Decision problems for groups - survey and reflections

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Contents

1 Introduction

This is a survey of decision problems for groups, that is of algorithms for answering various questions about groups and their elements. The general objective of this area can be formulated as follows:

> **Objective:** To determine the existence and nature of algorithms which decide
>
> - *local properties* – whether or not elements of a group have certain properties or relationships;
> - *global properties* – whether or not groups as a whole possess certain properties or relationships.

The groups in question are assumed to be given by finite presentations or in some other explicit manner.

Historically the following three fundamental decision problems formulated by Max Dehn in 1911 have played a central role:

word problem: Let G be a group given by a finite presentation. Does there exist an algorithm to determine of an arbitrary word w in the generators of G whether or not $w =_G 1$?

conjugacy problem: Let G be a group given by a finite presentation. Does there exist an algorithm to determine of an arbitrary pair of words u and v in the generators of G whether or not u and v define conjugate elements of G?

isomorphism problem: Does there exist an algorithm to determine of an arbitrary pair of finite presentations whether or not the groups they present are isomorphic?

In terms of the general objective, the word and conjugacy problems are decision problems about local properties while the isomorphism problem is a decision problem about a global relationship.

Motivation for studying these questions can be found in algebraic topology. For one of the more interesting algebraic invariants of a topological space is its fundamental group. If a connected topological space T is reasonably nice, for instance if T is a finite complex, then its fundamental group $\pi_1(T)$ is finitely presented and a presentation can be found from any reasonable description of T. The word problem for $\pi_1(T)$ then corresponds to the problem of determining whether or not a closed loop in T is contractible. The conjugacy problem for $\pi_1(T)$ corresponds to the problem of determining whether or not two closed loops are freely homotopic (intuitively whether one can be deformed into the other). Since homeomorphic spaces have isomorphic fundamental groups, a solution to the isomorphism problem would give a method for discriminating between spaces (the homeomorphism problem).

Following the development of the theory of algorithms in the 1930's (recursive functions and Turing machines), it was reasonable to expect that Dehn's fundamental problems might be recursively unsolvable. It turns out that not only these problems but a host of local and global decision problems are unsolvable. These developments are discussed in the next two sections. In subsequent sections we consider decision problems restricted to classes of groups enjoying particular algebraic properties, the problem of computing invariants (largely homological) of groups and some measures of computational complexity.

The purpose of this survey is to give some picture of what is known about decision problems in group theory. While a number of references are given for various results, historical matters have been largely neglected. Naturally the choice of material reported on reflects the author's interests and many worthy contributions to the field will unfortunately go without mention. A number of relatively straight forward proofs have been included; usually they are not too difficult, or illustrate the concepts involved or even, occasionally, have a novel aspect. Many concepts and results from mathematical logic, particularly recursive function theory, are explained in an informal manner and occasionally at some length. Hopefully this will make these concepts more accessible for a wide audience.

2 Basic local unsolvability results

A finite presentation π of a group is a piece of notation such as

$$< x_1, \ldots, x_n \mid R_1 = 1, \ldots, R_r = 1 >$$

where the x_i are letters in some fixed alphabet and the R_j are words in the x_i and their inverses x_i^{-1}. The group presented by π, denoted $gp(\pi)$, is the quotient group of the free group on the x_i by the normal closure of the R_j. Usually it is not necessary to distinguish so carefully between a group and its presentation and we often write simply

$$G =< x_1, \ldots, x_n \mid R_1 = 1, \ldots, R_r = 1 >$$

to mean the G is the group defined by the given presentation.

It is convenient to introduce some notation for several decision problems we will consider. Suppose that G is a finitely presented group defined by a presentation as above. Then the *word problem* for G is the decision problem

$$WP(G) = (?w \in G)(w =_G 1).$$

Here the "?" is intended as a sort of quantifier and should be read as "the problem of deciding for an arbitrary word w in G whether or not ...." A closely related problem is the *equality problem*:

$$EqP(G) = (?w_1, w_2 \in G)(w_1 =_G w_2).$$

Of course, $w_1 =_G w_2$ if and only if $w_1 w_2^{-1} =_G 1$ so that an algorithm for solving either of $WP(G)$ or $EqP(G)$ easily yields an algorithm for solving the other. On the other hand, from the viewpoint of computational complexity, these problems are subtly different.

Again using this "?" quantifier, the *conjugacy problem* for G is

$$CP(G) = (?u, v \in G)(\exists x \in G)(x^{-1}ux =_G v).$$

If H is a finitely generated subgroup of G and if H given by say a finite set of words which generate it, then the *generalized word problem* for H in G is the problem of deciding for an arbitrary word w in G whether or not w lies in the subgroup H, that is

$$GWP(H, G) = (?w \in G)(w \in H).$$

When the subgroup H is an arbitrary finitely generated subgroup rather than a fixed one we write simply $GWP(G)$.

On the face of it, each of these algorithmic problems appears to depend on the given presentation. We will show below that the solvability of each of these problems is independent of the finite presentation chosen. It can happen that for a particular finitely presented group each of the above problems is solvable. For instance, if G is a finite group given by a multiplication table presentation, it is easy to describe algorithms for solving $WP(G)$, $CP(G)$ and $GWP(G)$. Similarly, if $F =< x_1, \ldots, x_n \mid \ >$ is a finitely generated free group $WP(F)$ is solved by freely reducing and $CP(F)$ is solved by cyclically permuting and freely reducing. The $GWP(H, F)$ for finitely generated subgroups H of F is more difficult and its solution is due to Nielsen (see [71]).

Finally, in terms of the "?" notation, the *isomorphism problem* for finitely presented groups is

$$IsoP = (?\pi_1, \pi_2 \text{ finite presentations})(gp(\pi_1) \cong gp(\pi_2)).$$

We assume the reader is familiar with the rudiments of the theory of algorithms and recursive functions. Thus a set of objects is *recursive* if there is an algorithm for deciding membership in the set. A set S of objects is *recursively enumerable* if there is an algorithm for listing all the objects in S. It is easy to see that every recursive set is recursively enumerable. Moreover, a set S is recursive if and only if both S and its complement are recursively enumerable. A diagonal argument can be used to prove the important result that there exists a set which is recursively enumerable but not recursive. This fact is in a sense the source of all undecidability results in mathematics.

Each of the above decision probems is recursively enumerable in the sense that the collection of questions for which the answer is "Yes" is recursively enumerable. For instance, the set of words w of G such that $w =_G 1$ is recursively enumerable. For it is the set of words freely equal to a product of conjugates of the given finite set of defining relations and this set can (in principle) be systematically listed. Thus $WP(G)$ is recursively enumerable. Now $WP(G)$ is recursively solvable (decidable) exactly when the set of words $\{w \in G \mid w =_G 1\}$ is recursive. So $WP(G)$ is recursively solvable if and only if $\{w \in G \mid w \neq_G 1\}$ is receursively enumerable.

Similarly, one can systematically list all true equations between words of G and all true conjugacy equations so that $EqP(G)$ and $CP(G)$ are recursively enumerable. $GWP(H, G)$ is recursively enumerable since one can list the set of all true equations between words of G and words in the generators of H. Finally, if two presentations present isomorphic groups, then one can be obtained from the other by a finite sequence of Tietze transformations. Since the set of presentations obtainable from a given one by a finite sequence of Tietze transformations is recursively enumerable, it follows that $IsoP$ is recursively enumerable.

We recall the notion of Turing reducibility. If A and B are two sets of objects, we write $A \leq_T B$ if an (hypothetical) algorithm to answer questions about membership in B would yield an algorithm to answer questions about A. Thus the decision problem for A is reducible to that for B. One way to make this precise is through the theory of recursive functions. Recursive functions can be defined as the collection of functions obtained from certain base functions (like multiplication and addition) by closing under the usual operations of composition, minimalization and recursion. A function is said to be B-recursive if it is among the functions obtained from the base functions together with the characteristic function for B by closing under the usual operations. Then $A \leq_T B$ is defined to mean that that the characteristic function of A is B-recursive. Of course, if B is already recursive (that is, membership in B is decidable) and if $A \leq_T B$ then A is also recursive.

Now the relation $\leq_T$ is a partial order so we can form the corresponding equivalence relation. Two sets of objects A and B are *Turing equivalent* $A \equiv_T B$ if each is Turing reducible to the other, that is both $A \leq_T B$ and $B \leq_T A$. In terms of this notation there are some obvious relationships among our decision problems:

$$EqP(G) \equiv_T WP(G) \leq_T CP(G)$$

$$WP(G) \equiv_T GWP(1,G) \leq_T GWP(G).$$

We have already observed the first equivalence. Since $w =_G 1$ if and only if w and 1 are conjugate in G it follows that $WP(G) \leq_T CP(G)$. The other assertions are clear.

A *recursive presentation* is a presentation of the form

$$< x_1, \ldots, x_n \mid R_1 = 1, R_2 = 1, \ldots >$$

where $R_1, R_2, \ldots$ is a recursively enumerable set of words. A finitely generated group G is *recursively presented* if it has a recursive presentation. Of course finitely presented groups are recursively presented but the converse is false. The word problem and conjugacy problem are defined for recursively presented groups as before and they are still recursively enumerable problems.

Lemma 2.1 *Let G be a finitely generated group given by a recursive presentation*

$$G =< x_1, \ldots, x_n \mid R_1 = 1, R_2 = 1, \ldots > .$$

Suppose that H is a finitely generated group with generators $y_1, \ldots, y_m$ and that $\phi : H \to G$ is an injective homomorphism. Then H has a recursive presentation of the form

$$H =< y_1, \ldots, y_n \mid Q_1 = 1, Q_2 = 1, \ldots >$$

where $Q_1, Q_2, \ldots$ is a recursively enumerable set of words in $y_1, \ldots, y_m$. Moreover, $WP(H) \leq_T WP(G)$.

Proof: Let $F =< y_1, \ldots, y_m \mid \ >$ be the free group with basis $y_1, \ldots, y_m$. Now we can write $\phi(y_i) = u_i$ $(i = 1, \ldots, m)$ where the u_i are certain words on $x_1, \ldots, x_n$. There is then a unique homomorphism $\psi : F \to G$ such that $\psi(y_i) = u_i$ $(i = 1, \ldots, m)$ and since ϕ is injective we have $H \cong F/ker\ \psi$. Now the set of all formal products of the words u_i and their inverses is a recursively enumerable set of words of G. The set of words of G equal to the identity is also recursively enumerable. Hence the intersection of these two sets is a recursively enumerable set of words, and it follows that $ker\ \psi$ is a recursively enumerable set of words on $y_1, \ldots, y_m$. The first claim follows by taking $Q_1, Q_2, \ldots$ to be a recursive enumeration of $ker\ \psi$.

For the second claim, suppose that we have an algorithm A_G to solve the word problem for G. We describe an algorithm to solve the word problem for H as follows: let $w(y_1, \ldots, y_m)$ be an arbitrary word in the generators of H. Since ϕ is injective, $w =_H 1$ if and only if $\phi(w) =_G 1$. Now $\phi(w) = w(u_1, \ldots, u_m)$ so we can apply the algorithm A_G to decide whether or not $w(u_1, \ldots, u_m) =_G 1$. If so, then $w =_H 1$; if not, then $w \neq_H 1$. This algorithm solves the word problem for H. Thus $WP(H) \leq_T WP(G)$ completing the proof.

Lemma 2.2 *For finitely presented groups (respectively finitely generated, recursively presented groups), the word problem, conjugacy problem and generalized word problem are algebraic invariants. That is, for any two presentations π_1 and π_2 of the same group on a finite set of generators, $WP(\pi_1) \equiv_T WP(\pi_2)$, $CP(\pi_1) \equiv_T CP(\pi_2)$ and $GWP(\pi_1) \equiv_T GWP(\pi_2)$.*

Proof: The proof is in each case similar to the proof of the second part of the previous lemma except that ϕ is an isomorphism. We omit the details.

The main local unsolvability result is the following:

Theorem 2.3 (Novikov [87], Boone [21]) *There exists a finitely presented group whose word problem is recursively unsolvable.*

The original proofs of this result proceed along the following lines: start with a Turing machine T whose halting problem is unsolvable. That is, the problem of deciding whether the machine started with an arbitrary tape in a certain state will eventually halt is unsolvable. Constructions of Markov and of Post, associate to such a Turing machine a certain semigroup $S(T)$ whose defining relations mimic the transition rules defining the Turing machine T. They show a code word incorporating a tape and state of T is equal in $S(T)$ to a particular fixed halting word, say q_0, if and only if T halts when started with that tape and state.

Groups $G(T)$ having unsolvable word problem are constructed by in turn mimicking the defining relations of $S(T)$ inside a group. The construction is not so direct as the Markov-Post construction and involves starting with free groups and performing a number of HNN-extensions and/or free products with amalgamation. Nevertheless, there is a direct coding of a tape and state of T as a word w of $G(T)$ so that $w =_{G(T)} 1$ if and only if the machine T halts when started with that tape and state. Since T has an unsolvable halting problem, it follows that $G(T)$ has unsolvable word problem.

A readable account of the Novikov-Boone Theorem along these lines can be found in the textbook by Rotman [92].

In view of the previously noted relationships among our various decision problems, the Novikov-Boone Theorem has the following immediate corollary:

Corollary 2.4 *There exists a finitely presented group G such that $WP(G)$, $CP(G)$ and $GWP(G)$ are all recursively unsolvable.*

In contrast to the difficulties encountered for finitely presented groups, it is easy to give examples of finitely generated, recursively presented groups with unsolvable word problem. For example, let $S \subset \mathbf{N}$ be a recursively enumerable set of natural numbers which is not recursive. Define the recursively presented group

$$H_S =< a, b, c, d \mid a^{-i}ba^i = c^{-i}dc^i \ \forall i \in S > .$$

Now H_S can be described as the free product with amalgamation of the free group $< a, b \mid \ >$ and the free group $< c, d \mid \ >$ amalgamating the subgroup (freely) generated by the left hand sides of the indicated equations with the subgroup (freely) generated by the right hand sides. It follows from the normal form theorem for amalgamated free products that $a^{-i}ba^ic^{-i}d^{-1}c^i =_{H_S} 1$ if and only if $i \in S$. Thus $S \leq_T WP(H_S)$ and so $WP(H_S)$ is recursively unsolvable.

