned} \tag{435}$$

This proves(428). ∎

Remark 25. For a connection between the Rogers-Ramanujan identities, statistical mechanics, thermodynamics and elliptic functions, the reader is referred to [88]. The physicist Baxter independently derived the Rogers-Ramanujan identities in 1979 when working with the so-called hard hexagon model.

The Rogers-Ramanujan identities are also useful in finite type invariant theory [571]. In this case the problem is to find an analytic continuation on the set $|z| = 1$ of a function defined for $|z| < 1$. This analytic continuation only exists for $z = e^{i\theta}$, $\theta \in \mathbb{Q}$.

The Rogers-Ramanujan identities can easily be derived from vertex operator algebras [119].

2.25. **A new q-Taylor formula.** Our next aim is to prove two q-analogues of Taylors formula. The following q-generalization of Taylors formula was introduced by Jackson 1909 [459]:

Theorem 2.30. *If the function $f(x)$ is capable of expansion as a convergent power series and if $q \neq$ root of unity, then*

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-a)^{(n)}}{\{n\}_q!} (D_q^n f)(a). \tag{436}$$

Proof. Assume that

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^{(k)}, \tag{437}$$

with certain constants $\{c_k\}_{k=0}^{\infty}$. Term by term q-differentiation of (255) gives

$$D_q (x-a)^{(n)} = \{n\}_q (x-a)^{(n-1)}. \tag{438}$$

Hence, from (437),

$$D_q^n f(x) = \sum_{k=0}^{\infty} c_{k+n} (x-a)^{(k)} \prod_{m=0}^{n-1} \{k+m+1\}_q, \; n = 1, 2, \ldots. \tag{439}$$

Let us put $x = a$ in the system of equations (439) and we obtain

$$c_k = \frac{D_q^k f(a)}{\{k\}_q!}, \; k = 0, 1, \ldots, \tag{440}$$

which gives (436). □

The (q, n)-exponential function $E_{q,n}(x)$ is defined by

$$E_{q,n}(x) = \sum_{k=0}^{n-1} \frac{x^k}{\{k\}_q!}, \; n \geq 1. \tag{441}$$

When the series on the right of (441) is q-differentiated term by term, we obtain a q-analogue of the differential equation for the exponential function in the form

$$D_q E_{q,n}(ax) = aE_{q,n-1}(ax). \tag{442}$$

To give the reader a hint to the following q-Taylor theorem, we present another q-exponential function:

$$E_{\frac{1}{q},n}(x) = \sum_{k=0}^{n-1} \frac{x^k q^{\binom{k}{2}}}{\{k\}_q!},\ n \geq 1. \tag{443}$$

The q-difference equation for (443) is

$$D_q E_{\frac{1}{q},n}(ax) = aE_{\frac{1}{q},n-1}(qax). \tag{444}$$

We need the following lemma for the proof of (448):

Lemma 2.31.

$$D_{q,t}\left(-\frac{(x-\widetilde{q^{m-2}t})^{(m-1)}}{q^{\binom{m-1}{2}}\{m-1\}_q!}\right) = \frac{(x-\widetilde{q^{m-2}t})^{(m-2)}}{q^{\binom{m-2}{2}}\{m-2\}_q!},\ m = 2, 3, \ldots. \tag{445}$$

Proof. The lemma is true for $m = 2$. Assume that it is true for $m-1$ and use (111) and (114) to prove by induction that it is also true for m. By the induction hypothesis

$$D_{q,t}\left(-\frac{(x-\widetilde{q^{m-3}t})^{(m-2)}}{q^{\binom{m-2}{2}}\{m-2\}_q!}\right) = \frac{(x-\widetilde{q^{m-3}t})^{(m-3)}}{q^{\binom{m-3}{2}}\{m-3\}_q!}, \tag{446}$$

we obtain

(447)

$$D_{q,t}\left(-\frac{(x-\widetilde{q^{m-2}t})^{(m-1)}}{q^{\binom{m-1}{2}}\{m-1\}_q!}\right) = D_{q,t}\left(-\frac{(x-\widetilde{q^{m-2}t})^{(m-2)}}{q^{\binom{m-2}{2}}\{m-2\}_q!}\frac{x-t}{\{m-1\}_q}\right) =$$

$$\overset{\text{by}(111),(114),(446)}{=}$$

$$= \frac{1}{\{m-1\}_q}\left(\frac{(x-\widetilde{q^{m-2}t})^{(m-2)}}{q^{\binom{m-2}{2}}\{m-2\}_q!} + \frac{q(x-qt)(x-\widetilde{q^{m-2}t})^{(m-3)}}{q^{\binom{m-3}{2}}\{m-3\}_q!}\right) =$$

$$= \frac{(x-\widetilde{q^{m-2}t})^{(m-3)}}{q^{\binom{m-2}{2}}\{m-2\}_q!\{m-1\}_q}\left(q^{m-3}(x-qt) + q^{m-2}(x-qt)\{m-2\}_q\right) =$$

$$= \frac{(x-\widetilde{q^{m-2}t})^{(m-2)}}{q^{\binom{m-2}{2}}\{m-2\}_q!\{m-1\}_q}(1+q\{m-2\}_q) = \frac{(x-\widetilde{q^{m-2}t})^{(m-2)}}{q^{\binom{m-2}{2}}\{m-2\}_q!}.$$

□

Theorem 2.32. *Let $0 < |q| < 1$ and let $f(x)$ be n times q-differentiable. Then the following generalization of* (436) *holds for $n = 1, 2, \ldots$:*
(448)
$$f(x) = \sum_{k=0}^{n-1} \frac{(x-a)^{(k)}}{\{k\}_q!}(D_q^k f)(a) + \int_{t=a}^{x} \frac{(x-qt)^{(n-1)}}{\{n-1\}_q!}(D_q^n f)(t)\, d_q(t).$$

Proof. The theorem is true for $n = 1$ by inspection. Assume that the theorem is true for $n = m - 1$ and use (166) and (445) to prove by induction that the theorem is also true for $n = m$.

(449)
$$f(x) = \sum_{k=0}^{m-2} \frac{(x - \widetilde{q^{k-1}a})^{(k)}}{q^{\binom{k}{2}}\{k\}_q!}(D_q^k f)(a) +$$
$$+ \int_{t=a}^{x} \frac{(x - \widetilde{q^{m-2}t})^{(m-2)}}{q^{\binom{m-2}{2}}\{m-2\}_q!}(D_q^{m-1} f)(t)\, d_q(t) \overset{\text{by}(445)}{=}$$
$$= \sum_{k=0}^{m-2} \frac{(x - \widetilde{q^{k-1}a})^{(k)}}{q^{\binom{k}{2}}\{k\}_q!}(D_q^k f)(a) + \left[- \frac{(x - \widetilde{q^{m-2}t})^{(m-1)}}{q^{\binom{m-1}{2}}\{m-1\}_q!}(D_q^{m-1} f)(t) \right]_{t=a}^{t=x} +$$
$$+ \int_{t=a}^{x} \frac{(x - \widetilde{q^{m-1}t})^{(m-1)}}{q^{\binom{m-1}{2}}\{m-1\}_q!}(D_q^{m} f)(t)\, d_q(t) =$$
$$= \sum_{k=0}^{m-1} \frac{(x - \widetilde{q^{k-1}a})^{(k)}}{q^{\binom{k}{2}}\{k\}_q!}(D_q^k f)(a) + \int_{t=a}^{x} \frac{(x - \widetilde{q^{m-1}t})^{(m-1)}}{q^{\binom{m-1}{2}}\{m-1\}_q!}(D_q^{m} f)(t)\, d_q(t) =$$
$$\overset{\text{by}(226)}{=} \sum_{k=0}^{m-1} \frac{(x-a)^{(k)}}{\{k\}_q!}(D_q^k f)(a) + \int_{t=a}^{x} \frac{(x-qt)^{(m-1)}}{\{m-1\}_q!}(D_q^m f)(t)\, d_q(t).$$

□

Remark 26. A similar formula with a remainder expressed as a multiple sum was presented in [481].

Remark 27. Wallisser [916] has found a criterion for an entire function to be expanded in a q-Taylor series for the special case $a = 1$ and $q < 1$.

Theorem 2.33. [916] *Put $M_{E_{\frac{1}{q}}}(r) = max_{|x|=r}|E_{\frac{1}{q}}(x)|$.*

If the maximum of the absolute value of an entire function f on $|x| = r$ satisfies the inequality

(450) $$M_f(r) \leq C M_{E_{\frac{1}{q}}}(r\tau),\ q \in \mathbb{R},\ q < 1,\ \tau < (\frac{1}{q} - 1)^{-1},$$

then f can be expanded in the q-Taylor series (436) *for the special case* $a = 1$.

3. The recent history of q-calculus with some physics.

3.1. The connection between q-calculus, analytic number theory and combinatorics. Some of the greatest stars in this area are Nörlund, Vandiver, Carlitz, Gould, Apostol, Andrews, Koblitz, Satoh and H.M. Srivastava. Let

$$\triangle_q^n f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}_q q^{\binom{k}{2}} f(x+n-k). \tag{451}$$

In 1924 Nörlund [684], [369, p. 91] proved the following finite q-Taylor expansion

$$f(x+y) = \sum_{k=0}^{n} \binom{x}{k}_q \triangle_q^k f(y). \tag{452}$$

Nörlund [684, ch. 2], [917, p. 265] also introduced the *Nörlund q-Bernoulli numbers* $b_{n,q}$ and the *Nörlund q-Bernoulli polynomials* $b_{n,q}(x)$, which are defined by [917, p. 265]

$$b_{0,q} = 1,\ (b_q \oplus_q\ 1)^k - b_q^k = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1, \end{cases} \tag{453}$$

where b_q^k is replaced by $b_{k,q}$.

$$b_{n,q}(x) = (x \oplus_q b_q)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q b_{k,q} x^{n-k}. \tag{454}$$

By (311) the following result obtains [684, ch. 2], [917, p. 265]

$$b_{n,q}(x \oplus_q y) = \sum_{k=0}^{n} \binom{n}{k}_q b_{n-k,q}(y) x^k \tag{455}$$

In 1933 [142] Carlitz (1907-) introduced *q-Stirling numbers*. In 1937 Vandiver [904] and in 1941 Carlitz [143] discussed generalized Bernoulli and Euler numbers. The final breakthrough came in the 1948 article by Carlitz [144]. Based on the Nörlund q-Taylor expansion (452) Carlitz defined new q-analogues of *Bernoulli numbers*, *Bernoulli polynomials* and *Euler numbers*. The Carlitz' q-Bernoulli numbers can be defined inductively by [144], [530], [779]

$$B_{0,q} = 1,\ q(qB_q + 1)^k - B_q^k = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1, \end{cases} \tag{456}$$

where as usual, B_q^k is replaced by $B_{k,q}$. The Carlitz q-Bernoulli polynomials are defined by [144], [530], [779]

$$B_{n,q}(x) = (q^x B_q + \{x\}_q)^n = \sum_{k=0}^{n} \binom{n}{k} B_{k,q} q^{kx} \{x\}_q^{n-k}. \tag{457}$$

The generating function of the Carlitz q-Bernoulli numbers is given by [779, p. 347]:

$$\sum_{n=0}^{\infty} B_{n,q} \frac{z^n}{n!} = \sum_{n=0}^{\infty} q^n e^{\{n\}_q z}(1 - q - q^n z),\ 0 < |q| < 1. \tag{458}$$

The generating function of the Carlitz q-Bernoulli polynomials is given by [779, p. 350]:

$$\sum_{n=0}^{\infty} B_{n,q}(x) \frac{z^n}{n!} = \sum_{n=0}^{\infty} q^n e^{\{n\}_q z q^x + \{x\}_q z}(1-q-q^{n+x} z),\ 0 < |q| < 1. \tag{459}$$

In 1953 Carlitz [145] generalized a result of Apostol [40] and expressed generalized *Dedekind sums* by *Eulerian numbers.* In 1953 Carlitz [146] gave a simple proof of the *reciprocity theorem* for Dedekind sums. In 1954 Carlitz [147] generalized a result of Frobenius [331] and showed that many of the properties of the q-Bernoulli numbers and other related q-numbers can be derived from the q-Eulerian numbers. The Carlitz' q-Eulerian numbers $H_{n,q}(\lambda)$ can be defined inductively by [779, p. 351]

$$H_{0,q}(\lambda) = 1,\ (qH_q + 1)^n - \lambda {H_q}^n(\lambda) = 0,\ n \geq 1,\ |\lambda| > 1. \tag{460}$$

where as usual, $H_q^n(\lambda)$ is replaced by $H_{n,q}(\lambda)$. The Carlitz' q-Eulerian polynomials are defined by [779, p. 351]

$$H_{n,q}(\lambda, x) = (q^x H_q + \{x\}_q)^n. \tag{461}$$

In 1964 Carlitz [157], [808] extended the Bernoulli, Eulerian and Euler numbers and corresponding polynomials as formal Dirichlet series. In 1965 Andrews [24] proved some identities for *Mock theta functions* by q-calculus technique. In 1978 Milne [631] used the finite operator calculus of Rota [756] to obtain a q-analogue of the Charlier polynomials and of Dobinski's equality for the Bell numbers. The q-Bell numbers were expressed as the n:th moments of a q-Poisson distribution. A q-Stirling number of the second kind is defined by [631, p. 93]

$$S_q(n, k) = \frac{\triangle_q^k}{\{k\}_q!}(\{x\})^n | x = 0, \tag{462}$$

The q-Bell numbers [631, p. 93] $\mathcal{B}_{n,q}$ are defined by

$$\mathcal{B}_{n,q} = \sum_{k=0}^{n} S_q(n,k). \tag{463}$$

In 1982 Koblitz [530] found a q-analogue of p-adic Dirichlet series and in 1989 Satoh [779] constructed a *q-analogue of Riemanns $\zeta-$ function.* In 1991 Bracken, McAnally, Zhang and M.D.Gould [123] mentioned a q-Fibonacci sequence satisfying a linear difference equation. In 1992 Satoh [780] constructed a *q-analogue of Dedekind sums* with the help of q-Bernoulli numbers [753] and at the same time solved a first order q-difference equation. Also in 1992 Cigler [207] found the following new q-analogues of Stirling numbers. A q-analogue of Stirling numbers of the first kind $s(n,k)$ is [207, p. 101]

$$\left[{n \atop k} \right]_q = \frac{D_q^k}{\{k\}_q!}(x+1)_n|x=-1. \tag{464}$$

A q-analogue of Stirling numbers of the second kind $S(n,k)$ is [207, p. 99]

$$\left\{ {n \atop k} \right\}_q = \sum_{m=0}^{n-1} \binom{n-1}{m}_q S(m,k-1) q^{\binom{n-m}{2}} \tag{465}$$

In the same year Sagan [769] derived various arithmetic properties of the q-binomial coefficients and q-Stirling numbers. In 1999 Satoh [782] extended a well-known formula for sums of products of two Bernoulli numbers to that of q-Bernoulli numbers.

Remark 28. In some applications [779, p. 347] q can be a p-adic number $q \in \mathbb{C}_p$, where $\mathbb{C}_p$ denotes the p-adic completion of the algebraic closure of $\mathbb{Q}_p$ [385, p. 5].

In her thesis from 1946, which was written under the direction of Earl Rainville, sister Celine [706]

> showed how one can find recurrence relations that are satisfied by sums of hypergeometric terms in a purely mechanical way. She used the method in her thesis [268] to find pure recurrence relations that are satisfied by various hypergeometric polynomial sequences. In two later papers she developed the method further, and explained its workings to a broad audience in her paper [269].

In 1991 Zeilberger [958] found a method that did the same job as sister Celine's algorithm, but a great deal faster. In 1992 Wilf & Zeilberger [937] showed that every q-identity, with a fixed number of summations and/or integration signs, possesses a short, computer-constructible proof, and gave a fast algorithm for finding such proofs. In 1996 Yen [951] showed that q-hypergeometric identities can be proved by checking that they are correct for only finitely many, N say, values of n, and gave a specific polynomial formula for N.

In 1956 Gould [363] generalized *Vandermonde's convolution*, and in 1957 [364] showed that this result is equivalent to a theorem stated by Rothe 1793 [759]. In 1960 [367] and in 1962 [372] Gould [367] generalized a *theorem of Jensen on convolutions.* In [372] the so-called *Gould polynomials* were also defined, which have later been used in statistics [177]. In 1961 Gould [368] further generalized Vandermonde's convolution and found a q-analogue [369] of a formula of Sparre Andersen [365]. In 1964 Gould [374] applied his previous results to orthogonal polynomials. In 1966 Gould [375] generalized a previous result [366] on q-series transformations, where the *q-Vandermonde transformation* was defined. In 1966 Gould and Kaucky [376] evaluated a class of binomial coefficient summations. In 1971 Bender [94] found a *q-Vandermonde convolution.* In 1973 Gould and Hsu [379] published an important inverse series relation, which had been communicated by Hsu in a personal letter to Gould in 1965. In the same year Carlitz [160] found a *q-analogue of the Gould-Hsu inverse relations.* In 1981 Sulanke [847] generalized Bender's formula to the q-multinomial case.

In 1988 H.M. Srivastava and Todorov [827]

> derived an explicit representation for the *generalized Bernoulli polynomials* as a double sum involving ordinary hypergeometric functions. An application is made to the *generalized Bernoulli numbers*, and consequently a result of Gould [378] for the ordinary Bernoulli numbers is generalized.

In 1991 Chu Wenchang [193] found a multivariable analogue of the Gould-Hsu inverse relations and as a corollary a multivariable analogue of the Gould polynomial convolution formula and of the Abel generalization of the binomial formula. In [194], [196], Chu Wenchang found a remarkable connection between this Carlitz q-analogue and a *chain reaction* between q-series identities. In 1996 A. Garsia and M. Haiman [341] introduced a new bivariate q-analogue of the Catalan numbers, which was used in 1998 by Haiman [408] in connection with the *Hilbert scheme.* All this work is heavily dependent on the Macdonald polynomials. In 1998 Chu Wenchang [199] systematically showed how the q-Saalschütz summation formula, special cases of the summation formula (7) in [342] and other summation formulae can be used to derive around 200 identities involving almost poised basic series and bilateral almost poised basic series. Some of the resulting identities are employed to derive q-Whipple transformations, Rogers-Ramanujan type identities, and q-Clausen and q-Orr product formulae. In 1997 Cigler [208] found *q-Gould polynomials*, which are of combinatorial interest in the context of counting lattice paths. In the same year Cigler [209] showed how the q-Gould polynomials are related to certain rooted trees and further derived some new results for q-Gould polynomials and q-Catalan polynomials. In 1998 Cigler [210]

> shows how operator calculus can be effectively used to derive generating functions and recurrences for q-Catalan numbers.

In 1997 Gould and H.M. Srivastava [381]

> first present a unification (and generalization) of some combinatorial identities associated with the familiar Vandermonde convolution. A basic (or q-) extension of this general combinatorial identity is then obtained. An interesting open problem, relevant to the discussion in this paper, is also presented.

In 1999 Álvarez-Nodarse, Quintero and Ronveaux [21] presented a q-Stirling polynomial.

3.2. **Some aspects of the development of q-calculus in the second half of the twentieth century.** During the 1950's some of the most important contributers to the subject were Lucy Joan Slater and D.B. Sears (1918-1999). L.J. Slater attended Bailey's lectures on q-hypergeometric series in 1947-50 at Bedford College, London University and in 1966 published the book [810], which covers the great advances made in the subject since 1936 when Bailey's book [71] was published. Sears proved many new transformation formulas for ${}_3\phi_2$ series, balanced ${}_4\phi_3$ series, and very-well-poised ${}_{n+1}\phi_n$ series [343]. In 1954 Alder [16] generalized the Rogers-Ramanujan identities.

The so-called Sturm-Liouville q-difference equation has been studied in [96] and [575]. This equation is a q-analogue of the Sturm-Liouville differential equation. In [432] an algorithm was presented for computing a standard form for second order linear q-difference equations. This standard form is useful for determining the q-difference Galois group [432] and the set of Liouvillian solutions of a given equation. In 1997 Van der Put & Singer [902] studied Galois theory of difference equations.

Swarttouw found contiguous relations for the general q-hypergeometric function (122) in 1990 [851]. See also Horikawa 1992 [438]. In 1983 Philip Feinsilver [272] derived q-analogues of the binomial, negative binomial, Poisson, geometric and gamma distributions via an extension of the standard Bernoulli counting scheme. In 1989 Feinsilver [275] constructed measures that act as q-analogues of Lebesgue measure. Feinsilver also studied q-analogues of Fourier and Laplace transforms. The so-called *q-Fourier transform* has also been explored in [49], [284]. In [273] Feinsilver studied the q-analogue of the canonical Heisenberg-Weyl algebra and constructed analogues of the standard boson Fock space. Abdi [2] has solved linear q-difference equations with the help of a q-analogue of Laplace transform. In 1996 Thuswaldner [868] made an asymptotic analysis of the solutions of q-difference equations by using the Mellin transform.

In 1995 Singer [803] proved

> a q-analogue of Lagrange inversion formula which places into a common setting and extends the work of Andrews, Gessel, Garsia and Remmel. Examples of q-Lagrange inversion are given, including new Rogers-Ramanujan type identities.

In 1996 V.N. Singh and Rai [807] proved two general basic hypergeometric series identities which contain as special cases all the Rogers-Ramanujan type identities. In 1992 Verma and Jain [911] published a survey of summation formulae for basic hypergeometric series.

In 1975 Dashen and Frishman [221] found an important special case of the *Knizhnik-Zamolodchikov (KZ) equation*, which was rediscovered 1984 [526] in the general framework of the WZNW model [62]. In 1992 I.Frenkel and Reshetikin [325] started to study quantum affine algebras and holonomic difference equations, in particular the KZ equation in connection with conformal field theory. [325, pp. 1-2]

> For mathematicians conformal field theory is a representation of certain geometric categories of Riemann surfaces or a regular representation of a Lie algebra depending on a parameter (vertex operator algebra.) For physicists, it is first of all the theory that characterizes the critical behaviour of two dimensional physical systems.

In this paper new holonomic q-difference equations for the matrix coefficients of the products of intertwining operators for quantum affine algebra $U_q(\widehat{g})$ representations of level k were derived. In the special case of spin $-\frac{1}{2}$ representations, the authors demonstrate that the connection matrix yields a famous Baxter solution of the Yang-Baxter equation corresponding to the solid-on-solid model of statistical mechanics. The Yang-Baxter equation is the q-analogue of the Jacobi identity for a quantum group. Using a classical result of Birkhoff from 1913 [106] from the theory of q-difference equations, the authors show that matrix coefficients of connection matrices can be expressed in terms of ratios of elliptic theta functions, or in other words, sections of a line bundle on an elliptic curve. [325, p. 55]

> Results of this paper provide only the very first steps towards the understanding of a q-analogue of conformal field theory, of an elliptic generalization of the quantum group, the relations between them and the mathematical and physical implications of these new structures. One of the most subtle properties of quantum groups is its behaviour when the parameter q is equal to a root of unity. At these points the naive parallel with the classical theory breaks down and one encounters new arithmetic phenomena. In the case of quantum affine algebras and associated elliptic algebras, the role of two special parameters (namely the deformation parameter q and the level k in the case of the quantum affine algebras) can be even more significant. The arithmetic of the circle is now being replaced by the arithmetic of the elliptic curve. It is interesting to note that the values of q which are roots of unity correspond to the cusps for the modular group $PSL(2,\mathbb{Z})$. The representation theory of quantum affine algebras may bring a new meaning to formulas of the arithmetic theory of elliptic curves.

This paper was a revolution to the development of mathematical physics and has been followed by many publications in the same spirit. Two recent examples are [255] and [90]. The q-Knizhnik-Zamolodchikov equation (q-KZ) has been studied in [907],[182], [478], [636], [857], [858], [639]. In [182] the case $|q| = 1$ was studied in connection with the Heisenberg xxz model. In [857] the q-KZ associated with $g\ell_{N+1}$ was solved, in [858] the q-KZ associated with $s\ell_2$ was solved. In [330]

> the q-KZ for 2-point functions in the $U_q(\mathrm{sl}_2)$ case $(0 < q < 1)$ at arbitrary non-negative level k can be reduced

> to two decoupled second-order linear difference equations. The authors observe that solutions of these equations can be expressed in terms of continuous q-Jacobi polynomials.

In [639] the q-KZ associated with C_n was solved.

The following quotation is from Julius Borcea's thesis 1998 [119]:

> The structure and representation theories of affine Kac-Moody Lie algebras encode a wealth of deep connections between many different areas of both mathematics and physics. For instance, the nature of the celebrated Macdonald identities was clarified in the midseventies when Kac derived an analogue of Weyl's character formula for standard modules for affine Kac-Moody algebras. It is by now a well-known fact that suitably specialized characters of standard modules for affine Lie algebras can be expressed as certain infinite products. In exchange, these products - which usually correspond to fermionic representations - may be interpreted as generating functions of partition functions for coloured partitions defined by congruence conditions. It would therefore be possible to obtain combinatorial identities of Rogers-Ramanujan type by constructing bases for these modules which are parametrized by coloured partitions satisfying difference conditions instead. The study of infinite-dimensional Lie algebras in general and affine Lie algebras in particular has been greatly influenced by the emergence of the theory of representations of vertex (operator) algebras in the mid-eighties [324]. Vertex operators themselves first appeared in the mathematical literature in [574]. Vertex operators had been previously used by physicists in dual resonance models and - since then - in conformal field theories as well. Within a few years, vertex operators formed a solid bridge between mathematics and physics.

In [846] summation formulae for q-series with independent bases were obtained and used to derive transformation and expansion of q-series involving independent bases.

In [732], [733], Rahman and Suslov made a unified approach to the summation and integration formulae for q-hypergeometric series from a Pearson-type difference equation on a q-linear lattice without the benefit of a single transformation formula. In [730] a q-quadratic lattice gives an extension of Askey's integral on the real line.

3.3. **Umbral calculus.** In the following three sections we say something about three parallell approaches: Umbral calculus, quantum groups and quantum algebras.

The umbral calculus is a modern version for the symbolic method of the theory of invariants which was invented by Clebsch and Gordan [212] 1872. Combinatorics, representation theory and invariant theory are intimately connected as was first observed by Alfred Young (1873-1940) [758]. Already in 1959 [18] Al-Salam used a kind of umbral calculus in connection with q-Bernoulli numbers and polynomials. In this paper a q-Euler-Maclaurin expansion was derived. In 1960 Carlitz [154] found an operational formula for Laguerre polynomials and and in 1962 Gould and Hopper [373] generalized Carlitz' formula and found operational formulas connected with two generalizations of Hermite polynomials. In 1971 Andrews [25] introduced the concept of an Eulerian family of polynomials.

A few years after the umbral calculus was developed by G.C. Rota and his collaborators [757], [752], the *q-umbral calculus* was explored in [203], [205], [447], [753], [754]. In 1982 [753] a *q-Euler-Maclaurin expansion* and a new q-Leibniz formula were derived. In [148], [150], [203], [126] and [206] a *q-Mehler formula* was derived. In 1981 Cigler [205] used the q-umbral calculus to derive some q-analogues of the Laguerre polynomials. See also the very important paper by Moak [647] from the same year, where orthogonality relations, some basic properties of the roots and a q-difference equation for $L_n^\alpha(x;q)$ were presented. In 1980 Cigler [204] found a new q-analogue of the Lagrange inversion formula, which was predicted by Carlitz in 1973 [159], and generalized Jackson's 1910 [463] q-analogue of Abel's series.

In 1987 Khan [513] found q-analogues of the formulas of Carlitz and Gould & Hopper for the Laguerre polynomials. With the help of the umbral calculus, the method of *parameter augmentation* for basic hypergeometric series was developed [178],[179], which is an important tool to derive q-hypergeometric series identities.

With the development of the umbral calculus, the foundations for quantum groups were built [112].

3.4. **Quantum groups.** The late Professor Forsyth has suggested to Jackson [466] that if the base q in the q-difference operator is replaced by $1+\epsilon$, q-analysis could be used to deal with physical problems in which reality is never in exact accord with physical equations. This is exactly what this section is about except that the most important case is $q =$ root of unity.

In 1978 Santilli [777] mentioned q-deformed algebras in hadron physics [112]. Santilli's article was written in a time when some physicists had put the validity of the Pauli principle in doubt. Finally this problem was solved by letting every quark exist in three different disguises or colours.

Quantum groups were introduced in the mid-eighties by Drinfeld [243], Jimbo [475] 1985, [476] 1986 (connection between trigonometric solutions of the Yang-Baxter equation and the Hecke algebra and the corresponding symmetric representation of the q-analogue of $g\ell_{N+1}$), [477] 1987, where the explicit form of the quantum R-matrix—the solution of the so-called Yang-Baxter equation—in the fundamental representation for the generalized Toda model associated with nonexceptional affine Lie algebras is obtained. This R-matrix depends on the spectral parameter through trigonometric functions. The q-analogue of the harmonic oscillator was derived by Kuryshkin [561] (1980), Macfarlane [596] (1989), Biedenharn [104] (1989) and others. Biedenharn showed its connection to the $SU(2)_q$ algebra. In 1990 Soni [812] found q-analogues of some prototype Berry phase calculations. In 1990 Gray & Nelson [384] proved a completeness relation for the q-analogue of the coherent states. In 1991 Bracken, McAnally, Zhang and M.D.Gould [123]

> constructed the Bargmann space, the space of a class of entire functions of a complex variable z for q-bosons. Here the q-boson annihilation operator is represented by a q-differential operator while the creation operator is represented by multiplication by z. Furthermore a q-integral analogue of the scalar product in this Bargmann space and a q-analogue of the completeness relation for q-boson coherent states are found.

In 1992 Chiu & Gray & Nelson [186] studied the classical limit of the q-analogue quantized radiation field paralleling conventional quantum optics analyses.

Connes [215] developed non-commutative geometry and Woronowicz [941], [942] elaborated the framework of the non-commutative differential calculus [442]. In 1988 Manin [601] introduced a general construction for quantum groups as linear transformations on the quantum superplane. In 1990 Wess and Zumino [931] developed the differential calulus on the quantum hyperplane. Differential operators on quantum Minkowski space were first presented 1992 by O. Ogievetski et.al. in

[687]. The following year a general theory of braided differential operators was developed by S. Majid [598], which allowed for a more systematic presentation of the Poincaré algebra from [687]. In 1994 Pillin [710] constructed wave equations in momentum space by using irreducible representations of the q-Poincaré group. In the same year Meyer [620] gave a systematic account of a component approach to the algebra of forms on q-Minkowski space, introducing the corresponding exterior derivative, Hodge star operator, coderivative, Laplace-Beltrami operator and Lie-derivative. Using this (braided) differential geometry, Meyer gave a detailed exposition of the q-d'Alembert and q-Maxwell equations. Also in 1994 Dobrev [232] introduced new q-Maxwell equations which were the first members of an infinite hierarchy of q-difference equations and at the same time proposed new q-Minkowski space-time coordinates. This was important because it is very difficult to change coordinates in the q-integral.

The so-called *q-Schrödinger equation* has been studied in [167] and [168], [234]. Via a generalized version of the *Ehrenfest theorem*, the resulting momentum operator expressed as q-difference operators lead in the limit $q \to 1$ to the *Doebner-Goldin equation*, a nonlinear Schrödinger equation derived in [236] [884]. In [234] a q-deformation of the Witt algebra was made. In 1997 Dobrev [233] presented a new form of the q-d'Alembert equation. In the footsteps of Dobrev, Klimek [525] studied the integrals of motion for some equations on q-Minkowski space.

A simple q-Klein-Gordon equation in (2+1)-dimensional quantum gravity has been derived in [612]. A q-analogue of the Dirac equation that iterates to a q-Klein-Gordon equation and its covariance under the quantum Lorentz group was derived in [788].

A q-Green formula has been derived in [174] and a q-Stokes theorem has been derived in [840].

Quantum groups are also used in quantum gravity, which is connected to general relativity, phase space representations of quantum mechanics and M-theory. One example is the Spin Foam Models of Quantum Gravity', where the authors 'review the Ponzano-Regge model of 3-dimensional quantum gravity and explain how its infrared divergences are eliminated in the Turaev-Viro model by replacing the group $SU(2)$ by the quantum group $SU_q(2)$, which corresponds to introducing a nonzero cosmological constant. We also describe how these models and their generalizations to other groups and quantum groups fit into the framework of 'higher-dimensional algebra' (or in other words, the theory of n-categories). Then we discuss recent extensions of these ideas to 4-dimensional quantum gravity and topological quantum field theories. [66]

Two binomial formulae for two variables satisfying a quadratic relation have recently been published in [93] and [755]. These relations have applications in quantum group theory and non-commutative geometry.

A recent trend in modern physics is the study of the quantum anti-de Sitter space [967], [539] possibly in connection with $q =$ root of unity [841]. An introduction to the de Sitter model of the universe is given in [45].

The following notion has recently been introduced in q-calculus:

A *spider* is an axiomatization of the representation theory of a group, quantum group, Lie algebra, or other group or group-like object. It is also known as a spherical category, or a strict, monoidal category with a few extra properties, or by several other names. A recently useful point of view, developed by other authors, of the representation theory of $sl(2)$ has been to present it as a spider by generators and relations. That is, one has an algebraic spider, defined by invariants of linear representations, and one identifies it as isomorphic to a combinatorial spider, given by generators and relations. We generalize this approach to the rank 2 simple Lie algebras, namely A_2, B_2, and G_2. Our combinatorial rank 2 spiders yield bases for invariant spaces which are probably related to Lusztig's canonical bases, and they are useful for computing quantities such as generalized $6j$-symbols and quantum link invariants. Their definition originates in definitions of the rank 2 quantum link invariants that were discovered independently by the author and Francois Jaeger [558].

A Hopf algebra is called spherical if it has a grouplike element with certain properties related to the square of the antipode and some trace maps [78].

At the moment, Jesper Thorén [867] in Lund has just published his dissertation on q-Kac-Moody Lie algebras connected to quantum groups. Lars Hellström and Sergei Silvestrov have recently published a paper called Centralizers in the q-deformed Heisenberg algebras [430].

3.5. **Quantum algebras.** The q-algebra approach, due to Kalnins, Miller and coworkers, and Floreanini and Vinet uses only the q-analogue of the Lie algebra [531, p. 95].

The factorization method of Darboux-Schrödinger-Infeld-Hull [219], [453], [500] makes it possible to compute the eigenvalues and eigenfunctions for families of Schrödinger operators.

The theory of *commutative ordinary differential operators*, which predates the work of Lax, started in 1879 when Floquet [300] studied commutative first order differential operators. The Italian Pincherle was investigating a ring of pseudo-differential operators in one variable 1897

[657]. In 1903 Wallenberg [915] used elliptic functions to study when two differential operators commute, and in 1905 Schur proved that a set of differential operators which commute with a given operator is itself commutative [657].

In a series of wellwritten papers all entitled Commutative ordinary differential operators [130], [131], [132] Burchnall and Chaundy used the Darboux process to find a connection between differential operators and algebraic-geometric constructions. Burchnall and Chaundy restricted themselves to operators with scalar-valued coefficients.

The theory of non-commutative polynomials was explored by Ore 1933 [701] and by Jacobson 1937 [469].

In 1955 Weisner [926] explored the group-theoretic origin of certain generating functions.

Let $D = \frac{d}{dx}$ and let $f(D) = \sum_{k=0}^{n} a_k D^k$ be a general differential operator of degree n. In 1955 Flanders proved that the commutator $C[f]$ of f is a finitely generated ring over the algebra of all polynomials in $f(D)$ with constant coefficients [22]. In 1958 Amitsur [22] simplified Flanders' proof in a completely algebraic way, but his result is restricted to the case of a derivation acting on a field of characteristic zero.

In 1977 Krichever [549] allows matrix-valued coefficients, but restricts himself to the study of commuting pairs L and T both of whose leading coefficients must be constant nonsingular diagonal matrices [165]. In 1991 Chowdhury & Gupta [188] used commutative differential operators to construct a new class of solutions for the BKP equation. In 1994 Nakayashiki [665] generalized the Burchnall-Chaundy construction to commuting partial differential operators and found differential equations for all theta functions (not just Jacobians, as does the KP theory), in the same vein as a paper by I. Barsotti [79] 1983.

In 1974 two experts on soliton theory, Zakharov & Shabat [955] presented a powerful generalization of Darboux transformations, called the *dressing operator* [814]. In 1994 Mukhopadhaya and Chowdhury [651] presented a q-deformed dressing-operator.

In 1964 [622], 1968 [623],[624], 1969 [625] and in 1972 [627] Miller used the above-mentioned factorization method to connect the representation theory of certain Lie algebras with the computation of addition formulas and Clebsch-Gordan coefficients. In 1970 [626] Miller adopted this method to the q-case. A factorization method is established for systems of second order linear q-difference equations. The factorization types are shown to correspond to irreducible representations of infinite-dimensional Lie algebras. If the q-difference equations degenerate to differential equations (as q approaches 1), then a Lie theory of hypergeometric and related functions is obtained in the limit. If the q-difference equations degenerate to ordinary difference equations, then a Lie theory of special functions of a discrete variable is obtained in the limit.

In 1977 Miller [629] explored the relationship between symmetries of a linear second-order partial differential equation of mathematical physics, the coordinate systems in which the equation admits solutions via separation of variables, and the properties of the special functions that arise in this manner. He also introduced the R-separation of variables technique.

In 1978 [492] Kalnins and Miller proved that the Klein-Gordon equation is separable in 261 orthogonal coordinate systems. In each case the coordinate systems presented are characterized in terms of three symmetric second order commuting operators in the enveloping algebra of the Poincaré group. In [493] Kalnins and Miller discussed the R-separation of variables of the wave equation in space-time and found 106 coordinates to given a total of 368 conformally inequivalent orthogonal coordinates which admits an R-separation of variables.

