Impulse Response of Second-Order Systems

INTRODUCTION

This document discusses the response of a second-order system, like the mass-spring-dashpot system shown in Fig. 1, to an impulse.

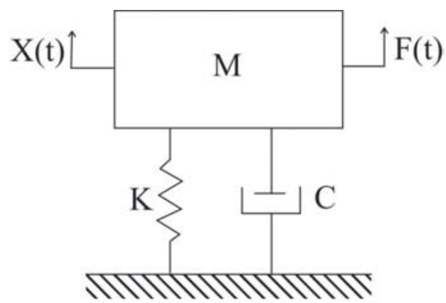

Fig. 1. Second-order mass-spring-dashpot system.

IMPULSE

An impulse is a large force applied over a very short period of time. In practice, an example of an impulse would be a hammer striking a surface. Mathematically, a unit impulse is referred to as a Dirac delta function, denoted by $\delta(t)$. It is called a unit impulse because its area is 1. As shown in Fig. 2, the force is applied over the time from 0 to t_1. Therefore, as t_1 approaches zero, in order for the area to remain equal to 1 the height must approach infinity.

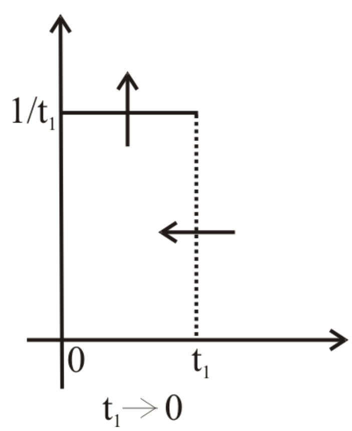

Fig. 2. Dirac delta function.

A general non-unit impulse function can be represented as $A\delta(t)$, where A is its area.

EQUATIONS DESCRIBING SYSTEM RESPONSE

The equation of motion describing the behavior of a second-order mass-spring-dashpot system with a unit impulse input is

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \delta(t). \tag{1}$$

The form of the system response will depend on whether the system is under-damped, critically damped, or over-damped. The most straightforward way to solve this differential equation and determine the system response is to use the Laplace transform. The Laplace transform of a Dirac delta function is

$$L\{\delta(t)\} = 1. \tag{2}$$

Under-Damped

For an under-damped system ($\zeta<1$), assuming zero initial conditions, the form of the response is

$$x(t) = \frac{1}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t. \tag{3}$$

Critically Damped

For a critically damped system ($\zeta=1$), and again assuming zero initial conditions, the response is given by

$$x(t) = te^{-\omega_n t}. \tag{4}$$

Over-Damped

For an over-damped system ($\zeta>1$), with zero initial conditions, the response is

$$x(t) = \frac{1}{2\omega_n\sqrt{\zeta^2-1}}\left[e^{-\omega_n\left(\zeta-\sqrt{\zeta^2-1}\right)t} - e^{-\omega_n\left(\zeta+\sqrt{\zeta^2-1}\right)t}\right]. \tag{5}$$

FORM OF SYSTEM RESPONSE

The response of a system to an impulse looks identical to its response to an initial velocity. The impulse acts over such a short period of time that it essentially serves to give the system an initial velocity.

Fig. 3 shows the impulse response of three systems: under-damped, critically damped, and over-damped.

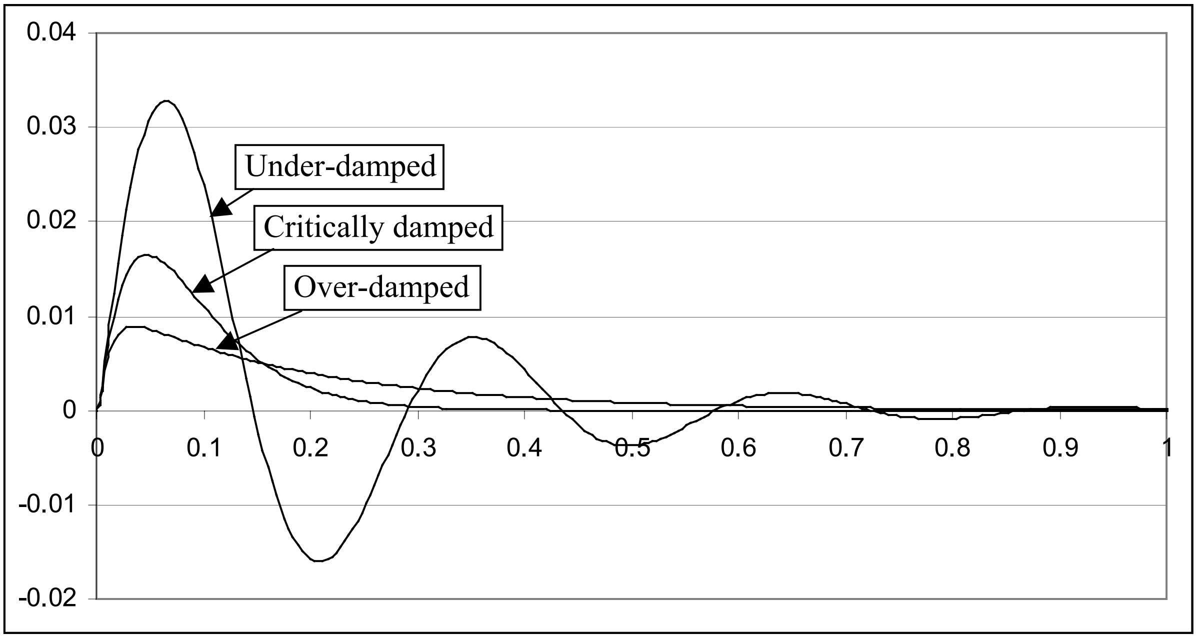

Fig. 3. Impulse response of under-damped, critically damped, and over-damped systems.

Table 1 lists the damping ratios of the three systems whose response is shown in Fig. 3.

Table 1. Damping ratios for three example systems.

System type	**Damping ratio (ζ)**
Under-damped	0.22
Critically damped	1.0
Over-damped	2.2