Using this observation Higman [54] gave a very different proof of the unsolvability of the word problem. Indeed he proved the following remarkable result:

Theorem 2.5 (Higman Embedding Theorem) *A finitely generated group H can be embedded in a finitely presented group if and only if H is recursively presented.*

That finitely generated subgroups of finitely presented groups are recursively presented is contained in our first lemma above. The difficult part of this theorem is to show that a recursively presented group can be embedded in a finitely presented group.

The Novikov-Boone Theorem is an easy corollary. For let H_S be the finitely generated, recursively presented group with unsovable word problem constructed above. By Higman's Embedding Theorem, H_S can be embedded in a finitely presented group, say G_S. Then by an earlier lemma, $WP(H_S) \leq_T WP(G_S)$ and so G_S has unsolvable word problem.

Higman's Embedding Theorem has a number of other remarkable aspects. It provides a complete characterization of the finitely generated subgroups of finitely presented groups - namely they are the recursively presented groups. It also provides a direct connection between a purely algebraic notion and a notion from recursive function theory. Another consequence is the existence of universal finitely presented groups.

Corollary 2.6 ([54]) *There exists a universal finitely presented group; that is, there exists a finitely presented group G which contains an isomorphic copy of every finitely presented group.*

To prove this one systematically enumerates all finite presentations on a fixed countable alphabet. The free product of all of these can be embedded in a two generator group which will be recursively presented. This group can then be embedded in a finitely presented group which is the desired universal group.

It is known from recursive function theory that the relation $\equiv_T$ partitions sets into equivalence classes called *degrees of unsolvability*. Those degrees of unsolvability which contain a recursively enumerable set are called *r.e. degrees of unsolvability* and are of particular interest. The r.e. degrees are then partially ordered by $\leq_T$. There is a smallest r.e. degree denoted $\mathbf{0}$ which consists of the recursive sets and a largest r.e. degree denoted $\mathbf{0}'$ which is essentially the general halting problem for all Turing machines. However, a particular Turing machine can have a halting problem with degree lying strictly between these two. There are infinitely many r.e. degrees and they have a rich structure; for example, they are dense with respect to the partial order $\leq_T$. In view of this varied collection of r.e. degrees, it is natural to ask which degrees arise from word problems of finitely presented groups. The answer is the following strengthening of the Novikov-Boone Theorem:

Theorem 2.7 (Fridman [35], Clapham [27], Boone [21]) *Let* $\mathbf{D}$ *be an r.e. degree of unsolvability. Then there is a finitely presented group whose word problem has degree* $\mathbf{D}$*. In more detail, there is an explicit, uniform construction which when applied to a Turing machine* T *having halting problem of degree* $\mathbf{D}$ *yields a finitely presented group* $G(T)$ *such that* $WP(G(T))$ *is Turing equivalent to the halting problem for* T*.*

The arguments used to prove the Novikov-Boone Theorem already constructed a group $G(T)$ such that the halting problem for T is $\leq_T WP(G(T))$. The difficulty is in showing the word problem isn't any harder than the halting problem for T. The proofs of this result are technically rather difficult.

It is easy to see that the amalgamated free product of two free groups H_S described above has $WP(H_S) \equiv_T S$. Clapham's approach to the previous theorem is to show that Higman's embedding theorem can be made "degree preserving". More precisely, he shows the following:

Theorem 2.8 (Clapham [28]) *If* H *is a finitely generated, recursively presented group, then* H *can be embedded in a finitely presented group* G *such that* $WP(H) \equiv_T WP(G)$*. In particular, a finitely generated group with solvable word problem can be embedded in a finitely presented group with solvable word problem.*

As we shall see in the next section, the precise control of the word problem implicit in this result enables one to obtain additional unsolvability results of a global nature.

Since one always has $WP(G) \leq_T CP(G)$, one can ask whether they are always the same. They can in fact be of any two appropriate r.e. degrees of unsolvability. The most general result in this direction is the following:

Theorem 2.9 (Collins [29]) *Let $\mathbf{D}_1$ and $\mathbf{D}_2$ be r.e. degrees of unsolvability such that $\mathbf{D}_1 \leq_T \mathbf{D}_2$. Then there is a finitely presented group G such that $WP(G)$ has degree $\mathbf{D}_1$ and $CP(G)$ has degree $\mathbf{D}_2$. In particular, there is a finitely presented group with solvable word problem but unsolvable conjugacy problem.*

We turn now to briefly consider other local decision problems concerning elements in a group.

The structure of finitely generated abelian groups can be completely determined from a finite presentation of such a group, and in particular one can solve the word problem for such groups. Consequently, if G is an arbitrary finitely presented group one can effectively determine the structure of its abelianization $G/[G,G]$. So for instance, there is an algorithm to decide whether G is perfect. Moreover, since one can solve the word problem for $G/[G,G]$ it follows that one can decide of a arbitrary word w of G whether or not $w \in [G,G]$.

However, it would seem that any property of elements a finitely presented group which is not determined by the abelianization $G/[G,G]$ will be recursively unrecognizable. The following result show a few common properties of elements are not recognizable.

Theorem 2.10 (Baumslag, Boone and Neumann [10]) *There is a finitely presented group G such that there is no algorithm to determine whether or not a word in the given generators represents*

1. *an element of the center of G;*
2. *an element permutable with a given element of G;*
3. *an n-th power, where $n > 1$ is a fixed integer;*
4. *an element whose class of conjugates is finite;*
5. *a commutator;*
6. *an element of finite order > 1.*

Proof: Fix a finitely presented group U having unsolvable word problem. Define G to be the ordinary free product of U with a cyclic group of order 3 and an infinite cyclic group, that is,

$$G = U * < s \mid \ > * < t \mid t^3 = 1 > .$$

We use the commutator notation $[x, y] = x^{-1}y^{-1}xy$. In the following, w is a variable for an arbitrary word in the generators of U.

The center of G is trivial so w lies in the center of G if and only if $w =_U 1$. So there is no algorithm to determine whether an arbitrary word of G lies in the center. This gives the first assertion. Similarly, w is permutable with s if and only if $w =_U 1$ which establishes the second assertion. The element $s^n[t, w]$ is an n-th power if and only if $w =_U 1$ establishing the third assertion. The conjugacy class of w is finite if and only if $w =_U 1$ since if $w \neq_U 1$ the conjugates $s^{-i}ws^i$ would all be distinct. This gives the fourth assertion. For the fifth assertion, note that $[s, t]w$ is a commutator if and only if $w =_U 1$. Finally for the sixth assertion, observe that tw has infinite order if and only if $w \neq_U 1$, while if $w =_U 1$ then tw has order 3. This completes the proof.

In the next section we will present some related unsolvability results concerning the subgroups of a finitely presented group generated by finite sets of elements. These are a sort of mixture between local and global unsolvability results.

3 Basic global unsolvability results

In this section the existence of a finitely presented group with unsovable word problem is applied to obtain a number of global unsolvability results.

Consider the problem of recognizing whether a finitely presented group has a certain property of interest. For example, can one determine from a presentation whether a group is finite? or abelian? It is natural to require that the property to be recognized is *abstract* in the sense that whether a group G enjoys the property is independent of the presentation of G.

An abstract property P of finitely presented groups is *recursively recognizable* if there is an effective method which when applied to an arbitrary finite presentation π determines whether or not $gp(\pi)$ has the property P. More formally, P is *recursively recognizable* if $\{\pi \mid gp(\pi) \in P\}$ is a recursive set of finite presentations.

It turns out that very few interesting properties of groups are recursively recognizable. To formulate the key result we need the following definition.

Definition 3.1 *An abstract property P of finitely presented groups is said to be a* Markov property *if there are two finitely presented groups G_+ and G_- such that*

1. *G_+ has the property P; and*
2. *if G_- is embedded in a finitely presented group H then H does not have property P.*

These groups G_+ and G_- will be called the positive *and* negative witnesses *for the Markov property P respectively.*

It should be emphasized that if P is a Markov property then the negative witness does not have the property P, nor is it embedded in any finitely presented group with property P.

For example the property of being finite is a Markov property. For G_+ one can take $< a \mid a^2 = 1 >$ which is a finite group. For G_- one can take the group $< b, c \mid b^{-1}cb = c^2 >$ which is an infinite group and therefore not embedded in any finite group.

Similarly, the property of being abelian is a Markov property. Indeed the two groups chosen as witnesses for the property of being finite will also serve as witnesses for the property of being abelian.

An example of a property which is not a Markov property is the property of being perfect, that is $G/[G,G] \cong 1$. For it is not hard to show (and indeed will follow from the constructions given below) that any finitely presented group can be embedded in a perfect finitely presented group. Hence there can be no negative witness G_- for the property of being perfect.

An abstract property P of finitely presented groups is *hereditary* if H embedded in G and $G \in P$ imply that $H \in P$, that is, the property P is inherited by finitely presented subgroups. A property of finitely presented groups P is *non-trivial* if it is neither the empty property nor is it enjoyed by all finitely presented groups. Suppose P is a non-trivial, hereditary property of finitely presented groups. Then, since P is non-trivial, there are groups $G_+ \in P$ and $G_- \notin P$. But if G_- is embedded in a finitely presented group H, then $H \notin P$ because P is hereditary. Thus P is a Markov property with witnesses G_+ and G_-. This proves the following:

Lemma 3.1 *If P is a non-trivial hereditary property of finitely presented groups, then P is a Markov property.*

Another useful observation is the following:

Lemma 3.2 *If $\emptyset \neq P_1 \subseteq P_2$ are properties of finitely presented groups and if P_2 is a Markov property, then P_1 is also a Markov property.*

For if G_- is a negative witness for P_2 and if $K \in P_1$, then P_1 is a Markov property with positive and negative witnesses K and G_-.

Recall from the previous section that Higman has constructed a universal finitely presented group, say U. If P is a Markov property with positive and negative witnesses G_+ and G_-, then G_- is embedded in U so $U \notin P$. Moreover, if U is embedded in a finitely presented group H then so is G_- and hence $H \notin P$. Thus P is a Markov property with positive and negative witnesses G_+ and U. Hence U is a negative witness for every Markov property.

The main unsolvability result concerning the recognition of properties of finitely presented groups is the following:

Theorem 3.3 (Adian [2], [3], Rabin [88]) *If P is a Markov property of finitely presented groups, then P is not recursively recognizable.*

Before indicating a proof of this result, we note the following easy corollaries:

Corollary 3.4 *The following properties of finitely presented groups are not recursively recognizable:*

1. *being the trivial group;*
2. *being finite;*
3. *being abelian;*
4. *being nilpotent;*
5. *being solvable;*
6. *being free;*
7. *being torsion-free;*
8. *being residually finite;*
9. *having a solvable word problem;*
10. *being simple;*
11. *being automatic.*

For each of (1) through (9) is a non-trivial, hereditary property and hence is a Markov property. For (10), it is known (see below) that finitely presented, simple groups have solvable word problem and hence, by the above lemma, being simple is a Markov property. Similarly for (11), automatic groups have solvable word problem and so being automatic is a Markov property.

Corollary 3.5 *The isomorphism problem for finitely presented groups is recursively unsolvable.*

For by (1) in the previous corollary there is no algorithm to determine of an arbitrary presentation π whether or not $gp(\pi) \cong 1$.

Proof of the Adian-Rabin Theorem: We are going to give a simple proof of the Adian-Rabin Theorem which is our modification of one given by Gordon [40]. The construction is quite straightforward and variations on the details can be applied to obtain further results. So suppose that P is a Markov property and that G_+ and G_- are witnesses for P. We also have available a finitely presented group U having unsolvable word problem.

Using these three items of initial data, we construct a recursive family of finite presentations $\{\pi_w \mid w \in U\}$ indexed by the words of U so that if $w =_U 1$ then $gp(\pi_w) \cong G_+$ while if $w \neq_U 1$ then G_- is embedded in U. Thus $gp(\pi_w) \in P$ if and only if $w =_U 1$. Since U has unsolvable word problem, it follows that P is not recursively recognizable.

The family $\{\pi_w \mid w \in U\}$ is rather like a collection of buildings constructed from playing cards standing on edge. Such a building can be rather unstable so that if an essential card is removed (corresponding to $w =_U 1$) then the entire structure will collapse. The main technical result needed is the following.

Lemma 3.6 (Main Technical Lemma) *Let K be a group given by a presentation on a finite or countably infinite set of generators, say*

$$K =< x_1, x_2, \ldots \mid R_1 = 1, R_2 = 1, \ldots > .$$

For any word w in the given generators of K, let L_w be the group with presentation obtained from the given one for K by adding three new generators a, b, c together with defining relations

$$\begin{aligned}
a^{-1}ba &= c^{-1}b^{-1}cbc && (1)\\
a^{-2}b^{-1}aba^2 &= c^{-2}b^{-1}cbc^2 && (2)\\
a^{-3}[w,b]a^3 &= c^{-3}bc^3 && (3)\\
a^{-(3+i)}x_iba^{(3+i)} &= c^{-(3+i)}bc^{(3+i)} \quad i = 1, 2, \ldots && (4)
\end{aligned}$$

where $[w, b]$ is the commutator of w and b. Then

1. *if $w \neq_K 1$ then K is embedded in L_w by the inclusion map on generators;*
2. *the normal closure of w in L_w is all of L_w; in particular, if $w =_K 1$ then $L_w \cong 1$, the trivial group;*
3. *L_w is generated by the two elements b and ca^{-1}.*

If the given presentation of K is finite, then the specified presentation of L_w is also finite.

Proof: Suppose first that $w \neq_K 1$. In the free group $< b, c \mid \ >$ on generators b and c consider the subgroup C generated by b together with the right hand sides of the equations (1) through (4). It is easy to check that the indicated elements are a set of free generators for C since in forming the product of two powers of these elements or their inverses some of the conjugating symbols will remain uncancelled and the middle portions will be unaffected.