Then in 1980 Kalnins, Manocha and Miller [494] developed a theory which enabled the powerful use of Lie algebraic methods for the solutions of partial differential equations by two-variable hypergeometric series. As shown in [494] the canonical differential equations for the Appell function (93) are

$$(\partial u_1 \partial u_2 - \partial u_3 \partial u_4)\mathfrak{F}_2 = 0,\ (\partial u_1 \partial u_5 - \partial u_6 \partial u_7)\mathfrak{F}_2 = 0 \tag{466}$$

with eigenvalue equations

$$\begin{aligned} &D_1 + D_3 + D_6 \sim -a,\ D_2 + D_3 \sim -b, \\ &D_4 - D_3 \sim c - 1,\ D_5 + D_6 \sim -b',\ D_7 - D_6 \sim c' - 1. \end{aligned} \tag{467}$$

Here $A \sim a$ stands for $A\mathfrak{F}_2 = a\mathfrak{F}_2$, and $D_j = u_j \partial u_j$. Furthermore,

$$\mathfrak{F}_2 = F_2(a, b, b'; c, c'; \frac{u_3 u_4}{u_1 u_2}, \frac{u_6 u_7}{u_1 u_5}) u_1^{-a} u_2^{-b} u_5^{-b'} u_4^{c-1} u_7^{c'-1}. \tag{468}$$

Related problems have also been treated by Sasaki [778]. In 1992 Kalnins, Manocha and Miller [495] used the factorization method to find q-algebra representations of $U_q(su_2)$ and q-oscillator algebras. In 1993 Kalnins, Miller and Mukherjee [496] used the factorization method to find representations of the q-oscillator algebra.

In 1994 Kalnins and Miller [497] used the factorization method to find matrix elements of oscillator algebra representations.

In 1994 Kalnins, Miller and Mukherjee [498] used the factorization method to find representations of the Lie algebra of plane motions.

In 1994 Kalnins and Miller [500] used the factorization method to find representations of the Euclidean, pseudo-Euclidean and oscillator algebras, and their tensor products.

In 1997 Kalnins, Miller and Chung [201] used the factorization method to find q-superalgebra representations.

In 1991 [301] Floreanini and Vinet studied the matrix elements of the representation of the two-dimensional Euclidean quantum algebra and q-Bessel functions. In 1993 [304] Floreanini and Vinet studied the connection between the matrix elements of the representation of the quantum group $U_q(sl_2)$ and the q-series ${}_2\phi_1$. In 1993 [303] Floreanini and Vinet studied the two-dimensional Euclidean quantum algebra and the q-oscillator algebra. In 1993 [305] Floreanini and Vinet studied certain automorphisms of the q-oscillator algebra and the q-Charlier polynomials. In 1995 [309] Floreanini and Vinet studied the $Su_q(1,1)$ and the q-Gegenbauer polynomials. In 1994, 1995 [306], [310] Floreanini and Vinet studied the q-heat equation.

In 1995 Floreanini and Vinet [308] presented a general method to determine the symmetry operators of linear q-difference equations and applied it to q-difference analogues of the Helmholtz, heat and wave equations in diverse dimensions. In 1996 [312] Floreanini, LeTourneux and Vinet studied the q-deformed $e(4)$ and continuous q-Jacobi polynomials.

Yet another q-algebra approach due to Feinsilver [274], [276],[277] uses raising and lowering operators that close under commutation. In [278] representations of the Euclidean group and Bessel functions were studied.

3.6. **Chevalley groups and q-Schur algebras.** We start with a fascinating story taken from [324, p. xxi]:

> The discovery of the Monster was preceded by a long history of development of another branch of mathematics- the theory of finite groups, a subject originally associated with Galois. It is natural to ask for the classification of all finite groups, yielding the enumeration of all kinds of finite symmetries, although this problem is even nowadays considered too difficult. The building blocks of an arbitrary finite group are simple groups, and the core of the problem is the classification of the finite simple groups. By the end of the nineteenth century, thanks to the work of Jordan, Dickson and others, several infinite families of simple groups were known. In addition, already in 1861, Mathieu had discovered five

> strange finite groups [610]. The Mathieu groups were called 'sporadic' for the first time in the book of Burnside, who noted that they 'would probably repay a closer examination than they have yet received' [137]. The pioneer of the field in our century was Brauer, who made several crucial contributions to the classification problem [124]. Most of the finite simple groups, now called group of Lie type or Chevalley groups, admit a uniform construction in terms of simple Lie algebras, via a systematic treatment discovered in [185]. But it was not clear at that time how many sporadic groups besides the Mathieu groups might exist.
>
> The modern classification race started with the work of Feit and Thompson in 1962, who proved that every nonabelian finite simple group has even order, or equivalently, contains an involution [279]. This work made feasible the tremendous classification project led primarily by Gorenstein, resulting, after two decades of work by a large group of mathematicians, in the classification theorem [360], [361]. The classification of the finite simple groups was unprecedented in the history of mathematics by virtue of the length of its proof-over 10 000 pages. The result itself is no less fascinating. Besides 16 infinite families of groups of Lie type and the additional family of alternating groups on n letters, $n \geq 5$, there exists exactly 26 sporadic simple groups, each of which owes its existence to a remarkable combination of circumstances.

The monster is the biggest of these 26 sporadic simple groups.

Let $q = p^s$ be a prime power. The Galois field is denoted $GF(q)$. Further let $B(v, w)$ be a non-degenerate form over $GF(q)$ in $(GF(q))^n$. A Chevalley group is the group of linear transformations of $(GF(q))^n$ preserving the form $B(v, w)$ [913]. The spherical functions for Chevalley groups over $GF(q)$ can be expressed by q-orthogonal polynomials [832, p. 87].

Let F be an infinite field of arbitrary characteristic. Schur reduced the study of those polynomial representations of $GL(n, F)$, which are homogeneous of a given degree n to the representation theory of the *Schur algebra* $S_F(n, m)$. In 1989 Dipper and James [229] presented a q-Schur algebra. For a recent account on this see the 1998 book by Stephen Donkin [239], and [350], [46].

A q-analogue of homological algebra was presented in [244].

3.7. **Some aspects on q-orthogonal polynomials and q-polynomials in several variables.** In a series of papers during the years 1903-1905 [533] Jackson introduced the two related q-Bessel functions $J_\nu^{(j)}(x,q)$, where $j=1,2$, see also Hahn [401], Ismail [455]. A third q-Bessel function was introduced by Hahn [404] (in a special case; we thank G.Gasper for this reference) and by Exton [261] (5.3.1.11) [533].

> For H. Exton's q-analogue of the Bessel function we derive Hansen-Lommel type orthogonality relations, which, by a symmetry, turn out to be equivalent to orthogonality relations which are q-analogues of the Hankel integral transform pair [533].
>
> These functions also arise in the study of an eigenvalue problem on the quantum Lobachevskii space [693].

The *q-Wronskian*

$$W_q(f_1(x),f_2(x)) \equiv \begin{vmatrix} f_1(x) & f_2(x) \\ D_q f_1(x) & D_q f_2(x) \end{vmatrix} \tag{469}$$

was introduced in [852], where a second solution of the q-difference equation of the Hahn-Exton q-Bessel function, corresponding to the classical $J_\nu^{(j)}(x,q)$, is found. We introduce a q-extension of the Wronskian to determine that the two solutions form a fundamental set [852].

In 1938, H.Krall [542], [543] attacked the problem of classifying the orthogonal polynomials satisfying differential equations. In particular, he found that the equation had to be of even order. The *Krall polynomials* can be obtained as the Darboux transformation of some classical orthogonal polynomials [388].

This method was adapted to the q-case in [387], see also the papers of Allan Krall [540], [541].

In [320] a fourth order q-difference equation for the first associated of the q-classical orthogonal polynomials was derived. The coefficents of this equation are given in terms of the polynomials σ and τ which appear in the q-Pearson difference equation $D_q(\sigma\rho)=\tau\rho$ defining the weight ρ of the q-classical orthogonal polynomials inside the q-Hahn tableau [320].

The theory of *q-polynomials in several variables* is not yet fully developed. The basic analogues of Appell hypergeometric functions were first studied and defined by F. H. Jackson in 1942 [465] and in 1944 [466]. In 1959 Carlitz [152] studied symmetric q-polynomials in three variables, and he was followed by Reid who investigated the four variable case in 1968 [744] and the n-variable case in 1971 [745]. In 1981 [822] H.M. Srivastava

> uses Dixon's theorem to establish a multivariable expansion which looks different from the known ones and has not been published before in its present form.

In 1983 H.M. Srivastava [823]

> gives and proves several very general formulae which imply identities expressing a basic hypergeometric series in one or several variables as a sum of products of similar basic hypergeometric series. These are basic analogues of results given previously by the author.

In 1984 H.M. Srivastava [824]

> derives a *bibasic analogue* of his earlier result [823]. As a corollary of his general theorem he obtains the main bibasic result of [20].

In 1992 Gupta [394] derived some new bibasic hypergeometric transformations. In 1992 Rassias & Singh [739] found a new transformation formula for the q-analogue of the Kampé de Fériet function. In 1995 B. Srivastava [819]

> gave certain transformations of q-Appell and q-Lauricella series on two bases. It is interesting to note that the q-Appell series on two bases can be reduced to an expression with only one base. Limiting cases of q-Appell series are closely connected with Ramanujan's $G(a, \lambda)$ function and so can be expressed in terms of continued fractions.

In 1996 Bowman [122] found a multivariate generalization of Heine's transformation formula and characterized the symmetric polynomials introduced by Rogers [751] in 1893. In 1996 U.B. Singh [809] obtained two new bibasic q-hypergeometric transformations with the help of Bailey's 1947 [69] transformation.

In 1999 Karlsson and H.M. Srivastava [505] derived certain new transformations and reduction formulas for multiple q-series by exploiting certain known q-hypergeometric series transformations. Also in 1999 Sahai [771]

> discuss a few models of the quantum universal enveloping algebra of sl(2) from the special function point of view. Two sets of such models are given, one acting on the space of ${}_1\phi_0$ functions and the other on the space of q-Appell functions. These models are closely related through a q-integral transformation. Some interesting identities are obtained.

3.8. **The symmetric and nonsymmetric Macdonald polynomials.** We have divided this section into two parts. We first present the *Macdonald symmetric polynomials*, which are symmetrizations of the nonsymmetric Macdonald polynomials [339]. Let the index μ range over integer partitions and let X denote the vector $x_1, x_2, \ldots$. In 1988 Macdonald [592] introduced a new class of symmetric functions, $P_\mu(X; q, t)$ which generalize the Hall-Littlewood polynomials, the Schur functions and the Jack and zonal polynomials of harmonic analysis [339]. In 1991 Salam and Wybourne [776] presented a q-deformation of Macdonald polynomials and q-analogues of the characters of the symmetric group. In 1992 Jing & Jozefiak [479] derived a Jacobi-Trudy type raising operator formula for the Macdonald symmetric functions. In 1994 Jing [480] connected basic hypergeometric functions with Macdonald functions and introduced two parameter vertex operators to construct a family of symmetric functions generalizing the Hall-Littlewood functions. In 1994 van Diejen [903] diagonalized the difference Calogero-Sutherland system by Koornwinder's multivariable generalization of the Askey-Wilson polynomials [844]. In 1994 Olshanetski & Rogov [692] used the Macdonald symmetric polynomials to describe the one-dimensional Liouville quantum field theory. In 1995 Lapointe and Vinet [563] presented a Rodrigues formula for the Jack polynomials, which is connected to the dynamical algebra of the Calogero-Sutherland model. The wave functions of Calogero-Sutherland model are expressible in terms of Jack polynomials [564]. In 1996 Sahi [772] presented a generalization of Macdonald symmetric polynomials. In 1996 Awata, Odake and Shiraishi [59] found an integral representation of the Macdonald operator regarded as a natural deformation of the Calogero-Sutherland Hamiltonian. In 1996 Konno [532] diagonalized the trigonometric Ruijsenaars-Schneider model by means of the Macdonald symmetric polynomials. In 1996 Etingof and Kirillov Jr. [254] gave a representation- theoretic proof of the inner product formula for the Macdonald symmetric polynomials conjectured by Macdonald 1988 [592]. In 1996 Kaneko [503] extended the Jacobi triple product identity to the Macdonald symmetric polynomials. In 1997 Okounkov [689] proved a multivariate generalization of the q-binomial theorem in the context of the *shifted Macdonald polynomials* and a binomial formula. By using the q-Wronskian Olshanetskiĭ & Rogov [693] defined q-analogues of modified Bessel functions and Macdonald functions. In 1997 Lapointe and Vinet [565] presented formulae of Rodrigues-type for the Macdonald polynomials and discussed the limiting case of the Jack polynomials. In 1998 Mimachi [640] gave a new derivation of the inner product formula for the Macdonald symmetric polynomials. In

1999 Kuznetsov & Sklyanin [562] found a non-local factorisation operator for the Macdonald symmetric polynomials by using a result of Fox [321].

In 1992 P.Freund [327] explained how the Macdonald polynomials can be used to express the eigenfunctions of the Cartier Laplacian on a tree.

The nonsymmetric Macdonald polynomials are certain generalizations of Schur functions [59].

> The Jackson integrals of Jordan-Pochhammer type are the simplest multivariable generalizations of Heine's basic hypergeometric function. They satisfy a system of first order q-difference equations, whose connection problem was solved by Mimachi in 1989 [634]. Recently Aomoto and others [36], [37], [38], [39] showed that the connection matrix determined by Mimachi is related to the ABF-solution of the quantum Yang-Baxter equation [29]. On the other hand, Frenkel and Reshetikin [325] studied a q-analogue of the chiral vertex operators of the WZNW model, along the line of Tsuchiya and Kanie [881]. In particular, they introduced a q-difference system called the quantum Knizhnik-Zamolodchikov equation, and discussed the relation of the connection matrix with elliptic solutions of the quantum Yang-Baxter equation. Then it seems possible to understand the result of [38] in the framework of Frenkel and Reshetikin [613].

In the same papers [36], [37] Aomoto also found a q-analogue of de Rham cohomology associated with Jackson integrals.

In 1995 Macdonald [595] introduced *nonsymmetric Macdonald polynomials*, which are generalizations of Weyl's formula for the characters of a compact Lie group. The Macdonald polynomials were related to certain induced representations of affine Hecke algebras by Cherednik and to harmonic analysis on certain quantizations of homogeneous spaces by Noumi and Sugitani [844]. The Hecke algebra is isomorphic to the group algebra for the symmetric group and is a q-deformation of this group algebra [358]. In 1996 Sahi [773] found a new scalar product for nonsymmetric Jack polynomials. In 1997 Mimachi and Noumi [637] found an integral representation of eigenfunctions for Macdonald's q-difference operator. In 1998 Mimachi and Noumi [638] found a reproducing kernel for nonsymmetric Macdonald polynomials of type A_{n-1}. In the same year Sahi [774] extended the q-binomial theorem to the nonsymmetric

Macdonald polynomials. In 1999 Ujino and Wadati [891] presented a Rodrigues formula for the nonsymmetric multivariable Hermite polynomial by using the Dunkl-Cherednik operators. In 1999 Nishino, Ujino and Wadati [679] presented a Rodrigues formula for the nonsymmetric multivariable Laguerre polynomial by using the Dunkl-Cherednik operators. In 1999 Nishino, Ujino and Wadati [678] presented a Rodrigues formula for the nonsymmetric Macdonald polynomial by using the Dunkl-Cherednik operators.

4. Generalized Vandermonde determinants.

4.1. Applications of Schur functions and Schur polynomials.

The Schur functions are particularly relevant to discussions of the quantum Hall effect [845], [785]. The quantum Hall effect [845], [717] was discovered on about the hundredth anniversary of Hall's original work in 1980 by von Klitzing, Dorda and Pepper. The quantum Hall effect is a two-dimensional electron gas in a strong magnetic field which shows a universal and exact quantization of the electric conductance at low temperature. The Schur functions are also relevant to the characters of irreducible representations of $U(n)$ [960, p. 213], [845]; to the characters of $Gl(n,\mathbb{C})$, which can be expressed in terms of Schur functions [750],[724],[p. 237][81], [932, ch. VII.6] ; to the characters of $Sp(2n+1,\mathbb{C})$ [599]; to the characters of $Sp(2n,\mathbb{R})$ [397]; and to the characters of the simple Lie algebras $sl(n,\mathbb{C})$ and $su(n)$, which have the same representations [332]. Quite recently, Schur functions have been used in q-calculus [509].

Furthermore generalized Vandermonde determinants occur in Sato theory, where the variables are partial differential operators [688], [227].

A new determinantal expression for Schur functions called the ribbon determinant formula for Schur functions was discussed in [567], [886].

In [724] Proctor & Wilson gave a combinatorial interpretation of a four parameter terminating sum identity for the basic hypergeometric series ${}_6\phi_5$. This identity was first produced by taking the principal specialization of a tensor product identity for two $GL(n)$ characters expressed by Schur functions.

In [737] Ram & Remmel & Whitehead described the q-analogues of the combinatorial interpretations of the entries of the transition matrices between the classical bases of symmetric functions. In [454] Ishikawa & Wakayama found some new product identities for certain summations of Schur functions. And in [617] Mendèz described the connection between umbral shifts and symmetric functions of Schur type. These examples show that Schur functions is a very active area of research.

The *elementary Schur polynomial* p_n is defined by the following two equivalent equations:

$$p_n(x_1, x_2, \ldots) = \sum_{\substack{k_1+k_2+\ldots=n \\ k_1,k_2,\ldots\geq 0}} \prod_{l=1,2,\ldots} \frac{x_l^{k_l}}{k_l!}. \tag{470}$$

$$exp(\sum_{l=1}^{\infty} x_l z^l) = \sum_{n=0}^{\infty} p_n(x_1, x_2, \ldots) z^n. \tag{471}$$

These polynomials satisfy the equation

$$\frac{\partial p_n}{\partial x_m} = p_{n-m}, \ (p_n = 0 \text{ for } n < 0). \tag{472}$$

Let λ be an arbitrary partition of m. The Schur polynomial X_λ is defined by [945, p. 374]:

$$X_\lambda \equiv \begin{vmatrix} p_{\lambda_1} & \cdots & p_{\lambda_1+j-1} & \cdots & p_{\lambda_1+n-1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ p_{\lambda_i+1-i} & \cdots & p_{\lambda_i+j-i} & \cdots & p_{\lambda_i+n-i} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ p_{\lambda_n-n+1} & \cdots & p_{\lambda_n-n+j} & \cdots & p_{\lambda_n} \end{vmatrix}, \tag{473}$$

where $n = l(\lambda)$. The Schur polynomials play an essential role in the Fock representations of the Virasoro algebra [945].

In 1997 Haine L. & Iliev P. [410] used a q-analogue of the Schur polynomials to provide rational solutions of the N:th q-KDV hierarchy.

4.2. **Generalized Vandermonde determinants with two deleted rows.** In this section we will prove a general equation for a generalized Vandermonde determinant with two deleted rows in terms of the elementary symmetric polynomials e_n. We will henceforth use k_j as summation indices and we will use both λ and l_j (which denotes the deleted rows) to characterize the generalized Vandermonde determinant.

Lemma 4.1.

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ \widehat{x_1^l} & \cdots & \widehat{x_n^l} \\ \vdots & \ddots & \vdots \\ x_1^n & \cdots & x_n^n \end{vmatrix} = \left(\prod_{1\le j<i\le n}(x_i-x_j)\right) e_{n-l}(x_1,\ldots,x_n). \tag{474}$$

Proof. By the properties of the roots of an equation [843], we know that the Vandermonde determinant

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n+1} \\ \vdots & \ddots & \vdots \\ x_1^n & \cdots & x_{n+1}^n \end{vmatrix} = \left(\prod_{1\le j<i\le n}(x_i-x_j)\right)\left(\prod_{1\le j\le n}(x_{n+1}-x_j)\right) =$$

$$= \left(\prod_{1\le j<i\le n}(x_i-x_j)\right)(-1)^n\sum_{l=0}^{n}(-1)^l e_{n-l}(x_1,\ldots,x_n)x_{n+1}^l =$$

$$= \sum_{l=0}^{n}(-1)^{l+n}\left(\prod_{1\le j<i\le n}(x_i-x_j)e_{n-l}(x_1,\ldots,x_n)\right)x_{n+1}^l.$$

On the other hand, an expansion with respect to column $n+1$ gives

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n+1} \\ \vdots & \ddots & \vdots \\ x_1^n & \cdots & x_{n+1}^n \end{vmatrix} = \sum_{l=0}^{n}(-1)^{l+n}(-1)^{\binom{n}{2}}a_{\lambda+\delta}x_{n+1}^l.$$

Equating coefficients of x_{n+1}^l, we are done. $\square$

Expand the determinant

$$\Xi = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ y & x_1 & \cdots & x_{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ y^{n+1} & x_1^{n+1} & \cdots & x_{n+1}^{n+1} \end{vmatrix} \tag{475}$$

with respect to column $n+2$. The minors $\Xi_{0,m}$ are defined by (476):

$$\Xi = \sum_{m=0}^{n+1} x_{n+1}^m(-1)^{m+n+1}\Xi_{m+1} \quad . \tag{476}$$

Starting with (474), the strategy in this chapter will be to express these minors in two different ways to obtain an equation which gives an expression for the generalized Vandermonde determinants in terms of multiple sums of elementary symmetric functions.

Remark 29. The y in (475) is a dummy variable, which is used in the computations. The x_j are the variables that will enter in the generalized Vandermonde determinants.

To simplify notation, we introduce the following operator $\Theta_{n,l}^{N}$:

Definition 7. Let $0 \leq N \leq l \leq n$, put $k = (k_1, \ldots, k_N)$, $1 \leq k_j \leq n, 1 \leq j \leq N$ and let $U_{n,N}$ be the subset of $\{1, \ldots, n\}^N$, where no repetitions are allowed. Then

$$\begin{aligned}\Theta_{n,l}^{N} \equiv &\sum_{k \in U_{n,N}} \\ &\left[\prod_{j=1}^{N-1}(x_1 \ldots \prod_{i=1}^{j}\widehat{x_{k_i}} \ldots x_n)\right](x_1 \ldots \prod_{i=1}^{N}\widehat{x_{k_i}} \ldots x_n)^2 \times \\ &\times e_{n-l}(x_1, \ldots, \widehat{x_{k_1}}, \ldots, \widehat{x_{k_N}}, \ldots, x_n) \times \\ &\times (-1)^{k_1+\ldots+k_N+I(\pi)_N}\left(\prod_{\substack{1 \leq j < i \leq n, \\ i,j \neq \{k_1,\ldots,k_N\}}}(x_i - x_j)\right),\end{aligned} \tag{477}$$

where $I(\pi)_N$ is the number of inversions of the permutation [356] $\pi = (k_1, \ldots, k_N)$, where the k_j are counted in increasing order as $1, \ldots, N$. In particular,

$$\Theta_{n,l}^{0} \equiv -\left(\prod_{1 \leq j < i \leq n}(x_i - x_j)\right) x_1 \ldots x_n e_{n-l}(x_1, \ldots, x_n); \tag{478}$$

$$\begin{aligned}\Theta_{n,l}^{1} \equiv &\sum_{k=1}^{n}(-1)^k (x_1 \ldots \widehat{x}_k \ldots x_n)^2 \times \\ &\times \left(\prod_{1 \leq j < i \leq n, i,j \neq \{k\}}(x_i - x_j)\right) e_{n-l}(x_1, \ldots, \widehat{x}_k, \ldots, x_n);\end{aligned} \tag{479}$$

$$\Theta_{n,n}^{n} \equiv (-1)^n \left(\prod_{1 \leq j < i \leq n}(x_i - x_j)\right). \tag{480}$$

Theorem 4.2. *Let $2 \leq l \leq n+1$. Then*

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1^2 & \cdots & x_n^2 \\ \vdots & \ddots & \vdots \\ \widehat{x_1^l} & \cdots & \widehat{x_n^l} \\ \vdots & \ddots & \vdots \\ x_1^{n+1} & \cdots & x_n^{n+1} \end{vmatrix} = \sum_{k=1}^{n} (x_1 \ldots \widehat{x_k} \ldots x_n)^2 \times \tag{481}$$
$$\times \left(\prod_{1 \leq j < i \leq n, i,j \neq \{k\}} (x_i - x_j) \right) e_{n-l+1}(x_1, \ldots, \widehat{x_k}, \ldots, x_n) \times$$
$$\times (-1)^{k+1} \equiv -\Theta_{n,l-1}^1.$$

Proof. Our aim is first to expand the minor Ξ_2 with respect to row 1, and then expand all the minors but the first one with respect to column 1 and use (474).

(482)

$$\Xi_2 \equiv \begin{vmatrix} 1 & 1 & \cdots & 1 \\ y^2 & x_1^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ y^{n+1} & x_1^{n+1} & \cdots & x_n^{n+1} \end{vmatrix} = \left(\prod_{1\le j<i\le n}(x_i-x_j)\right)(x_1\ldots x_n)^2 +$$

$$+\sum_{k=1}^{n}(-1)^k \begin{vmatrix} y^2 & x_1^2 & \cdots & \widehat{x_k^2} & \cdots & x_n^2 \\ y^3 & x_1^3 & \cdots & \widehat{x_k^3} & \cdots & x_n^3 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ y^{n+1} & x_1^{n+1} & \cdots & \widehat{x_k^{n+1}} & \cdots & x_n^{n+1} \end{vmatrix} =$$

$$= \left(\prod_{1\le j<i\le n}(x_i-x_j)\right)(x_1\ldots x_n)^2 + \sum_{k=1}^{n}(x_1\ldots\widehat{x_k}\ldots x_n)^2 \times$$

$$\times \sum_{l=2}^{n+1}(-1)^{l+k}y^l \begin{vmatrix} 1 & \cdots & \widehat{1} & \cdots & 1 \\ x_1 & \cdots & \widehat{x_k} & \cdots & x_n \\ x_1^2 & \cdots & \widehat{x_k^2} & \cdots & x_n^2 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \widehat{x_1^{l-2}} & \cdots & \widehat{x_k^{l-2}} & \cdots & \widehat{x_n^{l-2}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_1^{n-1} & \cdots & \widehat{x_k^{n-1}} & \cdots & x_n^{n-1} \end{vmatrix} =$$

$$\overset{\text{by}(474)}{=} \left(\prod_{1\le j<i\le n}(x_i-x_j)\right)(x_1\ldots x_n)^2 + \sum_{l=2}^{n+1}\sum_{k=1}^{n} y^l e_{n-l+1}$$

$$(x_1,\ldots,\widehat{x_k},\ldots,x_n)(-1)^{k+l}(x_1\ldots\widehat{x_k}\ldots x_n)^2 \left(\prod_{\substack{1\le j<i\le n,\\ i,j\neq\{k\}}}(x_i-x_j)\right).$$

Now expand the determinant Ξ_2 with respect to column 1.

$$\Xi_2 = \left(\prod_{1\le j<i\le n}(x_i-x_j)\right)(x_1\ldots x_n)^2 +$$

$$+\sum_{l=2}^{n+1}(-1)^{l+1}y^l \begin{vmatrix} 1 & \cdots & 1 \\ x_1^2 & \cdots & x_n^2 \\ \vdots & \ddots & \vdots \\ \widehat{x_1^l} & \cdots & \widehat{x_n^l} \\ \vdots & \ddots & \vdots \\ x_1^{n+1} & \cdots & x_n^{n+1} \end{vmatrix}.$$

Finally equate the coefficients of y^l. □

Theorem 4.3. *Let* $3 \leq l \leq n+1$. *Then*

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ x_1^3 & \cdots & x_n^3 \\ \vdots & \ddots & \vdots \\ \widehat{x_1^l} & \cdots & \widehat{x_n^l} \\ \vdots & \ddots & \vdots \\ x_1^{n+1} & \cdots & x_n^{n+1} \end{vmatrix} = \tag{483}$$

$$= \sum_{k\in U_{n,2}} (x_1 \ldots \widehat{x_{k_1}} \ldots x_n) e_{n-l+1}\left(x_1, \ldots, \widehat{x_{k_1}}, \ldots\right.$$
$$\left. \ldots, \widehat{x_{k_2}}, \ldots, x_n\right) (-1)^{1+k_1+k_2+I(\pi)_2} (x_1 \ldots \widehat{x_{k_1}} \ldots \widehat{x_{k_2}} \ldots x_n)^2 \times$$
$$\times \left(\prod_{1\leq j<i\leq n, i,j\neq\{k_1,k_2\}} (x_i - x_j) \right) \equiv -\Theta^2_{n,l-1}.$$

Proof. Expand the determinant Ξ_3 with respect to row 1.

$$\Xi_3 \equiv \begin{vmatrix} 1 & 1 & \cdots & 1 \\ y & x_1 & \cdots & x_n \\ y^3 & x_1^3 & \cdots & x_n^3 \\ \vdots & \vdots & \ddots & \vdots \\ y^{n+1} & x_1^{n+1} & \cdots & x_n^{n+1} \end{vmatrix} = (x_1 \ldots x_n) \begin{vmatrix} 1 & \cdots & 1 \\ x_1^2 & \cdots & x_n^2 \\ \vdots & \ddots & \vdots \\ x_1^n & \cdots & x_n^n \end{vmatrix} +$$

$$+\sum_{k_1=1}^{n}(-1)^{k_1} \begin{vmatrix} y & x_1 & \cdots & \widehat{x_{k_1}} & \cdots & x_n \\ y^3 & x_1^3 & \cdots & \widehat{x_{k_1}^3} & \cdots & x_n^3 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ y^{n+1} & x_1^{n+1} & \cdots & \widehat{x_{k_1}^{n+1}} & \cdots & x_n^{n+1}. \end{vmatrix}.$$

Expand this last determinant with respect to column 1 and use (474) and (481).

$$\Xi_3=\left(\prod_{1\le j<i\le n}(x_i-x_j)\right)x_1\ldots x_n e_{n-1}(x_1,\ldots,x_n)+$$
$$+\,y\sum_{k_1=1}^{n}(x_1\ldots\widehat{x_{k_1}}\ldots x_n)^3\left(\prod_{1\le j<i\le n,i,j\ne\{k_1\}}(x_i-x_j)\right)(-1)^{k_1}+$$
$$+\sum_{l=3}^{n+1}y^l\sum_{k_1=1}^{n}(-1)^{k_1+l}\begin{vmatrix} x_1 & \cdots & \widehat{x_{k_1}} & \cdots & x_n\\ x_1^3 & \cdots & \widehat{x_{k_1}}^3 & \cdots & x_n^3\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \widehat{x_1^l} & \cdots & \widehat{x_{k_1}^l} & \cdots & \widehat{x_n^l}\\ \vdots & \cdots & \vdots & \vdots & \cdots\\ x_1^{n+1} & \cdots & \widehat{x_{k_1}^{n+1}} & \cdots & x_n^{n+1}\end{vmatrix}=$$
$$=\left(\prod_{1\le j<i\le n}(x_i-x_j)\right)x_1\ldots x_n e_{n-1}(x_1,\ldots,x_n)+$$
$$+\,y\sum_{k_1=1}^{n}(x_1\ldots\widehat{x_{k_1}}\ldots x_n)^3\left(\prod_{1\le j<i\le n,i,j\ne\{k_1\}}(x_i-x_j)\right)(-1)^{k_1}+$$
$$+\sum_{l=3}^{n+1}y^l\sum_{k_1=1}^{n}(-1)^{k_1+l}x_1\cdots\widehat{x_{k_1}}\cdots x_n\times \tag{484}$$
$$\times\begin{vmatrix} 1 & \cdots & \widehat{1} & \cdots & 1\\ x_1^2 & \cdots & \widehat{x_{k_1}}^2 & \cdots & x_n^2\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \widehat{x_1^{l-1}} & \cdots & \widehat{x_{k_1}^{l-1}} & \cdots & \widehat{x_n^{l-1}}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ x_1^n & \cdots & \widehat{x_{k_1}^n} & \cdots & x_n^n\end{vmatrix}\overset{\text{by}(481)}{=}$$
$$=\left(\prod_{1\le j<i\le n}(x_i-x_j)\right)x_1\ldots x_n\ e_{n-1}(x_1,\ldots,x_n)+$$
$$+\,y\sum_{k_1=1}^{n}(x_1\ldots\widehat{x_{k_1}}\ldots x_n)^3\left(\prod_{1\le j<i\le n,i,j\ne\{k_1\}}(x_i-x_j)\right)(-1)^{k_1}+$$
$$+\sum_{l=3}^{n+1}y^l\sum_{k\in U_{n,2}}(x_1\ldots\widehat{x_{k_1}}\ldots x_n)\times$$
$$\times\, e_{n-l+1}(x_1,\ldots,\widehat{x_{k_1}},\ldots,\widehat{x_{k_2}},\ldots,x_n)(-1)^{l+k_1+k_2+I(\pi)_2}\times$$
$$\times\,(x_1\ldots\widehat{x_{k_1}}\ldots\widehat{x_{k_2}}\ldots x_n)^2\left(\prod_{1\le j<i\le n,i,j\ne\{k_1,k_2\}}(x_i-x_j)\right).$$

The factor $(-1)^{I(\pi)_2+1}$ comes from a renumbering of the summation indices. By expanding the determinant Ξ_3 with respect to column 1 we get

$$\begin{aligned}\Xi_3 = &\left(\prod_{1\leq j<i\leq n}(x_i-x_j)\right)x_1\ldots x_n\ e_{n-1}(x_1,\ldots,x_n)+\\ &+(-1)y\begin{vmatrix}1 & \cdots & 1\\ x_1^3 & \cdots & x_n^3\\ \vdots & \ddots & \vdots\\ x_1^{n+1} & \cdots & x_n^{n+1}\end{vmatrix}+\\ &+\sum_{l=3}^{n+1}(-1)^{l+1}y^l\begin{vmatrix}1 & \cdots & 1\\ x_1 & \cdots & x_n\\ x_1^3 & \cdots & x_n^3\\ \vdots & \ddots & \vdots\\ \widehat{x_1^l} & \cdots & \widehat{x_n^l}\\ \vdots & \ddots & \vdots\\ x_1^{n+1} & \cdots & x_n^{n+1}\end{vmatrix}.\end{aligned}\tag{485}$$

The theorem now follows by equating the coefficients of y^l. □

We can now state a general theorem for a generalized Vandermonde determinant with two deleted rows.

Theorem 4.4. *Let* $0<l_1<l_2<n+1$. *Then*

$$\begin{vmatrix}1 & \cdots & 1\\ x_1 & \cdots & x_n\\ \vdots & \ddots & \vdots\\ \widehat{x_1^{l_1}} & \cdots & \widehat{x_n^{l_1}}\\ \vdots & \ddots & \vdots\\ \widehat{x_1^{l_2}} & \cdots & \widehat{x_n^{l_2}}\\ \vdots & \ddots & \vdots\\ x_1^{n+1} & \cdots & x_n^{n+1}\end{vmatrix} = (-1)^{\binom{l_1+1}{2}}\Theta_{n,l_2-1}^{l_1}.\tag{486}$$

For the proof we need to prove the following lemma for Ξ_l by induction.

Lemma 4.5. *Let* $2\leq l\leq n+2$. *Then*

$$\Xi_l=\sum_{k_1=0}^{l-2}y^{k_1}(-1)^{k_1+\binom{k_1+1}{2}}\Theta_{n,l-2}^{k_1}+\sum_{k_1=l}^{n+1}y^{k_1}(-1)^{k_1+1+\binom{l}{2}}\Theta_{n,k_1-1}^{l-1}.\tag{487}$$

Proof. The lemma is true for $l = 2$ by (482) and for $l = 3$ by (484). Assume that the induction hypothesis is true for Ξ_{l-1}.

(488)
$$\Xi_{l-1} = \sum_{k_1=0}^{l-3} y^{k_1}(-1)^{k_1+\binom{k_1+1}{2}}\Theta_{n,l-3}^{k_1} + \sum_{k_1=l-1}^{n+1}(-1)^{k_1+1+\binom{l-1}{2}}y^{k_1}\Theta_{n,k_1-1}^{l-2}.$$

On the other hand an expansion of Ξ_{l-1} with respect to column 1 gives

(489)
$$\begin{aligned}\Xi_{l-1} = &\left(\prod_{1\le j<i\le n}(x_i - x_j)\right) x_1 \ldots x_n \; e_{n-l+3}(x_1, \ldots, x_n)\\ &+ \sum_{k_1=1}^{l-3}(-1)^{k_1}y^{k_1}\begin{vmatrix} 1 & \cdots & 1\\ x_1 & \cdots & x_n\\ \vdots & \ddots & \vdots\\ \widehat{x_1^{k_1}} & \cdots & \widehat{x_n^{k_1}}\\ \vdots & \ddots & \vdots\\ \widehat{x_1^{l-2}} & \cdots & \widehat{x_n^{l-2}}\\ \vdots & \ddots & \vdots\\ x_1^{n+1} & \cdots & x_n^{n+1}\end{vmatrix}\\ &+ \sum_{k_1=l-1}^{n+1}(-1)^{k_1+1}y^{k_1}\begin{vmatrix} 1 & \cdots & 1\\ x_1 & \cdots & x_n\\ \vdots & \ddots & \vdots\\ \widehat{x_1^{l-2}} & \cdots & \widehat{x_n^{l-2}}\\ \vdots & \ddots & \vdots\\ \widehat{x_1^{k_1}} & \cdots & \widehat{x_n^{k_1}}\\ \vdots & \ddots & \vdots\\ x_1^{n+1} & \cdots & x_n^{n+1}.\end{vmatrix}.\end{aligned}$$

Equating the coefficients for y^{k_1} of the two last equations gives first (478) and (481), then for $1 < k_1 < l-2$

$$\tag{490} \begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{k_1}} & \cdots & \widehat{x_n^{k_1}} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l-2}} & \cdots & \widehat{x_n^{l-2}} \\ \vdots & \ddots & \vdots \\ x_1^{n+1} & \cdots & x_n^{n+1} \end{vmatrix} = (-1)^{\binom{k_1+1}{2}} \Theta_{n,l-3}^{k_1},$$

and finally for $l-2 < k_1 < n+2$

$$\tag{491} \begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l-2}} & \cdots & \widehat{x_n^{l-2}} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{k_1}} & \cdots & \widehat{x_n^{k_1}} \\ \vdots & \ddots & \vdots \\ x_1^{n+1} & \cdots & x_n^{n+1} \end{vmatrix} = (-1)^{\binom{l-1}{2}} \Theta_{n,k_1-1}^{l-2}.$$

The verification of the induction hypothesis is completed by expanding Ξ_l with respect to row 1.

$$\Xi_l \equiv \begin{vmatrix} 1 & 1 & \cdots & 1 \\ y & x_1 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{y^{l-1}} & \widehat{x_1^{l-1}} & \cdots & \widehat{x_n^{l-1}} \\ \vdots & \vdots & \ddots & \vdots \\ y^{n+1} & x_1^{n+1} & \cdots & x_n^{n+1} \end{vmatrix} = (x_1 \ldots x_n) \begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l-2}} & \cdots & \widehat{x_n^{l-2}} \\ \vdots & \ddots & \vdots \\ x_1^{n} & \cdots & x_n^{n} \end{vmatrix} +$$

$$+ \sum_{k_2=1}^{n} (-1)^{k_2} \begin{vmatrix} y & x_1 & \cdots & \widehat{x_{k_2}} & \cdots & x_n \\ y^2 & x_1^2 & \cdots & \widehat{x_{k_2}^2} & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \widehat{y^{l-1}} & \widehat{x_1^{l-1}} & \cdots & \widehat{x_{k_2}^{l-1}} & \cdots & \widehat{x_n^{l-1}} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ y^{n+1} & x_1^{n+1} & \cdots & \widehat{x_{k_2}^{n+1}} & \cdots & x_n^{n+1} \end{vmatrix}.$$

Expand the last determinant with respect to column 1 and use (481), (490) and (491).

$$\Xi_l = \sum_{k_1=0}^{1} y^{k_1}(-1)^{k_1+\binom{k_1+1}{2}}\Theta_{n,l-2}^{k_1} + \sum_{k_2=1}^{n}\sum_{k_1=2}^{l-2}(-1)^{k_2+k_1+1}y^{k_1}\times$$

$$\times x_1\cdots\widehat{x_{k_2}}\cdots x_n \begin{vmatrix} 1 & \cdots & \widehat{1} & \cdots & 1 \\ x_1 & \cdots & \widehat{x_{k_2}} & \cdots & x_n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \widehat{x_1^{k_1-1}} & \cdots & \widehat{x_{k_2}^{k_1-1}} & \cdots & \widehat{x_n^{k_1-1}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \widehat{x_1^{l-2}} & \cdots & \widehat{x_{k_2}^{l-2}} & \cdots & \widehat{x_n^{l-2}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_1^n & \cdots & \widehat{x_{k_2}^n} & \cdots & x_n^n \end{vmatrix} + \sum_{k_2=1}^{n}\sum_{k_1=l}^{n+1}$$

$$(-1)^{k_2+k_1}y^{k_1}x_1\cdots\widehat{x_{k_2}}\cdots x_n \times \begin{vmatrix} 1 & \cdots & \widehat{1} & \cdots & 1 \\ x_1 & \cdots & \widehat{x_{k_2}} & \cdots & x_n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \widehat{x_1^{l-2}} & \cdots & \widehat{x_{k_2}^{l-2}} & \cdots & \widehat{x_n^{l-2}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \widehat{x_1^{k_1-1}} & \cdots & \widehat{x_{k_2}^{k_1-1}} & \cdots & \widehat{x_n^{k_1-1}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_1^n & \cdots & \widehat{x_{k_2}^n} & \cdots & x_n^n \end{vmatrix} =$$

$$= \sum_{k_1=0}^{1} y^{k_1}(-1)^{k_1+\binom{k_1+1}{2}}\Theta_{n,l-2}^{k_1} + \sum_{k_2=1}^{n}\sum_{k_1=2}^{l-2}(-1)^{k_2+1+k_1+\binom{k_1+1}{2}}\times$$

$$\times y^{k_1}x_1\cdots\widehat{x_{k_2}}\cdots x_n\Theta_{n-1,l-3}^{k_1-1}[x_1,\ldots,\widehat{x_{k_2}},\ldots,x_n]+$$

$$+\sum_{k_2=1}^{n}\sum_{k_1=l}^{n+1}(-1)^{k_2+k_1+\binom{l-1}{2}}y^{k_1}x_1\cdots\widehat{x_{k_2}}\cdots x_n\times$$

$$\times\Theta_{n-1,k_1-2}^{l-2}[x_1,\ldots,\widehat{x_{k_2}},\ldots,x_n] = \sum_{k_1=0}^{l-2} y^{k_1}(-1)^{k_1+\binom{k_1+1}{2}}\Theta_{n,l-2}^{k_1}+$$

$$+\sum_{k_1=l}^{n+1} y^{k_1}(-1)^{k_1+1+\binom{l}{2}}\Theta_{n,k_1-1}^{l-1}.$$

Finally equation (486) follows from (490) or (491). □

We are now going to prove an equation for a generalized Vandermonde determinant with an arbitrary number of rows deleted. This equation will be a natural generalization of theorem 4.4!

4.3. **Generalized Vandermonde determinants with any number of deleted rows.** We start with a definition of some numbers which will be used throughout this chapter:

Definition 8. Let $\{b_{\lambda,j,t,u}\}_{j=1}^{t}$ be natural numbers, $t, u \in \mathbb{N}$, which satisfy the equation

$$b_{\lambda,t,t,u} - 2 + \sum_{j=1}^{t-1}(b_{\lambda,j,t,u} - 1) = u. \tag{492}$$

These numbers also satisfy the inequalities

$$b_{\lambda,j,t,u} \geq 1, 1 \leq j \leq t-1;\ b_{\lambda,t,t,u} \geq 2. \tag{493}$$

The maximum value of $b_{\lambda,j,t,u}$ is in fact equal to the jumps in degree (for $j = 1, \dots, t$) of the generalized Vandermonde determinant as the following equation shows:

$$max(b_{\lambda,j,t,u}) = \lambda_{n-j} + 1 - \lambda_{n-j+1},\ j = 1, \dots, t. \tag{494}$$

To compute the $b_{\lambda,j,t,u}$, we apply the following procedure: First $b_{\lambda,1,t,u}$ gets a maximal value, then $b_{\lambda,2,t,u}$, etc. until the 'increment' u is exhausted.

Remark 30. The following theorem defines an equivalence relation E on the set $a_{\lambda+\delta}$ of all generalized Vandermonde determinants. The equivalence class $E_{n+s,l_{s-1}+1}$ is defined by the following two criteria:

(1) The highest power in the determinant is $n + s - 1$.
(2)

$$\sum_{j=1}^{l_{s-1}-s+2} b_{\lambda,j,l_{s-1}-s+2,u} = l_{s-1} + 1. \tag{495}$$

Any two generalized Vandermonde determinants which belong to the same equivalence class $E_{n+s,l_{s-1}+1}$ can be transformed to each other by the method shown in the following proof.

Example 4. The following table gives an example of how to compute $I(\pi)_3 + \sum_{i=1}^{3} k_i \bmod 2$ (n=5):

k_1	k_2	k_3	$I(\pi) + \sum_i k_i$	k_1	k_2	k_3	$I(\pi) + \sum_i k_i$
1	2	3	1	3	4	1	1
1	2	4	-1	3	4	2	-1
1	2	5	1	3	4	5	1
1	3	2	-1	3	5	1	-1
1	3	4	1	3	5	2	1
1	3	5	-1	3	5	4	-1
1	4	2	1	4	1	2	-1
1	4	3	-1	4	1	3	1
1	4	5	1	4	1	5	-1
1	5	2	-1	4	2	1	1
1	5	3	1	4	2	3	-1
1	5	4	-1	4	2	5	1
2	1	3	-1	4	3	1	-1
2	1	4	1	4	3	2	1
2	1	5	-1	4	3	5	-1
2	3	1	1	4	5	1	1
2	3	4	-1	4	5	2	-1
2	3	5	1	4	5	3	1
2	4	1	-1	5	1	2	1
2	4	3	1	5	1	3	-1
2	4	5	-1	5	1	4	1
2	5	1	1	5	2	1	-1
2	5	3	-1	5	2	3	1
2	5	4	1	5	2	4	-1
3	1	2	1	5	3	1	1
3	1	4	-1	5	3	2	-1
3	1	5	1	5	3	4	1
3	2	1	-1	5	4	1	-1
3	2	4	1	5	4	2	1
3	2	5	-1	5	4	3	-1

The following examples show how to compute the $b_{\lambda,j,t,u}$.

Example 5. We start with the determinant defined by $\lambda = (5,5,0)$. To compute it we move backwards from the determinant defined by $\lambda = (2,2,0,0,0,0)$, which has $u = 0$, and the bs are easy to calculate. Transform to $\lambda = (3,3,0,0,0)$, which has $b_{\lambda,1,3,1} = 1$, $b_{\lambda,2,3,1} = 1$, $b_{\lambda,3,3,1} = 3$. Transform to $\lambda = (4,4,0,0)$, which has $b_{\lambda,1,2,2} = 1$, $b_{\lambda,2,2,2} = 4$. And finally transform to $\lambda = (5,5,0)$, which has $b_{\lambda,1,1,3} = 5$.

Example 6. We start with the determinant defined by $\lambda = (4, 2, 0)$. To compute it we move backwards from the determinant defined by $\lambda = (2, 0, 0, 0, 0)$, which has $u = 0$, and the bs are easy to calculate. Transform to $\lambda = (3, 1, 0, 0)$, which has $b_{\lambda,1,3,1} = 1$, $b_{\lambda,2,3,1} = 2$, $b_{\lambda,3,3,1} = 2$. And finally transform to $\lambda = (4, 2, 0)$, which has $b_{\lambda,1,2,2} = 3$, $b_{\lambda,2,2,2} = 2$.

Theorem 4.6. *Let $0 < l_1 < l_2 < \ldots < l_s < n + s - 1$. Further assume that we have deleted s rows from the original Vandermonde determinant. For convenience put*

$$N \equiv l_{s-1} - s + 2. \tag{496}$$

Then

$$\begin{aligned}(-1)^{\binom{n}{2}} a_{\lambda+\delta} \equiv \begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_1}} & \cdots & \widehat{x_n^{l_1}} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_s}} & \cdots & \widehat{x_n^{l_s}} \\ \vdots & \ddots & \vdots \\ x_1^{n+s-1} & \cdots & x_n^{n+s-1} \end{vmatrix} &= (-1)^{\binom{N+1}{2}} \times \\ \times \sum_{k \in U_{n,N}} \prod_{j=1}^{N} (x_1 \ldots \prod_{i=1}^{j} \widehat{x_{k_i}} \ldots x_n)^{b_{\lambda,j,N,s-2}} &\times \\ \times\, e_{n-l_s+s-1}(x_1, \ldots, \widehat{x_{k_1}}, \ldots, \widehat{x_{k_N}}, \ldots, x_n) &\times \\ \times\, (-1)^{k_1+\ldots+k_N+I(\pi)_N} \left(\prod_{\substack{1 \le j < i \le n, \\ i,j \neq \{k_1,\ldots,k_N\}}} (x_i - x_j) \right). \end{aligned} \tag{497}$$

Proof. We introduce an 'induction variable' m, which goes from 1 to $s - 2$. This variable counts the number of times we move backwards in the same equivalence class $E_{n+s,l_{s-1}+1}$. For each m the exponent of the extracted monomial decreases and the number of variables in the new determinant decreases by one. The induction hypothesis is true for $m = 0$ by theorem (4.4).

Assume that the theorem is true for $m-1$.

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n+s-1-m} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_{s-m}}} & \cdots & \widehat{x_{n+s-1-m}^{l_{s-m}}} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_s}} & \cdots & \widehat{x_{n+s-1-m}^{l_s}} \\ \vdots & \ddots & \vdots \\ x_1^{n+s-1} & \cdots & x_{n+s-1-m}^{n+s-1} \end{vmatrix} = \tag{498}$$

$$= (-1)^{\binom{l_{s-1}+2-m}{2}} \sum_{k\in U_{n+s-1-m,l_{s-1}+1-m}}$$

$$\prod_{j=1}^{l_{s-1}+1-m} (x_1 \ldots \prod_{i=1}^{j} \widehat{x_{k_i}} \ldots x_{n+s-1-m})^{b_{\lambda,j,l_{s-1}+1-m,m-1}} \times$$

$$\times\, e_{n-l_s+s-1}(x_1, , \ldots, \widehat{x_{k_1}}, \ldots, \widehat{x_{k_{l_{s-1}+1-m}}}, \ldots, x_{n+s-1-m}) \times$$

$$\times (-1)^{k_1+\ldots+k_{l_{s-1}+1-m}+I(\pi)_{l_{s-1}+1-m}} \left(\prod_{\substack{1\le j<i\le n+s-1-m,\\ i,j\neq\{k_1,\ldots,k_{l_{s-1}+1-m}\}}} (x_i - x_j) \right).$$

An expansion of the same determinant with respect to the last column gives

$$\sum_{l=0}^{l_{s-1-m}} (-1)^{n+s-m+l} x_{n+s-1-m}^{l} \begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n+s-2-m} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l}} & \cdots & \widehat{x_{n+s-2-m}^{l}} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_s}} & \cdots & \widehat{x_{n+s-2-m}^{l_s}} \\ \vdots & \ddots & \vdots \\ x_1^{n+s-1} & \cdots & x_{n+s-2-m}^{n+s-1} \end{vmatrix} + \tag{499}$$

$$+\, G(x_{n+s-1-m}),$$

where $G(x_{n+s-1-m})$ are the terms with $x_{n+s-1-m}$ of order $l_{s-1-m}+1$ and higher.

We now have to pick out the terms which contain $x_{n+s-1-m}^{l_{s-1-m}}$ from (498) i.e. we have to solve the equation

$$l_{s-1-m}=\sum_{h=1}^{j} b_{\lambda,h,l_{s-1}+1-m,m-1}. \tag{500}$$

This implies that

$$k_{j+1}=n+s-1-m. \tag{501}$$

Further

$$b_{\lambda,j,l_{s-1}+1-m,m-1}+b_{\lambda,j+1,l_{s-1}+1-m,m-1}\mapsto b_{\lambda,j,l_{s-1}-m,m}. \tag{502}$$

When $m=1$, the bs have the following values:

$$b_{\lambda,1,l_{s-1}-1,1}=1,\ldots,b_{\lambda,l_{s-2},l_{s-1}-1,1}=2,b_{\lambda,l_{s-2}+1,l_{s-1}-1,1}=1,$$

$$,\ldots,b_{\lambda,l_{s-1}-2,l_{s-1}-1,1}=1,b_{\lambda,l_{s-1}-1,l_{s-1}-1,1}=2.$$

If $l_{s-2}+1=l_{s-1}$, the last term shall be $b_{\lambda,l_{s-2},l_{s-1}-1,1}=3$.

This is accomplished by putting $j=l_{s-2}+1$ and $k_j=n+s-2$ followed by suppression of the index k_j, which results in a reordering of the k_is. By equating equations (498) and (499) , cancelling the factor

$(-1)^{n+s-m+l_{s-1-m}}$ and equating the coefficients of $x_{n+s-1-m}^{l_{s-1-m}}$, we obtain

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n+s-2-m} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_{s-m-1}}} & \cdots & \widehat{x_{n+s-2-m}^{l_{s-m-1}}} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_s}} & \cdots & \widehat{x_{n+s-2-m}^{l_s}} \\ \vdots & \ddots & \vdots \\ x_1^{n+s-1} & \cdots & x_{n+s-2-m}^{n+s-1} \end{vmatrix}=$$

$$=(-1)^{\binom{l_{s-1}+2-m}{2}+n+s-m+l_{s-1-m}}\sum_{k\in U_{n+s-2-m,l_{s-1}-m}}$$

$$\prod_{j=1}^{l_{s-1}-m}(x_1\ldots\prod_{i=1}^{j}\widehat{x_{k_i}}\ldots x_{n+s-2-m})^{b_{\lambda,j,l_{s-1}-m,m}}\times$$

$$\times e_{n-l_s+s-1}(x_1,\ldots,\widehat{x_{k_1}},\ldots,\widehat{x_{k_{l_{s-1}-m}}},\ldots,x_{n+s-2-m})\times$$

$$\times(-1)^{k_1+\ldots+k_{l_{s-1}-m}+(n+s-1-m)+I(\pi)_{l_{s-1}+1-m}}\left(\prod_{\substack{1\le j<i\le n+s-2-m,\\ i,j\neq\{k_1,\ldots,k_{l_{s-1}-m}\}}}(x_i-x_j)\right).$$

Obviously, $I(\pi)_{l_{s-1}+1-m}=I(\pi)_{l_{s-1}-m}(-1)^{l_{s-1}-m-l_{s-1-m}}$, and a computation shows that the last determinant is equal to

$$(-1)^{\binom{l_{s-1}+1-m}{2}}\sum_{k\in U_{n+s-2-m,l_{s-1}-m}}$$

$$\prod_{j=1}^{l_{s-1}-m}(x_1\ldots\prod_{i=1}^{j}\widehat{x_{k_i}}\ldots x_{n+s-2-m})^{b_{\lambda,j,l_{s-1}-m,m}}\times$$

$$\times e_{n-l_s+s-1}(x_1,\ldots,\widehat{x_{k_1}},\ldots,\widehat{x_{k_{l_{s-1}-m}}},\ldots,x_{n+s-2-m})\times$$

$$\times(-1)^{k_1+\ldots+k_{l_{s-1}-m}+I(\pi)_{l_{s-1}-m}}\left(\prod_{\substack{1\le j<i\le n+s-2-m,\\ i,j\neq\{k_1,\ldots,k_{l_{s-1}-m}\}}}(x_i-x_j)\right).$$

Now put $m=s-2$ to obtain equation (497). $\square$

4.4. **Generalized Vandermonde determinants again.** In this section we will prove a general equation for a generalized Vandermonde determinant with an arbitrary number of rows deleted in terms of elementary symmetric polynomials. It turns out that this equation is equivalent to Aitken's equation (37) for the Schur function which was stated in the introduction. This is due to the fact that the elementary symmetric functions of negative index are zero. We will henceforth use k_j as summation indices and we will use l_j to keep track of deleted rows in generalized Vandermonde determinants. In the following theorem we tacitly assume that our symmetric polynomials are functions of the variables $x_1,\ldots,x_n$ or $x_1,\ldots,x_{n+1}$ depending on the circumstances.

Theorem 4.7. *Let n and $l_1,\ldots,l_s$ be integers such that $0<l_1<l_2<\ldots<l_s<n+s-1$. For convenience put*

$$\sum_{k_0,\ldots,k_{s-1}}\equiv\sum_{k_0=0}^{s-1}\sum_{k_1=0,k_1\neq k_0}^{s-1}\cdots\sum_{\substack{k_{s-1}=0,k_{s-1}\neq k_0\ldots,k_{s-1}\neq k_{s-2},\\ \sum_{j=0}^{s-1}k_j=\sum_{j=0}^{s-1}j}}^{k_{s-1}=s-1}\tag{503}$$

and let $I(\pi)_s$ *be the number of inversions of the permutation* $\pi = (k_0, \ldots, k_{s-1})$.

Then

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_1}} & \cdots & \widehat{x_n^{l_1}} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_s}} & \cdots & \widehat{x_n^{l_s}} \\ \vdots & \ddots & \vdots \\ x_1^{n+s-1} & \cdots & x_n^{n+s-1} \end{vmatrix} = \left(\prod_{1 \le j < i \le n} (x_i - x_j) \right) \times \tag{504}$$

$$\times \sum_{k_0, \ldots, k_{s-1}} (-1)^{\binom{s}{2} + I(\pi)_s} \prod_{t=0}^{s-1} e_{n+k_t-l_{s-t}}.$$

Proof. The theorem is true for $s = 1$ by (474). Assume that the statement is true for $s-1$ deleted rows. A simplification shows that

(505)

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n+1} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_2}} & \cdots & \widehat{x_{n+1}^{l_2}} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_s}} & \cdots & \widehat{x_{n+1}^{l_s}} \\ \vdots & \ddots & \vdots \\ x_1^{n+s-1} & \cdots & x_{n+1}^{n+s-1} \end{vmatrix} =$$

$$\left(\prod_{1 \le j < i \le n} (x_i - x_j) \right) \left(\prod_{1 \le j \le n-1} (x_{n+1} - x_j) \right) \sum_{k_0, \ldots, k_{s-2}} (-1)^{\binom{s-1}{2} + I(\pi)_s}$$

$$\prod_{t=1}^{s-1} e_{n+1+k_t-l_{s+1-t}} = \left(\prod_{1 \le j < i \le n} (x_i - x_j) \right) \sum_{l_1=0}^{l_1=l_2-1} (-1)^{n+l_1} x_{n+1}^{l_1}$$

$$\sum_{k_0, \ldots, k_{s-1}} (-1)^{\binom{s}{2} + I(\pi)_s} \prod_{t=1}^{s} e_{n+k_t-l_{s+1-t}} + G(x_{n+1}),$$

where $G(x_{n+1})$ are the terms with x_{n+1} of order l_2 and higher. An expansion of the left hand side determinant with respect to the last column gives

$$\sum_{l_1=0}^{l_2-1}(-1)^{n+l_1}x_{n+1}^{l_1}(-1)^{\binom{n}{2}}a_{\lambda+\delta}+G_6(x_{n+1}), \tag{506}$$

where $G_6(x_{n+1})$ are the terms with x_{n+1} of order l_2 and higher. The induction proof is completed by equating the right hand sides of equations (505) and (506) , cancelling the factor $(-1)^{n+l_1}$ and equating the coefficients of $x_{n+1}^{l_1}$. □

Remark 31. The l_j are given by the following expression:

$$\{l_j\}_{j=1}^{\lambda_1}=b_0\star b_1\star\ldots\star b_{n-2}, \tag{507}$$

where $\star$ means concatenation and

$$\begin{aligned}&\{b_k\}=(\lambda_{n-k}+k)\{I_k\}+\{a_k\},\ k=0,\ldots,n-2,\\ &\{a_k\}=\{1,2,\ldots,\lambda_{n-k-1}-\lambda_{n-k}\},\ \{I_k\}=\{1,\ldots,1\},\end{aligned} \tag{508}$$

where $\{a_k\}_{k=0}^{k=n-2}>0$ is an increasing sequence for each fixed k, $\{I_k\}$ contains $\lambda_{n-k-1}-\lambda_{n-k}$ elements and λ is the usual partition $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n\geq 0$ of m.

5. Generalized Vandermonde determinants, difference equations, symmetric polynomials and representations of the symmetric group.

The purpose of this section is to point out a connection between solutions of difference equations, symmetric polynomials and representation theory of the symmetric group. An important role is played by determinants generalizing the Vandermonde determinant. To our knowledge this connection has not been investigated previously. We hope that some ideas and results presented in this article should be useful both for the further development of these three subjects and for their applications.

The equations considered in this section involve the operator A on the formal series (sums) $\sum_{k=0}^{\infty}c_kv_k$, where $c_0,c_1,\ldots$ are complex numbers, and $v_0,v_1,\ldots$ are some fixed elements in a complex linear space. We will require that two formal series $\sum_{k=0}^{\infty}c_kv_k$, and $\sum_{k=0}^{\infty}\tilde{c_k}v_k$, are equal if and only if $c_k=\tilde{c_k}$ for all non-negative integers k. The formal series form a linear space with respect to term-wise addition and multiplication by complex scalars:

$$\sum_{k=0}^{\infty} c_k v_k + \sum_{k=0}^{\infty} \tilde{c_k} v_k = \sum_{k=0}^{\infty} (c_k + \tilde{c_k}) v_k, \tag{509}$$

$$\alpha(\sum_{k=0}^{\infty} c_k v_k) = \sum_{k=0}^{\infty} \alpha c_k v_k. \tag{510}$$

The operator A is linear, acts term-wise on the formal series and is defined on $v_0, v_1, \ldots$ as a shift annihilating v_0:

$$Av_0 = 0,\ Av_k = v_{k-1},\ 1 \leq k < \infty. \tag{511}$$

The equations involving A will be called difference equations. Polynomials of A, or in other words difference operators with constant coefficients, are defined by

$$F_n(A) = \sum_{j=0}^{n} a_j A^j,\ a_n \neq 0,\ a_0 A^0 = a_0 I, \tag{512}$$

where I is the unit operator, n is a positive integer , and $\{a_j\}_{j=0}^n$ is a set of complex numbers.

Throughout this section we will use the notation

$$[\varphi]_k = c_k \tag{513}$$

for $\varphi = \sum_{k=0}^{\infty} c_k v_k$. With this notation

$$[\varphi]_k = [A^k \varphi]_0. \tag{514}$$

Let us consider formal series solutions

$$\varphi_\rho = \sum_{k=0}^{+\infty} c_k v_k \tag{515}$$

to the eigenvalue equation

$$F_n(A)\varphi_\rho = \rho\varphi_\rho, \tag{516}$$

where ρ and $\{c_k\}_{k=0}^{\infty}$ are some complex numbers. In order that a series φ_ρ be a solution to (516) it is necessary and sufficient that

$$\left(\sum_{j=0}^{n} a_j A^j\right)\left(\sum_{k=0}^{+\infty} c_k v_k\right) = \sum_{k=0}^{+\infty}(\sum_{j=0}^{n} a_j c_{k+j}) v_k = \rho \sum_{k=0}^{+\infty} c_k v_k, \tag{517}$$

or equivalently that the coefficients c_k satisfy the following recurrence equation

$$\rho c_k = \sum_{j=0}^{n} a_j c_{j+k},\ 0 \leq k < \infty. \tag{518}$$

This is a recurrence equation of order n with constant coefficients, and hence all its solutions can be expressed in terms of the roots $r_1, \ldots, r_n$ of the corresponding characteristic equation

$$r^n + \sum_{j=1}^{n-1} \frac{a_j}{a_n} r^j + \frac{a_0 - \rho}{a_n} = 0. \tag{519}$$

If $n = 1$, then the recurrence equation (518) takes the form

$$(a_0 - \rho)c_k + a_1 c_{k+1} = 0.$$

The characteristic equation (519) takes the form

$$r + \frac{a_0 - \rho}{a_1} = 0,$$

and has only one root $r_1 = -\frac{a_0 - \rho}{a_1}$. Thus $c_k = d_1 r_1^k = d_1 (\frac{\rho - a_0}{a_1})^k$ for some complex parameter d_1. So, when $n = 1$, any formal series solution φ_ρ of the equation (516) has the form

$$\varphi_\rho = \sum_{k=0}^{\infty} d_1 r_1^k v_k = \sum_{k=0}^{\infty} d_1 (\frac{\rho - a_0}{a_1})^k v_k.$$

The parameter d_1 can assume specific values for instance when φ_ρ satisfies some initial conditions.

Henceforth, without any further notice, we will always assume that $n > 1$.

5.1. **The simple root case.** In this and the next section we will consider only the case when all the roots of (519) are simple, that is $r_i \neq r_j$ for $i, j \in \{1, \cdots, n\}$ such that $i \neq j$.

If this is the case, then all solutions of (518) can be expressed in the form

$$c_k = \sum_{l=1}^{n} d_l r_l^k, \tag{520}$$

where $\{d_l\}_{l=1}^n$ are some complex parameters which can assume specific values if, for example, some initial conditions are to be satisfied by φ_ρ.

Suppose that φ_ρ satisfies the following n initial conditions

$$[\varphi_\rho]_{n-i} = [A^{n-i}\varphi_\rho]_0 = y_{n-i}, \;\; i \in \{1, \ldots, n\}. \tag{521}$$

These conditions are equivalent to the following linear system of equations for $d_1, \ldots, d_n$:

$$\begin{pmatrix} r_1^{n-1} & \ldots & r_j^{n-1} & \ldots & r_n^{n-1} \\ \vdots & & \vdots & & \vdots \\ r_1^{n-i} & \ldots & r_j^{n-i} & \ldots & r_n^{n-i} \\ \vdots & & \vdots & & \vdots \\ 1 & \ldots & 1 & \ldots & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ \vdots \\ d_j \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} y_{n-1} \\ \vdots \\ y_{n-i} \\ \vdots \\ y_0 \end{pmatrix}, \tag{522}$$

where the matrix of the system $V(r_1, \cdots, r_n) = (r_j^{n-i})_{1 \le i,j \le n}$ is the well-known Vandermonde matrix, whose determinant

$$\det(V(r_1, \cdots, r_n)) = \prod_{1 \le i < j \le n} (r_i - r_j) \tag{523}$$

is non-zero since we consider the case when $r_1, \ldots, r_n$ are pairwise distinct. By Cramer's rule the system (522) has the unique solution

$$d_l = \frac{\det(\Xi_l)}{\prod\limits_{1 \le i < j \le n} (r_i - r_j)}, \; l \in \{1, \ldots, n\}, \tag{524}$$

where Ξ_l is the matrix obtained by replacing the l:th column of the matrix $V(r_1, \cdots, r_n)$ by the column vector on the right hand side of (522). Expanding $\det(\Xi_l)$ with respect to the l:th column gives

$$\det(\Xi_l) = \sum_{m=1}^{n} (-1)^{m+l} y_{n-m} \det\left(V_{Y(n,\ldots,1;m)}(r(l))\right), \tag{525}$$

where $r(l) = (r_1, \ldots, \widehat{r_l}, \ldots, r_n)$ denotes the vector obtained from $r = (r_1, \ldots, r_n)$ by removing the l:th coordinate, $Y(n, \ldots, 1; m)$ denotes the vector obtained from the vector $(n, \ldots, 1)$ by removing the m:th coordinate, and the $(n-1) \times (n-1)$ matrix $V_{Y(n,\ldots,1;m)}(r(l))$ is obtained from the matrix $V(r_1, \ldots, r_n)$ by deleting the l:th column and the m:th row.

The determinant of the matrix $V_{Y(n,\ldots,1;m)}(r(l))$ can be expressed using the elementary symmetric polynomials e_k as follows:

$$\det\left(V_{Y(n,\ldots,1;m)}(r(l))\right) =$$

$$\Big(\prod_{\substack{1 \le i < j \le n \\ (i \ne l) \wedge (j \ne l)}} (r_i - r_j)\Big) \;\; e_{m-1}(r(l)). \tag{526}$$

Substituting subsequently (526) into (525), (525) into (524), (524) into (520) and (520) into (515) we get the following result.

Theorem 5.1. *When the characteristic equation* (519) *has n pairwise distinct roots* $r_1, \ldots, r_n$*, the difference equation* (516) *has only one formal series solution satisfying the initial conditions* (521). *This solution is given by the formula*

$$\varphi_\rho = \sum_{k=0}^{+\infty}\left(\sum_{l=1}^{n} \frac{\det(\Xi_l)}{\prod\limits_{1\le i<j\le n}(r_i - r_j)} r_l^k\right) v_k = \tag{527}$$

$$= \sum_{k=0}^{+\infty}\Big(\sum_{l=1}^{n}\sum_{m=1}^{n} \frac{(-1)^{m+l} y_{n-m}}{\prod\limits_{\substack{1\le i<j\le n\\(i=l)\vee(j=l)}}(r_i - r_j)} e_{m-1}(r(l)) r_l^k\Big) v_k.$$

5.2. **Simple root case with general initial conditions.** Suppose now that φ_ρ satisfies the following n initial conditions

$$[\varphi_\rho]_{\lambda_i+n-i} = [A^{\lambda_i+n-i}\varphi_\rho]_0 = y_{\lambda_i+n-i},\ i \in \{1, \ldots, n\}, \tag{528}$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a vector of integers such that $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n \ge 0$. If $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$, then (528) is reduced to (521). The initial conditions (528) are equivalent to the following linear system of equations for $d_1, \ldots, d_n$:

$$\begin{pmatrix} r_1^{\lambda_1+n-1} \ldots r_j^{\lambda_1+n-1} \ldots r_n^{\lambda_1+n-1} \\ \vdots \qquad \vdots \qquad \vdots \\ r_1^{\lambda_i+n-i} \ldots r_j^{\lambda_i+n-i} \ldots r_n^{\lambda_i+n-i} \\ \vdots \qquad \vdots \qquad \vdots \\ r_1^{\lambda_n} \ \ldots r_j^{\lambda_n} \ \ldots r_n^{\lambda_n} \end{pmatrix} \begin{pmatrix} d_1 \\ \vdots \\ d_j \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} y_{\lambda_1+n-1} \\ \vdots \\ y_{\lambda_i+n-i} \\ \vdots \\ y_{\lambda_n} \end{pmatrix}. \tag{529}$$

Put $Y(\lambda) = (\lambda_1+n-1, \ldots, \lambda_i+n-i, \ldots, \lambda_n)$, and denote by $V_{Y(\lambda)}(r_1, \ldots, r_n)$ the matrix of the system (529). In the case when $\lambda_1 = \ldots = \lambda_n = 0$, this matrix becomes the Vandermonde matrix, and hence it is sometimes called a *generalized Vandermonde matrix*. Its determinant might be zero even when $r_1, \ldots, r_n$ are pairwise different. If this is the case, we will say that the pair $r = (r_1, \ldots, r_n)$, $\lambda = (\lambda_1, \ldots, \lambda_n)$ and the initial value problem (516), (528) is *Vandermonde-zero*. Otherwise (r, λ) and the initial value problem (516), (528) will be called Vandermonde-nonzero. When $\lambda_1 = \ldots = \lambda_n = 0$ the initial value problem (516), (528) is always Vandermonde-nonzero, since the Vandermonde matrix $V(r_1, \ldots, r_n)$ has nonzero determinant if and only if $r_1, \ldots, r_n$ are pairwise different, which is exactly what happens when $r_1, \ldots, r_n$ are simple roots of the characteristic equation (519).

Example 7. If $n=2$, then the initial value problem (516), (528) is Vandermonde-zero if and only if

$$\det\begin{pmatrix} r_1^{\lambda_1+1} & r_2^{\lambda_1+1} \\ r_1^{\lambda_2} & r_2^{\lambda_2} \end{pmatrix} = r_1^{\lambda_1+1}r_2^{\lambda_2} - r_1^{\lambda_2}r_2^{\lambda_1+1} = 0. \tag{530}$$

By assumption $r_1 \neq r_2$ as simple roots of the characteristic equation (519). Thus (530) can not take place when $\lambda_1 = \lambda_2 = 0$, since it becomes $r_1 - r_2 = 0$. However, when $\lambda_2 \neq 0$ there are other possibilities. Namely, if $\lambda_2 \neq 0$ and $r_1 \neq r_2$, then (530) is satisfied if and only if either $r_1 = 0$ or $r_2 = 0$, or $r_1 \neq 0$, $r_2 \neq 0$ and $(\frac{r_1}{r_2})^{\lambda_1-\lambda_2+1} = 1$.

In this example the condition for the generalized Vandermonde determinant to be zero is formulated very explicitely in terms of the roots of the characteristic equation when $n=2$, and moreover has clear geometrical interpretation. Such conditions for the generalized Vandermonde determinant to be zero would be very useful to have for $n \geq 2$ not only with the results of this paper in mind.

If $\det(V_{Y(\lambda)}(r_1,\ldots,r_n)) \neq 0$, the system (529) has a unique solution, which by Cramer's rule is given by

$$d_l = \frac{\det(\Xi_{Y(\lambda),l})}{\det(V_{Y(\lambda)}(r_1,\ldots,r_n))},\ l \in \{1,\ldots,n\}, \tag{531}$$

where $\Xi_{Y(\lambda),l}$ is the matrix obtained by replacing the l:th column of $V_{Y(\lambda)}(r_1,\ldots,r_n)$ by the column vector on the right hand side of (529). Expansion of $\det(\Xi_{Y(\lambda),l})$ with respect to the l:th column gives

$$\det(\Xi_{Y(\lambda),l}) = \sum_{m=1}^{n}(-1)^{m+l}y_{\lambda_m+n-m}\det\left(V_{Y(\lambda;m)}(r(l))\right), \tag{532}$$

where $Y(\lambda;m)$ is the vector obtained from $Y(\lambda)$ by removing the m:th coordinate $\lambda_m + n - m$.

Substituting subsequently (532) into (531), (531) into (520) and (520) into (515) we get the following result.

Theorem 5.2. *When all n roots $r_1,\ldots,r_n$ of the characteristic equation* (519) *are pairwise distinct, and the initial value problem* (516), (528) *is Vandermonde-nonzero, it has only one formal series solution. This solution is given by the formula*

$$\begin{aligned} \varphi_\rho &= \sum_{k=0}^{+\infty}\left(\sum_{l=1}^{n}\frac{\det(\Xi_{Y(\lambda),l})}{\det(V_{Y(\lambda)}(r_1,\ldots,r_n))}r_l^k\right)v_k = \\ &= \sum_{k=0}^{+\infty}(\sum_{l=1}^{n}\sum_{m=1}^{n}(-1)^{m+l}y_{\lambda_m+n-m}\frac{\det\left(V_{Y(\lambda;m)}(r(l))\right)}{\det(V_{Y(\lambda)}(r))}r_l^k)v_k. \end{aligned} \tag{533}$$

The generalized Vandermonde determinants

$$\det\left(V_{Y(\lambda)}(r)\right),\ \det\left(V_{Y(\lambda;m)}(r(l))\right)$$

and hence the solution φ_ρ to the initial-value problem (516), (528) can be expressed using the elementary and the complete symmetric polynomials.

We will use some notation from [594]. With this notation, $Y(\lambda)=\lambda+\delta$ and

$$a_{\lambda+\delta}=\det\left(V_{Y(\lambda)}(r)\right).$$

By (37) the generalized Vandermonde determinant $a_{\lambda+\delta}$ can be expressed using the elementary symmetric polynomials as follows:

$$a_{\lambda+\delta}=a_\delta\det\left(e_{\lambda'_i-i+j}\right)_{1\leq i,j\leq n'}=a_\delta\sum_{w\in S_{n'}}\varepsilon(w)e_{\lambda'+\delta-w\delta}. \tag{534}$$

where $n'\geq l(\lambda')$. The identities (37) imply for $n>1$ the following formulas:

$$\begin{aligned}&\det\left(V_{Y(\lambda;m)}(r(l))\right)=\\ &=\prod_{\substack{1\leq i<j\leq n\\(i\neq l)\wedge(i\neq l)}}(r_i-r_j)\sum_{w\in S_{n-1}}\varepsilon(w)h_{Y(\lambda;m)+\delta^{(n-1)}-w\delta^{(n-1)}}(r(l))=\\ &\prod_{\substack{1\leq i<j\leq n\\(i\neq l)\wedge(i\neq l)}}(r_i-r_j)\sum_{w\in S_{n'}}\varepsilon(w)e_{Y(\lambda;m)+\delta^{(n')}-w\delta^{(n')}}(r(l)),\end{aligned} \tag{535}$$

where $\delta^{(n)}=(n-1,\ldots,0)$. and $n'\geq l(Y(\lambda;m)')$. Consequently, in the Vandermonde-nonzero case, the solution to the initial value problem (516), (528) can be expressed in terms of the complete symmetric polynomials and the elementary symmetric polynomials as follows:

$$
\begin{aligned}
\varphi_\rho = &\sum_{k=0}^{+\infty}(\sum_{l=1}^{n}\sum_{m=1}^{n}(-1)^{m+l}y_{\lambda_m+n-m}\frac{1}{\prod\limits_{\substack{1\le i<j\le n\\ (i=l)\vee(j=l)}}(r_i-r_j)}\\
&\frac{\sum\limits_{w\in S_{n-1}}\varepsilon(w)h_{Y(\lambda;m)-w\delta^{(n-1)}}(r(l))}{\sum\limits_{w\in S_n}\varepsilon(w)h_{\lambda+\delta-w\delta}(r)}r_l^k)v_k =\\
= &\sum_{k=0}^{+\infty}(\sum_{l=1}^{n}\sum_{m=1}^{n}(-1)^{m+l}y_{\lambda_m+n-m}\frac{1}{\prod\limits_{\substack{1\le i<j\le n\\ (i=l)\vee(j=l)}}(r_i-r_j)}\\
&\frac{\sum\limits_{w\in S_{n'}}\varepsilon(w)e_{Y(\lambda;m)+\delta^{(n')}-w\delta^{(n')}}(r(l))}{\sum\limits_{w\in S_{n''}}\varepsilon(w)e_{\lambda'+\delta-w\delta}(r)}r_l^k)v_k,
\end{aligned}
\tag{536}
$$

where $n''\ge l(\lambda')$, and $n'\ge l(Y(\lambda;m)')$.