Similarly, in the ordinary free product $K* < a, b \mid \ >$ of K with the free group on generators a and b consider the subgroup A generated by b together with the left hand sides of the equations (1) through (4). Using the assumption that $w \neq_K 1$ it is again easy to check that the indicated elements are a set of free generators for A.

Thus assuming $w \neq_K 1$, the indicated presentation for L_w together with the equation identifying the symbol b in each the two factors is the natural presentation for the free product with amalgamation

$$\begin{array}{ccc} (K* < a, b \mid \ >) & * & < b, c \mid \ > . \\ A & = & C \end{array}$$

So if $w \neq_K 1$, then K is embedded in L_w establishing the first claim.

Now let N_w denote the normal closure of w in L_w. Clearly $[w, b] \in N_w$ so by equation (3), $b \in N_w$. But equations (1) and (2) ensure that a, b, c are all conjugate and so a, b, c all belong to N_w. Finally, since each of the system of equations (4) can be solved to express x_i in terms of a, b, c, it follows that $x_i \in N_w$ for $i = 1, 2, \ldots$. Thus each of the generators of L_w belongs to N_w and so $L_w = N_w$. This verifies the second assertion.

Finally, let M be the subgroup of L_w generated by b and ca^{-1}. Equation (1) can be rewritten as $b(ca^{-1})b(ca^{-1})^{-1}b^{-1} = c$ so that $c \in M$. But then from $ca^{-1} \in M$ it follows that $a \in M$. Finally from the system of equations (4) which can be solved for the x_i in terms of a, b, c it follows that $x_i \in M$ for $i = 1, 2, \ldots$ and so $M = L_w$. (For later use we note that neither equation (2) nor equation (3) was used in the proof of the final assertion). This completes the proof of the lemma.

Using this technical lemma it is easy to complete the proof of the Adian-Rabin Theorem. We are given the three finitely presented groups U, G_+ and G_- which can be assumed presented on disjoint alphabets as follows:

$$U = < y_1, \ldots, y_k \mid Q_1 = 1, \ldots, Q_q = 1 >$$

$$G_- = < u_1, \ldots, u_m \mid S_1 = 1, \ldots, S_s = 1 >$$

$$G_+ = < v_1, \ldots, v_n \mid T_1 = 1, \ldots, T_t = 1 >$$

Let $K = U * G_-$ the ordinary free product of U and G_- presented as the union of the presentations of its factors. Since U has unsolvable word problem, K also has unsolvable word problem. Also both U and G_- are embedded in K by the inclusion map on generators. For any word w in the generators of U (these are also generators of K) form the presentation L_w as in the Main Technical Lemma. Finally we form the ordinary free product $L_w * G_+$.

A presentation π_w for these groups $L_w * G_+$ can be obtained by simply writing down all of the above generators together with all of the above defining equations. Such a presentation is defined for any word w in U whether or not $w \neq_U 1$. But it follows from the lemma that if $w \neq_U 1$ then the group G_- is embedded in $gp(\pi_w) = L_w * G_+$ and so $gp(\pi_w) \notin P$ by the definition of a Markov property. On the other hand, if $w =_U 1$ then by the lemma $L_w \cong 1$ and so $gp(\pi_w) \cong G_+$ and hence $gp(\pi_w) \in P$.

Thus we have shown that the recursive collection of presentations

$$\{\pi_w \mid w \text{ a word in } U\}$$

has the property that $gp(\pi_w) \in P$ if and only if $w =_U 1$. Since U has unsolvable word problem, it follows that P is not recursively recognizable. This completes the proof of the Adian-Rabin Theorem.

The Main Technical Lemma (and minor variations on it) can be used to establish a number other results. Here are some well-known results which follow easily:

Corollary 3.7 (Higman, Neumann, and Neumann [55]) *Every countable group K can be embedded in a two generator group L. If K can be presented by n defining relations, then L can be chosen to have n defining relations.*

Proof: Since K is countable it can be presented as in the statement of the Main Technical Lemma. Form L as in the lemma except omit the two defining relations (2) and (3). Only equation (3) involved the parameter w and neither equation (2) nor equation (3) were used in the proof that L is two generator. Then K is embedded in L which is a two generator group with generators b and ca^{-1}. This proves the first assertion. (At the expense of considering cases, one could instead use the lemma as stated by choosing a fixed $w \neq_K 1$.)

Equation (1) of the lemma defines c in terms of these two generators. Then using a as an abbreviation for $(ca^{-1})^{-1}c$, the system of equations (4) define the x_i in terms of the given generators. Hence equation (1) and all of the equations of (4) can be eliminated, leaving only the relations R_j rewritten in terms of the generators b and ca^{-1}. This completes the proof.

Combining the proof of this corollary with the Higman Embedding Theorem we obtain the following result which is frequently useful.

Corollary 3.8 *If the group K can be presented by a recursive set of generators subject to a recursively enumerable set of defining relations, then K can be embedded in a two generator, finitely presented group L. Under this embedding the given generators of K are represented by a recursive set of words in the generators of L.*

Proof: As in the proof of the previous corollary, the group K can be embedded in a two generator recursively presented group, say L_1. By the Higman Embedding Theorem this recursively presented group L_1 can be embedded in a finitely presented group, say K_1. Now again applying the previous corollary, K_1 can be embedded in a two generator finitely presented group L as desired. The final assertion follows from the explicit nature of the embedding of K into the two generator group L_1.

Mixing these constructions with results from recursive function theory, we obtain the following result which has a number of applications, for instance to the study of algebraically closed groups.

Corollary 3.9 (Miller [78]) *There exists a finitely presented group G with unsolvable word problem such that every non-trivial quotient group of G also has unsolvable word problem.*

Proof: A pair of disjoint sets of natural numbers P and Q is said to be *recursively inseparable* if there is no recursive set R such that $P \subseteq R$ and $Q \cap R = \emptyset$. The result we need from recursive function theory is that there exists a disjoint pair of recursively enumerable sets P and Q which are recursively inseparable. We may suppose that these sets are chosen so that $0 \in P$ and $1 \in Q$. Define K_0 to be the group presented by

$$K_0 =< e_0, e_1, e_2, e_3, \dots \mid e_0 = e_i \ \forall i \in P, \quad e_1 = e_j \ \forall j \in Q > .$$

Since P and Q are recursively enumerable, the previous corollary implies that K_0 can be embedded in a two generator, finitely presented group which we will denote by K. We continue to use the symbol e_k as an abbreviation for the word in K which is the image of image of e_k under the embedding. Now apply the Main Technical Lemma above to K and the word $e_0e_1^{-1}$ to obtain the finitely presented group $G = L_{e_0e_1^{-1}}$. Since P and Q were disjoint, $e_0 \neq_K e_1$ or equivalently $e_0e_1^{-1} \neq_K 1$.

Now suppose that H is a non-trivial quotient group of G. We view H as being presented on the same set of generation symbols as G. Since H is non-trivial, by the second assertion of the Main Technical Lemma $e_0 \neq_H e_1$. Put $R = \{i \mid e_0 =_H e_i\}$. Since H is a quotient group of G it follows that $P \subseteq R$. But since $e_0 \neq_H e_1$ it follows that $Q \cap R = \emptyset$. Because P and Q are recursively inseparable, R is not recursive. Now the $\{e_k\}$ are a recursive set of words in the generators of H so if H had a solvable word problem then R would be recursive. Hence H must have an unsolvable word problem. This completes the proof.

Corollary 3.10 (P. Hall [52], Goryushkin [42], Schupp [97]) *Every countable group K can be embedded in a 2-generator simple group.*

Proof: If $K \cong 1$ the result is clear. Suppose $K \not\cong 1$ and let $x_1, x_2, \ldots$ be a list of the non-trivial elements of K and take K to be presented on these generators. Form the two generator group L as in the Main Technical Lemma except that the equations (3) and (4) are to be replaced by the two systems of equations

$$a^{-(3+2i)}[x_i, b]a^{(3+2i)} = c^{-(3+2i)}dc^{(3+2i)} \quad i = 1, 2, \ldots \tag{3}$$

$$a^{-(4+2i)}x_i ba^{(4+2i)} = c^{-(4+2i)}dc^{(4+2i)} \quad i = 1, 2, \ldots. \tag{4}$$

Note that the commutator $[x_i, b]$ has infinite order since each $x_i \neq_K 1$ and that K is embedded in L.

Using Zorn's lemma, choose a normal N subgroup of L maximal with respect to the property $K \cap N = 1$. The normal closure in L of any x_i contains $[x_i, b]$ and hence d and is thus clearly all of L. It follows that L/N is a two generator simple group containing an isomorphic copy of K. This completes the proof.

The Adian-Rabin Theorem asserts that a large number of properties of finitely presented groups are not recursive. Recursive function theory provides a number of methods for classifying the difficulty of decision problems so it is natural to ask how difficult are various properties of groups to recognize?

Many properties of interest are recursively enumerable. For instance, in the case of the property "being trivial", the collection of all finite presentations of the trivial group is recursively enumerable. To see this one simply observes that each such presentation can be obtained from the obvious one $< x \mid x = 1 >$ by a finite sequence of Tietze transformations and the set of all Tietze transformations of any finite presentation is recursively enumerable.

Similarly, the property of "being abelian" is recursively enumerable. For we know a canonical form in which to present a finitely presented abelian group; that is, these nice presentations are recognizable and every finitely presented abelian group has such a presentation. Since every other presentation can be obtained from a canonical one by Tietze transformations, it follows that "being abelian" is a recursively enumerable property. Similarly being finite, nilpotent, polycyclic and free are all recursively enumerable properties.

An appropriate method for trying to classify familiar group theoretic properties is to try to locate them in the *arithmetic hierarchy*. Recall that a property or relation P is recursively enumerable (r.e.) if and only if it can be expressed in the form $\exists x R$ where R is a recursive relation involving an additional variable. (A coding device can be used to reduce the apparently more general form $\exists x_1 \ldots \exists x_n R_n$ to a single existential quantifier.) The set of relations expressible in this form is denoted Σ_1^0. The superscript "0" here is to distinguish the arithmetic hierarchy we are going to describe from others. The set of relations expressible in the form $\forall x R$ where R is a recursive relation is denoted Π^0_1. Now the complement of an r.e. relation lies in Π^0_1. For if P has the form $\exists x R$ where R is a recursive relation then $\neg P$ has the form $\forall x \neg R$. Since $\neg R$ is also recursive, it follows that $\neg P$ is in Π^0_1. Recall that a relation P is recursive if and only if both P and $\neg P$ are recursively enumerable. Thus P is recursive if and only if $P \in \Sigma^0_1 \cap \Pi^0_1$.

More generally, a relation P is said to be in Σ^0_n if it is expressible in the form $\exists\forall\ldots R$ where there are n alternations of quantifiers. (As mentioned before, adjacent quantifiers of the same type can be collapsed into a single quantifier; only the number of alternations matters.) Thus for example $P \in \Sigma^0_3$ means that $P \leftrightarrow \exists x_1 \forall x_2 \exists x_3 R$ for some recursive relation R. Similarly P is said to be in Π^0_n if it is expressible in the form $\forall\exists\ldots R$ where there are n alternations of quantifiers. Thus $P \in \Pi^0_2$ means that $P \leftrightarrow \forall x_1 \exists x_2 R$ for some recursive relation R.

We need a few fundamental facts about this arithmetic hierarchy. First, if P is a recursive relation then P is in Σ^0_n and in Π^0_n for all $n \geq 0$. If P is in Σ^0_m or in Π^0_m, then P is in Σ^0_n and in Π^0_n for all $n > m$. In particular $\Sigma^0_m \cup \Pi^0_m \subseteq \Sigma^0_{m+1} \cap \Pi^0_{m+1}$. Also P is in Σ^0_n if and only if its complement $\neg P$ is in Π^0_n.

Finally there is the result called the Arithmetical Hierarchy Theorem due to Kleene which asserts these classes are increasingly difficult: for each $n \geq 1$ there is a unary Σ^0_n relation P which is not Π^0_n and hence not Σ^0_n or Π^0_m for any $m < n$. Hence also $\neg P$ is Π^0_n but not Σ^0_n or Π^0_m for any $m < n$. This is of course a generalization of the existence of r.e. but non-recursive sets. In addition, one can show that each Σ^0_n contains certain "most difficult" relations not in Π^0_n which are said to be Σ^0_n-*complete* (and similarly there are Π^0_n-complete relations).

While many properties of groups may be recursively enumerable, it turns out that the property of "having a solvable word problem" is very far from recursively enumerable. Recall that there are constructions which yield groups with word problem equivalent to the halting problem for given Turing machines (Theorem 1.7). Combining this with some results from recursive function theory, Boone and Rogers have shown the following:

Theorem 3.11 (Boone and Rogers [24]) *For finitely presented groups, the property of* having a solvable word problem *is* Σ^0_3*-complete.*

This result has a number of striking consequences about the general enterprise of solving the word problem for finitely presented groups, some of which are the following:

Corollary 3.12 ([24]) *There is no recursive enumeration*

$$G_0, G_1, G_2, \ldots$$

of all finitely presented groups with solvable word problem.

For if there were such an enumeration, then all finite presentations of groups having solvable word problem would be recursively enumerable, that is, Σ^0_1. But by the theorem this is impossible since having a solvable word problem is Σ^0_3-complete.

Similarly combining the Theorem 1.7 with a construction applied to Turing machines Boone and Rogers establish the following:

Theorem 3.13 ([24]) *There is no uniform partial algorithm which solves the word problem in all finitely presented groups with solvable word problem.*

Combining this last corollary with an enumeration of homomorphisms one can show the following result.

Corollary 3.14 (Miller [77]) *There is no universal solvable word problem group. That is, if G is a finitely presented group which contains an isomorphic copy of every finitely presented group with solvable word problem, then G itself must have unsolvable word problem.*

More generally, one can ask what is the level in the arithmetic hierarchy of any Markov property of interest. If it is r.e. then the Adian-Rabin result can be used to show it is Σ^0_1-complete. Upper bounds can be found for those not apparently r.e., but the status of several properties remains unresolved. The table below indicates some of the unresolved issues.