So far we have discussed the connection between solutions of difference equations and symmetric polynomials. We will continue by pointing out the way the representations of the symmetric group become involved into solutions of difference equations. The relevant material on representations of the symmetric group can be found for example in [573], [579], [661], [663], [802]. By the Frobenius character formula we obtain

$$
\det(V_{Y(\lambda)}(r)) = \det(V(r))\left(\sum_{\mu\in S^*_{|\lambda|}}\chi_\lambda(\mu)c_\mu^{-1}\mathcal{S}_\mu\right), \tag{537}
$$

where $S^*_{|\lambda|}$ denotes the set of all conjugacy classes of $S_{|\lambda|}$.

Combining this result with (533), (523) one gets an expression for φ_ρ in terms of the characters of the symmetric group. In the Vandermonde-nonzero case it follows from (537) that

$$\frac{\det\left(V_{Y(\lambda;m)}(r(l))\right)}{\det(V_{Y(\lambda)}(r))} = \tag{538}$$

$$= \frac{1}{\prod\limits_{\substack{1\le i<j\le n\\(i=l)\vee(j=l)}}(r_i-r_j)} \frac{\sum\limits_{\tilde{\mu}\in S^*_{|\lambda|-\lambda_m}} \chi_{\lambda(m)}(\tilde{\mu})c_{\tilde{\mu}}^{-1}\mathcal{S}_{\tilde{\mu}}(r(l))}{\sum\limits_{\mu\in S^*_{|\lambda|}}\chi_\lambda(\mu)c_\mu^{-1}\mathcal{S}_\mu(r)},$$

where $\lambda(m)$ is the vector obtained from the vector λ by removing the m:th coordinate λ_m, and $\chi_{\lambda(m)}$ is the character of the corresponding to $\lambda(m)$ irreducible representation of the symmetric group $S^*_{|\lambda|-\lambda_m}$ corresponding to $\lambda(m)$. This implies that

$$\varphi_\rho = \sum_{k=0}^{+\infty}(\sum_{l=1}^{n}\sum_{m=1}^{n}(-1)^{m+l}y_{\lambda_m+n-m}\times \tag{539}$$

$$\frac{1}{\prod\limits_{\substack{1\le i<j\le n\\(i=l)\vee(j=l)}}(r_i-r_j)} \frac{\sum\limits_{\tilde{\mu}\in S^*_{|\lambda|-\lambda_m}} \chi_{\lambda(m)}(\tilde{\mu})c_{\tilde{\mu}}^{-1}\mathcal{S}_{\tilde{\mu}}(r(l))}{\sum\limits_{\mu\in S^*_{|\lambda|}}\chi_\lambda(\mu)c_\mu^{-1}\mathcal{S}_\mu(r)} r_l^k v_k.$$

In the Vandermonde-zero case we only get solutions if the vector y of initial conditions on the right hand side of (529) belongs to the linear space $V_{Y(\lambda)}(r)(\mathbb{C}^n)$, or equivalently if it satisfies a system of linear equations defining $V_{Y(\lambda)}(r)(\mathbb{C}^n)$. When solutions of (529) exist, they form an affine subspace of $\mathbb{C}^n$ of dimension $n-T$, where T denotes the rank of $V_{Y(\lambda)}(r)$.

Remark 32. A similar technique was used in quantum chromodynamics in the article [28] by Andrews & Onofri, where a simple closed form for a particular instance of Wilson's loop variables is derived both via group theory and via q-hypergeometric series.

6. A new Vandermonde-related determinant and its connection to difference equations.

6.1. Introduction. The following famous equation

$$
\begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_0 & r_1 & \cdots & r_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ r_0^{n-1} & r_1^{n-1} & \vdots & r_{n-1}^{n-1} \end{vmatrix} = \prod_{0\le i<j\le n-1} (r_j - r_i) \tag{540}
$$

was found by Vandermonde.

The purpose of this section is to present a new Vandermonde-related determinant and to present a new proof of a Vandermonde-related determinant due to Flowe & Harris [315]. For extensive reference on this subject see [546]. These determinants arise when solving difference equations by Cramer's rule, as is outlined in subsection 2. We will use the notation of Macdonald [594] for symmetric functions.

6.2. Formal series solutions of difference equations. In this subsection we will consider the multiple root case, that is $r_i = r_j$ for some $i, j \in \{0, \cdots, n-1\}, i \neq j$. Assume that the integers $\{n_k\}_{k=0}^{s-1}$ have been chosen in such a way that

$$
\sum_{t=0}^{s-1} n_t = n,\ n_0 \geq \ldots \geq n_{s-1} > 0. \tag{541}
$$

Then if the characteristic equation (519) has s roots $r_0, \ldots r_{s-1}$ of multiplicity $n_0, \ldots n_{s-1}$ respectively, then the coefficients c_k of φ_ρ in (515) are

$$
c_k = \sum_{t=0}^{s-1} P_t(k) r_t^k, \tag{542}
$$

where $P_t(k)$ are polynomials in k of degree $\leq n_t - 1$. Suppose that φ_ρ satisfies the initial conditions (521). The initial conditions (521) give a new system of linear equations:

$$
\sum_{t=0}^{s-1} P_t(k) r_t^k = y_k,\ k = 0, \ldots n-1. \tag{543}
$$

Of course, this system can be solved by Cramer's rule. First put [489]

$$
P_t(k) = \sum_{h=0}^{n_t-1} d_{h,t} \binom{k}{h}. \tag{544}
$$

This produces a natural basis for the solutions of the difference equation (516). The equations (543) and (544) then give a system of n linear equations for the coefficients $d_{h,t}$:

$$\sum_{t=0}^{s-1} \sum_{h=0}^{min(k,n_t-1)} d_{h,t} \binom{k}{h} r_t^k = y_k,\ k = 0, \dots n-1. \tag{545}$$

6.3. **A new Vandermonde-related determinant.**

Definition 9. Let $0 \le t \le s-1$. Then

$$V(r_t) \equiv \begin{pmatrix} \binom{0}{0} & \dots & \binom{0}{j} & \dots & \binom{0}{n_t-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{i}{0} r_t^i & \dots & \binom{i}{j} r_t^{i-j} & \dots & \binom{i}{n_t-1} r_t^{i+1-n_t} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{n-1}{0} r_t^{n-1} & \dots & \binom{n-1}{j} r_t^{n-1-j} & \dots & \binom{n-1}{n_t-1} r_t^{n-n_t} \end{pmatrix}. \tag{546}$$

The following two lemmata will be necessary for the proof of the following theorem:

Lemma 6.1.

$$\begin{aligned} &h_k(r_1, \dots, r_l, r_m) - h_k(r_1, \dots, r_l, r_n) \equiv \\ &\equiv \sum_{\sum_j a_j = k,\ j \in \{1,2,\dots,l,m\}} \prod_j r_j^{a_j} - \sum_{\sum_j a_j = k,\ j \in \{1,2,\dots,l,n\}} \prod_j r_j^{a_j} = \\ &= (r_m - r_n) \sum_{\sum_j a_j = k-1,\ j \in \{1,2,\dots,l,m,n\}} \prod_j r_j^{a_j} \equiv \\ &\equiv (r_m - r_n) h_{k-1}(r_1, \dots, r_l, r_m, r_n),\ \forall l. \end{aligned} \tag{547}$$

Proof.

$$\begin{aligned} &\sum_{\sum_j a_j = k,\ j \in \{1,2,\dots,l,m\}} \prod_j r_j^{a_j} - \sum_{\sum_j a_j = k,\ j \in \{1,2,\dots,l,n\}} \prod_j r_j^{a_j} = \\ &= \sum_{t=0}^{k} \sum_{\sum_j a_j = k-t,\ j \in \{1,2,\dots,l\}} \prod_j r_j^{a_j} (r_m^t - r_n^t) = \\ &= (r_m - r_n) \sum_{t=0}^{k} \sum_{\sum_j a_j = k-t,\ j \in \{1,2,\dots,l\}} \prod_j r_j^{a_j} (\sum_{p=0}^{s-1} r_m^p r_n^{s-1-p}) = \\ &= (r_m - r_n) \sum_{\sum_j a_j = k-1,\ j \in \{1,2,\dots,l,m,n\}} \prod_j r_j^{a_j},\ \forall l. \end{aligned} \tag{548}$$

□

Lemma 6.2. *The complete symmetric polynomial* $h_k(r_1, \ldots, r_m)$ *contains* $\binom{k+m-1}{m-1}$ *terms.*

Proof. We first make the following observation, which is proved by induction:

$$\sum_{l=0}^{k} \binom{l+m-2}{m-2} = \binom{k+m-1}{m-1}. \tag{549}$$

The induction proof of the lemma goes as follows. The statement is true for $m = 1$. Assume true for $m - 1$ variables. Then

$$h_k(r_1, \ldots, r_m) = \sum_{l=0}^{k} r_1^l h_{k-l}(r_2, \ldots, r_m). \tag{550}$$

The proof is completed by comparison with (549) and computation of the number of terms. □

Theorem 6.3. *Let* $r_0, \ldots, r_{s-1}$ *be all the* s *roots of the characteristic equation* (519) *and let* $r_0, \ldots, r_{m-1}$ *be the roots with multiplicity* > 1. *If there exists a formal series solution of the difference equation* (516) *satisfying the initial conditions* (521) *then it can be written in the form:*

$$\varphi_\rho = \sum_{k=0}^{+\infty} (\sum_{t=0}^{s-1} \sum_{h=0}^{n_t-1} d_{h,t} \binom{k}{h} r_t^k) v_k, \tag{551}$$

where $\{d_{h,t} | h, t \in \mathbb{Z},\ 0 \le h \le n_t - 1,\ 0 \le t \le s-1$ *satisfy the linear system*

$$(V(r_0) \ldots V(r_{s-1})) \begin{pmatrix} d_{0,0} \\ \vdots \\ d_{n_0-1,0} \\ \vdots \\ d_{0,m-1} \\ \vdots \\ d_{n_{m-1}-1,m-1} \\ d_{0,m} \\ \vdots \\ d_{0,s-1} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}, \tag{552}$$

whose coefficient determinant has the following value:

(553)
$$D(r_0, \ldots, r_{s-1}) \equiv |(V(r_0) \ldots V(r_{s-1}))| = \prod_{0 \le i < j \le s-1} ((r_j - r_i)^{n_j n_i}).$$

Proof. We start with an $n \times n$ Vandermonde determinant, which has a wellknown value given by (540). We divide this determinant into s blocks of width n_t i.e. $D = \Big| \ (a_0, \ldots a_{s-1}) \ \Big|$, where

$$a_t \equiv \begin{pmatrix} 1 & \cdots & 1 \\ r_{t,1} & \cdots & r_{t,n_t} \\ r_{t,1}^2 & \cdots & r_{t,n_t}^2 \\ \vdots & \ddots & \vdots \\ r_{t,1}^{n-1} & \cdots & r_{t,n_t}^{n-1} \end{pmatrix} \tag{554}$$

The first index t $(0 \leq t \leq s-1)$ of r denotes the number of the root and the second index takes the values $1, \ldots, n_t$. We are now going to perform certain transformations of this determinant. To save space, we are only going to write down block number t in each calculation. We now explain all transformations of a_t. The result after all transformations will be denoted $\hat{a}_t$. The process of transformations of a_t contains $n_t - 1$ similar steps. Let u count the number of these steps, i.e. $u = 1, \ldots, n_t - 1$. We now describe the first three steps. Each time we subtract column u from columns $u+1, \ldots, n_t$. Then by lemma 6.1 we divide by $\displaystyle\prod_{l=u+1}^{n_t} (r_{t,l} - r_{t,u})$.

$$\prod_{k=0}^{m-1} \prod_{1 \leq j < l \leq n_k} (r_{k,l} - r_{k,j}) \prod_{k > i, 1 \leq l \leq n_k, 1 \leq j \leq n_i} (r_{k,l} - r_{i,j}) = \Big| \ a_0, \ldots a_{s-1} \ \Big| =$$

$$= \begin{vmatrix} 1 & 0 & \cdots & 0 \\ r_{t,1} & r_{t,2} - r_{t,1} & \cdots & r_{t,n_t} - r_{t,1} \\ r_{t,1}^2 & r_{t,2}^2 - r_{t,1}^2 & \cdots & r_{t,n_t}^2 - r_{t,1}^2 \\ r_{t,1}^3 & r_{t,2}^3 - r_{t,1}^3 & \cdots & r_{t,n_t}^3 - r_{t,1}^3 \\ \vdots & \vdots & \ddots & \vdots \\ r_{t,1}^{n-1} & r_{t,2}^{n-1} - r_{t,1}^{n-1} & \cdots & r_{t,n_t}^{n-1} - r_{t,1}^{n-1} \end{vmatrix} \stackrel{\text{by}(6.1)}{=} \prod_{k=0}^{m-1} \prod_{l=2}^{n_k} (r_{k,l} - r_{k,1}) \times$$

$$\times \begin{vmatrix} 1 & 0 & \cdots & 0 \\ r_{t,1} & 1 & \cdots & 1 \\ r_{t,1}^2 & h_1(r_{t,1}, r_{t,2}) & \cdots & h_1(r_{t,1}, r_{t,n_t}) \\ r_{t,1}^3 & h_2(r_{t,1}, r_{t,2}) & \cdots & h_2(r_{t,1}, r_{t,n_t}) \\ \vdots & \vdots & \ddots & \vdots \\ r_{t,1}^{n-1} & h_{n-2}(r_{t,1}, r_{t,2}) & \cdots & h_{n-2}(r_{t,1}, r_{t,n_t}) \end{vmatrix} = \prod_{k=0}^{m-1} \prod_{l=2}^{n_k} (r_{k,l} - r_{k,1}) \times$$

$$\times\begin{vmatrix} 1 & 0 & 0 \\ r_{t,1} & 1 & 0 \\ r_{t,1}^2 & h_1(r_{t,1},r_{t,2}) & h_1(r_{t,1},r_{t,j})-h_1(r_{t,1},r_{t,2}) \\ r_{t,1}^3 & h_2(r_{t,1},r_{t,2}) & h_2(r_{t,1},r_{t,j})-h_2(r_{t,1},r_{t,2}) \\ \vdots & \vdots & \vdots \\ r_{t,1}^{n-1} & h_{n-2}(r_{t,1},r_{t,2}) & h_{n-2}(r_{t,1},r_{t,j})-h_{n-2}(r_{t,1},r_{t,2}) \end{vmatrix}, j=3,\ldots,n_t =$$

$$\overset{\text{by}(6.1)}{=} \prod_{k=0}^{m-1}\left(\prod_{l=2}^{n_k}(r_{k,l}-r_{k,1})\prod_{l=3}^{n_k}(r_{k,l}-r_{k,2})\right)\times$$

$$\times\begin{vmatrix} 1 & 0 & 0 \\ r_{t,1} & 1 & 0 \\ r_{t,1}^2 & h_1(r_{t,1},r_{t,2}) & 1 \\ r_{t,1}^3 & h_2(r_{t,1},r_{t,2}) & h_1(r_{t,1},r_{t,2},r_{t,j}) \\ \vdots & \vdots & \ddots \\ r_{t,1}^{n-1} & h_{n-2}(r_{t,1},r_{t,2}) & h_{n-3}(r_{t,1},r_{t,2},r_{t,j}) \end{vmatrix}, j=3,\ldots,n_t =$$

$$= \prod_{k=0}^{m-1}\left(\prod_{l=2}^{n_k}(r_{k,l}-r_{k,1})\prod_{l=3}^{n_k}(r_{k,l}-r_{k,2})\right)\times$$

$$\times\begin{vmatrix} 1 & 0 & 0 & 0 \\ r_{t,1} & 1 & 0 & 0 \\ r_{t,1}^2 & h_1(r_{t,1},r_{t,2}) & 1 & 0 \\ r_{t,1}^3 & h_2(r_{t,1},r_{t,2}) & h_1(r_{t,1},r_{t,2},r_{t,3}) & \ddots \\ \vdots & \vdots & \ddots & \ddots \\ r_{t,1}^{n-1} & h_{n-2}(r_{t,1},r_{t,2}) & h_{n-3}(r_{t,1},r_{t,2},r_{t,3}) & \ddots \end{vmatrix}$$

$$\begin{vmatrix} 0 \\ 0 \\ 0 \\ h_1(r_{t,1},r_{t,2},r_{t,j})-h_1(r_{t,1},r_{t,2},r_{t,3}) \\ \cdots \\ h_{n-3}(r_{t,1},r_{t,2},r_{t,j})-h_{n-3}(r_{t,1},r_{t,2},r_{t,3}) \end{vmatrix}, j=4,\ldots,n_t =$$

$$\overset{\text{by}(6.1)}{=} \prod_{k=0}^{m-1}\left(\prod_{l=2}^{n_k}(r_{k,l}-r_{k,1})\prod_{l=3}^{n_k}(r_{k,l}-r_{k,2})\prod_{l=4}^{n_k}(r_{k,l}-r_{k,3})\right)\times$$

$$\times\begin{vmatrix} 1 & 0 & 0 & 0 \\ r_{t,1} & 1 & 0 & 0 \\ r_{t,1}^2 & h_1(r_{t,1},r_{t,2}) & 1 & 0 \\ r_{t,1}^3 & h_2(r_{t,1},r_{t,2}) & h_1(r_{t,1},r_{t,2},r_{t,3}) & 1 \\ r_{t,1}^4 & h_3(r_{t,1},r_{t,2}) & h_2(r_{t,1},r_{t,2},r_{t,3}) & h_1(r_{t,1},r_{t,2},r_{t,3},r_{t,j}) \\ \vdots & \vdots & \ddots & \ddots \\ r_{t,1}^{n-1} & h_{n-2}(r_{t,1},r_{t,2}) & h_{n-3}(r_{t,1},r_{t,2},r_{t,3}) & h_{n-4}(r_{t,1},r_{t,2},r_{t,3},r_{t,j}) \end{vmatrix},$$

$$j=4,\ldots,n_t.$$

We continue this process until the determinant has been 'diagonalized'. In the end we obtain the equality

$$\begin{aligned}\prod_{k=0}^{m-1}\prod_{1\le j<l\le n_k}(r_{k,l}-r_{k,j})\prod_{k>i,1\le l\le n_k,1\le j\le n_i}(r_{k,l}-r_{i,j})=\\ =\prod_{k=0}^{m-1}\prod_{1\le j<l\le n_k}(r_{k,l}-r_{k,j})\left|\ \hat{a_0},\ldots\hat{a_{s-1}}\ \right|,\end{aligned}\tag{555}$$

where

$$\hat{a_t}\equiv\big(h_{i-j}(r_{t,1},\ldots,r_{t,j+1})\big),\ 0\le j\le n_t-1,\ 0\le i\le n-1.\tag{556}$$

Now divide (555) by

$$\prod_{k=0}^{m-1}\prod_{1\le j<l\le n_k}(r_{k,l}-r_{k,j}),$$

to obtain the equality

$$\prod_{k>i,1\le l\le n_k,1\le j\le n_i}(r_{k,l}-r_{i,j})=\left|\ \hat{a_0},\ldots\hat{a_{s-1}}\ \right|.\tag{557}$$

To conclude the proof we will delete the second index of r to obtain the determinant (553). The coefficient of r will then be given by the number of terms in the complete symmetric polynomial h_k, which is computed by lemma 6.2. □

6.4. **A new proof of a Vandermonde-related determinant.** In order to get another coefficient determinant we instead put

$$P_t(k)=\sum_{h=0}^{n_t-1}d_{h,t}k^h.\tag{558}$$

The equations (543) and (558) give a system of n linear equations for the coefficients $d_{h,t}$:

$$\sum_{t=0}^{s-1}\sum_{h=0}^{n_t-1} d_{h,t}k^h r_t^k = y_k,\ k=0,\dots n-1. \tag{559}$$

For notational purposes we need a few definitions.

Definition 10. Let $0 \le t \le s-1$. Then

$$U(r_t) \equiv \begin{pmatrix} \binom{0}{0} & \dots & \binom{0}{j} j! r_t^j & \dots & \binom{0}{n_t-1}(n_t-1)! r_t^{n_t-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{i}{0} r_t^i & \dots & \binom{i}{j} j! r_t^i & \dots & \binom{i}{n_t-1}(n_t-1)! r_t^i \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{n-1}{0} r_t^{n-1} & \dots & \binom{n-1}{j} j! r_t^{n-1} & \dots & \binom{n-1}{n_t-1}(n_t-1)! r_t^{n-1} \end{pmatrix}, \tag{560}$$

and

$$\widetilde{V(r_t)} \equiv \begin{pmatrix} 1 & 0 & \dots & 0 \\ r_t & r_t & \dots & r_t \\ r_t^2 & 2r_t^2 & \dots & 2^{n_t-1} r_t^2 \\ \vdots & \vdots & \vdots & \vdots \\ r_t^{n-1} & (n-1) r_t^{n-1} & \dots & (n-1)^{n_t-1} r_t^{n-1} \end{pmatrix}. \tag{561}$$

Consequently we get the following result:

Theorem 6.4. Let $r_0, \dots, r_{s-1}$ be all the s roots of the characteristic equation (519) and let $r_0, \dots, r_{m-1}$ be the roots with multiplicity > 1. If there exists a formal series solution of the difference equation (516) satisfying the initial conditions (521) then it can also be written in the form:

$$\varphi_\rho = \sum_{k=0}^{+\infty}\left(\sum_{t=0}^{s-1}\sum_{h=0}^{n_t-1} d_{h,t}k^h r_t^k\right) v_k, \tag{562}$$

where $\{d_{h,t}|h,t\in\mathbb{Z},\ 0\le h\le n_t-1,\ 0\le t\le s-1\}$ satisfy the linear system

$$\left(\widetilde{V(r_0)}\ldots\widetilde{V(r_{s-1})}\right)\begin{pmatrix} d_{0,0}\\ \vdots\\ d_{n_0-1,0}\\ \vdots\\ d_{0,m-1}\\ \vdots\\ d_{n_{m-1}-1,m-1}\\ d_{0,m}\\ \vdots\\ d_{0,s-1}\end{pmatrix}=\begin{pmatrix} y_0\\ y_1\\ \vdots\\ y_{n-1}\end{pmatrix}, \tag{563}$$

whose coefficient determinant has the following value [315],[546] *:*

$$\begin{gathered} D(\widetilde{r_0,\ldots,r_{s-1}})\equiv\left|(\widetilde{V(r_0)}\ldots\widetilde{V(r_{s-1})})\right|=\\ \prod_{0\le i<j\le s-1}((r_j-r_i)^{n_jn_i})\prod_{t=0}^{s-1}\left(r_t^{\binom{n_t}{2}}\prod_{k=1}^{n_t-1}k!\right). \end{gathered} \tag{564}$$

Proof. [619] Let

$$\{c_{ij}\}_{i=0,j=0}^{i=n-1,j=n_t-1}$$

denote the elements of the determinant $D(r_0,\ldots,r_{s-1})$ multiplied by $r_t^j j!$, i.e. by (553)

(565)

$$|((U(r_0)\ldots U(r_{s-1}))|=\prod_{0\le i<j\le s-1}((r_j-r_i)^{n_jn_i})\prod_{t=0}^{s-1}\left(r_t^{\binom{n_t}{2}}\prod_{k=1}^{n_t-1}k!\right).$$

and let

$$\widetilde{\{c_{ij}\}}_{i=0,j=0}^{i=n-1,j=n_t-1}$$

denote the elements of the determinant $D(\widetilde{r_0,\ldots,r_{s-1}})$, where j denotes the column number for each root $r_t,\ 0\le t\le s-1$ and i denotes the row number. We must prove by column addition, that

$$c_{ij}=\sum_{k=1}^{j}a_{jk}\widetilde{c_{ik}}\ ,a_{jj}=1,\ a_{jk}\in\mathbb{Z}. \tag{566}$$

But this is clear from the following wellknown result, which is called Lagrange interpolation [249]:

$$j!\binom{i}{j} = \sum_{k=1}^{j} a'_{jk} i^k,\ a'_{jk} \in \mathbb{Z}. \tag{567}$$

It is easily checked that $a'_{jj} = 1$, because the highest order coefficients are equal. □

6.5. **The multiple root case with general initial conditions.** Suppose now that φ_ρ satisfies the initial conditions (528). Let $\{n_k\}_{k=1}^{s}$ be integers satisfying $\sum_{l=1}^{s} n_l = n$ and $n_1 \geq \ldots \geq n_s > 0$. Then if the characteristic equation (519) has s roots $r_1, \ldots, r_s$ of multiplicities $n_1, \ldots, n_s$ respectively, then the coefficients c_k of φ_ρ in (515) can be written in the form

$$c_k = \sum_{l=1}^{s} P_l(k) r_l^k,$$

where $P_l(k) = \sum_{h=0}^{n_l-1} d_{h,l} k^h$ are polynomials in k of degree $\leq n_l - 1$ with complex coefficients $d_{h,l}$. The initial conditions (528) become

$$\sum_{l=1}^{s} P_l(\lambda_i + n - i) r_l^{\lambda_i+n-i} = y_{\lambda_i+n-i},\ i \in \{1, \ldots, n\}. \tag{568}$$

The conditions (568) are equivalent to the following system of linear equations for the coefficients $d_{h,l}$:

$$\sum_{l=1}^{s} \sum_{h=0}^{n_l-1} d_{h,l} (\lambda_i + n - i)^h r_l^{\lambda_i+n-i} = y_{\lambda_i+n-i},\ i \in \{1, \ldots, n\}. \tag{569}$$

Consequently we get the following result.

Theorem 6.5. *Let $r_1, \ldots, r_n$ be all the n roots of the characteristic equation* (519)*. If there exists a formal series solution φ_ρ of the difference equation* (516) *satisfying the initial conditions* (528)*, then*

$$\varphi_\rho = \sum_{k=0}^{+\infty} (\sum_{l=1}^{s} \sum_{h=0}^{n_l-1} d_{h,l} k^h r_l^k) v_k, \tag{570}$$

with $\{d_{h,l} \mid h, l \in \mathbb{Z},\ 0 \le h \le n_l - 1,\ 1 \le l \le s\}$ *satisfying the linear system*

$$V((r_1,n_1),\dots,(r_s,n_s);\lambda)\begin{pmatrix} d_{0,1} \\ \vdots \\ d_{n_1-1,1} \\ \vdots \\ d_{0,s} \\ \vdots \\ d_{n_s-1,s} \end{pmatrix} = \begin{pmatrix} y_{\lambda_1+n-1} \\ \vdots \\ y_{\lambda_i+n-i} \\ \vdots \\ y_{\lambda_n} \end{pmatrix}, \tag{571}$$

whose matrix has the form

$$V((r_1,n_1),\dots,(r_s,n_s);\lambda) = (V(r_1,n_1;\lambda),\dots,V(r_s,n_s;\lambda)),$$

where $V(r_l,n_l;\lambda)$ *denotes the* $n \times n_l$ *matrix*

$$\begin{pmatrix} r_l^{\lambda_1+n-1} \dots (\lambda_1+n-1)^{j_l} r_l^{\lambda_1+n-1} \dots (\lambda_1+n-1)^{n_l-1} r_l^{\lambda_1+n-1} \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ r_l^{\lambda_i+n-i} \dots (\lambda_i+n-i)^{j_l} r_l^{\lambda_i+n-i} \dots (\lambda_i+n-i)^{n_l-1} r_l^{\lambda_i+n-i} \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ r_l^{\lambda_n} \quad \dots \quad \lambda_n^{j_l} r_l^{\lambda_n} \quad \dots \quad \lambda_n^{n_l-1} r_l^{\lambda_n} \end{pmatrix}$$

for all $l \in \{1,\dots,s\}$.

If the linear system (571) *has no solutions, then there are no formal series solutions to* (516) *with initial conditions* (528).

When the determinant of the matrix of the system (571) is not zero the initial value problem (516), (528) will be also called Vandermonde-nonzero. Otherwise it will be called Vandermonde-zero. It would be interesting and useful to have a formula for the determinant of the matrix of the system (571) generalizing the corresponding formulas for the case of simple roots.

7. q-Calculus and physics.

Applications of q-calculus to problems in physics abound. We have tried to describe this briefly in five separate sections.

7.1. **The q-Coulomb problem and the q-hydrogen atom.** In 1935 V.Fock [317] studied the $O(4)$ (the group of motions of $O(3)$) symmetry of the hydrogen atom. In 1967 R.J.Finkelstein [285] showed how the complete dynamics of the hydrogen atom is related to $O(3)$ and the

Schrödinger equation in the momentum representation may be interpreted as an integral equation on $O(3)$. The solutions of this integral equation are the Wigner functions D^j_{mn}.

In q-field theory, the momentum operator is replaced by

$$p_q\Psi(x) = \frac{\hbar}{i} D_q\Psi(x). \tag{572}$$

The replacement of a differential operator by a difference operator then suggests the replacement of the usual continuum by an underlying lattice or alternatively, by the use of a space with noncommuting coordinates whose spectra define this lattice [290].

The q-Coulomb problem has been studied by Finkelstein in [175], [141], [294] and by Feigenbaum & Freund [270]. It is shown that the new wave functions are the matrix elements of the irreducible representations of the quantum group $SU_q(2)$. This new integral equation is formulated in terms of the Woronowicz integral. The eigenvalues are given by a modified Balmer formula which replaces n by $\{n\}_q$. This is called the q-hydrogen atom. In [288] the orthogonality relations for $SU_q(2)$ were formulated as a Woronowicz integral and a Green function was expressed by a q-analogue of Wigner functions. In [747] representations of $sU_q(2)$ on a real two-dimensional sphere were constructed in terms of q-special functions, which include q-spherical harmonics. In [289] a q-delta distribution was obtained by a deformation of quantum mechanics. In 1991 Van Isacker [906] generalized the Pauli principle to four-dimensional space by establishing a connection between O(4) tensor character and Bose or Fermi statistics.

7.2. **General relativity.** In [180], a canonical quantization of Yang-Mills theories was made. In 1996 Finkelstein [291] made a q-deformation of general relitivity by replacing the Lorentz group by the q-Lorentz group. The *q-Yang-Mills equation* has been studied by Kamata and Nakamula 1999 [501].

7.3. **Molecular and nuclear spectroscopy.** In 1990 [743] the quantum algebra $SU_q(2)$ was applied to rotational spectra of deformed nuclei. In 1991 [114] it was also shown that spectra of superdeformed bands in even-even nuclei, as well as rotational bands with normal deformation are described approximatively by $SU_q(2)$. In 1991 [964], [966], [965] a quantum group theoretic application to vibrating and rotating diatomic molecules was made. This theoretical model coincides approximatively with the vibrational Raman spectra. In 1992 [670] a q-deformed Aufbau principle working for both atoms and monoatomic ions was proposed from the q-deformed chain $SO(4) > SO(3)_q$ In 1994

[603] a q-algebra technique was applied to molecular backbending in AgH. In 1996 [649] a q-Heisenberg algebra technique was applied to the superfluidity of ^4He.

7.4. **Elementary particle physics and chemical physics.** In 1994 [349] q-analogues of hadron mass sum rules were obtained through Alexander polynomials of certain knots. In 1996 [293] a q-gauge theory was used to modify the Möller formula for electron positron annihilation and a q-analogue of Planck's blackbody spectrum was anticipated (compare [61]).

In 1997 [504] q-deformed bosons and fermions were used to derive a generalized Fokker-Planck equation, which easily can be integrated under stationary conditions and which reproduces the statistical distributions for these particles for both real and complex q-values. Also in 1997 [452] a consecutive description of functional methods of quantum field theory for systems of interacting q-particles was given. These particles obey exotic statistics and appear in many problems of condensed matter physics, magnetism and quantum optics.

An introduction to the Lipkin model was given in [523]. The q-deformed Lipkin model was presented 1995 [56]. In 1999 [870] a q-deformation of the NJL model for quantum chromodynamics (QCD) was made.

7.5. **String theory.** String theory unites quantum theory and general relativity. A particle's world line in general relativity is its x-coordinate as a function of time. A string is a 1-dimensional object in space which sweeeps out a 2-dimensional world sheet as it moves through space-time [800]. A tachyon is a superluminal particle which can propagate with superluminal velocity in gravitational fields [237].

> A review of a wide class of papers on applications of p-adic numbers in mathematical physics is presented.
>
> A major part is devoted to p-adic string theory. p-adic open string amplitudes (4- and N-tachyon) in terms of the Tate-Gelfand-Graev gamma function are considered. An effective tachyon Lagrangian is suggested. Nonlocality of the Lagrangian leads to acausality of the theory. It is shown that the field equation corresponding to the Lagrangian admits a spherically symmetric (soliton-like) Euclidean solution.
>
> The world sheet of the p-adic open string is the discrete homogeneous Bethe lattice, or the Bruhat-Tits

> tree. p-adic strings thus correspond to a specific discretization of the world sheet.
>
> The open string world sheet action is constructed in terms of the Cartier Laplacian on the tree. The p-adic string amplitudes can be obtained in this way. Adelic formulas for p-adic string amplitudes are considered both for open and closed strings. Other p-adic systems, such as strings with Chan-Paton factors, superstrings, applications of ultrametric topology in spin glasses, and p-adic quantum mechanics are briefly discussed. Of special interest is the p-adic-quantum group connection, which appears in consideration of scattering on real and p-adic symmetric spaces.
>
> A self-contained mathematical introduction includes the theory of p-adic numbers, p-adic integration and special functions, the theory of adeles and algebraic extensions of the field of p-adic numbers [125]. The author considers the scattering on the p-adic Bruhat-Tits tree $H^{(p)} = \mathrm{SL}(2, Q_p)/\mathrm{SL}(2, Z_p)$. p-adic zonal spherical functions $\phi_\lambda^{(p)}$ on $H^{(p)}$ are defined as functions on the tree $H^{(p)}$, whose value at a vertex depends only on the distance from this vertex to the origin and satisfies the conditions: (i) $\phi_\lambda^{(p)}(0) = 1$, (ii) $\phi_\lambda^{(p)}$ is an eigenfunction of the Cartier Laplacian on the tree.
>
> The explicit expression for $\phi_\lambda^{(p)}$ is given in terms of Harish-Chandra's c-function.
>
> The expressions for $\phi_\lambda^{(p)}$ in terms of Rogers-Askey-Ismail polynomials and Macdonald polynomials are also given and connections with spherical functions on $\mathrm{SL}(2)_q$ are discussed [327].

An introduction to the Tate-Gelfand-Graev gamma function is given in [328].

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References

[1] Abanov A.G. & Talstra J.C. & Wiegmann P.B.: Hierarchical structure of Azbel-Hofstadter problem: Strings and loose ends of Bethe ansatz. Nuclear Physics B 525 (1998), 571-596.

[2] Abdi W.H.: On certain q-difference equations and q-Laplace transform. Proceedings of the national academy of sciences, India A, (1962), 1-15.

[3] Abdi W.H.: Certain inversion and representation formulae for q-Laplace transforms. Math. Z. 83 1964 238–249.

[4] Adams C.R.: The general theory of a class of linear partial q-difference equations. Transactions of the American mathematical society 26 (1924), 283-312.

[5] Adams C.R.: Note on the existence of analytic solutions of the non-homogeneous linear q-difference equation, ordinary and partial. Annals of mathematics 27 (1925), 73-83.

[6] Adams C.R.: On the linear ordinary q-difference equation. Annals of mathematics 30, 195-205 (1928).