Markov property	recursion theoretic status
being trivial	r.e. (Σ^0_1-complete)
being finite	r.e.
being abelain	r.e.
being nilpotent	r.e.
being polycyclic	r.e.
being solvable	? ($\leq \Sigma^0_3$)
being free	r.e.
being torsion-free	? ($\leq \Pi^0_2$)
being residually finite	? ($\leq \Pi^0_2$)
having solvable word problem	Σ^0_3-complete
being simple	? ($\leq \Pi^0_2$)
being automatic	r.e.

We turn now to some unsolvability results concerning the recognition of certain properties of the subgroups of a finitely presented group generated by finite sets of elements. The following is a general result of this type motivated by the Adian-Rabin Theorem.

Theorem 3.15 (Baumslag, Boone and Neumann [10]) *Let P be an algebriac property of groups and assume that (i) there is a finitely presented group that has P and that (ii) there is an integer n such that no free group F_r of rank $r \geq n$ has P. Then there is a finitely presented group G_P such that there is no algorithm to determine whether or not the subgroup generated by an arbitrary finite set of words in the given generators of G_P has the property P.*

Proof: Fix a finitely presented group U having unsolvable word problem. Let G_+ be a finitely presented group which has property P. By hypothesis, there is an integer n such that the free groups F_r of rank $r \geq n$ do not have P. We may assume that G_+ is generated by at least n elements, say by $t_1, \ldots, t_r (r \geq n)$. Now form the ordinary free product G_P of U with G_+ and a free group of rank two, that is,

$$G_P = U * G_+ * < a, b \mid \; > .$$

We use w as a variable for words in the generators of U. If $w \neq_U 1$ then in G_P the elements

$$a^{-1}b^{-1}[a, w]ba, \ldots, a^{-r}b^{-1}[a, w]ba^r$$

freely generate a free subgroup of rank r. Hence also the elements

$$t_i a^{-i} b^{-1}[a, w]ba^i \quad \text{for } i = 1, \ldots, r$$

freely generate a free subgroup of rank r which does not have P. On the other hand, if $w =_U 1$ then $t_i a^{-i} b^{-1}[a, w]ba^i =_{G_P} t_i$ so in this case these elements just generate G_+ which has P. That is, the elements $t_i a^{-i} b^{-1}[a, w]ba^i$ for $i = 1, \ldots, r$ generate a subgroup having the property P if and only if $w =_U 1$. Since U has unsolvable word problem, the result follows.

Corollary 3.16 ([10]) *There are finitely presented groups G (depending on the property considered) such that there is no algorithm to determine whether or not the subgroup generated by an arbitrary finite set of words of G is*

1. *trivial;*
2. *finite;*
3. *free;*
4. *locally free;*
5. *cyclic;*
6. *abelian;*

7. *nilpotent;*
8. *soluble;*
9. *simple;*
10. *directly decomposable;*
11. *freely indecomposable;*
12. *a group with solvable word problem.*

Using similar constructions, Baumslag, Boone and Neumann further show the following which is similar to the above corollary but not an immediate consequence of the previous theorem:

Theorem 3.17 ([10]) *There are finitely presented groups G (depending on the property considered) such that there is no algorithm to determine whether or not the subgroup generated by an arbitrary finite set of words of G is*

1. *a finitely related subgroup;*
2. *a subgroup of finite index;*
3. *a normal subgroup;*
4. *a subgroup with finitely many conjugates.*

4 Decision problems and constructions

In this section we consider the basic decision problems for finitely presented groups which are built from more elementary groups by such operations as direct products, extensions, free products, amalagamated free products, and HNN-extensions.

The basic decision problems are well behaved with respect to (ordinary) free products. Suppose that A and B are finitely presented groups with solvable word problem (respectively, solvable conjugacy problem). Then their free product $A * B$ has solvable word problem (respectively, solvable conjugacy problem) by stantard results on normal forms and conjugacy in free products. Mihailova [76] has shown the more difficult result that if A and B have solvable generalized word problem, then $A * B$ also has solvable generalized word problem. Concerning the isomorphism problem, the Grushko-Neumann theorem easily implies the following: If C is a recursive class of freely indecomposable finitely presented groups and if there is an algorithm to decide the isomorphism problem for groups in C, then there is an algorithm to decide the isomorphism problem for the class of all free products of finitely many groups in C.

One might have thought the direct product was a rather benign construction. For example it is clear that if A and B are finitely presented groups with solvable word problem (respectively, solvable conjugacy problem) then their direct product $A \times B$ has solvable word problem (respectively, solvable conjugacy problem). However, Mihailova [75] has shown that solvability of the generalized word problem is not preserved by direct products. The proof is based on the following lemma.

Lemma 4.1 *Let M be any group with a given set of generators $\{s_1, \dots, s_n\}$ having quotient group H with presentation*

$$H =< s_1, \dots, s_n \mid R_1 = 1, \dots, R_m = 1 >$$

on the images of the given generators of M. Let $G = M \times M$ be the direct product of two copies of M and let L_H be the subgroup of G generated by the elements

$$(s_1, s_1), (s_2, s_2), \dots, (s_n, s_n)$$
$$(R_1, 1), (R_2, 1), \dots, (R_m, 1).$$

Then for any pair of word u and v in the given generators,

$$(u, v) \in L_H \text{ if and only if } u =_H v.$$

Before beginning the proof, we note that the subgroup L_H is just the pull-back or fibre-product of two copies of the quotient mapping from M onto H.

Proof: Clearly if $(u, v) \in L_H$ then $u =_H v$ since this is true for each of the generators of L_H and H is a quotient group of M.

For the converse, first suppose $w =_H 1$. Then

$$w =_M \prod_{k=1}^{r} X_k(s_i)^{-1} R_{j_k}^{\epsilon_k} X_k(s_i)$$

for suitable words X_k in the given generators. But then in $G = M \times M$ we have

$$(w, 1) =_G \prod_{k=1}^{r} X_k((s_i, s_i))^{-1} (R_{j_k}^{\epsilon_k}, 1) X_k((s_i, s_i)) \in L_H$$

as desired. More generally suppose $u =_H v$. Then $uv^{-1} =_H 1$ and so $(uv^{-1}, 1) \in L_H$. Since L_H contains the diagonal of $G = M \times M$, it contains (v, v) and hence we also have $(u, v) \in L_H$. This completes the proof.

Theorem 4.2 *Let M be a finitely presented group having a quotient group H with unsolvable word problem. Then the group $G = M \times M$ has a finitely generated subgroup L_H such that the generalized word problem for L_H in G is recursively unsolvable.*

Proof: According to the previous lemma using the same notation, $(w, 1) \in L_H$ if and only if $w =_H 1$. Since the word problem for H is unsolvable, the problem of deciding membership in L_H is recursively unsolvable. This proves the theorem.

Corollary 4.3 (Mihailova [75]) *Let F be a finitely generated free group of rank at least 2. Then the group $G = F \times F$ has a finitely generated subgroup L such that the generalized word problem for L in G is recursively unsolvable.*

The above lemma can also be used to obtain a number of other unsolvability results concerning the direct product of two free groups. The following result was proved for free groups of rank at least nine by Miller [77] and later improved to two generators by Schupp [97]

Theorem 4.4 (Miller [77]) *Let F be a free group of rank at least 2 and let $G = F \times F$. The the problem to determine of an arbitrary finite set of words whether or not they generate G is recursively unsolvable.*

Proof: To see this we apply the proof of the Adian-Rabin Theorem in the case of the Markov property "being trivial" so that $G_+ = 1$. In this case one obtains a recursive family of presentations $\{\pi_w \mid w \in U\}$ indexed by words in a group U with unsolvable word problem such that $\pi_w \cong 1$ if and only if $w =_U 1$. Moreover, if we suppose F has free basis $\{s_1, \ldots, s_n\}$, then each π_w can be written as a presentation on the same generating symbols. For by the Main Technical Lemma the presentations could be given on two generators, say s_1 and s_2. In case $n > 2$ the same group can then also be presented by adding additional generators $\{s_3, \ldots, s_n\}$ and additional defining relations $s_3 = 1, \ldots, s_n = 1$.

Now in the above lemma take $M = F$ and let L_w be the subgroup generated by the indicated elements using the presentation π_w as H. Let $H_w = gp(\pi_w)$. Then by the lemma and properties of the π_w we have

$$\begin{aligned} L_w = G &\leftrightarrow \forall u, v \in F((u, v) \in L_w) \\ &\leftrightarrow \forall u, v \in F(u =_{H_w} v) \\ &\leftrightarrow H_w \cong 1 \\ &\leftrightarrow w =_U 1. \end{aligned}$$

Since U has unsolvable word problem, this proves the result.

Continuing to use the notation of the previous proof, let N_w be the kernel of the natural homomorphism from F onto H_w. Now by the above lemma $(y, 1) \in L_w$ if and only if $y \in N_w$. Thus the intersection of L_w with the first factor $F \times \{1\}$ of G is N_w (or more precisely $N_w \times \{1\}$). If $w \neq_U 1$ then from the proof of the Adian-Rabin Theorem, we know U is embedded in H_w and so in particular H_w is infinite. So if $w \neq_U 1$ then N_w is not finitely generated.

Now in $F \times F$ one can check that the centralizer of any element is finitely generated. On the other hand the centralizer of an element $(1, z) \in L_w$ in L_w will have the form $N_w \times \mathbf{Z}$. So if $w \neq_U 1$ then the centralizer of an element of L_w need not be finitely generated. Consequently, if $w \neq_U 1$ then L_w and $G = F \times F$ are not isomorphic. This proves the following:

Theorem 4.5 *Let F be a free group of rank at least 2 and let $G = F \times F$. Then the problem to determine of an arbitrary finite set of words whether or not they generate a subgroup isomorphic to G is recursively unsolvable. Hence the isomorphism problem for subgroups of G given by finite sets of generators is recursively unsolvable.*

Variations on the above arguments can be used to show the following:

Theorem 4.6 (Miller [77]) *Let F be a free group of rank at least two. Put $G = F \times F$. Then G has a finitely generated subgroup L such that L has an unsolvable conjugacy problem. Moreover, the generalized word problem for L in G is unsolvable.*

Notice that the group $F \times F$ is residually free as are all of its subgroups. However, the finitely generated subgroups L constructed in the above are not finitely presented (see Baumslag and Roseblade [19]).

Recall that a property P of groups is a *poly-property* if, whenever N and G/N have P, so has G. Here N is a normal subgroup of G.

Lemma 4.7 *The following are poly-properties*

1. *being finitely generated*
2. *having a finite presentation*
3. *satisfying the maximum condition for subgroups*
4. *being finitely presented and having a solvable word problem.*

That the first three are poly-properties is shown in Hall [51] and that the last is a poly-property is easily verified. Note that it is not enough in the last item to assume finite generation because the extension might not then be even recursively presented (see [12]).

Since $F \times F$ where F is free can have unsolvable generalized word problem, the property "having solvable generalized word problem" is not a poly-property. Neither is "having solvable conjugacy problem". Indeed the following result shows groups with a surprisingly elementary structure can have an unsolvable conjugacy problem:

Theorem 4.8 (Miller [77]) *There is a finitely presented group G with the following properties:*

1. *G is the split extension of one finitely generated free group by another, that is, there is a split short exact sequence of groups $1 \to F \to G \to T \to 1$ where F and T are finitely generated free groups.*
2. *G is residually finite;*
3. *G has solvable word problem;*
4. *G has unsolvable conjugacy problem.*

Also, G is an HNN-extension of the finitely generated free group F with a finite number of stable letters (the generators of T) acting as automorphisms of F. The ordinary free product $G * T_0$ of G with a free group $T_0 \cong T$ has the structure of an amalgamated free product of two finitely generated free groups with finitely generated amalgamation.

Proof: (Sketch) Suppose that

$$U = < s_1, \ldots, s_n \mid R_1 = 1, \ldots, R_m = 1 >$$

is a finitely presented group with unsolvable word problem. Let $F =< q, s_1, \ldots, s_n \mid \;>$ be the free group of rank $n+1$ on the listed generators. The idea is to construct a group G in which the equations of U are mimicked by conjugations of certain words in F. Of course U can not be embedded in G since G is to have solvable word problem. G is defined as follows:

Generators: $q, s_1, \ldots, s_n, d_1, \ldots, d_n, t_1, \ldots, t_m$.

Defining relations:

$$\left.\begin{array}{rcl} t_i^{-1} q t_i & = & q R_i \\ t_i^{-1} s_j t_i & = & s_j \end{array}\right\} \begin{array}{l} 1 \le i \le m \\ 1 \le j \le n \end{array}$$

$$\left.\begin{array}{rcl} d_k^{-1} q d_k & = & s_k^{-1} q s_k \\ d_k^{-1} s_j d_k & = & s_j \end{array}\right\} \begin{array}{l} 1 \le k \le n \\ 1 \le j \le n \end{array}$$

For each d_k and t_i the right hand sides of the above defining relations generate the free group F and so by a theorem of Nielsen [71] they freely generate F. Hence these relations define an action of each d_k and t_i as an automorphism of F. If we denote by T the free group with basis $\{d_1, \ldots, d_n, t_1, \ldots, t_n\}$, then the quotient of G by its normal subgoup F is clearly isomorphic to T. A group with this sort of structure must be residually finite (see [77]). The assertions about HNN-extensions and amalgamated free products follow from general facts about those constructions.

To see that G has unsolvable conjugacy problem, one shows the following: if w is any word on $\{s_1, \ldots, s_n\}$, then qw is conjugate in G to q if and only if $w =_U 1$. Since the word problem for U is unsolvable, it follows that the conjugacy problem for G is unsolvable.