[7] Adler M. & van Moerbeke P.: Birkhoff strata, Bäcklund transformations, and regularization of isospectral operators. Adv. Math. 108 (1994), no. 1, 140–204.
[8] Adler M. & Horozov E.& van Moerbeke P.: The solution to the q-KdV equation. Physics letters A 242 (1998), 139-151.
[9] Agarwal A.K. & Kalnins E.G. & Miller W Jr.: Canonical equations and symmetry techniques for q-series. SIAM J.Math.Anal. 18, 6 (1987), 1519-1538.
[10] Agarwal R.P.: Difference equations and inequalities. Marcel Dekker 1992.
[11] Agarwal R.P.: Padé approximants, continued fractions and Heine's q-hypergeometric series. J. Math. Phys. Sci. 26 (1992), no. 3, 281–290.
[12] Ahlfors L.: Complex analysis. McGraw-Hill 1979.
[13] Aitchison I. J. R.: Berry phases, magnetic monopoles, and Wess-Zumino terms orhow the skyrmion got its spin. Acta Phys. Polon. B 18 (1987), no. 3, 207–235.
[14] Aitken A.C.: Note on dual symmetric functions. Proc. Edin. Math. Soc., (2) 2, 1931, 164-167.
[15] Alcaraz F. C. & Santos A. L.: Conservation laws for $Z(N)$ symmetric quantum spin models and their exact ground state energies. Nuclear Phys. B 275 (1986), no. 3, 436–458.
[16] Alder H. L.: Generalizations of the Rogers-Ramanujan identities. Pacific J. Math. 4, (1954). 161–168.
[17] Alekseev A. Y. & Faddeev L. D.: Involution and dynamics in the system "q-deformed top". (Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 200 (1992), Kraev. Zadachi Mat. Fiz. Smezh. Voprosy Teor. Funktsii. 24, 3–16, 187.
[18] Al-Salam W. A.: q-Bernoulli numbers and polynomials. Math. Nachr. 17 (1959), 239–260.
[19] Al-Salam W. A. & Ismail M.E.H.: Some operational formulas. J.Math. Anal. Appl. 51 (1975), 208-218.
[20] Al-Salam W. A. & Verma A.: On quadratic transformations of basic series. SIAM J. Math. Anal. 15 (1984), no. 2, 414–421.
[21] Álvarez-Nodarse R. & Quintero N.R. & Ronveaux A.: On the linearization problem involving Pochhammer symbols and their q-analogues. J. Comput. Appl. Math. 107 (1999), no. 1, 133–146.
[22] Amitsur S. A.: Commutative linear differential operators. Pacific J. Math. 8 1958 1–10.
[23] André D.: Développements de $secx$ et de $tanx$. C.R. Acad. Sc. Paris, vol. 88 (1879), pp.965-967.
[24] Andrews G.E.: q-identities of Auluck, Carlitz and Rogers. Duke mathematical journal 33 (1966), 575-581.
[25] Andrews G.E.: On the foundations of combinatorial theory. V. Eulerian differential operators. Studies in Appl. Math. 50 (1971), 345–375.
[26] Andrews G.E. & Gessel I.: Divisibility properties of the q-tangent numbers. Proc. Amer. Math. Soc. 68 (1978), no. 3, 380–384.
[27] Andrews G.E. & Foata D.: Congruences for the q-secant numbers. European J. Combin. 1 (1980), no. 4, 283–287.

[28] Andrews G.E. & Onofri E.: Lattice gauge theory, orthogonal polynomials and q-hypergeometric functions. Special functions: group theoretical aspects and applications, 163–188, Math. Appl., Reidel, Dordrecht-Boston, MA, 1984.

[29] Andrews G.E. & Baxter R.J. & Forrester P.J.: Eight-vertex SOS model and generalized Rogers- Ramanujan-type identities. J. Statist. Phys. 35 (1984), no. 3-4, 193–266.

[30] Andrews G.E. & Askey R.: Classical orthogonal polynomials. Orthogonal polynomials and applications (Bar-le-Duc, 1984), 36–62, Lecture Notes in Math., 1171, Springer, Berlin-New York, 1985.

[31] Andrews G.E.: q-series: their development and applications in analysis, number theory, combinatorics, physics and computer algebra. AMS 1986.

[32] Andrews G.E.: The theory of partitions. Reprint of the 1976 original. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998.

[33] Andrews G.E. & Askey R. & Roy R.: Special functions. Encyclopedia of Mathematics and its Applications, 71. Cambridge University Press, Cambridge, 1999. xvi+664 pp.

[34] Antonowicz M.: Gelfand-Dickey hierarchies with sources and Lax representations for restricted flows. Phys. Lett. A 165 (1992), no. 1, 47–52.

[35] Aoki K. & D'Hoker E.: Geometrical origin of integrability for Liouville and Toda theory. Perspectives in mathematical physics, International press 1994.

[36] Aomoto K. & Kato Y.: A q-analogue of de Rham cohomology associated with Jackson integrals. Special functions (Okayama, 1990), 30–60, ICM-90 Satell. Conf. Proc., Springer, Tokyo, 1991.

[37] Aomoto K.: Finiteness of a cohomology associated with certain Jackson integrals. Thoku Math. J. (2) 43 (1991), no. 1, 75–101.

[38] Aomoto K. & Kato Y. & Mimachi K.: A solution of the Yang-Baxter equation as connection coefficients of a holonomic q-difference system. Internat. Math. Res. Notices 1992, no. 1, 7–15.

[39] Aomoto K. & Kato Y.: Connection formula of symmetric A-type Jackson integrals. Duke Math. J. 74 (1994), no. 1, 129–143.

[40] Apostol T.M.: Generalized Dedekind sums and transformation formulae of certain Lambert series. Duke Math.J. 17 (1950), 147-157.

[41] Apostol T.M.: Introduction to analytic number theory. Undergraduate texts in Mathematics. Springer 1976.

[42] Appell P. & Kampé de Fériet J.: Fonctions hypergéométriques et hypersphériques. Paris 1926.

[43] Aratyn H. & Das A.: The sAKNS hierarchy. Modern Phys. Lett. A 13 (1998), no. 15, 1185–1199.

[44] Arbarello E. & De Concini C.: Geometrical aspects of the Kadomtsev-Petviashvili equation. Lecture notes in math. 1451, (1990), 95-137.

[45] Arcidiacono G.: Projective relativity cosmology and gravitation. Hadronic press 1986.

[46] Ariki S. Cyclotomic q-Schur algebras as quotients of quantum algebras. J. Reine Angew. Math. 513 (1999), 53–69.

[47] Arnold V.I. & Novikov S.P.: Dynamical systems VII, Springer 1994.

[48] Askey R.A.: The world of q. Quarterly/Centrum voor Wiskunde en Informatica 5 (1992):4, 251-269.

[49] Askey R.A. & Rahman M. & Suslov S.: On a general q-Fourier transformation with nonsymmetric kernels. Journal of computational and applied mathematics 68 (1996), 25-55.
[50] Atakishiyev N.M.: Ramanujan-type continuous measures for classical q-polynomials. Theoret. and Math. Phys. 105 (1995), no. 3, 1500–1508 (1996).
[51] Atakishiyev N.M.: On a one-parameter family of q-exponential functions. J. Phys. A., 29 (1996), L223-L227.
[52] Atakishiyeva M.K. & Atakishiyev N.M.: Fourier-Gauss transforms of the continuous big q-Hermite polynomials. J. Phys. A., 39 (1997), 559-565.
[53] Athanasiu G.G.& Floratas E.G.: The lightcone $SU_q(2)$ quantum algebra as dynamical symmetry of the Azbel-Hofstadter problem. Physics Letters B 352 (1995), 105-110.
[54] Avan J. & Babelon O. & Billey E.: The Gervais-Felder equation and the quantum Calogero-Moser systems. Commun. Math. Phys. 178, (1996), 281-299.
[55] Avancini S. S. & Krein, G.: Many-body problems with composite particles and q-Heisenberg algebras. J. Phys. A 28 (1995), no. 3, 685–691.
[56] Avancini S. S. & Eiras A. & Galetti D. & Pimentel B. M.& Lima C. L.: Phase transition in a q-deformed Lipkin model. J. Phys. A 28 (1995), no. 17, 4915–4923
[57] Awada M.A. & Sin S.J.: The string difference equation of the $D = 1$ matrix model and $W_{1+\infty}$ symmetry of the KP hierarchy. International Journal of modern physics A 7, 20 (1992), 4791-4802.
[58] Awata H. & Fukuma M. & Matsuo, Y. & Odake, S.: Character and determinant formulae of quasifinite representation of the $W_{1+\infty}$ algebra. Comm. Math. Phys. 172 (1995), no. 2, 377–400.
[59] Awata H. & Odake S. & Shiraishi J.: Integral representations of the Macdonald symmetric polynomials. Comm. Math. Phys. 179 (1996), no. 3, 647–666.
[60] Azbel M. Ya.: Energy spectrum of a conduction electron in a magnetic field. Soviet Physics Jetp 19, 3 (1964), 634-645.
[61] Babinec P.: On the q-analogue of a black body radiation. Acta Phys. Polon. A 82 (1992), no. 6, 957–960.
[62] Babujian H. M. & Flume R.: Knizhnik-Zamolodchikov equations and the algebraic Bethe ansatz. Symmetries and integrability of difference equations (Estérel, PQ, 1994), 29–35, CRM Proc. Lecture Notes, 9, Amer. Math. Soc., Providence, RI, 1996.
[63] Bachelot A. & Motet-Bachelot, A.: Les résonances d'un trou noir de Schwarzschild. Ann. Inst. H. Poincaré Phys. Thor. 59 (1993), no. 1, 3–68.
[64] Bäcklund A.V.: Einiges über Curven und Flächentransformationen. Lund Universitets Årsskrift 10 (1875), 1-12.
[65] Bäcklund A.V.: Concerning surfaces with constant negative curvature. New Era printing Co. Lancaster, PA 1905.
[66] Baez J.C.: Spin foam models. Classical Quantum Gravity 15,(1998), no. 7, 1827–1858.
[67] Bailey W.N.: Generalized hypergeometric series. Cambridge 1935, reprinted by Stechert-Hafner, New York, 1964.
[68] Bailey W.N.: A note on certain q-identities. Quarterly. J. Math. (Oxford) 12 (1941) 173-175.

[69] Bailey W.N.: Some identities in combinatory analysis. Proc. London Math. Soc. (2) 49, (1947). 421–425.

[70] Bailey W.N.: Identities of the Rogers-Ramanujan type. Proc. London Math. Soc. (2) 50 (1949),1-10.

[71] Bailey W.N.: Hypergeometric series. reprinted by Hafner, New York, 1972.

[72] Barnes E.W.: The theory of G-function. Quarterly J. Math., 31 (1899), 264-314.

[73] Barnes E.W.: Genesis of the double gamma function. Proc. London Math. Soc. 31 (1900) 358-381.

[74] Barnes E.W.: The theory of the double gamma function. Phil. Trans. Royal Soc. (A) 196 (1900), 265-388.

[75] Barnes E.W.: On the theory of the multiple gamma function. Trans. Cambridge Phil. Soc. 19 (1904), 374-425.

[76] Barnes E.W.: The asymptotic expansions of integral functions defined by general hypergeometric series. Proc. London Math. Soc. (2) 5 (1907) 59-116.

[77] Barnes E.W.: A new development of the theory of the hypergeometric functions. Proc. London Math. Soc. (2) 6 (1908) 141-177.

[78] Barrett J. W. & Westbury B. W.: Invariants of piecewise-linear 3-manifolds. Trans. Amer. Math. Soc. 348 (1996), no. 10, 3997–4022.

[79] Barsotti I.: Differential equations of theta functions. (Italian) Rend. Accad. Naz. Sci. XL Mem. Mat. (5) 7 (1983), 227–276.

[80] Bartz K.M.: On Carlitz theorem for Bernoulli polynomials. Number theory (Cieszyn, 1998). Ann. Math. Sil. No. 12 , (1998), 9–13.

[81] Barut A.O. & Raczka R.: Theory of group representations and applications. World Scientific 1986.

[82] Basu-Mallick B.: Multi-parameter deformed and nonstandard $Y(\mathrm{gl}_M)$ Yangian symmetry in integrable variants of Haldane-Shastry spin chain. J. Phys. Soc. Japan 67 (1998), no. 7, 2227–2233.

[83] Baszak M. & Rauch-Wojciechowski S.A.: Generalized Hénon-Heiles system and related integrable Newton equations. J. Math. Phys. 35 (1994), no.4, 1693–1709.

[84] Baxter R.J.: Eight-vertex model in lattice statistics. Phys. Rev. Lett. 26 (1971), 832-833.

[85] Baxter R.J.: One-dimensional anisotropic Heisenberg chain. Phys. Rev. Lett. 26 (1971), 834-835.

[86] Baxter R.J.: Partition function of the eight-vertex lattice model. Ann. Physics 70 1972 193–228.

[87] Baxter R.J.: One-dimensional anisotropic Heisenberg chain. Ann. Physics 70 (1972), 323–337.

[88] Baxter R.J.: Exactly solved models in statistical mechanics. Academic press 1982.

[89] Beauville A.: Le problème de Schottky: Une introduction. Gazette des mathematiciens, 37 (1988), 54-63.

[90] Beck J. & Frenkel I.B. Jing N.: Canonical basis and Macdonald polynomials. Adv. Math. 140 (1998), no. 1, 95–127.

[91] Beerends R. J. & Opdam E. M.: Certain hypergeometric series related to the root system BC. Trans. Amer. Math. Soc. 339 (1993), no. 2, 581–609.

[92] Bellucci S. & Ivanov E. & Krivonos S. & Pichugin A.: $N = 2$ superBoussinesq hierarchy: Lax pairs and conservation laws. Phys. Lett. B 312 (1993), no. 4,463–470.

[93] Benaoum H.B.: (q, h)-analogue of Newton's binomial formula. J. Physics A 32 (1999), 2037-2040.

[94] Bender E.A.: A generalized q-binomial Vandermonde convolution. Discrete Math. 1 1971/1972 no. 2, 115–119.

[95] Bercovich L. M.: Direct methods of finding exact invariant solutions of Kolmogorov-Petrovsky-Piskunov equation and some other nonlinear heat and diffusion equations. Modern group analysis, V (Johannesburg, 1994). Lie Groups Appl. 1 (1994), no. 1,27–37.

[96] Berezanski J.M.: Expansions in eigenfunctions of selfadjoint operators. AMS 1968, page 501.

[97] Berman G. & Fryer K.D.: Introduction to combinatorics. Academic Press 1972.

[98] Bessenrodt C. & Kleshchev A.: On Kronecker products of complex representations of the symmetric and alternating groups. Pasific Journal of mathematics 190, 2 (1999) 201-223.

[99] Bethe H.: Zur Theorie der Metalle. I. Eigenwerte und Eigenfunktionen Atomkette, Z.Physik 71 (1931), 205-226.

[100] Bhaskara K.H. & Viswanath K.: Poisson algebras and Poisson manifolds. Pitman Research notes in Mathematics Series 1987.

[101] Bhati S. S. & Joshi S.: Certain transformation formulas for basic hypergeometric functions. Bull. Calcutta Math. Soc. 88 (1996), no. 2, 131–134.

[102] Bianchi L.: Lezioni di geometria differenziale, vol.1, Spoerri, Pisa, 1892.

[103] Bianchi L.: Applicazioni goemetriche del metodo delle approssimazioni successive di Picard. Opere, vol 1, Part 2, Edizioni Cremonese, Roma, 1953.

[104] Biedenharn L. C.: The quantum group $\mathrm{SU}_q(2)$ and a q-analogue of the boson operators. J. Phys. A 22 (1989), no. 18, L873–L878.

[105] Bilge A.H.: Singular travelling wave solutions of the fifth-order KdV, Sawada-Kotera and Kaup equations. J. Phys. A 29 (1996), no. 16,4967–4975.

[106] Birkhoff G.D.: The generalized Riemann problem for linear differential equations and the allied problems for linear difference- and q-difference equations. Proc. Amer. Acad. Arts and Sci 49, 521-568 (1913).

[107] Birkhoff G.D. & Trjitzinsky W.J.: Analytic theory of singular q-difference equations. Acta Math. 60, (1933),1-89.

[108] Birkhoff & Rota: Ordinary differential equations. Ginn and company 1962.

[109] Bobenko A. I. & Kuznetsov V. B.: Lax representation and new formulae for the Goryachev-Chaplygin top. J. Phys. A 21 (1988), no. 9, 1999–2006.

[110] Bobenko A. I. & Lorbeer B. & Suris Y. B.: Integrable discretizations of the Euler top. J. Math. Phys. 39 (1998), no. 12, 6668–6683.

[111] Bogdanov L. V. & Konopelchenko B. G.: Lattice and q-difference Darboux-Zakharov-Manakov systems via $\overline{\partial}$-dressing method. J. Phys. A 28 (1995), no. 5, L173–L178.

[112] Bogoliubov N. M. & Bullough R. K.: q-deformed completely integrable Bose gas model. J. Phys. A 25 (1992), no. 14, 4057–4071.

[113] Bonatsos D. & Argyres E. N. & Raychev P. P:$\mathrm{SU}_q(1, 1)$ description of vibrational molecular spectra. J. Phys. A 24 (1991), no. 8, L403–L408.

[114] Bonatsos D. & Dreska S.B. & Raychev P.P. & Roussev R.P. & Smirnov F.: Description of superdeformed bands by the quantum algebra $SU_q(2)$. J. Phys. G 17 (1991), L67-L73.

[115] Bonatsos D. & Brito L. & Menezes D.: The q-deformed Moszkowski model: RPA modes. J. Phys. A 26 (1993), no. 4, 895–904.

[116] Bonatsos D. & Brito L. & Menezes D. & Providência C. & da Providência D.: Corrigendum: "The q-deformed Moszkowski model: RPA modes" J. Phys. 4, 895–904; MR 94i:81150a. J. Phys. A 26 (1993), no. 19, 5185.

[117] Bonatsos D. & Brito L. & Menezes D. & Providência C. & da Providência D.: Corrigenda: "The q-deformed Moszkowski model: RPA modes" [J. Phys. A 26 (1993), no. 4, 895–904; MR 94i:81150a; corrigendum, MR 94i:81150b] by Bonatsos, Brito and Menezes. J. Phys. A 28 (1995), no. 6, 1787.

[118] Bonatsos D. & Daskaloyannis C. & Kolokotronis P. & Lenis D.: Quantum algebras in nuclear structure. Romanian J. Phys. 41 (1996), no. 1-2, 109–192.

[119] Borcea J.: Dualities, affine vertex operator algebras, and geometry of complex polynomials. Thesis Lund 1998.

[120] Bordag L. A. & Yanovski A. B.: Polynomial Lax pairs for the chiral O(3)-field equations and the Landau-Lifshitz equation. J. Phys. A 28 (1995), no. 14, 4007–4013.

[121] Bordag L. A. & Yanovski A. B.: Algorithmic construction of O(3) chiral field equation hierarchy and the Landau-Lifshitz equation hierarchy via polynomial bundle. J. Phys. A 29 (1996), no. 17, 5575–5590.

[122] Bowman D.: A general Heine transformation and symmetric polynomials of Rogers. Analytic number theory, Vol. 1 (Allerton Park, IL, 1995), 163–171, Progr. Math., 138, Birkhäuser Boston, Boston, MA, 1996.

[123] Bracken A. J. & McAnally D. S. & Zhang R. B. & Gould M. D.: A q-analogue of Bargmann space and its scalar product. J. Phys. A 24 (1991), no. 7, 1379–1391.

[124] Brauer R.: On the structure of groups of finite order, in: Proc. Internat. Congr. Math., 1, Noordhoff, Groningen, North-Holland, Amsterdam, 1954, 209-217.

[125] Brekke L. & Freund P. G. O.: p-adic numbers in physics. Phys. Rep. 233 (1993), no. 1, 1–66.

[126] Bressoud D.M.: A simple proof of Mehler's formula for q-Hermite polynomials. Indiana Univ. Math. J. 29 (1980), no. 4, 577–580.

[127] Bressoud D.M.: Almost poised basic hypergeometric series. Proc. Indian Acad. Sci. Math. Sci. 97 (1987), no. 1-3, 61–66 (1988).

[128] Brout R.: Statistical mechanical theory of ferromagnetism. High density behavior. Phys. Rev. (2) 118 1960 1009–1019.

[129] Bucchianico A.D. & Loeb D.E. & Rota G.C.: Umbral calculus in Hilbert space. Mathematical essays in honor of Gian-Carlo Rota, Birkhäuser 1998.

[130] Burchnall J.L. & Chaundy T.W.: Commutative ordinary differential operators. Proc. Lond. Math. Soc. 21 (1922), 420-440.

[131] Burchnall J.L. & Chaundy T.W.: Commutative ordinary differential operators. Proc. Roy. Soc. Lond. 118 (1928), 557-583.

[132] Burchnall J.L. & Chaundy T.W.: Commutative ordinary differential operators II The identity $P^n = Q^m$. Proc. Roy. Soc. Lond. 134 (1932), 471–485.

[133] Burchnall J.L. & Chaundy T.W.: Expansions of Appell's double hypergeometric functions. Quart. J. Math., Oxford Ser. 11, (1940). 249–270.
[134] Burchnall J.L. & Chaundy T.W.: Expansions of Appell's double hypergeometric functions. II. Quart. J. Math., Oxford Ser. 12, (1941). 112–128.
[135] Burchnall J.L.: A note on the polynomials of Hermite. Quart. J. Math., Oxford Ser. 12, (1941). 9–11.
[136] Burchnall J.L.: Differential equations associated with hypergeometric functions. Quart. J. Math., Oxford Ser. 13, (1942). 90–106.
[137] Burnside W.: Theory of groups of finite order. Cambridge 1911.
[138] Buschman R. G.: Constant coefficient recurrences for q-hypergeometric functions. Gaṇita Sandesh 7 (1993), no. 1, 1–4 (1994).
[139] Butler F.A. & Brown E.: Model calculations of magnetic band structure. Physical review 166, 3 (1968), 630-636.
[140] Cadavid, A. C. & Finkelstein R. J.: Integration over the group space of $\mathrm{SU}_q(2)$, Clebsch-Gordan coefficients, and related identities. Lett. Math. Phys. 33 (1995), no. 3, 221–229.
[141] Cadavid A. C. & Finkelstein R. J.: The q-Coulomb problem in configuration space. J. Math. Phys. 37 (1996), no. 8, 3675–3683.
[142] Carlitz L.: On abelian fields. Transactions of the American mathematical society (1933), 122-136.
[143] Carlitz L.: Generalized Bernoulli and Euler numbers. Duke math. Jour. 8 (1941), 585-589.
[144] Carlitz L.: q-Bernoulli numbers and polynomials. Duke math. Jour. 15 (1948), 987-1000.
[145] Carlitz L.: Some theorems on generalized Dedekind sums. Pasific J. Math.3 (1953), 513-522.
[146] Carlitz L.: The reciprocity theorem for Dedekind sums. Pasific J. Math.3 (1953), 523-527.
[147] Carlitz L.: q-Bernoulli and Eulerian numbers. Trans. Amer.Soc. 76 (1954), 332-350.
[148] Carlitz L.: The expansion of certain products. Proceedings of the American mathematical society (1956), 558-564.
[149] Carlitz L.: A degenerate Staudt-Clausen theorem. Arch. Math. 7 (1956), 28–33.
[150] Carlitz L.: Some polynomials related to theta functions. Duke math. Jour. 40 (1957), 521-527.
[151] Carlitz L.: Expansions of q-Bernoulli numbers. Duke Math. J. 25 (1958) 355–364.
[152] Carlitz L.: A class of symmetric q-polynomials. Riv. Mat. Univ. Parma 10 (1959), 183–200.
[153] Carlitz L.: Some congruences involving sums of binomial coefficients. Duke math. Jour. 27 (1960), 77-79.
[154] Carlitz L.: A note on the Laguerre polynomials. Michigan Math. J. 7 (1960) 219–223.
[155] Carlitz L.: Generating functions for powers of certain sequences of numbers. Duke math. Jour. 29 (1962), 521-538.
[156] Carlitz L.: A q-identity. Mh. Math. 67 (1963), 305-310.

[157] Carlitz L.: Extended Bernoulli and Eulerian numbers. Duke math. Jour. 31 (1964), 667-689.
[158] Carlitz L. & Scoville R.: Tangent numbers and operators. Duke Math. J. 39 (1972), 413–429.
[159] Carlitz L.: Some q-expansion formulas. Glasnik Matematicki 8 (28), 2 (1973), 205-214.
[160] Carlitz L.: Some inverse relations. Duke math Jour. 40 (1973), 893-901.
[161] Carlitz L.: Fibonacci notes. III. q-Fibonacci numbers. Fibonacci Quart. 12 (1974), 317–322.
[162] Carlitz L.: Fibonacci notes. IV. q-Fibonacci polynomials. Fibonacci Quart. 13 (1975), 97–102.
[163] Carlitz L.: Degenerate Stirling, Bernoulli and Eulerian numbers. Utilitas Math. 15 (1979), 51–88.
[164] Carlitz L.: Weighted Stirling numbers of the first and second kind -II. The Fibonacci quarterly 18, 3 (1980), 242-257.
[165] Carlson R.C. & Goodearl K.R.: Commutants of ordinary differential operators. J. Differential Equations 35 (1980), no. 3, 339–365.
[166] Carmichael R.D.: The general theory of linear q-difference equations. American journal of mathematics 34, 147-168 (1912).
[167] Carow-Watamura U. & Schlieker M. & Watamura S.: $SO_q(N)$ covariant differential calculus on quantum space and quantum deformation of Schrödinger equation. Particles and fields 49 (1991), 439-446.
[168] Carow-Watamura U. & Watamura S.: The quantum group as a symmetry. Progress of theoretical physics supplement 118 (1995), 375-389.
[169] Carroll R.W.: On the ubiquitous Gelfand-Levitan-Marvcenko (GLM) equation. Acta Appl. Math. 18 (1990), no. 2, 99–144.
[170] Carroll R.W.: Topics in soliton theory. North-Holland 1991.
[171] Carroll R.W. & Kodama, Y. Solution of the dispersionless Hirota equations. J. Phys. A 28 (1995), no. 22, 6373–6387.
[172] Cauchy A.L.: Memoirè sur les Fonctions dont plusiers valeurs... Comp. Rend. Acad Sci. Paris 17 (1843), 526-534.
[173] Cauchy A.L.: Comp. Rend. Acad Sci. Paris 17 (1843), 567-572.
[174] Cerchiai B.L. & Hinterding R. & Madore J. & Wess J.: A calculus based on a q-deformed Heisenberg Algebra. preprint math.QA/9809160 (1998).
[175] Chan F. L. & Finkelstein R. J.: q-deformation of the Coulomb problem. J. Math. Phys. 35 (1994), no. 7, 3273–3284.
[176] Chan F. L. & Finkelstein R. & Oganesyan V.: The q-isotropic oscillator. J. Math. Phys. 38 (1997), no. 5, 2132–2147.
[177] Charalambides C. A.: Gould series distributions with applications to fluctuations of sums of random variables. J. Statist. Plann. Inference 14 (1986), no. 1, 15–28.
[178] Chen W.Y.C.: Parameter augmentation for basic hypergeometric series, II. Journal of combinatorial theory, A 80 (1997), 175-195.
[179] Chen W.Y.C. & Liu Z. G.: Parameter augmentation for basic hypergeometric series, I. Mathematical essays in honor of Gian-Carlo Rota, 111-129, Birkhäuser 1998.
[180] Cheng H.: Canonical quantization of Yang-Mills theories. Perspectives in mathematical physics, International press 1994.

[181] Cherednik I.V.: An analogue of the character formula for Hecke algebras. (Russian) Funktsional. Anal. i Prilozhen. 21 (1987), no. 2, 94–95.
[182] Cherednik I.: Quantum Knizhnik-Zamolodchikov equations and affine root systems. Comm. Math. Phys. 150 (1992), no. 1, 109–136.
[183] Cherednik I.: Double affine Hecke algebras and Macdonald's conjectures. Annals of mathematics 141 (1995), 191-216.
[184] Cherednik I.: Nonsymmetric Macdonald polynomials. Internat. Math. Res. Notices 1995, no. 10, 483–515.
[185] Chevalley C.: Sur certain groupes simples, Tohoko Math. J. 7 (1955), 14-66.
[186] Chiu S.H. & Gray R.W. & Nelson C.A.: The q-analogue quantized radiation field and its uncertainty relations. Phys. Lett. A 164 (1992), no. 3-4, 237–242.
[187] Choi M.D. & Elliott G.A. & Yui N.: Gauss polynomials and the rotation algebra. Invent. math. 99 (1990), 225-246.
[188] Chowdhury A.R. & Gupta N.D.: Commutative differential operator and a class of solutions for the BKP equation on the singular rational curve. J. Math. Phys. 32 (1991), no. 12, 3473–3475.
[189] Chu S. C.: 1303.
[190] Chu W.C. & Hsu L.C.: Some new applications of Gould-Hsu inversion. J. Combin. Inform. System Sci. 14 (1989), no. 1, 1–4.
[191] Chu Wenchang: Multifold analogues of Gould-Hsu inversions and their applications. Acta Math. Appl. Sinica (English Ser.) 5 (1989), no. 3, 262–268.
[192] Chu Wenchang : Some multifold reciprocal transformations with applications to series expansions. European Journal of combinatorics 12 (1991), 1-8.
[193] Chu Wenchang : Some new multiple series inversions and their applications. Systems Sci. Math. Sci. 4 (1991), no. 3, 208–214.
[194] Chu Wenchang : Inversion techniques and combinatorial identities—strange evaluations of basic hypergeometric series. Compositio Math. 91 (1994), no. 2, 121–144.
[195] Chu Wenchang : Inversion techniques and combinatorial identities. Basic hypergeometric identities, Publ. Math. Debrecen 44 (1994), no. 3-4, 301–320.
[196] Chu Wenchang : Basic hypergeometric identities: an introductory revisiting through the Carlitz inversions. Forum Math. 7 (1995), no. 1, 117–129.
[197] Chu Wenchang : Symmetry on q-Pfaff-Saalschutz-Sheppard-series. Pure Math. Appl. 6 (1995), no. 1, 31–39.
[198] Chu Wenchang : Many identities from one. Le matematiche LI (1996), 113-118.
[199] Chu Wenchang : Basic almost-poised hypergeometric series. Mem. Amer. Math. Soc. 135 (1998), no. 642, x+99 pp.
[200] Chung K.S. & Chung W.S. & Nam S.T. & Kang H.J.: New q-derivative and q-logarithm. International Journal of theoretical physics, 33, 10 (1994), 2019-2029.
[201] Chung W.S. & Kalnins E.G. & Miller W Jr.: Tensor products of q-superalgebra representations and q-series identities. J. Phys. A. 30, (1997), 7147-7166.
[202] Churchill R. C. & Falk G. T.: Lax pairs in the Hénon-Heiles and related families. Hamiltonian dynamical systems (Cincinnati, OH, 1992), 89–98, IMA Vol. Math.Appl., 63, Springer, New York, 1995.

[203] Cigler J.: Operatormethoden für q-Identitäten. Monatshefte für Mathematik 88,(1979), 87-105.

[204] Cigler J.: Operatormethoden für q-Identitäten III: Umbrale Inversion und die Lagrangesche Formel. Archiv der Mathematik 35 (1980), 533-543.

[205] Cigler J.: Operatormethoden für q-Identitäten II: q-Laguerre-Polynome. Monatshefte für Mathematik 91,(1981) 105-117.

[206] Cigler J.: Elementare q-Identitäten. Publication de l'institute de recherche Mathématique avancée (1982) 23-57.

[207] Cigler J.: A new q-analog of Stirling numbers. Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II 201 (1992), no. 1-10, 97–109.

[208] Cigler J.: Operatormethoden für q-Identitäten IV: Eine Klasse von q-Gould-Polynomen. (German) [Operator methods for q-identities. IV. A class of q-Gould polynomials] Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II 205 (1996), 169–174 (1997).

[209] Cigler J.: Operatormethoden für q-Identitäten V: q-Catalan-Bäume. (German) [Operator methods for q-identities. V. q-Catalan trees] Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II 205 (1996), 175–182 (1997).

[210] Cigler J.: Operatormethoden für q-Identitäten VI: Geordnete Wurzelbäume und q-Catalan-Zahlen. Sitzungsber. II 206 (1997), 253–266 (1998).

[211] Clausen: Über die Fälle, wenn die Reihe von der Form… J. reine angew. Math 3 (1828) 89-91.

[212] Clebsch A.: Theorie dir binären algebraischen Formen. Leipzig 1872.

[213] Cohen M.E. & Sun H.S.: On generating functions and double series expansions. The Fibonacci quarterly 19, 1 (1981), 69-73.

[214] Conforto F.: Abelsche Funktionen und algebraische Geometrie. Springer 1956.

[215] Connes A.: Publ. Math. IHES 62 41 1986.

[216] Conte R. & Musette M. & Grundland A. M.: Bäcklund transformation of partial differential equations from the PainlevéGambier classification. II. Tzitzeica equation. J. Math. Phys. 40 (1999), no. 4, 2092–2106.

[217] Cristofano G. & Maiella G. & Musto R. & Nicodemi F.: FQHE and Jain's approach on the torus. International journal of modern physics B, vol.9, No.8 (1995), 961-971.

[218] Damiani I. & De Concini C.: Quantum groups and Poisson groups. European school of group theory, Pitman Research Notes 311, 1994.

[219] Darboux G.: Leçons sur la théorie générale des surfaces, 2e partie, Gauthier-Villars, Paris, 1889.

[220] Darboux G.: Leçons sur les systémes orthogonaux et les coordonneés curvlignes. Paris 1910.

[221] Dashen R. & Frishman Y.: Four-fermion interactions and scale invariance. Phys. Rev. D 11 (1975), 2781-2802.

[222] Date E. & Horiuchi E.: On contiguity relations of Jackson's basic hypergeometric series $\Upsilon_1(a; b; c; x, y, 1/2)$ and its generalizations. Osaka J. Math. 34 (1997), no. 1, 215–221.

[223] Daum J.A.: The basic analog of Kummers theorem. Bull. Amer. Math. Soc. 48 (1942) 711-713.

[224] Davies B., Foda O., Jimbo M., Miwa T., Nakayashiki A.: Diagonalization of the xxz Hamiltonian by vertex operators. Commun. Math. Phys. 151 (1993), 89-153.

[225] Désarménien J.: Un analogue des congruences de Kummer pour les q-nombres d'Euler. (French) [An analogue of the Kummer congruences for the Euler q numbers] European J. Combin. 3 (1982), no. 1, 19–28.

[226] Dickey L.A.: Solition equations and hamiltonian systems. World Scientific 1991.

[227] Dickey L.A.: Lectures on classical W-algebras. Kluwer 1997.

[228] Dijkgraaf R, Verlinde H, Verlinde E.: Topological strings in $d < 1$. Nuclear Physics B352 (1991), 59-86.

[229] Dipper R. & James G.: The q-Schur algebra. Proc. London Math. Soc. (3) 59 (1989), no. 1, 23–50.

[230] Dixon A.C.: Summation of a certain series. Proc. London Math. Soc. (1) 35 (1903), 285-289.

[231] Dobinski G.: Summierung der Reihe $\sum_{n=1}^{\infty} \frac{n^m}{n!}$ für $m = 1, 2, \ldots$. Arch. Math. Phys. 61 (1877), 333-336.

[232] Dobrev V.K.: New q-Minkowski space-time and q-Maxwell equations hierarchy from q-conformal invariance. Phys. Lett. B 341 (1994), 133-138.

[233] Dobrev V.K.: q-Conformally covariant q-Minkowski space-time and invariant equations. Czechoslovak Journal of physics 47,11 (1997),1107-1114.

[234] Dobrev V.K. & Doebner H.D. & Twarock R.: Quantisation of kinematics and dynamics on S^1 with difference operators and a related q-deformation of the Witt algebra. J. Phys. A : Math. Gen. 30 (1997) 6841-6859.

[235] Dodd R.K. & Bullough R.K.: Proc. R. Soc. A 352 (1977) 481.

[236] Doebner H.D. & Goldin G.A.: On a general nonlinear Schrödinger equation admitting diffusion, currents. Phys. Lett. A 162 (1992), 397-401.

[237] Dolgov A. D. & Novikov I. D.: Superluminal propagation of light in gravitational field and non-causal signals. Phys. Lett. B 442 (1998), no. 1-4, 82–89.

[238] Doliwa A. & Santini P. M.: The integrable dynamics of a discrete curve. Symmetries and integrability of difference equations (Estérel, PQ, 1994), 91–102, CRM Proc. Lecture Notes, 9, Amer. Math. Soc., Providence, RI, 1996.

[239] Donkin S.: q-Schur algebras. Cambridge 1998.

[240] Dornhoff L.: Group representation theory, part A. Marcel Dekker 1971.

[241] Dougall J.: On Vandermonde's theorem and some more general expansions. Proc. Edin. Math. Soc. 25, (1907), 114-132.

[242] Drane F. B. & Roselle D. P.: A sequence of polynomials related to the Eulerian polynomials. Utilitas Math. 9 (1976), 33–37.

[243] Drinfeld V. G.: Quantum groups. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 798–820, Amer. Math. Soc., Providence, RI, 1987.

[244] Dubois-Violette M. & Kerner R.: Universal q-differential calculus and q-analogue of homological algebra. Acta Math. Univ. Comenianae LXV 2 (1996), 175-188.

[245] Dumont D.: Interprétations combinatoires des nombres de Genocchi. (French) Duke Math. J. 41 (1974), 305–318.

[246] Dutka J.: The early history of the hypergeometric function. Arch. Hist. Exact Sci. 31 (1984), no. 1, 15–34.

[247] Elworthy K. D. & Zhao H. Z.: The propagation of travelling waves for stochastic generalized KPP equations. With an appendix by J. G. Gaines. Lagrange geometry, Finsler spaces and noise applied in biology and physics. Math. Comput. Modelling 20 (1994), no. 4-5, 131–166.

[248] Enolskii V. Z. & Salerno M.: Lax representation for two-particle dynamics splitting on two tori. J. Phys. A 29 (1996), no. 17, L425–L431.

[249] Erdös P. & Turan P.: On interpolation I. Annals of mathematics 38,1 (1937), 142.

[250] Ernst F.J.: Phys.Rev. 167 (1968), 1175;168 (1968), 1415.

[251] Ernst T. & Silvestrov S. D.: Shift difference equations, symmetric polynomials and representations of the symmetric group, U. U. D. M. Report 1999:14, ISSN 1101-3591, Department of Mathematics, Uppsala University, 1999.

[252] Ernst T.: A new notation for q-calculus and a new q-Taylor formula. U. U. D. M. Report 1999:25, ISSN 1101-3591, Department of Mathematics, Uppsala University, 1999.

[253] Etingof P.I. & Kirilov A.A.: Macdonald's polynomials and representations of quantum groups. Math. Res. Lett. 1 (1994), no. 3, 279-296.

[254] Etingof P.I. & Kirilov A.A.: Representation-theoretic proof of the inner product and symmetric identities for Macdonald's polynomials. Compositio Math. 102 (1996), no. 2, 179-202.

[255] Etingof P.I. & Frenkel I.B. & Kirillov A.A.: Lectures on representation theory and Knizhnik-Zamolodchikov equations. AMS 1998.

[256] Euler L.: Methodus generalis summandi progressiones. Commentarii academiae Scientiarum Petropolitanae 6 (1732-1733) 68-97.

[257] Euler L.: Inventio summae cuiusque seriei ex dato termino generali. Commentarii academiae Scientiarum Petropolitanae 8 (1736), 9-22.

[258] Euler L.: Introductio in Analysin Infinitorum (1748), chapter VII.