In one direction this claim is easy. For observe that U is the homomorphic image of G obtained by mapping each s_i in G to the corresponding s_i in U and mapping all other generators to the identity 1_U. Denote this homomorphism by ϕ. Then if $y^{-1}qwy =_G q$ it follows that $\phi(y)^{-1}w\phi(y) =_U 1$ and hence $w =_U 1$. For the converse, suppose $w =_U 1$. Then

$$w =_U \prod_{k=1}^{r} X_k(s_i)^{-1} R_{j_k}^{\epsilon_k} X_k(s_i)$$

for suitable words X_k in the given generators. As an example of a conjugation in G, consider the following:

$$\begin{aligned}
(d_1 d_2 t_i^{\epsilon} d_2^{-1} d_1^{-1})^{-1} q d_1 d_2 t_i^{\epsilon} d_2^{-1} d_1^{-1} &=_G d_1 d_2 t_i^{-\epsilon} d_2^{-1} d_1^{-1} q d_1 d_2 t_i^{\epsilon} d_2^{-1} d_1^{-1} \\
&=_G d_1 d_2 t_i^{-\epsilon} d_2^{-1} s_1^{-1} q s_1 d_2 t_i^{\epsilon} d_2^{-1} d_1^{-1} \\
&=_G d_1 d_2 t_i^{-\epsilon} s_1^{-1} s_2^{-1} q s_2 s_1 t_i^{\epsilon} d_2^{-1} d_1^{-1} \\
&=_G d_1 d_2 s_1^{-1} s_2^{-1} q R_i^{\epsilon} s_2 s_1 d_2^{-1} d_1^{-1} \\
&=_G d_1 s_1^{-1} q s_2^{-1} R_i^{\epsilon} s_2 s_1 d_1^{-1} \\
&=_G q s_1^{-1} s_2^{-1} R_i^{\epsilon} s_2 s_1
\end{aligned}$$

Generalizing this calculation, one can find a word Y on $\{d_1, \ldots, d_n, t_1, \ldots, t_n\}$ determined by the representation of w as a product of conjugates of the defining relations of U such that $Y^{-1}qY =_G qw$ which is the desired result (see [77] for more details of the calculation). This completes our sketch of the proof.

The construction of the previous theorem can be combined with the construction used in proving the Adian-Rabin Theorem to show that the isomorphism problem for groups with such a very elementary structure is unsolvable. The details are somewhat more difficult.

Theorem 4.9 (Miller [77]) *Let U be a group with unsolvable word problem. Then there is a recursive class of finite presentations $\Omega = \{\pi_w \mid w \in U\}$ indexed by words of U such that*

1. *each $gp(\pi_w)$ is residually finite;*
2. *each $gp(\pi_w)$ is the split extension of one finitely generated free group by another;*
3. *the word problem for each of the groups $gp(\pi_w)$ is solvable by a uniform method;*
4. *$gp(\pi_w) \cong gp(\pi_1)$ if and only if $w =_U 1$.*

In particular, the isomorphism problem for Ω is unsolvable. Hence the isomorphism problem for finitely presented, residually finite groups is unsolvable.

The foregoing results show, among other things, that "having solvable generalized word problem" and having "solvable conjugacy problem" are not poly-properties. It is natural to ask whether these properties are at least preserved under finite extensions. For the generalized word problem the answer is not difficult.

Lemma 4.10 ([11]) *If G is a finite extension of the finitely generated group H having solvable generalized word problem, then G has solvable generalized word problem.*

In contrast to this, the conjugacy problem for a group G and for a subgroup of finite index in G can be quite different as the following result shows:

Theorem 4.11 ([30])

1. *There is a finitely presented group G_1 with unsolvable conjugacy problem that has a subgroup M of index 2 which has solvable conjugacy problem.*
2. *There is a finitely presented group G_2 with solvable conjugacy problem that has a subgroup L of index 2 which has unsolvable conjugacy problem.*

An example of the first type was given by Gorjaga and Kirkinskii [41], while examples of both of these phenomena were given by Collins and Miller [30]. The proofs are somewhat technical.

A further aspect of the above results is the following: denote by Ψ_0 the set of finitely generated free groups. Then denote by Ψ_1 the collection of groups formed from groups in Ψ_0 by either free product with finitely generated amalgamation or HNN-extension with finitely many stable letters and finitely generated associated subgroups. Similarly form Ψ_2 by applying these constructions to groups in Ψ_1. Note that all of these groups are finitely presented.

By a theorem of Nielsen [71], finitely generated free groups have solvable generalized word problem so each of the groups in Ψ_1 has a solvable word problem. However, the foregoing results show that the generalized word problem, the conjugacy problem and the isomorphism problem can be recursively unsolvable for groups in Ψ_1. Since the generalized word problem is unsolvable for a group in Ψ_1, it follows that the word problem for a suitable group in Ψ_2 is unsolvable.

For example, if F is free of rank at least two and L is a finitely generated subgroup such that the generalized word problem for L in $F \times F$ is unsolvable, one can form the HNN-extension

$$G =< F \times F, t \mid t^{-1}xt = x, x \in L > .$$

Then for any word $y \in F \times F$ we have $t^{-1}yt = y$ if and only if $y \in L$. Since membership in L is not decidable, the word problem for G is unsolvable. Of course $G \in \Psi_2$ and G is finitely presented.

Using the Mayer-Viettoris sequences for (co)homology of free products with amalgamation and for HNN-extensions, it is easy to check that groups in Ψ_1 have cohomological dimension ≤ 2 and that groups in Ψ_2 have cohomological dimension ≤ 3. The above observations give no information about the word problem for groups of cohomological dimension 2. Collins and Miller (unpublished) have verified that some of the groups constructed in the Boone-Britton proofs of the unsolvability of the word problem have cohomological dimension 2.

Theorem 4.12 *There exists a finitely presented group G of cohomological dimension 2 having unsolvable word problem. Indeed, G can be obtained from a free group by applying three successive HNN-extensions where the associated subgroups are finitely generated free groups.*

Of course the associated subgroups in the second and third HNN-extension are only free subgroups of the previous stage in the construction, not subgroups of the original free group. The proof is obtained by making minor variations to the one given in Rotman's textbook [92]. There a group $G = G(T)$ is constructed based on a Turing machine T which is first encoded into a semi-group and that in turn into G. The group G is obviously obtained by successive HNN-extensions. The only difficulty is to check that the associated subgroups in the final HNN-extension can be taken to be free. This can be done by arranging for the Turing machine T and the semigroup constructed to have a few special properties. The proof then uses the technical details of the proof in [92] and appeals to the deterministic nature and special properties of T.

5 Decision problems in algebraic classes of groups

While the fundamental decision problems are unsolvable for finitely presented groups in general, it is interesting to ask whether they can be solved for classes of finitely presented groups enjoying a particular algebraic property.

For example, it is easy to see that for the class of finite groups the word problem, the conjugacy problem, the generalized word problem and the isomorphism problem are all recursively solvable. Given any finite presentation of a finite group G, by enumerating Tietze transformations of the given presentation one can effectively find a multiplication table presentation for G. Each of the fundamental decision problems can then be effectively answered from knowledge of the multiplication table. Of course these are not practical algorithms and there is considerable interest in obtaining efficient practical algorithms for studying finite groups.

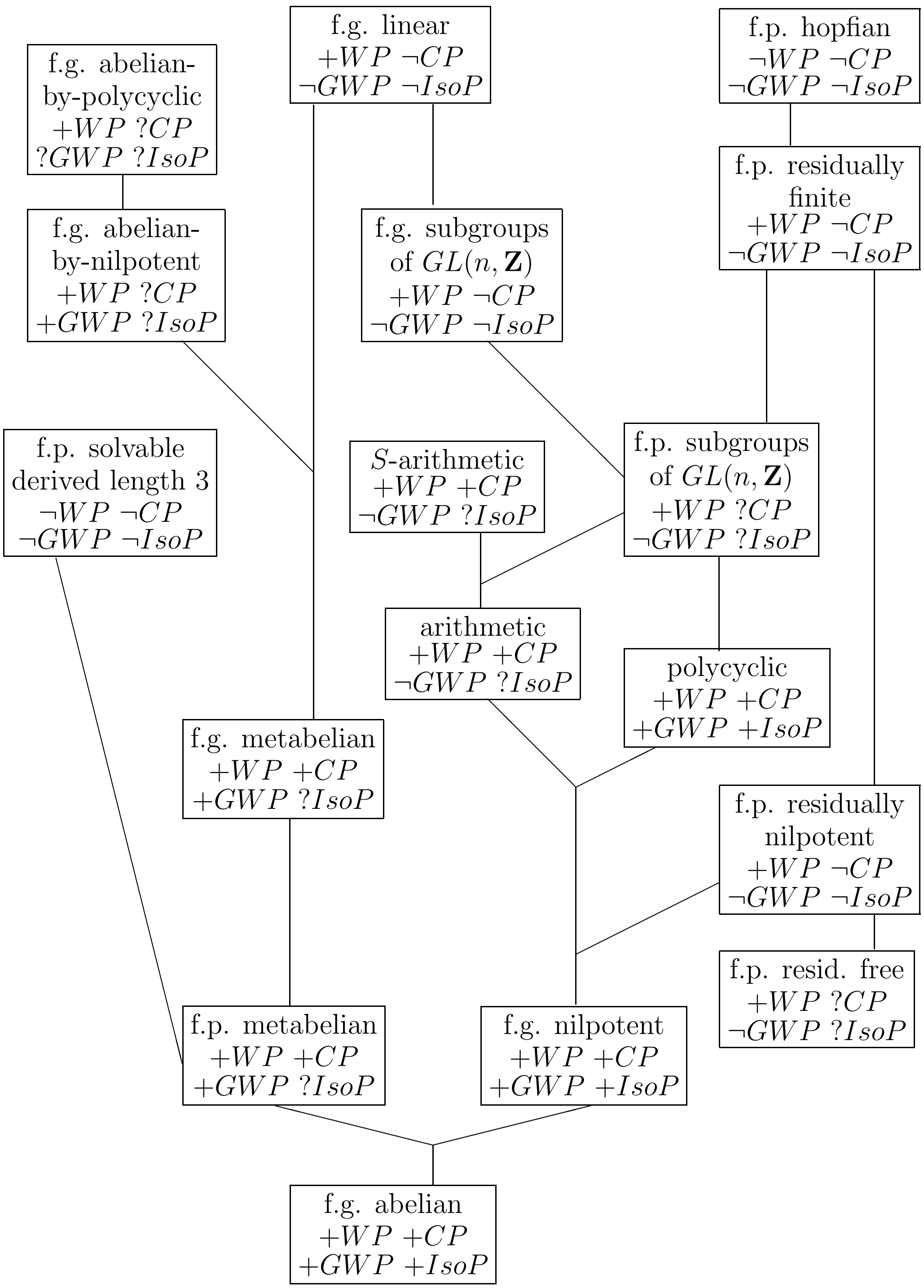
Similarly for the class of finitely generated free groups, the word and conjugacy problems are solvable by standard facts about equality and conjugacy. That the generalized word problem is solvable is a theorem of Nielsen [71]. The isomorphism type of a free group is determined by its rank which can be easily computed from any presentation by considering its abelianization. Hence all of the basic decision problems are solvable for finitely generated free groups.

In this section we survey what is known about the fundamental decision problems for classes of groups enjoying some of the more familiar algebraic properties, for example abelian, solvable, linear and so on. A diagram is included which summarizes the status of four fundamental problems and indicates some of the relationships between the various classes. Generally a class C_1 of groups is connected by a line to a class C_2 higher in the diagram if $C_1 \subseteq C_2$. Unfortunately not all containments and intersections can be accurately portrayed in the diagram.

The following notation is used for the various decision problems in the diagram: $+WP$ means that all groups in the class have solvable word problem; $\neg WP$ means that there exist groups in the class having unsolvable word problem; and $?WP$ means the solvability of the word problem seems to be an open question. Of course if a decision problem is solvable for a class of groups then it is solvable in every class contained that class. Likewise if a decision problem is unsolvable for a class of groups then it is also unsolvable for every larger class.

The following commentary, references and quoted results are intended to explain the status of the decision problems as indicated in the table.

As we have mentioned before, the structure of a finitely generated abelian group can be completely and effectively determined from a finite presentation for such a group. In particular this enables one to solve the word problem in each such group and to solve the ismorphism problem for the class of such groups. Now for abelian groups conjugacy is the same as equality so the conjugacy problem is also solvable. The generalized word problem for a subgroup H of an abelian group G is equivalent to the word problem for G/H so $GWP(G)$ is also solvable. Thus for finitely generated abelian groups


f.g. abelian-by-polycyclic
$+WP$ $?CP$
$?GWP$ $?IsoP$
f.g. linear
$+WP$ $\neg CP$
$\neg GWP$ $\neg IsoP$
f.p. hopfian
$\neg WP$ $\neg CP$
$\neg GWP$ $\neg IsoP$
f.p. residually finite
$+WP$ $\neg CP$
$\neg GWP$ $\neg IsoP$
f.g. abelian-by-nilpotent
$+WP$ $?CP$
$+GWP$ $?IsoP$
f.g. subgroups of $GL(n, \mathbf{Z})$
$+WP$ $\neg CP$
$\neg GWP$ $\neg IsoP$
f.p. solvable derived length 3
$\neg WP$ $\neg CP$
$\neg GWP$ $\neg IsoP$
S-arithmetic
$+WP$ $+CP$
$\neg GWP$ $?IsoP$
f.p. subgroups of $GL(n, \mathbf{Z})$
$+WP$ $?CP$
$\neg GWP$ $?IsoP$
arithmetic
$+WP$ $+CP$
$\neg GWP$ $?IsoP$
polycyclic
$+WP$ $+CP$
$+GWP$ $+IsoP$
f.g. metabelian
$+WP$ $+CP$
$+GWP$ $?IsoP$
f.p. residually nilpotent
$+WP$ $\neg CP$
$\neg GWP$ $\neg IsoP$
f.p. resid. free
$+WP$ $?CP$
$\neg GWP$ $?IsoP$
f.p. metabelian
$+WP$ $+CP$
$+GWP$ $?IsoP$
f.g. nilpotent
$+WP$ $+CP$
$+GWP$ $+IsoP$
f.g. abelian
$+WP$ $+CP$
$+GWP$ $+IsoP$

all of the basic decision problems are solvable.

Theorem 5.1 *Finitely generated linear groups have solvable word problem.*

For finitely generated groups which are linear over a field, this result is proved in Rabin [89] but also follows easily from an older result of Malcev [72]. For Malcev proved that if a finitely generated group G has a faithful representation as a group of matrices over a field, then G also has a faithful representation over a field which is a purely transcendental extension of finite degree of the prime field. As the arithmetic of such a field is clearly effective, the word problem for such a group G is solvable.

More generally, a finitely generated group G of matrices with entries from a commutative ring has a solvable word problem. For since G is finitely generated, the entries in its matrices all lie in some finitely generated commutative ring. Now the arithmetic of such a ring is effective and hence such a group has solvable word problem (see [11]).

In the group $SL(2, \mathbf{Z})$ the two matrices

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

freely generate a free subgroup of rank 2. Moreover, Sanov [93] has shown that an arbirtary 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with integer entries belongs to this subgroup if and only if the following three arithmetic conditions are satisfied:

1. $ad - bc = 1$
2. a and d are congruent to 1 mod 4
3. c and b are even.

It follows that the direct product $F_n \times F_n$ of two free groups of rank n has a faithful representation in $SL(4, \mathbf{Z})$. Combining this representation with the results discussed in the last section, one has the following:

Theorem 5.2 *The conjugacy problem, the generalized word problem and the isomorphism problem are all unsolvable for finitely generated subgroups of* $SL(4, \mathbf{Z})$. *Further, the generalized word problem is unsolvable for arithmetic groups.*

Note that the finitely generated groups in this theorem need not be finitely presented. Such a group can however be finitely described just by giving the finite set of integer matrrices which generate the group.

Recall that a group G is hopfian if $G/N \cong G$ implies $N = \{1\}$. Malcev [72] showed that finitely generated linear groups are residually finite. He further showed that finitely generated residually finite groups are hopfian.

Using the idea of "finite reducibility" which goes back to J. C. C. McKinsey [74] and was applied to groups by Dyson [33] and Mostowski [80] one can show the following result.

Theorem 5.3 *Finitely presented, residually finite groups have solvable word problem.*

Proof: (Sketch) For suppose we are given a finite presentation of a residually finite group G. To decide whether an arbitrary word w is equal to 1 in G we effectively enumerate two lists. The first list consists of all of the words equal to 1 in G, that is, all words which are freely equal to products of conjugates of the given defining relations.

The second list is more complicated: first systematicly enumerate all multiplication table presentations of finite groups. For any function f from the given generators of G to such a finite group K, one can effectively check whether f defines a homomorphism by checking to see whether the formal extension of f sends each of the finitely many defining relations of G to 1 in K. Thus we can effectively enumerate all homomorphisms f_i from G into finite groups K_i.

Now to decide whether $w =_G 1$ we start both listing processes. As each f_i is enumerated, evaluate $f_i(w)$ and check to see whether $f_i(w) = 1$ in K_i. Since G is residually finite, if $w \neq_G 1$ then for some i one eventually finds $f_i(w) \neq 1$ in K_i. On the other hand if $w =_G 1$ then w will appear in the first list. So by waiting until one of these two events occurs we can decide whether $w =_G 1$.

Despite the solvability of the word problem for finitely presented, residually finite groups the results explained in the previous section show the other fundamental decision problems are all unsolvable.

Theorem 5.4 *The conjugacy problem, the generalized word problem and the isomorphism problem are all unsolvable for finitely presented, residually finite groups.*

Recently, Baumslag [9] has varied these constructions to show that these problems are also unsolvable for the class of finitely presented, residually nilpotent groups.

In view of Malcev [72] one can ask whether more generally finitely presented hopfian groups have solvable word problem. That hopfian groups can have unsolvable word problem follows easily from an embedding theorem of Miller and Schupp [79].

Theorem 5.5 *A finitely presented group can be embedded into a finitely presented, hopfian group. In particular, there exist finitely presented hopfian groups with unsolvable word problem.*

One might have hoped that finitely presented solvable groups would have reasonable algorithmic properties. Any such hopes were destroyed by Kharlampovich [58] who constructed a finitely presented solvable group of derived length 3 with unsolvable word problem. Analysing and varying her construction, Baumslag, Gildenhuys and Strebel [18] have shown that the isomorphism problem is unsolvable for such groups. In summary:

Theorem 5.6 *The word problem and the isomorphism problem are unsolvable for finitely presented solvable groups of derived length 3.*

In contrast to this situation for solvable groups in general, a large number of decision problems are recursively solvable for polycyclic groups. A general reference for polycyclic groups is the book by Segal [98]. First it is clear that polycyclic groups are finitely presented and have solvable word problem since these are poly–properties. The conjugacy problem is solvable for polycyclic groups since Remmeslenikov [91] and Formanek [34] have shown they are conjugacy separable, that is, two elements of a polycyclic group which are non–conjugate remain non–conjugate in some finite quotient group. Malcev [73] has shown that polycyclic groups are subgroup separable which implies the generalized word problem is solvable. Alternatively, the generalized word problem for polycyclic groups can be solved by a direct inductive method (see [12]).

Quite remarkably, Grunewald and Segal [48] have shown that the isomorphism problem for finitely generated nilpotent groups is recursively solvable. And more recently Segal [99] has succeeded in solving the isomorphism problem for polycyclic groups. In fact all of the algorithms mentioned carry over to the larger class of polycyclic–by–finite groups. In summary:

Theorem 5.7 *The word problem, conjugacy problem, generalized word problem and isomoprphism problem for polycyclic–by–finite groups are recursively solvable.*

In fact a large number of other algorithmic questions about polycyclic–by–finite groups admit a positive solution (see [15]). Further, there is an algorithm to determine whether or not a given finitely presented solvable group is polycyclic (see [14]).

As part of their work leading to the solution of the isomorphism problem for finitely generated nilpotent groups, Grunewald and Segal solved the conjugacy problem for arithmetic groups (see [48]). Subsequently they extended this to S-arithmetic groups ([49]). They are further able to give a number of algorithms for determining orbits and constructing systems of generators for such groups.

Theorem 5.8 *The conjugacy problem for S-arithmetic groups is recursively solvable.*

Next we turn our attention to finitely generated, abelian–by–polycyclic groups. In [51] Hall has shown that if G is a polycyclic group, then any finitely generated right $\mathbf{Z}G$-module is Noetherian. Of course this is equivalent to the assertion that $\mathbf{Z}G$ is right Noetherian and is an extension of the Hilbert basis theorem. Now a finitely generated, abelian–by–polycyclic group E is an extension of normal abelian subgroup M by a polycyclic group G. Then M can be viewed as a $\mathbf{Z}G$-module which must be finitely generated as a module because E is a finitely generated group. A particular instance of this is the case of finitely generated metabelian groups E where $M = [E, E]$ and $G = E/[E, E]$. When E is metabelian $\mathbf{Z}G$ is actually a finitely generated commutative ring.

In studying such groups one is able to apply the techniques of commutative algebra including the Hilbert basis theorem, the Nullstellensatz and so on. The advantage for algorithmic questions is that large parts of commutative algebra can be carried out effectively. In particular, there is an effective version of the Hilbert basis theorem which carries over to group rings $\mathbf{Z}G$ where G is a polycyclic group. Such group rings are *submodule computable* which roughly means that operations in their finitely generated modules can be effectively carried out, that one can decide membership in finitely generated submodules and that one can find presentations for finitely generated submodules. See [11] for an account of these effective methods.

One easy consequence of these effective extensions of commutative algebra is the solvability of the word problem for finitely generated, abelian–by–polycyclic groups. The generalized word problem is apparently more difficult, but using Theorem 2.14 of [11] one can solve the generalized word problem for finitely generated, abelian–by–nilpotent groups. The problem is reduced to deciding whether two submodules over the same subgroup of a nilpotent group coincide. Whether this can be extended to the abelian–by–polycyclic case seems to be unknown. In summary:

Theorem 5.9 *The word problem for finitely generated, abelian–by–polycyclic groups is recursively solvable. The generalized word problem for finitely generated, abelian–by–nilpotent groups is recursively solvable.*

The particular case of metabelian groups is even more tractable. For finitely generated metabelian groups, Noskov [85] has solved the conjugacy problem. In summary:

Theorem 5.10 *For finitely generated metabelian groups the word problem, generalized word problem and conjugacy problem are all recursively solvable.*

Finitely generated metabelian groups need not be finitely presented. But from an algorithmic point of view there seems to be little advantage in assuming finite presentation. Decision problems are reduced to questions about modules over finitely generated commutative rings. The isomorphism problem for finitely generated metabelian groups seems to be open. Indeed the following important and related algorthmic problem seems to be open: Is the isomorphism problem for finitely presented modules over the integral polynomial ring $\mathbf{Z}[x_1, \ldots, x_n]$ recursively solvable?

This completes our commentary concerning the summary table. Decision problems for certain other algebraic classes will be considered in a subsequent section.

6 Algorithms for further classes of groups

In this section we review the status of the fundamental decision problems for some further classes of groups which did not fit conveniently into the previous two sections. Nevertheless algorithmic questions concerning these groups have played an important role in ongoing developments.

One relator groups: Consider a group G defined by a single defining relation, say

$$G =< x_1, \ldots, x_n \mid r = 1 >$$

where r is a cyclically reduced word on the x_i. Magnus [69] (see also [71] or [66]) initiated the study of such one relator groups by proving his Freheitsatz: the subgroup of G generated by a subset $S = \{x_{i_1}, \ldots, x_{i_k}\}$ of the given generators which omits at least one generator appearing in the cycllically reduced word r is a free group with basis S. His proof was by induction on the length of the defining relation r using a rewriting method which has become a powerful technique for studying one relator groups. Using this technique Magnus [70] succeeded in solving the word problem for such groups in the following strong sense: there is an algorithm to determine for an arbitrary word w of G and an arbitrary subset $S = \{x_{i_1}, \ldots, x_{i_k}\}$ of the given generators whether or not w belongs to the subgroup generated by S. Of course the ordinary word problem is just the case $S = \emptyset$. Note however that one can only decide membership in subgroups of a certain form. The full generalized word problem for one relator groups seems to be open.

In the case the defining relation is a proper power, say $r = u^n$ for some $n > 1$, Newman [84] proved a "Spelling Theorem" which provides a very sharp solution to the word problem (see [66]). Using this result he further solved the conjugacy problem for these one relator groups with torsion. Quite recently, Juhasz [57] has succeeded in solving the conjugacy problem for all one relator groups. In summary:

Theorem 6.1 *The word and conjugacy problems are recursively solvable for groups defined by a single defining relation.*

The isomorphism problem for one relator groups is open. Some modest progress has been made on classifying one relator groups with relation of a particular form, but progress seems difficult.

Simple groups: In [63] Kuznetsov observed the following result which hold more generally for a large number of algebraic systems.

Theorem 6.2 *A recursively presented simple group G has solvable word problem.*

Proof: Suppose $G =< x_1, \ldots, x_n \mid r_1 = 1, r_2 = 1, \ldots >$. If $G = 1$ the result certainly holds. So assume $G \neq 1$ and let u be a fixed word of G such that $u \neq_G 1$. Now for any word w on the generators of G, let G_w be the group obtained from G by adding w as a new defining relator, that is,

$$G_w < x_1, \ldots, x_n \mid w = 1, r_1 = 1, r_2 = 1, \ldots > .$$

Now if $w \neq_G 1$ then $G_w = 1$ since G is simple and in particular $u = 1$ in G_w. But if $w =_G 1$ then $G_w = G$ and of course $u \neq 1$ in G_w. Clearly G_w is again recursively presented.

To decide whether an arbitrary word w is equal to 1 in G begin recursively enumerating two lists of words. The first list consists of all word equal to 1 in G. The second list consists of all words equal to 1 in G_w. If $w =_G 1$ then w will appear in the first list. But $w \neq_G 1$ if and only if u appears in the second list. By examinig the lists until one of these events occurs we can determine whether or not w is equal to 1 in G. This completes the proof.

In particular, finitely presented simple groups have solvable word problem. In contrast to this, Scott [94] has shown the following:

Theorem 6.3 *There is a finitely presented simple group whose conjugacy problem is recursively unsolvable.*

See [95] for a recent survey concerning finitely presented infinite simple groups. Boone and Higman [23] have used a variation on Kuznetsov's argument and the Higman embedding theorem to give the following characterization of finitely presented groups with solvable word problem.

Theorem 6.4 (Boone-Higman Theorem) *A finitely presented group G has solvable word problem if and only if G can be embedded in a simple subgroup of a finitely presented group.*

Proof: (Sketch) Suppose $G \subseteq S \subseteq H$ where S is simple and H is finitely presented. Fix a word $u \in S$ with $u \neq_H 1$. For any word w of H let H_w be the presentation obtained from that of H by adding w as a new defining relator. Now to decide whether a word in the generators of G is equal to 1 in G we regard it as a word, say w, in the generators of H. As above, either w appears on the list of words equal to 1 in H or else since S is simple u appears on the list of words equal to 1 in H_w (in which case $w \neq_H 1$). So by enumerating these two lists we can decide whether w is equal to 1 in G.

For the converse, suppose G has solvable word problem. Then the set of all pairs of words (u, v) such that $u \neq_G 1$ and $v \neq_G 1$ is recursive and can be arranged in a recursive list as say $(u_i, v_i), i = 1, 2, \ldots$. Let $x, t_1, t_2, \ldots$ be new generating symbols and form the presentation

$$\sigma(G) =< G, x, t_1, t_2, \ldots \mid t_i^{-1}[u_i, x]t_i = v_i x^{-1} u_i x, i = 1, 2, \ldots > .$$

Then $\sigma(G)$ is an HNN-extension of the free product of G with the infinite cyclic group generated by x. The associated subgroups are just the infinite cyclic groups genrated by the $[u_i, x]$ and the $v_i x^{-1} u_i x$. A routine argument shows that the word problem for $\sigma(G)$ in the indicated presentation can be solved using the given solution to the word problem for G. Also observe that in $\sigma(G)$ each v_i lies in the normal closure of the corresponding u_i. Thus the normal closure in $\sigma(G)$ of any non-trivial element of G contains all of G.