[259] Euler L.: Institutiones calculi differentialis, vol. 2. S:t Petersburg 1755.

[260] Euler N. & Steeb W.H.: Nonlinear differential equations, Lie symmetries, and the Painlevé test. Modern group analysis: advanced analytical and computational methods in mathematical physics (Acireale, 1992), 209–215, Kluwer Acad. Publ., Dordrecht, 1993.

[261] Exton H.: q-hypergeometric functions and applications. Ellis Horwood 1983.

[262] Exton H.: New hypergeometric identities arising from Gauss' second summation theorem. J. Comput. Appl. Math. 88 (1998), no. 2, 269–274.

[263] Fack V. & Pitre S. N. & Van der Jeugt J.: New efficient programs to calculate general recoupling coefficients. II. Evaluation of a summation formula. Comput. Phys. Comm. 86 (1995), no. 1-2, 105–122.

[264] Fack V. & Pitre S. N. & Van der Jeugt J.: Racah coefficients, general recoupling coefficients and graph theoretical methods. Special functions and differential equations (Madras, 1997), 178–183, Allied Publ., New Delhi, 1998.

[265] Fack V. & Lievens S. & Van der Jeugt J.: On rotation distance between binary coupling trees and applications for $3nj$-coefficients. Comput. Phys. Comm. 119 (1999), no. 2-3, 99–114.

[266] Faddeev L.D. & Kashaev R.M.: Generalized Bethe ansatz for Hofstadter problem. Commun. Math. Phys. 169 (1995), 181-191.

[267] Falqui G. & Reina C. & Zampa A.: Krichever maps, Faà di Bruno polynomials, and cohomology in KP theory. Lett. Math. Phys. 42 (1997), no. 4, 349–361.

[268] Fasenmyer Sister Mary Celine: Some hypergeometric polynomials. Ph.D. dissertation, University of Michigan 1945.

[269] Fasenmyer Sister Mary Celine: A note on pure recurrence relations. Amer. Math. Monthly 56 (1949), 14-17.

[270] Feigenbaum J. & Freund P. G. O.: A q-deformation of the Coulomb problem. J. Math. Phys. 37 (1996), no. 4, 1602–1616.

[271] Feinsilver P.: Commutators, anti-commutators and Eulerian calculus. Rocky Mountain J. Math. 12 (1982), no. 1, 171–183.

[272] Feinsilver P.: q-probability distributions via an extension of the Bernoulli process. Proc. Amer. Math. Soc. 87 (1983), no. 3, 508–515.

[273] Feinsilver P.: Discrete analogues of the Heisenberg-Weyl algebra. Monatshefte für Mathematik 104, 89-108 (1987).

[274] Feinsilver P.: Lie algebras and recurrence relations. I. Acta Appl. Math. 13 (1988), no. 3, 291–333.

[275] Feinsilver P.: Elements of q-harmonic analysis. Journal of mathematical analysis and applications 141, 509-526 (1989).

[276] Feinsilver P.: Lie algebras and recurrence relations. II. Symmetries in science, III (Vorarlberg, 1988), 163–179, Plenum, New York, 1989.

[277] Feinsilver P.: Lie algebras and recurrence relations. III. q-analogs and quantized algebras. Acta Appl. Math. 19 (1990), no. 3, 207–251.

[278] Feinsilver P.: Lie algebras and recurrence relations. IV. Representations of the Euclidean group and Bessel functions. Acta Appl. Math. 43 (1996), no. 3, 289–316.

[279] Feit W. & Thompson J.: Solvability of groups of odd order. Pasific J. Math. 13 (1963), 775-1029.

[280] Feldheim E.: Nouvelles relations pour les polynômes d'Hermite. J. London Math. Soc. 13 (1938), 22-29.

[281] Fernandes R.L. & Santos J.P.: Integrability of the periodic KM system. Proceedings of the XXIX Symposium on Mathematical Physics (Toruń, 1996). Rep. Math. Phys. 40 (1997), no. 3, 475–484.

[282] Ferraro G.: Some aspects of Euler's theory of series: inexplicable functions and the Euler-Maclaurin summation formula. Historia Math. 25 (1998), no. 3, 290–317.

[283] Feuer A. & Goodwin, G.: Sampling in digital signal processing and control. Birkhäuser 1996.

[284] Fichtmüller M. & Weich W.: The limit transition $q \to 1$ of the q-Fourier transform. J. Math. Phys. 37, 9 (1996), 4683-4689.

[285] Finkelstein R. J.: Symmetry group of the hydrogen atom. J. Math. Phys. 8 (1967), no. 3, 443-449.

[286] Finkelstein R. J.: q deformations of the oscillator. J. Math. Phys. 33 (1992), no. 12, 4259–4266.

[287] Finkelstein R. J.: q deformations of the oscillator. J. Math. Phys. 33 (1992), no. 12, 4259–4266.

[288] Finkelstein R. J.: The orthogonality relations for $\mathrm{SU}_q(2)$. J. Math. Phys. 35 (1994), no. 11, 6123–6143.

[289] Finkelstein R. & Marcus E.: Transformation theory of the q-oscillator. J. Math. Phys. 36 (1995), no. 6, 2652–2672.

[290] Finkelstein R. J.: q-Field theory. Lett. Math. Phys. 34 (1995), no. 2, 169–176.

[291] Finkelstein R. J.: q-Gravity. Lett. Math. Phys. 38 (1996), no. 1, 53–62.

[292] Finkelstein R. J.: q-deformation of the Lorentz group. J. Math. Phys. 37 (1996), no. 2, 953–964.

[293] Finkelstein R. J.: q Gauge theory. Internat. J. Modern Phys. A 11 (1996), no. 4, 733–746.

[294] Finkelstein R. J.: The q-Coulomb problem. J. Math. Phys. 37 (1996), no. 6, 2628–2636.

[295] Finkelstein R. J.: A note on q-transformation theory. Lett. Math. Phys. 41 (1997), no. 2, 195–206.

[296] Finkelstein R. J.: q-Uncertainty relations. Internat. J. Modern Phys. A 13 (1998), no. 11, 1795–1803.

[297] Finkelstein R. J.: Observable properties of q-deformed physical systems. Lett. Math. Phys. 49 (1999), no. 2, 105–114.

[298] Flanders H.: Commutative linear differential operators. Report No. 1. University of California, Berkely 1955.

[299] Flanders H.: Differential forms with applications to the physical sciences. Dover 1963.

[300] Floquet G.: Théorie des équations différentielles linéaires. Ann. école normale (1879)

[301] Floreanini R. & Vinet L.: Representations of quantum algebras and q-special functions. University of Montreal preprint, UdeM-LPN-TH54, 1991.

[302] Floreanini R. & Vinet L.: Addition formulas for q-Bessel functions. J. Math. Phys. 33 (1992), no. 9, 2984–2988.

[303] Floreanini R. & Vinet L.: Quantum algebras and q-special functions. Ann. Physics 221 (1993), no. 1, 53–70.

[304] Floreanini R. & Vinet L.: On the quantum group and quantum algebra approach to q-special functions. Letters in mathematical physics 27: 179-190 1993.

[305] Floreanini R. & Vinet L.: Automorphisms of the q-oscillator algebra and basic orthogonal polynomials. Phys. Lett. A 180 (1993), no. 6, 393–401.

[306] Floreanini R. & Vinet L.: Symmetries of the q-difference heat equation. Lett. Math. Phys. 32 (1994), no. 1, 37–44.

[307] Floreanini R. & Vinet L.: Generalized q-Bessel functions. Canad. J. Phys. 72 (1994), no. 7-8, 345–354.

[308] Floreanini R. & Vinet L.: Quantum symmetries of q-difference equations. J. Math. Phys. 36, 6 (1995), 3134-3156.

[309] Floreanini R. & Vinet L.: Quantum algebra approach to q-Gegenbauer polynomials. Proceedings of the Fourth International Symposium on Orthogonal Polynomials and their Applications (Evian-Les-Bains, 1992). J. Comput. Appl. Math. 57 (1995), no. 1-2, 123–133.

[310] Floreanini R. & Vinet L.: Symmetries of the q-deformed heat equations. Nonlinear, deformed and irreversible quantum systems (Clausthal, 1994), 385–400, World Sci. Publishing, River Edge, NJ, 1995.

[311] Floreanini R. & LeTourneux J. & Vinet L.: More on the q-oscillator algebra and q-orthogonal polynomials. J, Phys. A., 28 (1995), L287-L293.

[312] Floreanini R. & LeTourneux J. & Vinet L.: A q-deformed $e(4)$ and continuous q-Jacobi polynomials. J. Math. Phys. 37 (1996), no. 8, 4135–4149.

[313] Floreanini R. & Vinet L.: q-gamma and q-beta functions in quantum algebra representation theory. J. Comput. Appl. Math. 68 (1996), no. 1-2, 57–68.

[314] Floreanini R. & Vinet L.: Basic Bessel functions and q-difference equations. Symmetries and integrability of difference equations (Esterel, PQ, 1994), 113–123, CRM Proc. Lecture Notes, 9, Amer. Math. Soc., Providence, RI, 1996.

[315] Flowe R.P. & Harris G.A.: A note on generalized Vandermonde determinants. Siam J. Matrix anal. appl. 14,4 (1993), 1146-1151.

[316] Foata D.: Further divisibility properties of the q-tangent numbers. Proc. Amer. Math. Soc. 81 (1981), no. 1, 143–148.

[317] Fock V.: Zur Theorie des Wasserstoffatoms. Zeitschrift für Physik 98 (1935), 145-154.

[318] Folguera A.: Davey-Stewartson type equations of the second and third order: derivation and classification. J. Math. Phys. 39 (1998), no. 5, 2947–2968.

[319] Foupouagnigni M. & Koepf W. & Ronveaux A.: Fourth-order difference equation for the associated classical discrete orthogonal polynomials. J. Comput. Appl. Math. 92 (1998), no. 2, 103–108.

[320] Foupouagnigni M. & Ronveaux A. & Koepf W.: Fourth order q-difference equation for the first associated of the q-classical orthogonal polynomials. J. Comput. Appl. Math. 101 (1999), no. 1-2, 231–236.

[321] Fox C.: The expression of hypergeometric series in terms of similar series. Proc. London Math. Soc. (2) 26 (1927). 201–210.

[322] Fractional calculus and its applications. Proceedings of the International Conference held at the University of New Haven, West Haven, Conn., June 15–16, 1974. Edited by Bertram Ross. Lecture Notes in Mathematics, Vol. 457. Springer-Verlag, Berlin-New York, 1975.

[323] Frenkel E.: Deformations of the KdV Hierarchy and related soliton equations. International mathematics research notices 2 (1996), 55-76.

[324] Frenkel I.B. & Lepowsky J. & Meurman A.: Vertex operator algebras and the monster. Pure and applied math. 134, Academic press 1988.

[325] Frenkel I.B. & Reshetikin N.Y.: Quantum Affine Algebras and Holonomic difference equations. Commun. Math. Phys. 146, (1992), no. 1, 1-60.

[326] Frenkel I.B. & Khovanov M.G. & Kirillov A.A.: Kazhdan-Lusztig polynomials and canonical basis. Transformation groups, vol.3, No.4 (1998), 321-336.

[327] Freund P. G. O.: p-adics and q-deformations. Quantum field theory, statistical mechanics, quantum groups and topology (Coral Gables, FL, 1991), 93–96, World Sci. Publishing, River Edge, NJ, 1992.

[328] Freund P. G. O.: 2-d adelic strings. Proceedings of the XXth International Conference on Differential Geometric Methods in Theoretical Physics, Vol. 1, 2 (New York, 1991), 780–783, World Sci. Publishing, River Edge, NJ, 1992.

[329] Freund P. G. O. & Zabrodin A. V.: The spectral problem for the q-Knizhnik-Zamolodchikov equation. Phys. Lett. B 311 (1993), no. 1-4, 103–109.

[330] Freund P. G. O. & Zabrodin A. V.: The spectral problem for the q-Knizhnik-Zamolodchikov equation and continuous q-Jacobi polynomials. Comm. Math. Phys. 173 (1995), no. 1, 17–42.

[331] Frobenius G.: Über die Bernoullischen Zahlen und die Eulerschen Polynome. Preuss. Akad. Wiss. Sitzungsber. (1910), 809-847.

[332] Fuchs, J.: Affine Lie Algebras and quantum groups. Cambridge 1992.

[333] Fulton W. & Lazarsfeld R.: Positive polynomials for ample vector bundles. Ann. of Math. (2) 118 (1983), no. 1, 35–60.

[334] Furlan P. & Hadjiivanov L. K. & Todorov I. T.: Operator realization of the SU(2) WZNW model. Nuclear Phys. B 474 (1996), no. 2, 497–511

[335] Gaffet B.: On the integration of the self-similar equations, and the meaning of the Cole-Hopf transformation. J. Math. Phys. 27 (1986), no. 10, 2461–2463.

[336] Gaffet B.: Physica 26D (1987) 123.

[337] Gardner C.S. & Green J.M. & Kruskal M.D. & Miura R.M.: Method for solving the Korteweg-de Vries equation. Phys. Rew. Lett., 19, no 19 (1967), 1095-97.

[338] Garnier R.: Sur une classe de systèmes différentiels abéliens déduits de la théorie des équations linéaires. Rend Circ. Mat. Palermo 43 (1919), 155-191.

[339] Garsia A.M. & Tesler G.: Plethystic formulas for Macdonald q,t-Kostka coefficients. Advances in mathematics 123 (1996) 2, 144-222.

[340] Garsia A.O. & Trinchero R.C.: Constructive building of the Lax pair in the non-linear sigma model. J. Math. Phys. 37 (1996), no. 8, 3973–3981.

[341] Garsia A.O. & Haiman M.: A remarkable q,t-Catalan sequence and q-Lagrange inversion. J. Algebraic Combin. 5 (1996), no. 3, 191–244.

[342] Gasper G.: SIAM J. Math. Anal. 12 (1981), no. 2, 196–200.

[343] Gasper G. & Rahman M.: Basic hypergeometric series. Cambridge 1990.

[344] Gasper G.: Elementary derivations of summation and transformation formulas for q-series. Special functions, q-series and related topics (Toronto, ON, 1995), 55–70, Fields Inst. Commun., 14, Amer. Math. Soc., Providence, RI, 1997.

[345] Gaudin M.: J.Physique 46 (7) (1985), 1027-1042.

[346] Gauss C.F.: Werke 2,(1876) 9-45.

[347] Gauss C.F.: Disquisitiones generales circa seriem infinitam... 1876.

[348] Gauss C.F.: Summatio quarundam serierum singularium. Opera, vol. 2,(1876) pp.16-17.

[349] Gavrilik A. M.: q-Serre relations in $U_q(u_n)$ and q-deformed meson mass sum rules. J. Phys. A 27 (1994), no. 3, L91–L94.

[350] Geck M.: Representations of Hecke algebras at roots of unity. Séminaire Bourbaki. Vol. 1997/98. Astérisque No. 252, (1998), Exp. No. 836, 3, 33–55.

[351] Gelfand, I. M. & Graev, M. I. & Retakh, V. S. Difference analogues and q-analogues of general hypergeometric systems of differential equations. (Russian) Dokl. Akad. Nauk 325 (1992), no. 2,215–220; translation in Russian Acad. Sci. Dokl. Math. 46 (1993), no. 1, 30–35.

[352] Gessel I. & Stanley R.P.: Stirling polynomials. J. Combinatorial Theory Ser. A 24 (1978), no. 1, 24–33.

[353] Gessel I.: A q-analog of the exponential formula. Discrete Math. 40 (1982), no. 1, 69–80.

[354] Gessel I. & Stanton D.: Applications of q-Lagrange inversion to basic hypergeometric series. Trans. Amer. Math. Soc. 277 (1983), no. 1, 173–201.

[355] Gesztesy F. & Holden H. & Simon B. & Zhao Z.: On the Toda and Kac-van Moerbeke systems. Trans. Amer. Math. Soc. 339 (1993), no. 2, 849–868.

[356] Gilbert W.J.: Modern algebra with applications. Wiley 1976.

[357] Goldman J. & Rota G.C.: The number of subspaces of a vector space. Recent progress in combinatorics (1969), 75-83.

[358] Goldschmidt D.M.: Group characters, symmetric functions and the Hecke algebra. AMS 1993.

[359] Goldstein P. P.: Testing the Painlevé property of the Maxwell-Bloch and reduced Maxwell-Bloch equations. Phys. Lett. A 121 (1987), no. 1, 11–14.

[360] Gorenstein D.: Finite simple groups. An introduction to their classification. Plenum press. New York 1982.

[361] Gorenstein D.: Classifying the finite simple groups. Colloquium Lectures. Anaheim, January 1985, American Math. Soc. Bull American Math. Soc. 14 (1986), 1-98.

[362] Gorsky A. & Marshakov A. & Mironov A. & Morozov A,: RG equations from Whitham hierarchy. Nuclear Phys. B527 (1998), no. 3, 690-716.

[363] Gould H.W.: Some generalizations of Vandermonde's convolution. Amer. Math. Monthly 63 (1956), 84-91.

[364] Gould H.W.: Final analysis of Vandermonde's convolution. Amer. Math. Monthly 64 (1957), 409-415.

[365] Gould H.W.: Note on a paper of Sparre Andersen. Math. Scand 6 (1958), 226–230.

[366] Gould H.W.: A q-binomial coefficient series transform. Bulletin of the American mathematical society 66 (1960), 383-387.

[367] Gould H.W.: Generalization of a theorem of Jensen concerning convolutions. Duke math J. 27 (1960), 71-76.

[368] Gould H.W.: A series transformation for finding convolution identities. Duke math J. 28 (1961), 193-202.

[369] Gould H.W.: The q-series generalization of a formula of Sparre Andersen. Math. Scand. 9 (1961), 90–94.

[370] Gould H.W.: The q-Stirling numbers of first and second kinds. Duke Math. J. 28 (1961), 281–289.

[371] Gould H.W.: A new convolution formula and some new orthogonal relations for inversion of series. Duke math J. 29 (1962), 393-404.

[372] Gould H.W.: Congruences involving sums of binomial coefficients and a formula of Jensen. Am. math. monthly 69 (1962), 400-402.

[373] Gould H.W. & Hopper A.T.: Operational formulas connected with two generalizations of Hermite polynomials. Duke Math. J. 29 (1962), 51–63.

[374] Gould H.W.: A new series transform with applications to Bessel, Legendre, and Tchebycheff polynomials. Duke math J. 31 (1964), 325-334.

[375] Gould H.W.: Note on a q-series transform. Journal of combinatorial theory 1 (1966), 397-399.

[376] Gould H.W. & Kaucky J.: Evaluation of a class of binomial coefficient summations. Journal of combinatorial theory 1 (1966), 233-247.

[377] Gould H.W.: The bracket function, q-binomial coefficients, and some new Stirling number formulas. Fibonacci Quart. 5 (1967), 401–422.

[378] Gould H.W.: Explicit formulas for Bernoulli numbers. Amer. Math. Monthly 79 (1972), 44–51.

[379] Gould H.W. & Hsu L.C.: Some new inverse series relations. Duke math J. 40 (1973), 885-891.

[380] Gould H.W.: Evaluation of sums of convolved powers using Stirling and Eulerian numbers. The Fibonacci quarterly 16, 6 (1978), 488-497.

[381] Gould H.W. & Srivastava H. M.: Some combinatorial identities associated with the Vandermonde convolution. Appl. Math. Comput. 84 (1997), no. 2-3, 97–102.

[382] Goursat M.: Memoiré sur les fonctions hypergeometric d'ordre superieur. Ann. de l'Ecole normale II, XII 1883.

[383] Grammaticos B & Nijhoff F.H. & Papageorgiou V. & Ramani A & Satsuma J.: Linearization and solutions of the discrete Painlevé III equation. Physics letters A 185 (1994), 446-452.

[384] Gray R. W. & Nelson C. A.: A completeness relation for the q-analogue coherent states by q-integration. J. Phys. A 23 (1990), no. 18, L945–L950.

[385] Grip K.: p-adiska tal. U. U. D. M. Report 1999:P14

[386] Groza V. A.: Quantum algebras, Kravchuk q-polynomials, and Kravchuk-Meixner q-functions. Ukrainian Math. J. 44 (1992), no. 7, 887–890 (1993).

[387] Grünbaum F. A. & Haine, L.: The q-version of a theorem of Bochner. J. Comput. Appl. Math. 68 (1996), no. 1-2, 103–114.

[388] Grünbaum F. A. & Haine, L.: Orthogonal polynomials satisfying differential equations: the role of the Darboux transformation. Symmetries and integrability of difference equations (Estérel, PQ, 1994), 143–154, CRM Proc. Lecture Notes, 9, Amer. Math. Soc., Providence, RI, 1996.

[389] Grünbaum F. A. & Haine, L.: A theorem of Bochner, revisited. Algebraic aspects of integrable systems, 143–172, Progr. Nonlinear Differential Equations Appl., 26, Birkhuser Boston, Boston, MA, 1997.

[390] Grünbaum F. A. & Haine, L.: Bispectral Darboux transformations: an extension of the Krall polynomials. Internat. Math. Res. Notices 1997, no. 8, 359–392.

[391] Grünbaum F. A. & Haine, L. & Horozov E.: On the Krall-Hermite and the Krall-Bessel polynomials. Internat. Math. Res. Notices 1997, no. 19, 953–966.

[392] Guilarte J. M. & Muñoz P.J.M. & de la Torre Mayado M.: Elliptic theta functions and the fractional quantum Hall effect. J. Geom. Phys. 27 (1998), no. 3-4, 297–332.

[393] Gunning C.: Riemann surfaces and genaralized Theta functions. Springer 1976.

[394] Gupta A.: On certain bilateral, bibasic q-hypergeometric series transformations and summations. Proc. Nat. Acad. Sci. India Sect. A 62 (1992), no. 1, 87-96.

[395] Gupta D. P. & Ismail M. E. H. & Masson D. R.: Contiguous relations, basic hypergeometric functions, and orthogonal polynomials. III. Associated continuous dual q-Hahn polynomials, J. Comput. Appl. Math. 68 (1996), no. 1-2, 115–149.

[396] Gustafson R. A. & Krattenthaler C.: Determinant evaluations and U(n) extensions of Heine's ${}_2\phi_1$-transformations. Special functions, q-series and related topics (Toronto, ON, 1995), 83–89, Fields Inst. Commun., 14, Amer. Math. Soc., Providence, RI, 1997.

[397] Haase R. W.& Johnson N. F.: Schur function analysis of the unitary discrete series representations of the noncompact symplectic group. J. Phys. A 26 (1993), no. 7, 1663–1672.

[398] Häggström O.: Positive-correlations in the fuzzy Potts model. Ann. Appl. Probab. 9 (1999), no. 4, 1149–1159.

[399] Hahn W.: Über Orthogonalpolynome, die q-Differenzengleichungen genügen. Mathematische Nachrichten 2 (1949),4-34.

[400] Hahn W.: Über Polynome, die gleichzeitig zwei verschiedenen Orthogonal systemen angehören. Mathematische Nachrichten 2. (1949),263-278.

[401] Hahn W.: Beiträge zur Theorie der Heineschen Reihen. Mathematische Nachrichten 2 (1949),340-379.

[402] Hahn W.: Über die höheren Heineschen Reihen und eine einheitliche Theorie der sogenannten speziellen Funktionen. Mathematische Nachrichten 3 (1950), 257-294.

[403] Hahn W.: Über uneigentliche Lösungen linearer geometrischer Differenzengleichungen. Math. Annalen, Bd. 125 (1952), 67-81.

[404] Hahn W.: Die mechanische Deutung einer geometrischen Differenzgleichung. Zeitung angewandte Mathematik Mechanik 33, 8-9 (1953), 270-272.

[405] Hahn W.: Über geometrische Differenzengleichungen mit einer Einheitswurzel als Parameter. Bollettino U.M.I. 4 11 (1975), 95-99.

[406] Hahn W.: On Nonlinear Geometric difference equations. Jour. Math. Phy. Sci. (India) 11 no.1 (1977), 89-94.

[407] Hahn W.: Lineare geometrische Differenzengleichungen. Forschungszentrum Graz, Bericht 169 (1981).

[408] Haiman M.: t, q-Catalan numbers and the Hilbert scheme. Selected papers in honor of Adriano Garsia (Taormina, 1994). Discrete Math. 193 (1998), no. 1-3, 201–224.

[409] Haine L. & Horozov E.: A Lax pair for Kowalevski's top. Phys. D 29 (1987), no. 1-2, 173–180.

[410] Haine L. & Iliev P.: The bispectral property of a q-deformation of the Schur polynomials and the q-KDV hierarchy. J. Phys. Math. Gen 30 (1997), 7217-7227.

[411] Haine L.: Beyond the classical orthogonal polynomials. The bispectral problem (Montreal, PQ, 1997), 47–65, CRM Proc. Lecture Notes, 14, Amer. Math. Soc., Providence, RI, 1998.

[412] Hamermesh M.: Group theory and its application to physical problems. Dover 1962.

[413] Han G.N. & Randrianarivony, A. & Zeng, J.: Un autre q-analogue des nombres d'Euler. (French) [Another q-analogue of the Euler numbers] The Andrews Festschrift (Maratea, 1998). Sem. Lothar. Combin. 42 (1999), Art. B42e, 22 pp. (electronic).

[414] Han G.N. & Zeng, J.: q-polynômes de Gandhi et statistique de Denert. (French) [Gandhi q-polynomials and Denert statistics] Discrete Math. 205 (1999), no. 1-3, 119–143.

[415] Han G.N. & Zeng, J.: On a q-sequence that generalizes the median Genocchi numbers. Ann. Sci. Math. Quebec 23 (1999), no. 1, 63–72.

[416] Hardy G.H.: On the expression of the double zeta-function and double gamma function in terms of elliptic functions. Trans. Cambridge Phil. Soc. 20 (1905), 395-427.

[417] Hardy G.H.: On double Fourier series, and especially these which represent the double zeta-function with real and incommensurable parameters. Quart. J. Math. 37 (1906), 53-79.
[418] Hardy G.H.: Summation of a series of polynomials of Laguerre. J. London Math. Soc. 7 (1932), 138-139 ; Addendum. Ibid. 7, 192.
[419] Hardy G.H.: The Indian mathematician Ramanujan. Amer. Math. Monthly 44 (1937), 137-155.
[420] Hasimoto H.: A solition on a vortex filament. J. Fluid Mech. 51 (1972), 477-485.
[421] Haukkanen P.: Formal power series for binomial sums of sequences of numbers. Fibonacci Quart. 31 (1993), no. 1, 28–31.
[422] Heckman G.J. & Opdam E. M.: Root systems and hypergeometric functions. I. Compositio Math. 64 (1987), 329-352.
[423] Heckman G.J.: Root systems and hypergeometric functions. II. Compositio Math. 64 (1987), 353-373.
[424] Heckman G.J.: An elementary approach to the hypergeometric shift operators of Opdam. Invent. Math. 103 (1991), no. 2, 341–350.
[425] Heine E.: Über die Reihe... J. reine angew. Math. 32, (1846), 210-212.
[426] Heine E.: Untersuchungen über die Reihe... J. reine angew. Math. 34, (1847), 285-328.
[427] Heine E.: Handbuch der Theorie der Kugelfunktionen. Bd.1. Berlin 1878.
[428] Heine E.: Handbuch der Theorie der Kugelfunktionen. Bd.2. Berlin 1878, reprinted Würzberg, 1961.
[429] Heisenberg W.: Zur Theorie des Ferromagnetismus. Z.Physik. 49 (1928), 619-636.
[430] Hellström L & Silvestrov, S: Centralisers in the q-deformed Heisenberg algebras. Preprint, Umeå 1999.
[431] Helminck G. F. & van de Leur J. W.: An analytic description of the vector constrained KP hierarchy. Comm. Math. Phys. 193 (1998), no. 3, 627–641.
[432] Hendriks P.A.: An algorithm for computing a standard form for second-order linear q-difference equations. Journal of pure and applied algebra 117, 118 (1997) 331-352.
[433] Higman B.: Applied group-theoretic and matrix methods. Oxford 1955.
[434] Hille E.: On Laguerre's series. I, II and III. Proc. Nat. Acad. Sci. USA 12 (1926), 261-269, 348-352.
[435] Hitchin N.: Stable bundles and integrable systems. Duke Math. J. 54 (1987), no. 1, 91–114.
[436] Hlavaty, L.: Painlevé analysis of the Calogero-Degasperis-Fokas equation. Phys. Lett. A 113 (1985), no. 4, 177–178.
[437] Hofstadter D.R.: Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields. Physical review B 14, 6 (1976), 2239-2249.
[438] Horikawa: Contiguity relations for q-hypergeometric functions and related quantum group. Proc. Japan Acad., 68, Ser A (1992), 157-160.
[439] Horn J.: Hypergeometrische Funktionen zweier veränderlichen. Mathematische Annalen 105 (1931), 381-407.
[440] Horozov E.: & van Moerbeke P.: The full geometry of Kowalewski's top and $(1, 2)$-abelian surfaces. Comm. Pure Appl. Math. 42 (1989), no. 4, 357–407.

[441] Hou B.Y. & Ding X.M. & Wang Y.S. & Hou B.Yuan: Exchange relations of Lax pair for nonlinear sigma models. Modern Phys. Lett. A 9 (1994), no. 17, 1521–1528.
[442] Hou B.Y. & Hou B.Y. & Ma Z. Q.: q-deformed Chern class, Chern-Simons and cocycle hierarchy. J. Phys. A 28 (1995), no. 3, 543–558.
[443] Howard F.T.: Applications of a recurrence for the Bernoulli numbers. J. Number Theory 52 (1995), no. 1, 157–172.
[444] Hsu W.Y. & Falicov L.M.: Level quantization and broadening for band electrons in a magnetic field: Magneto-optics throughout the band. Physical review B 13, 4 (1976), 1595-1606.
[445] Hu Z.N.: Heisenberg xyz Hamiltonian with integrable impurities. Physics Letters A 250 (1998), 337-343.
[446] Iguchi K.: Tight.binding model for DNA double chains: metal-insulator transition due to the formation of a double strand of DNA. International journal of modern physics B, vol.11, No.20 (1997), 2405-2423.
[447] Ihrig E. C. & Ismail M. E. H.: A q-umbral calculus. J. Math. Anal. Appl. 84 (1981), no. 1, 178–207.
[448] Iliev P.: Solutions to Frenkel's deformation of the KP hierarchy. J. Phys. Math. Gen 31 (1998), L241-L244.
[449] Iliev P.: Tau function solutions to a q-deformation of the KP hierarchy. Lett. Math. Phys. 44 (1998), no. 3, 187–200.
[450] Iliev P.: q-KP hierarchy, bispectrality and Calogero-Moser systems. preprint 1999.
[451] Iliev P.: Commutative alebras of q-difference operators and Calogero-Moser hierarchy. C.R. Acad. Sci. Paris, t. 329, Séries I, 1999.
[452] Ilinski K. N. & Kalinin G. V. & Stepanenko A. S.: q-functional field theory for particles with exotic statistics. Phys. Lett. A 232 (1997), no. 6, 399–408.
[453] Infeld L. & Hull T.E.: The factorization method, Rev. Modern Phys. 23 (1951), 21-68.
[454] Ishikawa M. & Wakayama M.: New Schur function series. J. Algebra 208 (1998), no. 2, 480–525.
[455] Ismail M. E. H.: The zeros of basic Bessel functions, the functions $J_{\nu+ax}(x)$, and associated orthogonal polynomials. J. Math. Anal. Appl. 86 (1982), no. 1, 1–19.
[456] Jackson F.H.: A basic-sine and cosine with symbolical solution of certain differential equations. Proc. Edinburgh math. Soc. 22 (1904), 28-39.
[457] Jackson F.H.: A generalization of the functions $\Gamma(n)$ and x^n. Proc. Roy Soc.London. 74, (1904), 64-72.
[458] Jackson F.H.: On q-functions and a certain difference operator. Trans. Roy Soc.Edin. 46, (1908), 253-281.
[459] Jackson F.H.: A q-form of Taylor's theorem. Mess. Math. 38 (1909), 62-64.
[460] Jackson F.H.: On q-definite integrals. Quart. J. Pure and Appl. Math. 41 (1910), 193-203.
[461] Jackson F.H.: On q-difference equations. American J. Math. 32 (1910), 305-314.
[462] Jackson F.H.: Transformations of q-series. Mess. Math. 39, (1910),145-153.
[463] Jackson F.H.: A q-generalization of Abel's series. Rendiconti Palermo 29 (1910), 340-346.

[464] Jackson F.H.: Summation of q-hypergeometric series. Mess. Math. 50, (1921),101-112.

[465] Jackson F.H.: Quart. J. Math., Oxford Ser. 13 (1942), 69–82;

[466] Jackson F.H.: Basic double hypergeometric functions Quart. J. Math., Oxford Ser. 15 (1944), 49–61;

[467] Jacobi C.G.J: Fundamenta Nova 1829.

[468] Jacobi C.G.J: De functionibus alternantibus earumque divisione per productum e differentiis elemantorum conflatum. Journal für die reine und angewandte Mathematik, 22, 1841, 360-371.

[469] Jacobson N.: Pseudo-linear transformations. Annals of Math. 38, 2 (1937), 484-507.

[470] James G.D.: The representation theory of the symmetric groups. Springer 1978.

[471] Jannussis, A.; Brodimas, G.; Sourlas, D.; Streclas, A.; Siafaricas, P.; Papaloucas, L.; Tsangas, N.: Foundations of the Lie-admissible Fock space of the hadronic mechanics. Proceedings of the First International Conference on Nonpotential Interactions and their Lie-Admissible Treatment, Part D (Orléans, 1982). Hadronic J. 5 (1981/82), no. 5, 1923–1947.

[472] Jaulent M. & Miodek I.: Nonlinear evolution equations associated with 'energy-dependent Schrödinger potentials'. Letters in Mathematical Physics 1 (1976), 243-250.

[473] Jensen J.L.W.: Sur une identité d'Abel et sur d'autre formules analogues. Acta mathematica. 26(1902), 307-318.

[474] Jiber A.V. & Shabat A.B.: Dokl. Akad. Nauk SSSR 247 (1979), 1103.

[475] Jimbo M.: A q-difference analogue of U and the Yang-Baxter equation. Lett. Math. Phys. 10 (1985), no. 1, 63–69.

[476] Jimbo M.: A q-analogue of $U(N+1)$, Hecke algebra, and the Yang-Baxter equation. Lett. Math. Phys. 11 (1986), no. 3, 247-252.

[477] Jimbo M.: Quantum R matrix for the generalized Toda system. Comm. Math. Phys. 102 (1986), no. 4, 537–547.

[478] Jimbo M. & Miwa T.: Quantum KZ equation with $|q| = 1$ and correlation functions of the XXZ model in the gapless regime. J. Phys. A 29 (1996), no. 12, 2923–2958.

[479] Jing N. H. & Jozefiak, T.: A formula for two-row Macdonald functions. Duke Math. J. 67 (1992), no. 2, 377–385. :

[480] Jing N. H.: q-hypergeometric series and Macdonald functions. J. Algebraic Combin. 3 (1994), no. 3, 291–305.

[481] Jing, Si-Cong & Fan, Hong-Yi: q-Taylor's formula with its q-remainder. Comm. Theoret. Phys. 23 (1995), no. 1, 117–120.

[482] Johnson W.P.: q-extensions of identities of Abel-Rothe type. Discrete Math. 159 (1996), no. 1-3, 161–177.

[483] Johnson W.P.: A q-analogue of Faà di Bruno's formula. J. Combin. Theory Ser. A 76 (1996), no. 2, 305–314.

[484] Johnson W.P.: Some applications of the q-exponential formula. Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics (New Brunswick, NJ, 1994). Discrete Math. 157 (1996), no. 1-3, 207–225.

[485] Jones G. & Singerman D.: Complex functions. Cambridge university press 1987.

[486] Jurvco B & Schupp P.: Adler-Kostant-Symes scheme for face and Calogero-Moser-Sutherland-type models. J. Math. Phys. 39 (1998), no. 7, 3577–3588.

[487] Kac V. & Wakimoto M.: A construction of generalized spin models. Perspectives in mathematical physics, 125–150, Conf. Proc. Lecture Notes Math. Phys., III, Internat. Press, Cambridge, MA, 1994.

[488] Kadomtsev B.B. & Petviashvili V.I.: On the stability of solitary waves in weakly dispersing media. Sov. Phys. Doklady 15 (1970), 539-541.

[489] Kaijser S.: private communication.

[490] Kajiwara K. & Ohta Y.: Determinant structure of the rational solutions for the Painlevé IV equation. J. Phys. A 31 (1998), no. 10, 2431–2446.

[491] Kalita N. C.: Quadratic and cubic transformations of basic hypergeometric series. Indian J. Math. 36 (1994), no. 3, 279–297.

[492] Kalnins E.G. & Miller W Jr.: Lie theory and the wave equation in space-time. 4. The Klein-Gordon equation and the Poincaré group. J. Math. Phys. 19, 6 (1978), 1233-1246.

[493] Kalnins E.G. & Miller W Jr.: Lie theory and the wave equation in space-time. R-separable solutions of the wave equation. J. Math. Phys. 19, 6 (1978), 1247-1257.

[494] Kalnins E.G. & Manocha H.L. & Miller W Jr.: The Lie theory of two-variable hypergeometric functions. Studies in applied mathematics 62:143-173 (1980).

[495] Kalnins E.G. & Manocha H.L. & Miller W Jr.: Models of q-algebra representations: tensor products of special unitary and oscillator algebras. J. Math. Phys. 33 (1992), no. 7, 2365–2383.

[496] Kalnins E.G. & Miller W Jr. & Mukherjee S.: Models of q-algebra representations: Matrix elements of the q-oscillator algebra. J. Math. Phys. 34, 11 (1993), 5333-5356.

[497] Kalnins E.G. & Miller W Jr.: Models of q-algebra representations: q-integral transforms and "addition theorems". J. Math. Phys. 35 (1994), no. 4, 1951–1975.

[498] Kalnins E.G. & Miller W Jr. & Mukherjee S.: Models of q-algebra representations: the group of plane motions. SIAM J. Math. Anal. 25 (1994), no. 2, 513–527.

[499] Kalnins E.G. & Miller W Jr.: A note on tensor products of q-algebra representations and orthogonal polynomials. Journal of computational and applied mathematics 68 (1996), 197-207.