Now iterating this construction $G_1 = \sigma(G), G_2 = \sigma(G_1), \ldots$ and forming the union

$$S = \bigcup_{j=1}^{\infty} G_j$$

we obtain a recursively presented simple group S in which G is embedded. By the Higman embedding theorem, S can be embedded in a finitely presented group H. This completes the proof.

Small cancellation groups: A subset R of a free group F is *symmetrized* if all of the elements of R are cyclically reduced and if $r \in R$ implies that all cyclically reduced conjugates of $r^{\pm 1}$ are also in R. Thus the words in R are all cyclically reduced and are closed under taking inverses and cyclic permutations. Let $N =< R >^F$ be the normal closure of R in F. Clearly any presentation (respectively, finite presentation) of a group can be converted to a presentation on the same set of generators with a symmetrized set (respectively, finite symmetrized set) of defining relators. One just cyclically reduces the given relators and then closes under inverses and cyclically permutations of words.

In *small cancellation theory* one consider various cancellation hypotheses on a symmetrized set of words R and uses them to deduce properties of the group $G = F/N$. In what follows we assume R is a symmetrized set of words.

If R contains two distinct words of the form $r_1 \equiv bc_1$ and $r_2 \equiv bc_2$ then the word b is called a *piece relative to* R or simply a *piece* when R is understood. Observe that, in forming the product $r_1^{-1}r_2$ and freely reducing, such a piece b is cancelled. Thus a piece is simply a subword of an element of R which can be cancelled by the multiplication of two non-inverse elements of R. Also note that an initial segment of a piece is again a piece.

There are two types of small cancellation hypotheses which assert that pieces are relatively small parts of elements of R. The first is a *metric condition* denoted $C'(\lambda)$ where λ is any positive real (for example $\frac{1}{6}$ or $\frac{1}{8}$). The set R is said to satisfy $C'(\lambda)$ if $r \equiv bc \in R$ where b is a piece implies that $|b| < \lambda|r|$. For instance, $C'(\frac{1}{6})$ means that in forming the product of any two non-inverse elements of R less that $\frac{1}{6}$ of either word is cancelled. Note that $C'(\frac{1}{8})$ implies $C'(\frac{1}{6})$ and that generally $C'(\lambda)$ is a stronger condition for smaller λ.

For any natural number p the *non-metric condition* $C(p)$ asserts that no element of R is a product of fewer than p pieces. Observe that $C'(\lambda)$ implies $C(p)$ for $\lambda \leq 1/(p-1)$. Thus $C'(\frac{1}{6})$ implies $C(7)$.

Another type of condition considered in small cancellation theory is the condition $T(q)$ for q a natural number. R satisfies $T(q)$ if for every sequence $r_1, \ldots, r_m$ $(3 \leq m \leq q)$ with no successive inverse pairs, at least one of the products $r_1r_2, \ldots, r_{m-1}r_m, r_mr_1$ is reduced without cancellation. (This condition turns out to be dual to to the condition $C(p)$ when $1/p + 1/q = 1/2$ in a suitable geometric sense. See [66].)

Small cancellation theory was initiated by Tartakovskii [101] [102] [103] who solved the word problem for groups whose defining relators R satisfy $C(7)$. Greendlinger [43] investigated the metric conditions $C'(\lambda)$ and showed that the word problem for groups with defining relators R satisfying $C'(\frac{1}{6})$ is solvable by *Dehn's algorithm* (see below). In addition he obtained quite a strong result for such groups called "Greedlinger's Lemma" which has a number of applications (see [66]). Further, Greendlinger [44] showed that the conjugacy problem for groups with defining relators R satisfying $C'(\frac{1}{8})$ is solvable by *Dehn's conjugacy algorithm* (see below).

Lyndon [65] introduced the geometric method of *Lyndon-van Kampen diagrams* into small cancellation theory and solved the word problem for groups whose defining relators satisfy $C(6)$ (and also in certain other cases). Schupp [96] used these geometric methods to obtain a solution to the conjugacy problem for these groups. In summary:

Theorem 6.5 *Let F be a free group, R a finite symmetrized subset of F and N the normal closure of R. Assume that R satisfies either $C(6)$, or $C(4)$ and $T(4)$, or $C(3)$ and $T(6)$. Then the word problem and the conjugacy problem for $G = F/N$ are recursively solvable.*

There are many applications of small cancellation theory and much more detailed information than we can present here. Moreover, the geometric methods using Lyndon-van Kampen diagrams provide significant insight into the word and conjugacy problems. A very readable account of small cancellation theory is given in Chapter V of the book [66] by Lyndon and Schupp.

Varieties of groups: In the previous section we noted Kharlampovich's result [58] that there is a finitely presented solvable group of derived length 3 with unsolvable word problem. In particular, this is an example of a finitely presented group satisfying a non-trivial varietal law having unsolvable word problem.

For groups in a non-trivial variety one often considers groups that are *relatively finitely presented*, that is, finitely generated groups which are defined by the laws of the variety together with finitely many additional relations. For example, a result of Hall [51] shows that finitely generated metabelian groups are always relatively finitely presented although they need not be "absolutely" finitely presented.

In a series of papers Kleiman [59], [61], [60], [62] has proved a number of remarkable results concerning varieties which answer a large number of questions. We mention only some of those results of an algorithmic nature.

Theorem 6.6 ([59],[61]) *There is a solvable variety defined by finitely many laws in which the non-cyclic free groups have unsolvable word problem.*

An easy consequence of the unsolvability of the word problem for these relatively free groups is the following:

Corollary 6.7 *There is a finite set S of identities such that the problem to decide whether an arbitrary word represents an identity which is a consequence of S is recursively unsolvable.*

As another consequence, Kleiman shows that the problem of recognizing whether a variety can be factored into a product is recursively unsolvable. In the context of varieties, the *identity problem* for a group is the problem of deciding whether or not an arbitrary identical relation (law) holds in the group. Kleiman [61] has also shown that there is a 3 generator group with solvable word problem in which the identity problem is unsolvable.

7 Geometry and complexity

Suppose that the group G has the presentation $P =< X \mid R >$ where R is a symmetrized set of words on X. Let F be the free group with basis X and N the normal closure of R. If w is a word in the group $G = F/N$ then $w =_G 1$ if and only if $w =_F u_1 r_1 u_1^{-1} \cdots u_m r_m u_m^{-1}$ where the u_i are words in F and the r_i are elements of R. The sequence $u_1 r_1 u_1^{-1}, \ldots, u_m r_m u_m^{-1}$ is said to be an R-sequence of length m for w. For any $w \in N$ we define $A_P(w)$ to be the minimium length of an R-sequence for w.

Using Lyndon-van Kampen diagrams one associates with any R-sequence for w a connected, simply connected *R-diagram* D (consisting of vertices, edges and regions) in the euclidean plane. The edges of D are labelled by elements of F in such a way that the label on the boundary of each region is an element of R and the label on the boundary circuit of D is w (see Chapter V of [66]). Conversely, R-diagrams with boundary label w are the diagrams of suitable R-sequences for w. Thus the $A_P(w)$ is the number of regions in the R-diagram of a minimal R-sequence for w.

For any word u of F we denote the length of u by $|u|$. A useful observation that is easily established using R-diagrams is the following (see [66] Lemma 1.2 of Chapter V, p. 239):

Lemma 7.1 *If $w \in N$ and $A_P(w) = m$ then there is an R-sequence*

$$u_1 r_1 u_1^{-1}, \ldots, u_m r_m u_m^{-1}$$

of length m for w such that

$$|u_j| \leq |w| + \sum_{i=1}^{m} |r_i| \quad j = 1, \ldots, m.$$

In particular, if the lengths of all the elements of R are bounded by some constant, say C_R, then the conjugating elements u_j in a minimal R-sequence can be chosen so that $|u_j| \leq |w| + m \cdot C_R$.

Consequently, when R is finite, in order to decide whether a word w lies in N or not, it suffices to have an upper bound for $A_P(w)$ in terms of $|w|$. For then one can systematically try the bounded number of R-sequences with conjugating elements of length at most the above estimate. Indeed it is easy to see that being able to compute such a bound is equivalent to solving the word problem in the case that R is finite.

Corollary 7.2 *Suppose that G has finite presentation $P =< X \mid R >$ where R is a symmetrized set of words on X. Let F be the free group with basis X and N the normal closure of R. Then the word problem for $G = F/N$ is recursively solvable if and only if there is a recursive function f such that $A_P(w) \leq f(|w|)$ for all $w \in N$.*

It is often helpful to use a more invariant form of the function A_P. Thus one defines [36] the *Dehn function* Ω_P as follows:

$$\Omega_P(n) = max\{A_P(w) \mid w \in N \text{ and } |w| \leq n\}.$$

It can be shown that if the finitely presented group G is defined by two finite presentations P and P_1, then there are constants c_1, c_2 and c_3 such that $\Omega_{P_1}(n) \leq c_1\Omega_P(c_2n) + c_3n$. In a different direction, suppose that the finite presentation $P_2 =< X \mid R_2 >$ of G is obtained from P by adding some elements of N to R so that $R \subseteq R_2$. Then $A_{P_2}(w) \leq A_P(w)$ for all $w \in N$ and $\Omega_{P_2}(n) \leq \Omega_P(n)$.

The above corollary can be restated now as follows: the word problem for G is solvable if and only if there is a recursive function f such that $\Omega_P(n) \leq f(n)$ for all $n > 0$. In fact, if Ω_P is bounded by such a recursive function, it then follows that Ω_P itself is recursive. So the corollary becomes: the word problem for G is solvable if and only if Ω_P is recursive.

Assuming the presentation $P =< X \mid R >$ is finite, there is a standard and familiar way to realize $G = F/N$ as the fundamental group of a 2-complex $K = K(P)$ consisting of a single 0-cell, one 1-cell for each free generator of F, and one 2-cell for each element of R. The 1-cells are attached as loops at the 0-cell giving a wedge of circles. The 2-cells are attached to the 1-skeleton by subdividing the boundary of the cell corresponding to $r \in R$ and sewing onto the 1-skeleton in accordance with r as a sequnce of generators and their inverses. The Seifert-van Kampen Theorem then shows that $\pi_1(K) \cong G$.

If an element $w \in N$ is represented as a loop $\tau(w)$ in the 1-skeleton of K and if D is the R-diagram of a (minimal) R-sequence for w, then there is clearly a continuous map from D to K which sends the boundary of D to $\tau(w)$, sends the edges of D into the 1-skeleton of K and sends the regions of D onto 2-cells of K. Since D is connected and simply connected this map lifts to a map of D into the universal covering space $\widetilde{K}$.

If we fix a 0-cell in the universal cover $\widetilde{K}$ as base point, then the 0-cells of $\widetilde{K}$ are in one-one correspondence with the elements of G. Now the 1-skeleton of $\widetilde{K}$ is a graph which we denote by $\Gamma(G)$ called the *Cayley graph of G*. Another way to view $\Gamma(G)$ is as follows: regard F as the fundamental group of the 1-skeleton K^1 of K. Then $\Gamma(G)$ can be identified with the covering space of K^1 corresponding to the normal subgroup N of F. Note that $\Gamma(G)$ depends only on the choice of generating set X and not on the choice of the defining relators. We write $\Gamma(G) = \Gamma(G, X)$ to show this dependence when necessary. Thus we think of the vertices of $\Gamma(G)$ as being labelled by the elements of G. For each generator in X there is an oriented edge entering and an edge leaving each vertex of $\Gamma(G)$. Paths in $\Gamma(G)$ correspond to words in X starting at the vertex labelled 1. In particular, loops starting at the vertex labelled 1 correspond to elements of N.

The Cayley graph $\Gamma(G) = \Gamma(G, X)$ is given the *word metric* defined by taking each edge to have unit length. A *geodesic* for an element $g \in G$ is a shortest path w in the Cayley graph from the vertex labelled 1 to the vertex labelled g. This w is a shortest word in the generators X representing the element G.

Dehn's algorithm and hyperbolic groups: Let G be a group with finite presentation $P =< X \mid R >$ and let F and N be as above. A *Dehn's algorithm for* G is a finite set of words $\Delta \subset N$ such that any non-empty word $w \in N$ can be shortened by applying a relator in Δ. That is, any non-empty $w \in N$ has the form $w \equiv ubv$ where there is an element of the form $bc \in \Delta$ with $|c| < |b|$. Thus applying the relator bc to w, we can deduce that $w =_G uc^{-1}v$ where $|uc^{-1}v| < |w|$.

If G has Dehn's algorithm Δ then one can solve the word problem for G in a particularly straight forward way: repeatedly try to shorten the word in question by replacing a subword using a relator in Δ. If Δ is a Dehn's algorithm for G as above, then $< X \mid \Delta >$ is also a finite presentation for G and $A_{<X \mid \Delta>}(w) \leq |w|$ for every $w \in N$. Thus also $\Omega_{<X \mid \Delta>}(n) \leq n$ for all n, so the Dehn function is bounded by a linear function. Observe this last property does not depend on the finite presentation of G.

If Δ is a Dehn's algorithm for G and Δ_1 is a larger finite set of words with $\Delta \subseteq \Delta_1 \subset N$, then Δ_1 is also a Dehn's algorithm for G. In particular, if c is a constant larger than the lengths of all the words in Δ then

$$\Delta_c = \{w \in N \mid |w| \leq c\}$$

is also a Dehn's algorithm. Using these observations and Theorem 7.3 below, one can check that having a Dehn's algorithm is independent of the choice of generating set and hence is an abstract property of the group.

The following are examples of groups having presentations with a Dehn's algorithm: free groups, finite groups (multiplication table presentation), and groups satisfying the cancellation condition $C'(\frac{1}{6})$ (Greendlinger's Lemma).