[500] Kalnins E.G. & Miller W Jr.: q-Algebra representations of the Euclidean, pseudo-Euclidean and oscillator algebras, and their tensor products. Symmetries and integrability of difference equations (Estérel, PQ, 1994), 173–183, CRM Proc. Lecture Notes, 9, Amer. Math. Soc., Providence, RI, 1996. 3

[501] Kamata M. & Nakamula A.: One-parameter family of selfdual solutions in classical Yang-Mills theory. preprint 1999.

[502] Kamvissis S.: Focusing nonlinear Schrödinger equation with infinitely many solitons. J.Math.Phys. 36 (8) (1995), 4175-4180.

[503] Kaneko J.: A triple product identity for Macdonald polynomials. J. Math. Anal. Appl. 200 (1996), no. 2, 355–367.

[504] Kaniadakis G. & Lavagno A.& Quarati P.: Kinetic model for q-deformed bosons and fermions. Phys. Lett. A 227 (1997), no. 3-4, 227–231.

[505] Karlsson P.W. & Srivastava H. M.: Transformations of multiple q-series with quasi-arbitrary terms. J. Math. Anal. Appl. 231 (1999), no. 1, 241–254.

[506] Karlsson P.: private communication at the workshop on special functions, Lund May 1999.

[507] Kassel C.: Quantum groups. Springer 1995.

[508] Katzarkov L. & Pantev T.: Stable G_2 bundles and algebraically completely integrable systems. Compositio Math. 92 (1994), no. 1, 43–60.

[509] Kawanaka N.: A q-series identity involving Schur functions and related topics. Osaka J.Math. 36 (1999), 157-176.

[510] Kel'bert M. Ya. & Suhov Yu. M.: The Markov branching random walk and systems of reaction-diffusion (Kolmogorov-Petrovskiĭ-Piskunov) equations. Comm. Math. Phys. 167 (1995), no. 3, 607–634.

[511] Kemp C. & David A.: q-logarithmic distribution. Advances in combinatorial methods and applications to probability and statistics, 465–470,Stat. Ind. Technol., Birkhuser Boston, Boston, MA, 1997.

[512] Kerov S.V.: Characters of Hecke and Birman-Wenzl algebras. Quantum groups (Leningrad, 1990), 142–147, Lecture Notes in Math., 1510.

[513] Khan M.A.: q-analogues of certain operational formulae. Houston J. Math. 13 (1987), no. 1, 75–82.

[514] Khesin B. & Lyubashenko V. & Roger C.: Extensions and contractions of the Lie algebra of q-pseudodifferential symbols on the circle. J. Funct. Anal. 143 (1997), no. 1, 55–97.

[515] Kibler M. & Négadi, T.: On the q-analogue of the hydrogen atom. J. Phys. A 24 (1991), no. 22, 5283–5289.

[516] King R.C.: Modification rules and products of irreducible representations of the unitary, Orthogonal, and symplectic groups. J. Math. Phys. 12,8 (1971), 1588-1598.

[517] King R.C. & El-Sharkaway N. G. I.: Standard Young tableaux and weight multiplicities of the classical Lie groups. J. Phys. A 16 (1983), no. 14, 3153–3177.

[518] King R.C. & El-Sharkaway N. G. I.: Standard Young tableaux and character generators of classical Lie groups. J. Phys. A 17 (1984), no. 1, 19–45.

[519] King R.C. & Wybourne, B. G. Characters of Hecke algebras $H_n(q)$ of type A_{n-1}. J. Phys. A 23 (1990), no. 23, L1193–L1197.

[520] Kirillov A.N.: Dilogarithm identities. Progress of theoretical physics supplement 118 (1995), 61-142.

[521] Kirillov A.N.: Lectures on affine Hecke algebras and Macdonald's conjectures. Bull. Amer. Math. Soc. (N.S.) 34 (1997), no. 3, 251–292.

[522] Kirillov N. & Noumi M.: Affine Hecke algebras and raising operators for Macdonald polynomials. Duke math. Journal 93,1 (1998), 1-38.

[523] Klein A. & Marshalek E. R.: Boson realizations of Lie algebras with applications to nuclear physics. Rev. Modern Phys. 63 (1991), no. 2, 375–558.

[524] Klein C. & Richter O.: Physically realistic solutions to the Ernst equation on hyperelliptic Riemann surfaces. Physical review D. 58, 124018 (1998).

[525] Klimek M.: Integrals of motion for some equations on q-Minkowski space. Czechoslovak Journal of physics 47,12 (1997),1199-1206.

[526] Knizhnik V. G. & Zamolodchikov, A. B.: Current algebra and Wess-Zumino model in two dimensions. Nuclear Phys. B 247 (1984), no. 1, 83–103.

[527] Koblitz N.: q-extension of the p-adic gamma function. Trans. Amer. Math. Soc. 260 (1980), no. 2, 449–457.

[528] Koblitz N.: q-extension of the p-adic gamma function. II. Trans. Amer. Math. Soc. 273 (1982), no. 1, 111–129.

[529] Koblitz N.: p-adic analog of Heine's hypergeometric q-series. Pacific J. Math. 102 (1982), no. 2, 373–383.

[530] Koblitz N.: On Carlitz's q-Bernoulli numbers. Journal of number theory 14 (1982), 332-339.

[531] Koelink E.: Eight lectures on quantum groups and q-special functions. Rev. Colombiana Mat. 30 (1996), no. 2, 93–180.

[532] Konno H.: Dynamical correlation functions and finite-size scaling in the Ruijsenaars-Schneider model. Nuclear Phys. B 473 (1996), no. 3, 579–600.

[533] Koornwinder T. & Swarttouw, R. F.: On q-analogues of the Fourier and Hankel transforms. Trans. Amer. Math. Soc. 333 (1992), no. 1, 445–461.

[534] Koornwinder T.: Compact quantum groups and q-special functions. European school of group theory, Pitman Research Notes 311, 1994.

[535] Koornwinder T.: Special functions and q-commuting variables. Instut Mittag-Leffler, Report no.1, 1995/96.

[536] Korteweg D.J. & de Vries G.: On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. Phil. Mag. 39 (1895), 422-443.

[537] Kowalski K. & Steeb W.H.: Symmetries and first integrals for nonlinear dynamical systems: Hilbert space approach. I. Ordinary differential equations. Progr. Theoret. Phys. 85 (1991), no. 4, 713–722.

[538] Kowalski K. & Steeb W.H.: Symmetries and first integrals for nonlinear dynamical systems, Hilbert space approach. II. Partial differential equations. Progr. Theoret. Phys. 85 (1991), no. 5, 975–983.

[539] Kowalski-Glikman J.: Black hole solution of quantum gravity. Physics Letters A 250 (1998), 62-66.

[540] Krall A. M.: Chebyshev sets of polynomials which satisfy an ordinary differential equation. SIAM Rev. 22 (1980), no. 4, 436–441.

[541] Krall A. M.: Orthogonal polynomials satisfying fourth order differential equations. Proc. Roy. Soc. Edinburgh Sect. A 87 (1980/81), no. 3-4, 271–288.

[542] Krall H. L.: Certain differential equations for Tchebycheff polynomials. Duke math. Journal 4 (1938), 705-718.

[543] Krall H. L.: On orthogonal polynomials satisfying a certain fourth order differential equation. Pennsylvania State College Studies, 1940 (1940). no. 6, 24 pp.

[544] Krattenthaler C.: HYP and HYPQ. Mathematica packages for the manipulation of binomial sums and hypergeometric series respectively q-binomial sums and basic hypergeometric series. Séminaire Lotharingien de Combinatoire (Gerolfingen, 1993), 61–76, Prépubl. Inst. Rech. Math. Av., 1993/34.

[545] Krattenthaler C. & Srivastava H. M.: Summations for basic hypergeometric series involving a q-analogue of the digamma function. Comput.Math. Appl. 32 (1996), no. 3, 73–91.

[546] Krattenthaler C.: Advanced Determinant calculus preprint.

[547] Kreft Ch. & Seiler R.: Models of the Hofstadter-type. J. Math. Phys. 37 (1996), no. 10, 5207–5243.

[548] Krichever I.M.: Methods of algebraic geometry in the theory of non-linear equations. Russian Math. Surveys 32:6 (1977), 185-213.

[549] Krichever I.M.: Integration of nonlinear equations by the method of algebraic geometry. Functional anal. appl. 11 (1977), 12-26.

[550] Kruskal M.D.: The birth of the solitions, in 'Nonlinear evolution equations solvable by the spectral transform', ed. Calogero F., Pitman, 1978.

[551] Kulish P.P. & Reshetikin N.Y.: Diagonalisation of $GL(n)$ invariant transfer matrices and quantum n-wave system (Lee model). J. Phys. A 16 (1983), no. 10, L591–L596.

[552] Kulish P.P. & Damaskinsky E.V.: On the q-oscillator and the quantum algebra $su_q(1,1)$. J. Phys. A. 23, (1990), L415-L419.

[553] Kumar A. & Maharana J.: Tetracritical Ising model coupled to the two-dimensional gravity. Internat. J. Modern Phys. A 7 (1992), no. 19, 4487–4499.

[554] Kummer E.E.: Über die hypergeometrische Reihe ... J. für Math. 15, (1836) 39-83 and 127-172.

[555] Kundu A. & Strampp W.: Derivative and higher-order extensions of Davey-Stewartson equation from matrix Kadomtsev-Petviashvili hierarchy. J. Math. Phys. 36 (1995), no. 8, 4192–4202.

[556] Kung J.P.S. & Rota G.: Bulletin of the American mathematical society new series 10 (1984), 27-85.

[557] Kuniba A. & Sakai K. & Suzuki J.: Continued fraction TBA and functional relations in XXZ model at root of unity. Nuclear Phys. B 525 (1998), no. 3, 597–626.

[558] Kuperberg G.: Spiders for rank 2 Lie algebras. q-alg/9712003.

[559] Kupershmidt B. A. & Mathieu P.: Quantum Korteweg-de Vries like equations and perturbed conformal field theories. Phys. Lett. B 227 (1989), no. 2, 245–250.

[560] Kurokawa N.: Gamma factors and Plancherel measures. Proc. Japan Acad. 68 (1992), 256-260.

[561] Kuryshkin V.: Annales Fondation Louis de Broglie 5 (1980), 111.

[562] Kuznetsov V.B. & Sklyanin E.K.: Factorisation of Macdonald polynomials. Symmetries and integrability of difference equations (Canterbury, 1996), 370–384, London Math. Soc. Lecture Note Ser., 255, Cambridge Univ. Press, Cambridge, 1999.

[563] Lapointe L. & Vinet L.: A Rodrigues formula for the Jack polynomials and the Macdonald-Stanley conjecture. Internat. Math. Res. Notices 1995, no. 9, 419–424.

[564] Lapointe L. & Vinet L.: Exact operator solution of the Calogero-Sutherland model. Commun. Math. Phys. 178, (1996), 425-452.

[565] Lapointe L. & Vinet L.: Creation operators for the Macdonald and Jack Polynomials. Letters in mathematical physics 40 (1997), 269-286.

[566] Larcombe, Peter On the history of the Catalan numbers: a first record in China. Math. Today (Southend-on-Sea) 35 (1999), no. 3, 89.

[567] Lascoux A. & Pragacz P.: Ribbon Schur functions. European Journal of combinatorics, 9 (1988), 561-574.

[568] Lascoux A. & Leclerc B. & Thibon J.Y.: Hecke algebras at roots of unity and crystal bases of quantum affine algebras. Commun. Math. Phys. 181, (1996), 205-263.

[569] Lau, Ka Sing: On the nonlinear diffusion equation of Kolmogorov, Petrovsky, and Piscounov. J. Differential Equations 59 (1985), no. 1, 44–70.
[570] Lax P.: Integrals of nonlinear equation of evolution and solitary waves. Comm. Pure Appl. Math. 21, 5 (1968), 467-90.
[571] Le Thang: private communication.
[572] Le Caine J.: The linear q-difference equation of the second order. American journal of mathematics 65, 585-600 (1943).
[573] Ledermann W.: Introduction to group characters, Cambridge University Press, 1987.
[574] Lepowsky J. & Wilson R.L.: Construction of the affine Lie Algebra $A_1^{(1)}$. Comm. Math.Phys. 62 (1978), 43-53.
[575] Lesky P.: Zweigliederige Rekursionen für die Koeffizienten von Polynomlösungen Sturm-Liouvillescher q-Differenzengleichungen. Zeitung angewandte Mathematik und Mechanik 74, 10, 497-500 (1994).
[576] Li D.: Hierarchical wave functions and fractional statistics in fractional quantum Hall effect on the torus. International journal of modern physics B, vol.7, No.15 (1993), 2779-2794.
[577] Li Y. & Mulase M.: Motohico Hitchin systems and KP equations. Internat. J. Math. 7 (1996), no. 2, 227–244.
[578] Liouville J.: Sur l'équation aux différences partiell. J. Math Pure Appl. 18 (1853) 71-72.
[579] Littlewood D.E., The theory of group characters and matrix representations of groups, Clarendon Press,Oxford, 1950.
[580] Liu Q. P.: New integrable systems related to a Kupershmidt's equation by Miura maps. J. Math. Phys. 35 (1994), no. 2, 816–820.
[581] Lorek A. & Weich W. & Wess J.: Non-commutative Euclidean and Minkowski structures. Zeitschrift für Physik, C 76 (1997), 375-386. Internat. J. Math. 7 (1996), no. 2, 227–244.
[582] Loris I. & Willox R.: Symmetry reductions of the BKP hierarchy. J. Math. Phys. 40 (1999), no. 3, 1420–1431.
[583] Lösch F. & Schoblik F.: Die Fakultät (Gammafunktion) und verwandte Funktionen mit besonderer Berücksichtigung ihrer Anwendungen. (German) B. G. Teubner Verlagsgesellschaft, Leipzig, 1951.
[584] Louw J. A. & Schwarz F. & Steeb W.-H.: First integrals and Yoshida analysis of Nahm's equations. J. Phys. A 19 (1986), no. 10, L569–L573.
[585] Lüscher M.: Quantum non-local charges and absence of particle production in the two-dimensional non-linear σ model. Nuclear physics B135 (1978), 1-19.
[586] Lusztig G.: Affine Hecke algebras and their graded version. J. Amer. Math. Soc. 2 (1989), no. 3, 599–635.
[587] Lusztig G.: Quantum groups at roots of 1. Geometriae Dedicata 35 (1990),89-114.
[588] Ma W.X. & Fuchssteiner B.: Explicit and exact solutions to a Kolmogorov-Petrovskii-Piskunov equation. Internat. J. Non-Linear Mech. 31 (1996), no. 3, 329–338
[589] Macdonald I.G.: Affine root systems and Dedekind's η-function. Invent. Math. 15 (1972), 91–143.
[590] Macdonald I.G.: The Poincaré series of a Coxeter group. Math. Ann. 199 (1972), 161–174.

[591] Macdonald I.G.: Some conjectures for root systems. SIAM J. Math. Anal. 13 (1982), no. 6, 988–1007.
[592] Macdonald I.G.: A new class of symmetric functions. Publ. I.R.M.A. Strasbourg 1988, 131-171.
[593] Macdonald I.G.: Orthogonal polynomials associated with root systems. Orthogonal polynomials (Columbus, OH, 1989), 311–318.
[594] Macdonald I.G.: Symmetric functions and Hall polynomials. Oxford, 1995.
[595] Macdonald I.G.: Affine Hecke algebras and orthogonal polynomials. Seminaire Bourbaki 797 (1994-1995), 189-206.
[596] Macfarlane A.J.: On q-analogues of the quantum harmonic oscillator and the quantum group $SU(2)_q$. J.Phys. A 22 (1989), 4581-4588.
[597] Magyar P.: Borel-Weil theorem for configuration varieties and Schur modules. Adv. Math. 134 (1998), no. 2, 328–366.
[598] Majid S.: Free braided differential calculus, braided binomial theorem, and the braided exponential map. J. Math. Phys. 34, 10 (1993), 4843-4856.
[599] Maliakas M.: On Odd Symplectic Schur Functions. Journal of Algebra, v 211, n 2, January 15, 1999, p 640-646.
[600] Manin Y. I. & Radul A.O.: A supersymmetric Extension of the Kadomtsev-Petviashvili Hierarchy. Communications in mathematical physics 98 (1985), 65-77.
[601] Manin Y. I.: Quantum groups and noncommutative geometry. Centre de Recherche Mathematique Universite de Montreal 1988.
[602] Manin Y. I.: Topics in noncommutative geometry. Princeton 1991.
[603] Marinova L. P. & Raychev P. P. & Maruani J.: Molecular backbending in AgH and its description in terms of q-algebras. Molecular Phys. 82 (1994), no. 6, 1115–1129.
[604] Marotte F & Zhang C.: Sur la sommabilité des séries entières solutions formelles d'une équation aux q-différences. II. (French) [On the summability of formal power series solutions of q-difference equations. II] C. R. Acad. Sci. Paris Sr. I Math. 327 (1998), no. 8, 715–718.
[605] Martinez-Moras F. & Ramos E.: Higher-dimensional classical W-algebras. Comm. Math. Phys. 157 (1993), no. 3, 573–589.
[606] Masaaki I.: Symmetries and conservation laws of the classical Boussinesq equation. Physics. Letters 104A, 5 (1984), 248-250.
[607] Maslov: Operational methods. Mir publishers 1976.
[608] Mason T. E.: On properties of the solutions of linear q-difference equations with entire function coefficients. American journal of mathematics 37 (1915), 439-444.
[609] Masuda T. & Sasaki T. & Suzuki H.: Seiberg-Witten theory of rank two gauge groups and hypergeometric series. Internat. J. Modern Phys. A 13 (1998), no. 18, 3121–3144.
[610] Mathieu E.: Mémoire sur l'étude des functions de plusieures quantités, sur la manière de les formes et sur les substitutions qui les laissent invariables. Crelle J. 6 (1861), 241-323.
[611] Matone M.: Liouville equation and Schottky problem. Letters in mathematical physics 33 (1995), 75-85.
[612] Matschull H.J. & Welling M.: Quantum mechanics of a point particle in (2+1)-dimensional gravity. Class. quantum gravity 15 (1998) 2981-3030.

[613] Matsuo A.: Jackson integrals of Jordan-Pochhammer type and quantum Knizhnik-Zamolodchikov equations. Comm. Math. Phys. 151 (1993), no. 2, 263–273.

[614] McIntosh R. J.: Asymptotic transformations of q-series. Canad. J. Math. 50 (1998), no. 2, 412–425.

[615] McKay J. & Strauss H.: The q-series of monstrous moonshine and the decomposition of the head characters. Comm. Algebra 18 (1990), no. 1, 253–278.

[616] Mehler F.G.: Über die Entwicklung einer Function von beliebig vielen Variabeln nach Laplaceschen Functionen höherer Ordnung. Journal für Mathematik 66 (1866), 161-176.

[617] Mendèz M.A.: Umbral shifts and symmetric functions of Schur type. Mathematical essays in honor of Gian-Carlo Rota 285-304, Birkhäuser 1998.

[618] Meschkowski H.: Differenzengleichungen. Göttingen 1959.

[619] Meurman A.: private communication.

[620] Meyer, U.: Wave equations on q-Minkowski space. Commun. Math. Phys. 174 (1996), 457-475.

[621] Mikhailov A.V.: Physica 1D (1981), 73.

[622] Miller W.Jr.: On Lie algebras and some special functions of mathematical physics, vol. 50, Mem. Amer. Math. Soc., Providence, RI, 1964.

[623] Miller W.Jr.: Lie theory and special functions, Academic Press 1968.

[624] Miller W.Jr.: Special functions and the complex Euclidean group in 3-space. I,II,III. J. Mathematical Phys. 9 (1968), 1163-1175; ibid. 9 (1968), 1175-1187; ibid. 9 1968 1434–1444.

[625] Miller W.Jr.: Lie theory and difference equations. J.Math. Anal. Appl. 28 (1969), 383-399.

[626] Miller W.Jr.: Lie theory and q-difference equations. SIAM J.Math. Anal. 1 (1970), 171-188.

[627] Miller W.Jr.: Lie theory and difference equations. II. J. Math. Anal. Appl. 39 (1972), 406–422.

[628] Miller W.Jr.: Lie theory and generalized hypergeometric functions. SIAM J.Math.Anal. 3 (1972), 31-44.

[629] Miller W.Jr.: Symmetry and separation of variables. Encyclopedia of mathematics and its applications, Addison-Wesley 1977.

[630] Miller W.Jr.: Symmetry techniques and orthogonality for q-series. q-series and partitions (Minneapolis, MN, 1988), 191–212, IMA Vol. Math. Appl., 18, Springer, New York-Berlin, 1989.

[631] Milne S. C.: q-analog of restricted growth functions, Dobinski's equality, and Charlier polynomials. Trans. Amer. Math. Soc. 245 (1978), 89–118.

[632] Milne S. C.: Restricted growth functions, rank row matchings of partition lattices, and q-Stirling numbers. Adv. in Math. 43 (1982), no. 2, 173–196.

[633] Milne S. C.: Summation theorems for basic hypergeometric series of Schur function argument. Progress in approximation theory (Tampa, FL, 1990), 51–77, Springer Ser. Comput. Math., 19, Springer, New York, 1992.

[634] Mimachi K.: Connection problem in holonomic q-difference system associated with a Jackson integral of Jordan-Pochhammer type. Nagoya Math.J. 116 (1989), 149-161.

[635] Mimachi K.: Holonomic q-difference system associated with the basic hypergeometric series ${}_{n+1}\phi_n$. Tohoku Math. J. (2) 45 (1993), no. 4, 485–490

[636] Mimachi K.: A solution to quantum Knizhnik Zamolodchikov equations and its application to eigenvalue problems of the Macdonald type. Duke mathematical journal 85, 3 (1996), 635-658.

[637] Mimachi K. & Noumi M.: An integral representation of eigenfunctions for Macdonald's q-difference operators. Tohoku Math. Journal 49 (1997), 517-525.

[638] Mimachi K. & Noumi M.: A reproducing kernel for nonsymmetric Macdonald polynomials. Duke Math. Journal 91,3 (1998), 621-634.

[639] Mimachi K.: A solution of the quantum Knizhnik Zamolodchikov equation of type C_n. Comm. Math. Phys. 197 (1998), no. 1, 229–246.

[640] Mimachi K.: A new derivation of the inner product formula for the Macdonald symmetric polynomials. Compositio Math. 113 (1998), no. 2, 117–122.

[641] Mimachi K.: Eigenfunctions of Macdonald's q-difference operator for the root system of type C_n. J. Funct. Anal. 163 (1999), no. 2, 272–278.

[642] Mishra R.: On certain generalized q-Laplace and Stieltjes transforms. J. Math. Sci. 5 (1970), no. 2, 22–37 (1979).

[643] Mishra R.: A few inversion formulae for generalized q-Laplace transforms. Indian J. Math. 14 (1972), 157–172.

[644] Mishra R.: Application of generalized q-Laplace transforms to the solution of certain q-integral equations. Indian J. Math. 23 (1981), no. 1-3, 115–120.

[645] Mishra R.: On a relation between a q-Laplace and two generalized q-Laplace transforms. Proc. Nat. Acad. Sci. India Sect. A 53 (1983), no. 4, 379–385.

[646] Mishra R.: A note on a relation between certain q-integral transform. Proceedings of National Symposium on Special Functions and their Applications (Gorakhpur, 1986), 146–152,

[647] Moak D.S.: The q-analogue of the Laguerre polynomials. J. Math. Anal. Appl. 81 (1981), no. 1, 20–47.

[648] Moak D.S.: The q-analogue of Stirling's formula. Rocky Mountain J. Math. 14 (1984), no. 2, 403–413.

[649] Monteiro M.R. & Rodrigues L.M.C.S.: Quantum algebraic nature of the phonon spectrum in ^{4}He. Physical review letters 76, 7 (1996), 1098-1100.

[650] Morosi C. & Pizzocchero L.: R-matrix theory, formal Casimirs and the periodic Toda lattice. J. Math. Phys. 37 (1996), no. 9, 4484–4513.

[651] Mukhopadhaya I. & Chowdhury A. R.: q-deformed dressing operators and modified integrable hierarchies. J. Nonlinear Math. Phys. 1 (1994),no. 4, 414–419.

[652] Mulase M.: Algebraic geometry of soliton equations. Proc. Japan Acad. Ser. A Math. Sci. 59 (1983), no. 6, 285–288.

[653] Mulase M.: Complete integrability of the Kadomtsevmhy Petviashvili equation. Adv. in Math. 54 (1984), no. 1, 57–66.

[654] Mulase M.: Cohomological structure in soliton equations and Jacobian varieties. J. Differential Geom. 19 (1984), no. 2, 403–430.

[655] Mulase M.: KP equations, strings, and the Schottky problem. Algebraic analysis, Vol. II, 473–492, Academic Press, Boston, MA, 1988.

[656] Mulase M.: A new super KP system and a characterization of the Jacobians of arbitrary algebraic super curves. J. Differential Geom. 34 (1991), no. 3, 651–680.

[657] Mulase M.: Algebraic theory of the KP equations. Perspectives in mathematical physics, International press 1994, 151-217.

[658] Müller U. & Weigt G.: Analytical solution of the SL$(2, \mathbf{R})$/U(1) WZNW black hole model. Phys. Lett. B 400 (1997), no. 1-2, 21–26.

[659] Mumford D.: Tata Lectures on Theta II, Birkhäuser 1984.

[660] Murakami H. & Ohtsuki T. & Yamada S.: Homfly Polynomial via an invariant of colored plane graphs 1998.

[661] Murnaghan F.D.: The theory of group representations. Baltimore, 1938.

[662] Musette M.: Nonlinear partial differential equations. An introduction to methods of complex analysis and geometry for classical mechanics and non-linear waves (Chamonix, 1993), 145–195,

[663] Naimark M. A.: Representation theory of groups, (Russian), Nauka, Moscow, 1976; English translation: Naimark M.A., Stern A.I.: Theory of group representations. Springer, 1982.

[664] Nakatani M.: A quantum analogue of the hypergeometric system $E_{3,6}$. Kyushu Journal of mathematics 49,1 (1995), 67-91.

[665] Nakayashiki A.: Commuting partial differential operators and vector bundles over abelian varieties. Amer. J. Math. 116 (1994), no. 1, 65–100.

[666] Nakayashiki A.: On the Thomae formula for Z_N curves. Publ. Res. Inst. Math. Sci. 33 (1997), no. 6, 987–1015.

[667] Narayana T.V. & Mathsen R.M. & Williams J.G.: Combinatorics, Representation theory and statistical methods in groups. Marcel Dekker 1980.

[668] Natanzon S. & Turaev V.: Systems of correlators and solutions of the WDVV equations. Comm. Math. Phys. 196 (1998), no. 2, 399–410.

[669] Needham J.: Science and Civilization in China, Vol 3. Mathematics and the Sciences of the Heavens and the earth, Cambridge 1959.

[670] Négadi, T. & Kibler M.: A q-deformed Aufbau Prinzip. J. Phys. A 25 (1992), no. 4, L157–L160.

[671] Netto: Combinatorik 1927.

[672] Niedermaier M. R.: GLM equations, tau function and scattering data. Comm. Math. Phys. 160 (1994), no. 2, 391–429.

[673] Nielsen N.: Traité élémentaire des nombres de Bernoulli. Gauthier Villars, Paris 1923.

[674] Nijhoff F.H. & Papageorgiou V.G. & Capel H.W.: Integrable time-discrete systems: Lattices and Mappings. Quantum groups (Leningrad, 1990), 142–147, Lecture Notes in Math.,312-325.

[675] Nijhoff F.H. & Papageorgiou V.G. & Capel H.W. & Quispel G. R. W.: The lattice Gelfand-Dickey hierarchy. Inverse Problems 8 (1992), no. 4, 597–621.

[676] Nijhoff F.W. & Pang G. D.: Discrete-time Calogero-Moser model and lattice KP equations. Symmetries and integrability of difference equations (Estrel, PQ, 1994), 253–264, CRM Proc. Lecture Notes, 9, Amer. Math. Soc., Providence, RI, 1996

[677] Nikiforov A F. & Uvarov V. B.: Polynomial solutions of hypergeometric type difference equations and their classification. Integral Transform, Spec. Funct. 1 (1993), no. 3, 223–249.

[678] Nishino A. & Ujino H. & Wadati M.: Rodrigues formula for the nonsymmetric Macdonald polynomial. J. Phys. Soc. Japan 68 (1999), no. 3, 701–704.

[679] Nishino A. & Ujino H. & Wadati M.: Rodrigues formula for the nonsymmetric multivariable Laguerre polynomial. J. Phys. Soc. Japan 68 (1999), no. 3, 797–802.

[680] Nishizawa M.: On a q-analogue of the multiple gamma functions. Lett. Math. Phys. 37 (1996), no. 2, 201-209.

[681] Nishizawa M. & Ueno K.: Integral solutions of q-difference equations of the hypergeometric type with $|q| = 1$, 1997.

[682] Nishizawa M.: On a solution of a q-difference analogue of Lauricella's D-type hypergeometric equation with $|q| = 1$. Publ. Res. Inst. Math. Sci. 34 (1998), no. 3, 277–290.

[683] Niven I. & Zuckerman. H & Montgomery H.: An introduction to the theory of numbers. Wiley 1991.

[684] Nörlund N.E.: Differenzenrechnung. Berlin 1924.

[685] Nuomi M.: Quantum Grassmannian and q-hypergeometric series. Quarterly CWI 5 (1992), 293-307.

[686] Odzijewicz A.: Quantum algebras and q-special functions related to coherent states maps of the disc. Comm. Math. Phys. 192 (1998), no. 1, 183–215.

[687] Ogievetsky O. & Schmidke W.B. & Wess J. & Zumino B.:: q-deformed Poincaré algebra. Commun. Math. Phys. 150 (1992), 495-518.

[688] Ohta Y. & Satsuma J. & Takahashi D. & Tokihiro T.: An elementary introduction to Sato theory. Recent developments in soliton theory. Progr. Theoret. Phys. Suppl. No. 94 , (1988), 210–241.

[689] Okounkov A.: Binomial formula for Macdonald polynomials and applications. Math. Res. Lett. 4 (1997), no. 4, 533–553.

[690] Olde D. A. B.: Asymptotic expansions for q-gamma, q-exponential, and q-Bessel functions. J. Math. Anal. Appl. 186 (1994), no. 3, 896–913.

[691] Oldham K.B. & Spanier J.: The fractional calculus. Theory and applications of differentiation and integration to arbitrary order. With an annotated chronological bibliography by Bertram Ross. Mathematics in Science and Engineering, Vol. 111. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1974.

[692] Olshanetski M. A. & Rogov V.-B. K.: Liouville quantum mechanics on a lattice from geometry of quantum Lorentz group. J. Phys. A 27 (1994), no. 13, 4669–4683.

[693] Olshanetski M. A. & Rogov V.-B. K.: Modified Bessel q-functions and Macdonald q-functions. Sb. Math. 187 (1996), no. 10, 1525–1544.

[694] Olshanetski M. A. & Rogov V.-B. K.: q-integral representations of modified Bessel q-functions and Macdonald q-functions. Sb. Math. 188 (1997), no. 8, 1235–1258.

[695] Opdam E.M.: Root systems and hypergeometric functions. III. Compositio Math. 67 (1988), no. 1, 21–49.

[696] Opdam E.M.: Root systems and hypergeometric functions. IV. Compositio Math. 67 (1988), no. 2, 191–209.

[697] Opdam E.M.: Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group. Compositio Math. 85 (1993), no. 3, 333–373.

[698] Opdam E. M.: An analogue of the Gauss summation formula for hypergeometric functions related to root systems. Math. Z. 212 (1993), no. 3, 313–336.

[699] Opdam E. M.: Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group. Compositio Math. 85 (1993), no. 3, 333–373.

[700] Opdam E.M.: Harmonic analysis for certain representations of graded Hecke algebras. Acta Math. 175 (1995), no. 1, 75–121.

[701] Ore O.: Theory of non-commutative polynomials. Annals of Math. 34 (1933), 480-508.

[702] Orr W. M.: Theorems relating to the products of two hypergeometric series. Trans. Camb. Phil. Soc. 17 (1899), 1-15.

[703] Paule P.: On identities of the Rogers- Ramanujan type. J. Math. Anal. Appl. 107 (1985), no. 1, 255–284.

[704] Pelinovsky D.: Rational solutions of the KP hierarchy and the dynamics of their poles. II. Construction of the degenerate polynomial solutions. J. Math. Phys. 39 (1998), no. 10, 5377–5395.

[705] Perk J. H. H.: Star-triangle equations, quantum Lax pairs, and higher genus curves. Theta functions—Bowdoin 1987, Part 1 (Brunswick, ME, 1987), 341–354, Proc. Sympos. Pure Math., 49, Part 1, Amer. Math. Soc., Providence, RI, 1989.

[706] Petkovsek M. & Wilf H. & Zeilberger D.: A=B. A.K. Peters 1996.

[707] Pfaff J.F.: Observationes analyticae ad L.Euleri Institutiones Calculi Integralis, Vol. IV, Supplem. II et IV, Nova Acta Academiae Scientiarum Petropolitanae, Tome XI (1797), Histoire, 38-57.

[708] Pickering A.: Lax pairs and exact solutions from Painlevé analysis. Nonlinear physics: theory and experiment (Lecce, 1995), 269–277.

[709] Piguet C.A. & Wang D.F. & Gruber C.: Quantum duality and Bethe ansatz for the Hofstadter problem on the hexagonal lattice. Physics Letters A 209 (1996) 110-114.

[710] Pillin, M.: q-deformed relativistic wave equations. J. Math. Phys. 35, 6 (1994), 2804-2817.

[711] Pincherle S.: Mémoire sur le calcul fonctionnel distributif. Math. Ann 49 (1897), 325-382.

[712] Pitre S. N. & Van der Jeugt J.: Wigner's 9-j coefficient and its relation to new summation theorems for hypergeometric series. Proceedings of the IV Wigner Symposium (Guadalajara, 1995), 270–273, World Sci. Publishing, River Edge, NJ, 1996.

[713] Plasma theory and nonlinear and turbulent processes in physics. Vol. 1, 2. Proceedings of the Third International Workshop held in Kiev, April 13–25, 1987. Edited by V. G. Bar yakhtar, V. M. Chernousenko, N. S. Erokhin, A. G. Sitenko and V. E. Zakharov. World Scientific Publishing Co., Singapore, 1988. Vol. 1: xvi+546 pp.; Vol. 2: pp. i–x and 547–998. ISBN: 9971-50-546-0

[714] Pochhammer L.: Über die Differentialgleichung der allgemeinen hypergeometrischen Reihe mit zwei endlichen singulären Punkten. J. reine angew. Math. 102 (1888), 76-159.

[715] Pochhammer L.: Zur Theorie der Eulers'chen Integrale. Math. Ann 35 (1890), 495-526.

[716] Polychronakos A. P.: Exchange operator formalism for integrable systems of particles. Phys. Rev. Lett. 69 (1992), no. 5, 703–705.

[717] Prange R.E. & Girvin S.M.: The quantum Hall effect. Springer 1987.

[718] Prasad S.: Certain q-transforms. Math. Scand. 30 (1972), 324–330.

[719] Pringsheim A.: Über Konvergenz und funktiontheoretischen Character gewisser limitär-periodischer Kettenbrüche. München 1910.

[720] Proctor R. A.: Equivalence of the combinatorial and the classical definitions of Schur functions. J. Combin. Theory Ser. A 51 (1989), no. 1, 135–137.

[721] Proctor R. A.: New symmetric plane partition identities from invariant theory work of De Concini and Procesi. European Journal Combinatorics 11 (1990), 289-300.

[722] Proctor R. A.: A generalized Berele-Schensted algorithm and conjectured Young tableaux for intermediate symplectic groups. Trans. Amer. Math. Soc. 324 (1991), no. 2, 655–692.

[723] Proctor R. A.: Young tableaux, Gelfand patterns, and branching rules for classical groups. J. Algebra 164 (1994), no. 2, 299–360.

[724] Proctor R. A. & Wilson D. C.: Interpretation of a basic hypergeometric identity with Lie characters and Young tableaux. Discrete Math. 137 (1995), no. 1-3, 297–302.

[725] Puta M.: Integrability and geometric prequantization of the Maxwell-Bloch equations. Bull. Sci. Math. 122 (1998), no. 3, 243–250.

[726] Qiao Z.: New hierarchies of isospectral and non-isospectral integrable NLEEs derived from the Harry-Dym spectral problem. Phys. A 252 (1998), no.3-4, 377–387.

[727] Quesne C.: Coherent states, K-matrix theory and q-boson realizations of the quantum algebra $\mathrm{su}_q(2)$. Phys. Lett. A 153 (1991), no. 6-7, 303–307.

[728] Ragnisco O. & Rauch-Wojciechowski S.: Integrable maps for the Garnier and for the Neumann system. J. Phys. A 29 (1996), no. 5, 1115–1124.

[729] Rahman M. & Verma A.: Quadratic transformation formulas for basic hypergeometric series. Trans. Amer. Math. Soc. 335 (1993), no. 1, 277–302.

[730] Rahman M. & Suslov S. K.: The Pearson equation and the beta integrals. SIAM J. Math. Anal. 25 (1994), no. 2, 646–693.

[731] Rahman M. & Suslov S. K.: Barnes and Ramanujan-type integrals on the q-linear lattice. SIAM J. Math. Anal. 25 (1994), no. 3, 1002–1022.

[732] Rahman M. & Suslov S. K.: A unified approach to the summation and integration formulas for q-hypergeometric functions. I. J. Statist. Plann. Inference 54 (1996), no. 1, 101–118.

[733] Rahman M. & Suslov S. K.: A unified approach to the summation and integration formulas for q-hypergeometric functions. II, III. Methods Appl. Anal. 5 (1998), no. 4, 399–412, 413–424.

[734] Rainville E. D.: The contiguous function relations for ${}_pF_q$ with applications to Bateman's $J_n^{u,v}$ and Rice's $H_n(\zeta, p, v)$. Bull. Amer. Math. Soc. 51, (1945). 714–723.

[735] Rajeswari V. & Srinivasa Rao K.: Four sets of ${}_3F_2(1)$ functions, Hahn polynomials and recurrence relations for the 3-j coefficients. J. Phys. A 22 (1989), no. 19, 4113–4123.

[736] Rajeswari V. & Srinivasa Rao K.: Generalized basic hypergeometric functions and the q-analogues of 3-j and 6-j coefficients. J. Phys. A 24 (1991), no. 16, 3761–3780.

[737] Ram A. & Remmel J.B. & Whitehead T.: Combinatorics of the q-basis of symmetric functions. Journal of combinatorial theory, series A 76 (1996), 231-271.

[738] Randrianarivony A.: Fractions continues, q-nombres de Catalan et q-polynômes de Genocchi. [Continued fractions, q-Catalan numbers and q-Genocchi polynomials] European J. Combin. 18 (1997), no. 1, 75–92.

[739] Rassias Th. M.& Singh & S. N.: Certain transformations involving basic hypergeometric series. Complex Variables Theory Appl. 18 (1992), no. 1-2, 3–6.

[740] Rauch H.E. & Lebowitz A.: Elliptic functions, Theta functions and Riemann surfaces. Baltimore 1973.

[741] Rauh A.: Degeneracy of Landau levels in crystals, phys. stat. sol. b 65, K131 (1974).

[742] Ravoson V. & Gavrilov L. & Caboz R.: Separability and Lax pairs for Hénon-Heiles system. J. Math. Phys. 34 (1993), no. 6, 2385–2393.

[743] Raychev P.P. & Roussev R.P. & Smirnov Yu.F.: The quantum algebra $SU_q(2)$ and rotational spectra of deformed nuclei. J.Phys. G 16 (1990) L137- 141.

[744] Reid L. J.: Operational equations for a class of symmetric q-polynomials. Duke Math. J. 35 (1968), 783–798.

[745] Reid L. J.: Some generalized symmetric q-polynomials. Math. Nachr. 49 (1971), 149–161.

[746] Reyman A. G. & Semenov-Tian-Shansky M. A.: Lax representation with a spectral parameter for the Kowalewski top and its generalizations. Lett. Math. Phys. 14 (1987), no. 1, 55–61.

[747] Rideau G. & Winternitz P.: Representations of the quantum algebra $\mathrm{su}_q(2)$ on a real two-dimensional sphere. J. Math. Phys. 34 (1993), no. 12, 6030–6044.

[748] Riordan J.: An introduction to combinatorial analysis. Wiley 1958.

[749] Riordan J.: Inverse relations and combinatorial identities. American math monthly 71 (1964), 485-498.

[750] Robinson G.: Representation theory of the symmetric group. University of Toronto press 1961.

[751] Rogers L.J.: On the expansion of some infinite products. Proc. London Math. Soc. 24 (1893), 337-352.

[752] Roman S. & Rota G.C.: The umbral calculus. Adv. in Math. 27 (1978), 95-188.

[753] Roman S.: The theory of the umbral calculus I. Journal of mathematical analysis and applications 87 (1982), 58-115.

[754] Roman S.: More on the Umbral calculus, with emphasis on the q-Umbral calculus. Journal of mathematical analysis and applications 107 (1985), 222-254.

[755] Rosengren H.: Multivariable orthogonal polynomials as coupling coefficients for Lie and quantum algebra representations. Dissertation, Lund 1999, page 167.

[756] Rota G.C.: The number of partitions of a set. Amer. Math. Monthly 71 (1964), 498–504.

[757] Rota G.C. & Kahaner D. & Odlyzko A.: Finite operator calculus. Journal math. anal. appl. 42 (1973), 684-760.

[758] Rota G.C.: Combinatorics, representation theory and invariant theory: the story of a ménage à trois. Selected papers in honor of Adriano Garsia (Taormina, 1994). Discrete Math. 193 (1998), no. 1-3, 5–16.

[759] Rothe H.A.: Formulae de serierum reversione demonstratio universalis signis localibus combinatorio-analyticorum vicariis exhibita, Leipzig, 1793.
[760] Rothe H.A.: Systematisches Lehrbuch der Arithmetik, Leipzig, 1811.
[761] Rudin W.: Real and complex analysis. McGraw-Hill 1987.
[762] Ruijsenaars S.N.M. & Schneider H.: A new class of integrable systems and its relation to solitons. Ann. Physics 170 (1986), no. 2, 370–405.
[763] Ruijsenaars S.N.M.: Relativistic Calogero-Moser systems and solitons. Topics in soliton theory and exactly solvable nonlinear equations (Oberwolfach, 1986), 182–190, World Sci. Publishing, Singapore, 1987.
[764] Ruijsenaars S.N.M.: Complete integrability of relativistic Calogero-Moser systems and elliptic function identities. Comm. Math. Phys. 110 (1987), no. 2, 191–213.
[765] Ruijsenaars S.N.M.: First order analytic difference equations and integrable quantum systems. J Math. Phys. 38 (2) (1997), 1069-1146.
[766] Ruijsenaars S.N.M.: Systems of Calogero-Moser type. Particles and fields (Banff, AB, 1994), 251–352, CRM Ser. Math. Phys., Springer, New York, 1999.
[767] Ryde F.: A contribution to the theory of linear homogeneous geometric difference equations (q-difference equations). Dissertation Lund 1921.
[768] Saalschütz L.: Eine Summationsformel. Zeitschr. Math. Phys. 35 (1890). 186-188.
[769] Sagan B. E.: Congruence properties of q-analogs. Adv. Math. 95 (1992), no. 1, 127–143.
[770] Sahai V.: On models of $U_q(\mathrm{sl}(2))$ and q-Euler integral transformation. Indian J. Pure Appl. Math. 29 (1998), no. 8, 819–827.
[771] Sahai V.: On models of $U_q(\mathrm{sl}(2))$ and q-Appell functions using a q-integral transformation. Proc. Amer. Math. Soc. 127 (1999), no. 11, 3201–3213.
[772] Sahi S.: Interpolation, integrality, and a generalization of Macdonald's polynomials. Internat. Math. Res. Notices (1996), no. 10, 457–471.
[773] Sahi S.: A new scalar product for nonsymmetric Jack polynomials. Internat. Math. Res. Notices (1996), no. 20, 997–1004.
[774] Sahi S.: The binomial formula for nonsymmetric Macdonald polynomials. Duke Math. J. 94 (1998), no. 3, 465–477.
[775] Saitoh N. & Saito S.: General solutions to the Bäcklund transformation of Hirota's bilinear difference equation. J. Phys. Soc. Japan 56 (1987), no. 5, 1664–1674.
[776] Salam M. A. & Wybourne, B. G. The q-deformation of symmetric functions and the symmetric group. J. Phys. A 24 (1991), no. 7, L317–L321.
[777] Santilli R.M.: Hadronic journal 1 (1978), 574-901.
[778] Sasaki T.: Contiguity relations of Aomoto-Gelfand Hypergeometric functions and applications to Appels' system... SIAM J. Math. anal. 22, 3 (1991), 821-846.
[779] Satoh J.: q-Analogue of Riemanns ζ-function and q-Euler numbers. Journal of number theory 31 (1989), 346-362.
[780] Satoh J.: A construction of q-analogue of Dedekind sums. Nagoya Math. J. 127(1992), 129-143.
[781] Satoh J.: Construction of q-analogue by using Stirling numbers. Japan. J. Math. (N.S.) 20 (1994), no. 1, 73–91.

[782] Satoh J.: Sums of products of two q-Bernoulli numbers. J. Number Theory 74 (1999), no. 2, 173–180.

[783] Saxena R. K. & Gupta R. K.: Certain multidimensional fractional q-integral operators involving Srivastava's generalized q-hypergeometric function of several variables. Pure Appl. Math. Sci. 31 (1990), no. 1-2, 127–136.

[784] Saxena R. K. & Kumar R.: Fractional q-differentiation and basic hypergeometric functions of two variables. Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 35(83) (1991), no. 3-4, 287–294.

[785] Scharf T. & Thibon J.Y. & Wybourne B.G.: Powers of the Vandermonde determinant and the quantum Hall effect. J, Phys. A., 27 (1994), 4211-4219.

[786] Schief W. K. & Rogers C.: Inverse problems 10 (1994), 711.

[787] Schief W. K.: The Tzitzeica equation: A Bäcklund transformation interpreted as truncated Painlevé expansion. J. Phys. A 29 (1996), no. 16, 5153–5155.

[788] Schirrmacher A.: quantum groups. quantum spacetime, and Dirac equation. Low-dimensional topology and quantum field theory 1993, 221-230.

[789] Schottky F.: Zur Theorie der Abelschen Funktionen von vier Variabeln. J. Reine Angew. Math. 102 (1888), 304-352.

[790] Schottky F.: Über die Module der Thetafunction. Acta Math 27 (1903), 235-288.

[791] Schur I.: Über vertauschbare lineare Differentialausdrücke. Sitzungsber. der Berliner Math. Gesel. 4 (1905), 2-8.

[792] Schützenberger M.P.: Une interpretation de certaines solution de l'equation functionelle. Comptes rendus 236 (1953), 352-353.

[793] Sears D.B.: On the transformation theory of basic hypergeometric functions. Proc. London Math. Soc (2) 53, (1951) 158-180.

[794] Semenov T. & Shansky M. A.: Quantum and classical integrable systems. Integrability of nonlinear systems (Pondicherry, 1996), 314–377, Lecture Notes in Phys., 495, Springer, Berlin, 1997.

[795] Sen Dolan Chapa & Chowdhury A.R.: On the Moyal quantized BKP type hierarchies. J. Nonlinear Math. Phys. 5 (1998), no. 1, 1–7.

[796] Shan J. & Levermore C.D. & McLaughlin D.W.: The semiclassical limit of the defocusing NLS hierarchy. Comm. Pure Appl. Math. 52 (1999), no. 5, 613–654.

[797] Sharma A. & Chak A. M.: The basic analogue of a class of polynomials. Riv. Mat. Univ. Parma 5 (1954), 325–337.

[798] Sharma A.: q-Bernoulli and Euler numbers of higher order. Duke Math. J. 25 (1958), 343–353.

[799] Shrivastava P. N.: On generalised Stirling numbers and polynomials. Riv. Mat. Univ. Parma (2) 11 (1970), 233–237.

[800] Siegel W.: Introduction to string field theory. Advanced Series in Mathematical Physics, 8. World Scientific Publishing Co., Inc., Teaneck, NJ, 1988.

[801] Silvestrov S.D.: private communication.

[802] Simon B.: Representation theory of finite and compact groups. AMS, 1996.

[803] Singer D.: q-analogues of Lagrange inversion. Adv. Math. 115 (1995), no. 1, 99–116.

[804] Singh S. N.: Certain transformations of q-hypergeometric functions. Math. Student 52 (1984), no. 1-4, 29–32 (1990).

[805] Singh S. P.: Certain transformation formulas involving basic hypergeometric functions. J. Math. Phys. Sci. 28 (1994), no. 4, 189–195.

[806] Singh S. P. Certain integrals involving basic hypergeometric functions. Math. Student 64 (1995), no. 1-4, 188–194 (1996).

[807] Singh S. P. & Rai P.: Two general basic hypergeometric identities. Ganita 47 (1996), no. 1, 35–43.

[808] Singh S. N. & Mishra S.S.: On the extension of Bernoulli, Euler and Eulerian polynomials. Tamkang Journal of mathematics 27, 3 (1996), 189-199.

[809] Singh U. B.: Certain bibasic hypergeometric transformations and their applications. J. Math. Anal. Appl. 201 (1996), no. 1, 44–56.

[810] Slater L.J.: Generalized hypergeometric functions. Cambridge 1966.

[811] Smith E.R.: Zur Theorie der Heineschen Reihe und ihrer Verallgemeinerung. Diss. Univ. München 1911.

[812] Soni S. K.: q-analogues of some prototype Berry phase calculations. J. Phys. A 23 (1990), no. 18, L951–L955.

[813] Sonine N.: Recherches sur les fonctions cylindriques et le développement des fonctions continues en séries. Mathematische Annalen 16 (1880), 1-80.

[814] Spiridonov V. & Vinet L. & Zhedanov A.: Discrete Schrödinger equation, Darboux transformations, and orthogonal polynomials. Symmetries and integrability of difference equations (Estrel, PQ, 1994), 363–369, CRM Proc. Lecture Notes, 9, Amer. Math. Soc., Providence, RI, 1996.

[815] Srinivasa Rao K. & Rajeswari V.: Quantum theory of angular momentum, Springer 1993.

[816] Srinivasa Rao K.: Quantum theory of angular momentum and generalized (ordinary & basic) hypergeometric functions. Symmetry methods in physics, Vol. 2 (Dubna, 1995), 465–474, Joint Inst. Nuclear Res., Dubna, 1996.

[817] Srinivasa Rao K.: Angular momentum coefficients and generalized hypergeometric functions. Special functions and differential equations (Madras, 1997), 165–170, Allied Publ., New Delhi, 1998.

[818] Srinivasa Rao K. & Van der Jeugt, J. Transformations of single and double hypergeometric series from the triple sum series for the 9-j coefficient. Internat. J. Theoret. Phys. 37 (1998), no. 2, 891–905.

[819] Srivastava B.: A note on certain bibasic q-Appell and Lauricella series. Glas. Mat. Ser. III 30(50) (1995), no. 1, 29–36.

[820] Srivastava B.: Some relations between contiguous bibasic q-Appell functions. Nepali Math. Sci. Rep. 16 (1997), no. 1-2, 37–44.

[821] Srivastava H. M.: On q-generating functions and certain formulas of David Zeitlin. Illinois J. Math. 15 (1971), 64–72.

[822] Srivastava H. M.: Some polynomial expansions for functions of several variables. IMA J. Appl. Math. 27 (1981), no. 3, 299–306.

[823] Srivastava H. M.: Certain q-polynomial expansions for functions of several variables. IMA J. Appl. Math. 30 (1983), no. 3, 315–323.

[824] Srivastava H. M.: Certain q-polynomial expansions for functions of several variables. II. IMA J. Appl. Math. 33 (1984), no. 2, 205–209.

[825] Srivastava H. M. & Manocha H. L.: A treatise on generating functions. Ellis Horwood Series: Mathematics and its Applications. Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], New York, 1984. 569 pp.

[826] Srivastava H. M. & Karlsson P.W.: Multiple Gaussian hypergeometric series. Ellis Horwood Series: Mathematics and its Applications. Ellis Horwood Ltd., Chichester; Halsted Press John Wiley & Sons, Inc., New York, 1985.

[827] Srivastava H. M. & Todorov P. G.: An explicit formula for the generalized Bernoulli polynomials. J. Math. Anal. Appl. 130 (1988), no. 2, 509–513.

[828] Srivastava H. M.: Some recent results on q-series and q-polynomials. Ramanujan International Symposium on Analysis (Pune, 1987), 139–149, Macmillan of India, New Delhi, 1989.

[829] Srivastava H. M.: Ordinary and basic (or q-) hypergeometric functions and associated orthogonal polynomial. Bull. Allahabad Math. Soc. 3 (1988), 45–91 (1990).

[830] Stamfort O.: Über die Bedeutung einiger Jenaer Mathematiker fr die Entwicklung der Mathematik. (German) Wiss. Z. Friedrich-Schiller-Univ. Jena/Thüringen 18 1969 Heft 2, 237–242.

[831] Stanley R. P.: Binomial posets, Möbius inversion, and permutation enumeration. J. Combinatorial Theory Ser. A 20 (1976), no. 3, 336–356.

[832] Stanton D.: Orthogonal polynomials and Chevalley groups. Special functions: group theoretical aspects and applications, 87–128, Math. Appl., Reidel, Dordrecht-Boston, MA, 1984.

[833] Starcher G.W.: On identities arising from solutions of q-difference equations and some interpretations in number theory (Thesis). Am. J. of Math. 53, 1931, 801-816.

[834] Steeb W.H. & Kloke M. & Spieker B. M.: Liouville equation, Painlevé property and Bäcklund transformation. Z. Naturforsch. A 38 (1983), 1054-1055.

[835] Steeb W.H. & Louw J. A.: Strampp, W. Lax formulation, Painlevé property and recursion operator for a space-dependent Burgers equation. Progr. Theoret. Phys. 75 (1986), no. 2, 455–457.

[836] Steeb W.H. & Louw J. A.: Nahm's equations, singular point analysis, and integrability. J. Math. Phys. 27 (1986), no. 10, 2458–2460.

[837] Steeb W.H. & Euler N.: Inviscid Burgers equation, Painlevé analysis and a Bäcklund transformation. Z. Naturforsch. A 45 (1990), no. 7, 929–930.

[838] Steeb W.H. & Euler N. & Mulser P.: On a hierarchy of nonlinear dynamical systems and Painlevé test. Found. Phys. Lett. 4 (1991), no. 5, 465–469.

[839] Steeb W.H.: Problems and solutions in theoretical and mathematical physics. Vol. II. Advanced level. World Scientific Publishing Co., Inc., River Edge, NJ, 1996. x+323 pp.

[840] Steinacker H.: Integration on quantum Euclidean space and sphere in N dimensions. q-alg/9506020 v3 (1995).

[841] Steinacker H.: Finite dimensional unitary representations of quantum Anti-de Sitter groups at roots of unity. Communications in mathematical physics 192 (1998), 687-706.

[842] Sternberg S.: Group theory and physics, Cambridge 1994.

[843] Stewart I.: Galois theory. Chapman Hall 1989.

[844] Stokman J.V.: Multivariable Orthogonal polynomials and quantum grassmanians. Thesis Amsterdam 1998 Universal Press.

[845] Stone M.H.: Schur functions, chiral bosons, and the quantum-Hall-effect edge states. Phys. Rev. B 42 (1990) 8399-8404.

[846] Subbarao M.V. & Verma A.: Some summations of q-series by telescoping. Pasific journal of mathematics 191, no.1 (1999), 173-182.

[847] Sulanke R.A.: A generalized q-multinomial Vandermonde convolution. J. Combin. Theory Ser. A 31 (1981), no. 1, 33–42.

[848] Sutherland B.: J. Math. Phys. 12 (1971), 246-256.

[849] Sutherland B.: Exact results for a quantum many-body problem in one dimension. Phys. Rev A 4 (1971), 2019-2021.

[850] Sutherland B.: Exact results for a quantum many-body problem in one dimension II. Phys. Rev A 5 (1972), 1372-1376.

[851] Swarttouw R. F.: The contiguous function relations for the basic hypergeometric series. J. Math. Anal. Appl. 149 (1990), no. 1, 151–159.

[852] Swarttouw R. F. & Meijer H. G. A q-analogue of the Wronskian and a second solution of the Hahn-Exton q-Bessel difference equation. Proc. Amer. Math. Soc. 120 (1994), no. 3, 855–864.

[853] Takahashi M. & Suzuki M.: One-dimensional anisotropic Heisenberg model at finite temeratures. Progress of theoretical physics 48, 6b (1972), 2187-2208.

[854] Takasaki K.: Gaudin model, KZ equation and an isomonodromic problem on the torus. Lett. Math. Phys. 44 (1998), no. 2, 143–156.

[855] Takhtajan L.A. & Faddev L.D.: The quantum method of the inverse problem and the Heisenberg xyz model. Russian math surveys 34:5 (1979), 11-68.

[856] Tamizhmani T. & Tamizhmani K. M. & Grammaticos B. & Ramani A.: Special function solutions for asymmetric discrete Painlevé equations. J. Phys. A 32 (1999), no. 24, 4553–4562.

[857] Tarasov V. & Varchenko A.: Asymptotic solutions to the quantized Knizhnik-Zamolodchikov equation and Bethe vectors. Mathematics in St. Petersburg, 235–273, Amer. Math. Soc. Transl. Ser. 2, 174, Amer. Math. Soc., Providence, RI, 1996.

[858] Tarasov V. & Varchenko A.: Geometry of q-Hypergeometric Functions as a Bridge between Yangians and Quantum Affine Algebras. Inventiones Mathematicae, 128 (1997), 501-588.

[859] Tasso H.: Hamiltonian formulation of odd Burgers hierarchy. J.Phys. A 29 (1996), 7779-7784.

[860] Taylor B.D.: Difference equations via the classical umbral calculus. Mathematical essays in honor of Gian-Carlo Rota, 397-411, Birkhäuser 1998.

[861] Theta functions—Bowdoin 1987. Part 1. Proceedings of the Thirty-fifth Summer Research Institute held at Bowdoin College, Brunswick, Maine, July 6–24, 1987.

[862] Thomae J.: Beiträge zur Theorie der durch die Heinische Reihe .... J. reine angew. Math. 70, (1869), 258-281.

[863] Thomae J.: Über die höheren hypergeometrischen Reihen, insbesondere die Reihe.... Math. Ann., Bd.2, (1870), 427-444.

[864] Thomae J.: Les séries Heinéennes supérieures. Ann Mat. pura appl. Bologna, II s 4 (1871). 105-139.

[865] Thomae J.: Über die Funktionen welche durch Reihen von der Form dargestellt werden: J. reine angew. Math. 87, (1879), 26-73.

[866] Thomae J.: J. reine angew. Math. 87, (1879), 222-349.

[867] Thorén J.: Irreducible representations of quantum affine algebras. Dissertation Lund 2000.

[868] Thuswaldner J.M.: q-Difference equations and their applications to combinatorial problems. Grazer Math. Ber. 328 (1996), 103-112.

[869] Tian F. & Ye J.: On the Whitham equations for the semiclassical limit of the defocusing nonlinear Schrödinger equation. Communications on pure and applied mathematics 52,6 (1999), 655-692.

[870] Timoteo V. S. & Lima C. L.: Effect of q-deformation in the NJL gap equation. Phys. Lett. B 448 (1999), no. 1-2, 1–5.

[871] Tinebra F.: The Lax pair of a generalized Thirring model. Internat. J. Modern Phys. A 14 (1999), no. 5, 659–682.

[872] Trjitzinsky W.J.: Analytic theory of linear q-difference equations. Acta Math. 61, (1933),1-38.

[873] Tsai T.-Y.: Pseudo Lax pair for the Veselov-Novikov equation. Chinese J. Math. 24 (1996), no. 3, 273–283.

[874] Tsiganov A. V.: A Lax triad for the Kowalewski top. Symmetry methods in physics, Vol. 2 (Dubna, 1995), 574–582, Joint Inst. Nuclear Res., Dubna, 1996.

[875] Tsiganov A. V.: The Kowalewski top in SUSY quantum mechanics. Phys. Lett. A 223 (1996), no. 1-2, 63–65.

[876] Tsiganov A. V.: The Kowalewski top: a new Lax representation. J. Math. Phys. 38 (1997), no. 1, 196–211.

[877] Tsiganov A. V.: On integrable deformations of the spherical top. J. Phys. A 32 (1999), no. 47, 8355–8363.

[878] Tsiganov A. V.: The Lax pairs for the Holt system. J. Phys. A 32 (1999), no. 45, 7983–7987.

[879] Tsiganov A. V.: Duality between integrable Stäckel systems. J. Phys. A 32 (1999), no. 45, 7965–7982.

[880] Tsuboi Z. & Kuniba A.: Solutions of a discretized Toda field equation for D_r from analytic Bethe ansatz. J.Phys. A 29 (1996), 7785-7796.

[881] Tsuchiya A. & Kanie Y.: Vertex operators in conformal field theory on P^1 and monodromy representations of braid group. Conformal field theory and solvable lattice models (Kyoto, 1986), 297–372, Adv. Stud. Pure Math., 16, Academic Press, Boston, MA, 1988.

[882] Tsumura H.: A note on q-analogues of the Dirichlet series and q-Bernoulli numbers. J. Number Theory 39 (1991), no. 3, 251–256.

[883] Tsumura H.: A note on q-analogues of Dirichlet series. Proc. Japan Acad. Ser.A Math. Sci. 75 (1999), no. 3, 23–25. 1

[884] Twarock R.: Derivation of a q-Schrödinger equation on S^1 via q-deformation of the kinematical algebra .... Arnold Sommerfeld Institut für Mathematische Physik Clausthal 1996.

[885] Tzitzeica G.: R. Acad. Sci. Paris 150 (1910) 955.

[886] Ueno K.: Lattice path proof of the ribbon determinant formula for Schur functions. Nagoya Math. J. 124 (1991), 55–59.

[887] Ueno K. & Shibukawa Y.: Character table of Hecke algebra of type A_{N-1} and representations of the quantum group $U_q(\mathrm{gl}_{n+1})$. Infinite analysis, Part A, B (Kyoto, 1991), 977–984, Adv. Ser. Math. Phys., 16, World Sci. Publishing, River Edge, NJ, 1992.

[888] Ueno K. & Nishizawa M.: Quantum groups and zeta-functions. Quantum groups (Karpacz, 1994), 115–126, PWN, Warsaw, 1995.

[889] Ueno K. & Nishizawa M.: The multiple gamma function and its q-analogue. Quantum groups and quantum spaces (Warsaw, 1995), 429–441, Banach Center Publ., 40, Polish Acad. Sci., Warsaw, 1997.

[890] Ueno K. & Nishizawa M.: Multiple gamma functions and multiple q-gamma functions. Publ. Res. Inst. Math. Sci. 33 (1997), no. 5, 813–838.

[891] Ujino H. & Wadati M.: Rodrigues formula for the nonsymmetric multivariable Hermite polynomial. J. Phys. Soc. Japan 68 (1999), no. 2, 391–395.

[892] Upadhyay M.: q-fractional differentiation and basic hypergeometric transformations. Ann. Polon. Math. 25 (1971/72), 109–124.

[893] Upadhyay M.: Fractional q-integration and integral representations of basic double hypergeometric series of higher order. Gaṇita 23 (1972), no. 1, 19–36.

[894] Upadhyay M.: On q-difference equations for basic hypergeometric series with unconnected bases. Collection of articles dedicated to the memory of the Balkan mathematicians Constantin Caratheodory, Josip Plemelj, Dimitrie Pompeiu and Gheorghe Ţiţeica in connection with the first centenary of their birth. Math. Balkanica 3 (1973), 559–564.

[895] Upadhyay M.: Certain fractional q-integral operators. Pure Appl. Math. Sci. 4 (1976), no. 1-2, 141–149.

[896] Vaklev Y.: Soliton solutions and gauge equivalence for the problem of Zakharov-Shabat and its generalizations. J. Math. Phys. 37 (1996), no. 3, 1393–1413.

[897] Van der Jeugt J. & Pitre S. N. & Srinivasa Rao, K.: Multiple hypergeometric functions and 9-j coefficients. J. Phys. A 27 (1994), no. 15, 5251–5264.

[898] Van der Jeugt J. & Pitre S. N. & Srinivasa Rao, K.: Transformation and summation formulas for double hypergeometric series. J. Comput. Appl. Math. 83 (1997), no. 2, 185–193.

[899] Van der Jeugt J. & Pitre S. N. & Srinivasa Rao, K.: Transformation and summation formulas for multiple hypergeometric series, and the 9-j coefficient. Special functions and differential equations (Madras, 1997), 171–177, Allied Publ., New Delhi, 1998.

[900] Van der Jeugt J. & Srinivasa Rao K.: Invariance groups of transformations of basic hypergeometric series. J. Math. Phys. 40 (1999), no. 12, 6692–6700.

[901] Vandermonde A.T.: Mèmoire sur des irrationelles de diffèrens ordre avec une application au cercle. Mèm. Acad. Roy.Sci.Paris, (1772), 489-498.

[902] Van der Put M & Singer M.F.: Galois theory of difference equations, Lecture notes in mathematics 1666 1997.

[903] van Diejen J. F.: On the diagonalization of difference Calogero-Sutherland systems. Symmetries and integrability of difference equations (Estrel, PQ, 1994), 79–89, CRM Proc. Lecture Notes, 9, Amer. Math. Soc., Providence, RI, 1996.

[904] Vandiver H.S.: On generalizations of the numbers of Bernoulli and Euler, Proceedings of the National Academy of sciences of the United... 23 (1937), 555-559.

[905] Vandiver H.S.: Simple explicit expressions for generalized Bernoulli numbers of the first order. Duke Math. Journ. 8 (1941), 575-584.

[906] Van Isacker P.: O(4) symmetry and angular momentum theory in four dimensions. Symmetries in science, V (Lochau, 1990), 323–340, Plenum, New York, 1991.

[907] Varchenko A. & Tarasov V.: Jackson integral representations of solutions of the quantized Knizhnik-Zamolodchikov equation. St. Petersburg math J. 6, 2 (1995), 275-313.

[908] Vekslerchik V. E.: Functional representation of the Ablowitz-Ladik hierarchy. J. Phys. A 31 (1998), no. 3, 1087–1099.

[909] Verde-Star L.: Biorthogonal Polynomial Bases and Vandermonde-like matrices. Studies in applied mathematics 95 (1995), 269-295.

[910] Verdoodt A.: p-adic q-umbral calculus. J. Math. Anal. Appl. 198 (1996), no. 1, 166–177.

[911] Verma A. & Jain V. K.: A survey of summation formulas for basic hypergeometric series. Gaṇita 43 (1992), no. 1-2, 1–31.

[912] Vilenkin N.J. & Klimyk A.U.: Representations of Lie groups and special functions, Vol 3, Kluwer Academic Publishers 1992.

[913] Vilenkin N.J. & Klimyk A.U.: Representations of Lie groups and special functions, Vol 2, Kluwer Academic Publishers 1993.

[914] Wadati M. & Olmedilla E. & Akutsu Y.: Lax pair for the one-dimensional Hubbard model. J. Phys. Soc. Japan 56 (1987), no. 4, 1340–1347.

[915] Wallenberg G.: Über die Vertauschbarkeit homogener linearer Differentialausdrücke. Archiv der Math. Phys., Dritte Reihe 4 (1903), 252-268.

[916] Wallisser R.: Über ganze Funktionen, die in einer geometrischen Folge ganze Werte annehmen. Monatshefte Mathematik 100, (1985), 329-335.

[917] Ward M.: A calculus of sequences. Amer. J. Math. 58 (1936), 255-266.

[918] Watson G.N.: The continuations of functions defined by generalized hypergeometric series. Trans. Camb. Phil. Soc. 21 (1910) 281-299.

[919] Watson G.N.: A note on generalized hypergeometric series. Proc. London Math. Soc. (2) 23 (1925), xiii-v

[920] Watson G.N.: A new proof of the Rogers-Ramanujan identities. J. London Math. Soc. 4, (1929) 4-9.

[921] Watson G.N.: Notes on generating functions of polynomials: (1) Laguerre polynomials. J. London Math. Soc. 8, (1933) 189-192.

[922] Watson G.N.: Notes on generating functions of polynomials: (2) Hermite polynomials. J. London Math. Soc. 8, (1933) 194-199.

[923] Watson G.N.: The final problem: an account of the mock theta functions. J. London Math. Soc. 11 (1936), 55-80.

[924] Watson G.N.: A note on the polynomials of Hermite and Laguerre. J. London Math. Soc. 13 (1938), 29-32.

[925] Wei Hong Zeng & Zhang Yi Bin & Wang & Chong Shou: A formula for the number of solutions to a symmetric matrix equation over a finite field of characteristic 2, and its q-hypergeometric series representation. (Chinese) Acta Math. Sinica 42 (1999), no. 1, 23–34

[926] Weisner L.: Group-theoretic origin of certain generating functions. Pasific J. Math. 5 (1955), 1033-1039.

[927] Weiss J. & Tabor M. & Carnevale G.: The Painlevé property for partial differential equations. J. Math. Phys. 24 (1983), 522-526.

[928] Weiss J.: Modified equations, rational solutions, and the Painlevé property for the Kadomtsev- Petviashvili and Hirota-Satsuma equations. J. Math. Phys. 26 (1985), no. 9, 2174–2180.

[929] Weiss J.: Bäcklund transformation and the Painlevé property. J. Math. Phys. 27 (1986), no. 5, 1293–1305.

[930] Weiss J.: Periodic fixed points of Bäcklund transformations and the Korteweg-de Vries equation. J. Math. Phys. 27 (1986), no. 11, 2647–2656.

[931] Wess J. & Zumino B.: Nuclear Physics (Proc. Suppl.) B 18 (1990), 302.

[932] Weyl H.: The Classical groups. Princeton 1939.

[933] Whipple F.J.W.: A group of generalized hypergeometric series: relations between 120 allied series of the type ${}_3F_2(a, b, c; d, e)$. Proc. London Math. Soc. (2) 23 (1925), 104-114.

[934] Whipple F.J.W.: Some transformations of generalized hypergeometric series. Proc. London Math. Soc. (2) 26 (1927), 257-272.

[935] Wigner E. P.: The collected works of Eugene Paul Wigner. Part A. The scientific papers. Vol. I. With a preface by Jagdish Mehra and Arthur S. Wightman. With a biographical sketch by Mehra, and annotation by Brian R. Judd and George W. Mackey. Edited by Wightman. Springer-Verlag, Berlin, 1993. xii+717 pp. I

[936] Wiegmann P.B. & Zabrodin A.V: Quantum group and magnetic translations Bethe ansatz, for the Asbel-Hofstadter problem. Nuclear Physics B 422 (1994), 495-514.

[937] Wilf H.S. & Zeilberger D.: An algorithmic proof theory for hypergeometric (ordinary and q) multisum/integral identities. Invent. math. 108 (1992), 575-633.

[938] Wilson G.: Collisions of Calogero-Moser particles and an adelic Grassmannian. With an appendix by I. G. Macdonald. Invent. Math. 133 (1998), no. 1, 1–41.

[939] Wimp J.: Siam Review 33 (1991), 489-493.

[940] Witten E.: On the structure of the topological phase of two-dimensional gravity. Nuclear Physics B340 (1990), 281-332.

[941] Woronowicz S.L.: Compact matrix Pseudogroups. Communications in Math. Physics 111 (1987), 613-665.

[942] Woronowicz S.L.: Differential calculus on compact matrix Pseudogroups (Quantum groups). Communications in Math. Physics 122 (1989), 125-170.

[943] Wright E.M.: Partitions of multi-partite numbers. Proceedings of the American mathematical society 7 (1956), 880-890.

[944] Wu Ying & Yang Xiao Xue: Soliton chain solutions to the two-dimensional higher order Korteweg-de Vries equations of Sawada-Kotera hierarchy.

[945] Yamada H.: The Virasoro algebra and the KP hierarchy. Infinite-dimensional groups with applications (Berkeley, Calif., 1984), 371–380, Math. Sci. Res. Inst. Publ., 4, Springer, New York, 1985.

[946] Yamamoto T.: Multicomponent Calogero model of B_N-type confined in a harmonic potential. Phys. Lett. A 208 (1995), no. 4-6, 293–302.

[947] Yang Huan-Xiong & Li You-Quan: Prolongation approach to Bäcklund transformation of Zhiber-Mikhailov-Shabat equation. J. Math. Phys. 37 (1996), no. 7,3491–3497.

[948] Yang, Kung-Wei: Solution of q-difference equations. Bull. London Math. Soc. 20 (1988), no. 1, 1–4.

[949] Yang, Kung-Wei: Matrix q-hypergeometric series. Discrete Math. 146 (1995), no. 1-3, 271–284.

[950] Yang, Qin Gzhu & Xu, Bo Wei: Energy spectrum of a q-analogue of the hydrogen atom. J. Phys. A 26 (1993), no. 7, L365–L368.

[951] Yen L.: A two-line algorithm for proving q-hypergeometric identities. Journal of mathematical analysis and applications 213 (1996), 1-14.

[952] Youjin Z. & Xinshen G.: A correspondence between the AKNS hierarchy and the JM hierarchy. Acta mathematicae applicatae sinica 4,4 (1988), 307-315.

[953] Yunbo Z.: On the use of the Painlevé property for obtaining Miura type transformation. Acta mathematicae applicatae sinica 7,2 (1991), 150-164.

[954] Yurov A. V.: The Bäcklund-Schlesinger transformation for Davey-Stewartson equations. (Russian) Teoret. Mat. Fiz. 109 (1996), no. 3,338–346; translation in Theoret. and Math. Phys. 109 (1996), no. 3, 1508–1514 (1997)

[955] Zakharov V.E. & Shabat A.B.: A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem.I, Functional Anal. Appl., 8 (1974), 226-235.

[956] Zakharov V.E. & Manakov S.V. & Novikov S.P. & Pitajevski L.P.: Theory of solitions. Nauka 1980 (Russian).

[957] Zappa G. & Casadio G.: The mathematical activity of Francesco Fa'a di Bruno from 1850 to 1859. (Italian) Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. (5) 16 (1992), no. 1-4.

[958] Zeilberger D.: The method of creative telescoping. J. Symbolic Composition 11 (1991), 195-204.

[959] Zeitlin D.: Generating functions for products of recursive sequences. Transactions of the American mathematical society 116 (1965), 300-315.

[960] Zelobenko D.P.: Compact Lie groups and their representations. AMS 1973.

[961] Zhang M.Q.: How to find the Lax Pair from the Yang-Baxter Equation. Commun.Math. Phys. 141, (1991), 523-531.

[962] Zhang X.D. & Tao C. Q.: The evaluation of basic hypergeometric series. I. J. Math. Res. Exposition 15 (1995), no. 3, 363–365.

[963] Zhang C.: Sur la sommabilité des séries entières solutions formelles d'une équation aux q-différences. I. (French) [On the summability of formal power series solutions of q-difference equations. I] C. R. Acad. Sci. Paris Sr. I Math. 327 (1998), no. 4, 349–352.

[964] Zhe Chang & Guo Han Ying & Yan Hong: The q-deformed oscillator model and the vibrational spectra of diatomic molecules. Phys. Lett. A 156 (1991), no. 3-4, 192–196.

[965] Zhe Chang & Yan Hong: The $\mathrm{SU}_q(2)$ quantum group symmetry and diatomic molecules. Phys. Lett. A 154 (1991), no. 5-6, 254–258.

[966] Zhe Chang & Yan Hong: Quantum group-theoretic approach to vibrating and rotating diatomic molecules. Phys. Lett. A 158 (1991), no. 5, 242–246.

[967] Zhe Chang: Quantum anti-de Sitter space. preprint 1999.

[968] Zhedanov A. S.: q rotations and other q transformations as unitary nonlinear automorphisms of quantum algebras. J. Math. Phys. 34 (1993), no. 6, 2631–2647.

[969] Zhu, Zuonong; Huang, Hongci; Xue, Weimin Lax representations and integrable time discretizations of the DDKdV, DDmKdV, and DDHOKdV. Phys. Lett. A 252 (1999), no. 3-4, 180–190.

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