After introducing the fundamental decision problems, Dehn [31] considered the fundamental groups of closed orientable surfaces of genus $g > 1$ having presentation

$$S_g =< a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 > .$$

He showed that the symmetrized closure R of the given defining relator is a Dehn's algorithm for S_g. Note that this presentation satisfies the small cancellation coditions $C(4g)$ and $C'(1/(4g-1))$.

Suppose the group G has a finite presentation $< X \mid R >$ where R is a Dehn's algorithm. A word u is said to be *R-reduced* if u can not be shortened by applying a relator in R. Also, u is *cyclically R-reduced* if every cyclic permutation of u is R-reduced. Clearly if u is any word on X, then by taking cyclic permutations and applying Dehn's algorithm one can effectively find a cyclically R-reduced word u' which is conjugate to u in G.

We say that *Dehn's conjugacy algorithm* solves the conjugacy problem for G if there is an integer constant c such that two non-trivial cyclically R-reduced words u and v are conjugate in G if and only if there is a word z on X such that $u =_G z^{-1}vz$ and $|z| \leq c \cdot (|u| + |v|)$. This clearly provides a recursive solution to the conjugacy problem for G since only finitely many conjugating elements z need to be tested. Also note that the number of regions in an annular R-diagram representing such a conjugacy equation is bounded by a linear function of $(|u| + |v|)$.

Dehn also solved the conjugacy problem for the groups S_g by showing what we have called Dehn's conjugacy algorithm applies. Actually, for S_g and more generally $C'(\frac{1}{8})$ groups a stronger condition holds. Namely, if u and v as before are conjugate, then they have cyclic permutations u' and v' such that $u' =_G z^{-1}v'z$ where z is a subword of an element of R. See [66] for details and generalizations to infinite sets of defining relators.

Gromov [45] has given several equivalent definitions of the notion of a *word hyperbolic group*. Let G be a finitely generated group and fix a finite generating set X for G. Let $\Gamma = \Gamma(G, X)$ be the corresponding Cayley graph with the word metric.

A triangle in Γ with geodesic sides is said to be *δ-thin* if any point on one side is at distance less than δ from some point on one of the other two sides. Γ is said to be *δ-hyperbolic* if every triangle in Γ with geodesic sides is δ-thin. Finally G is said to be *word hyperbolic* if it is δ-hyperbolic with respect to some generating set and some fixed $\delta \geq 0$. We remark that one should think of δ as a large integer rather than a small positive real in these definitions.

The following result gives some equivalent characterizations of word hyperbolic groups.

Theorem 7.3 *The following conditions on a finitely presented group G are equivalent:*

1. *G is word hyperbolic (in the sense of δ-thin triangles);*
2. *the Dehn function for G is bounded by a linear function;*
3. *G has a Dehn's algorithm.*

The above is a combination of results of Gromov [45] and of (independently) Lysënok [67] and Shapiro [5] (see also [8]). Not only do word hyperbolic groups have solvable word problem (by this theorem), but Gromov [45] has also shown they have solvable conjugacy problem:

Theorem 7.4 *Let G be a finitely presented word hyperbolic group. Then Dehn's conjugacy algorithm solves the conjugacy problem for G.*

Thus Dehn's algorithm for the word problem always implies that Dehn's conjugacy algorithm solves the conjugacy problem. In terms of R-diagrams this means the following: if the number of regions in a R-diagram which is a disk is bounded by a linear function of the length of the boundary, then the same is true of annular R-diagrams.

Automatic groups: The notion of an *automatic group* was introduced in [26]. Another reference is [17].

We will need the notion of a *finite state automaton*. Intuitively, a finite state automaton M is just a computing device with a fixed finite amount of storage (memory). M reads an input string (or word) from a free monoid Φ on a finite alphabet from a tape and eventually either "accepts" or "rejects" the input string. M reads only in one direction (no backups) and can write only in its fixed internal storage. (If arbitrarily large storage for scratch work were allowed, the resulting class of machines would be as powerful as Turing machines)

The set of strings (words) which M accepts is called the language recognized by M. A *regular language* is a set of words in a free monoid Φ which is recognized by some finite state automaton.

To compare two words using a finite state automaton one pads the shorter with a new symbol, say \$, on the end so the two words have the same length. Then intersperse these two words on the input tape taking symbols alternately from the two words. Equivalently, one can use a two tape automaton which reads its tapes with one (possibly padded) word on each at the same rate (synchronously). We call such a device a *synchronous two-tape automaton*. If instead such a two tape automaton is allowed to read its input tapes at different rates, we call such a device an *asynchronous two-tape automaton*. An asynchronous two tape automaton can recognize far more complicated sets of pairs than a synchronous one.

Let X be a set of generators for a group G. Let Φ be the free monoid with basis $X \cup X^{-1}$. For any word $w \in \Phi$ define $\mu(w) \in G$ to be the element represented by w. An *(synchronously) automatic structure for G* with respect to the generating set X is a regular language L in Φ such that $\mu(L) = G$ together with a synchronous two-tape automaton M which accepts the collection of pairs of elements of L which represent elements of G lying at most a unit apart in the Cayley graph $\Gamma(G, X)$. That is M accepts the set of pairs

$$\{(u, v) \mid u, v \in L \text{ and } \mu(u) = \mu(va) \text{ for some } a \in X \cup X^{-1} \cup \{1\}\}.$$

If in addition there is a finite state automaton M' which accepts the set of pairs

$$\{(u, v) \mid u, v \in L \text{ and } \mu(u) = \mu(bv) \text{ for some } b \in X \cup X^{-1} \cup \{1\}\}$$

the triple (L, M, M') is called a *biautomatic* or *two-sided automatic structure*. Finally, if instead M is an asychronously automaton, we say that (L, M) is an *asynchronously automatic structure*.

Note that while μ maps L onto G it need not be one-one; that is, and element of G may be represented by several elements of L.

The group G is said to be *automatic (respectively, biautomatic or asynchronously automatic)* if it has an automatic (respectively, biautomatic or asynchronously automatic) structure. Of course biautomatic groups are automatic, but it is not known whether these two notions coincide. Automatic groups are asynchronously automatic, but the class of asynchronously automatic groups is much larger.

One of the motivations for studying automatic groups was the observation [26] that word hyperbolic groups are automatic (even biautomatic [38]). Note that if R is a Dehn's algorithm for a word hyperbolic group, then the R-reduced words form a regular language. Moreover, it can be shown (see Gromov [45]) that if G is word hyperbolic with Cayley graph Γ, then the set of words on X corresponding to geodesic paths starting at $1 \in \Gamma$ is a regular language.

While not all of the non-metric small cancellation groups are word hyperbolic, Gersten and Short [37] have shown that groups satisfying any one of $C(6)$, or $C(4)$ and $T(4)$, or $C(3)$ and $T(6)$ have an automatic structure.

The word problem for these classes of automatic groups is solvable, and more detailed information is as follows (see [26] and [17]):

Theorem 7.5 *Asynchronously automatic groups all have solvable word problem. The Dehn function of an automatic group is bounded by a quadratic. The Dehn function Ω of an asynchronously automatic group is bounded by a simple exponential, that is, $\Omega(n) \leq c^n$ for some constant $c > 0$.*

Moreover, Gersten and Short [37] have shown the following result which generalizes previously mentioned results of Schupp and of Gromov:

Theorem 7.6 *The conjugacy problem for biautomatic groups is recursively solvable.*

Their proof uses various closure properties of regular languages and reformulates the conjugacy problem as a question about whether two regular languages have a non-empty intersection. This latter problem is known to be solvable [56].

For asynchronously automatic groups the conjugacy problem is no longer solvable. Indeed, it is not hard to show [17] that the split extension of one finitely generated free group by another is asychronously automatic. Consequently, the examples of [77] discussed in the earlier section on "Decision problems and constructions" show the following:

Theorem 7.7 *There exist asynchronously automatic groups with unsolvable conjugacy problem. The isomorphism problem for asynchronously automatic groups is recursively unsolvable.*

For more information on asynchronously automatic groups see [17]. These various classes of (bi)automatic groups have number of other interesting properties. Thurston has shown that automatic groups are of type FP_∞ (see [4]). Gersten and Short [39] have obtained useful information about subgroups of biautomatic and hyperbolic groups.

Normal forms and rewriting systems: Continuing with the above notations, one difficulty with Dehn's algorithm for solving the word problem is the following: if R is a Dehn's algorithm and we apply the process of R-reduction to a word $w \neq_G 1$ the resulting word, say $\rho(w)$ will be R-reduced but it is not unique. There may be other R-reduced words u with $u =_G \rho(w)$; for instance, one might have applied a completely different sequence of R-reductions.

One would like a solution to the word problem which transforms an arbitrary word into a sort of unique standard form, preferably by straight forward operations. Also it seems reasonable to expect that a subword of a word in standard form is also in standard form; otherwise one should continue trying to transform the subwords. Notice that the R-reduction process has all of these properties except uniqueness.

A set of words T in the free monoid Φ with basis $X \cup X^{-1}$ is a *set of normal forms* for G (with respect to the generating set X) if $\mu|_T$ is a bijection from T onto G. Thus every element of G is represented by a unique element of T. Equivalently, one can view T as a transversal in F of the normal subgroup N. If in addition T is closed under taking of subwords, then we call T a *hereditary set of normal forms*.

Observe that a set of words is closed under taking subwords if and only if it is closed under taking both initial and terminal segments of words. Thus T is a hereditary set of normal forms if and only if it is a two-sided Schreier transversal for N in F. One can well-order the words in Φ by ordering $X \cup X^{-1}$ and then ordering Φ by length and within the same length lexicographically. In [50] M. Hall shows the set T obtained by choosing the least element t_g representing each group element $g \in G$ is a two-sided Schreier transversal.

Thus every group G has a hereditary set of normal forms T with respect to a generating set X. Assuming X is finite, the set T constructed in this way is recursive if and only if G has solvable word problem.

So normal forms exist, but what of "transforming" words into normal form? One such notion has been investigated by computer scientists (see [64] or [68]). Define a *rewrite rule* to be an ordered pair (u, v) of words of Φ such that $u =_G v$. An *application* of the rewrite rule (u, v) consists of replacing a subword of the form u in a word w by v to obtain a new word w'. We write this as $w \to w'$. In more detail, an application of the rewrite rule (u, v) looks like $w \equiv yuz \to w' \equiv yvz$. One often writes $u \to v$ for the rewrite rule itself thereby emphasizing the ordered nature of the rule. The words u and v are called the left and right hand sides of the rewrite rule $u \to v$ respectively. Note that an application of a rewrite rule need not reduce the length of a word, and it may indeed lengthen the word.

Let Λ be a set of rewrite rules. A word w is Λ-*reduced* or Λ-*irreducible* if no subword of w is the left hand side of any rewrite rule in Λ. That is, w is Λ-irreducible if it is impossible to apply any of the rewrite rules in Λ to W.

A set of rewrite rules Λ is a *complete rewriting system* if it satisfies two conditions: (1) the set T of Λ-irreducible words is a hereditary set of normal forms; and (2) there are no infinite chains $w_1 \to w_2 \to \ldots$ of applications of rewrite rules from Λ.

If Λ is a complete rewriting system, it is clear that starting with any word w one can apply successively the rewrite rules of Λ to reach a unique normal form for w. If w_1 and w_2 are two words such that $w_1 =_G w_2$, then applying rewrite rules to each of w_1 and w_2 in any order will eventually lead to the same word. A rewrite system with this property is said to be *confluent*, a condition which could have been used in place of (1). Also, if Λ is a complete rewriting system then the collection of equations $u = v$ where $(u, v) \in \Lambda$ give a presentation for G.

Here is one (non-effective) way to obtain such a complete rewriting system. Choose a hereditary set T of normal forms as before. If w is a word of Φ, denote by $\overline{w}$ the unique element of T which is equal in G to w (the coset representative for wN in F). Take Λ to be the set of rewrite rules $ta \to \overline{(ta)}$ for all $t \in T$ and all $a \in X \cup X^{-1}$. We observe that if G has solvable word problem then this set of rewrite rules is recursive.

Of particular interest is the class of finitely generated groups which have a finite complete rewriting system. If G has a finite complete rewriting system Λ, then G is finitely presented and the set T of Λ-irreducible words is a regular language. The word problem for such a group is easily solved by repeatedly applying the finite set of rewrite rules until an irreducible word is obtained.

Examples of such groups are free groups and finite groups. One can also show that the class of groups with finite complete rewriting systems is closed under ordinary free products and under extensions (see [47]). In particular, this class includes all polycyclic–by–finite groups and groups which are extensions of one finitely generated free group by another. So from our previous results we can summarize the status of the fundamental decision problems as follows:

Theorem 7.8 *The word problem for groups with a finite complete rewriting system is recursively solvable. There exist groups groups with a finite complete rewriting system having unsolvable conjugacy problem and unsolvable generalized word problem. Moreover the isomorphism problem for such groups is unsolvable.*

It should be emphasized again that rewrite rules are not required to be length reducing. If a group G has a finite, length reducing, complete rewriting system, then that system gives a Dehn's algorithm and, moreover, it is known (see [68]) that G must be virtually free.

One interesting feature of groups with a finite complete rewriting system is that they are of type FP_∞ (see [6], [25], [46] and [100]). Moreover, one can in principle effectively calculate free resolutions and carry out certain homological calculations for such groups. The nature of subgroups of such groups remains to be explored, as do a number of generalizations. For instance, instead of finite systems one can consider "regular" complete rewriting systems. Exactly how these might be related to the various types of automatic groups is not yet clear.

Trivial words as a language: Another way of approaching the word problem is to consider the collection $N(\Phi)$ of all words $w \in \Phi$ such that $w =_G 1$. Of course words in $N(\Phi)$ represent elements of N but they may not be freely reduced. One measure of the complexity of the word problem for G is the complexity of $N(\Phi)$ as a language. Thus a group G is said to be *regular* (respectively, *context–free*) if $N(\Phi)$ is a regular (respectively, context–free) language.

Recall that a language is *context-free* if it is recognized by a pushdown automaton which is a non-deterministic finite state automaton with a pushdown stack ("first in, last out") storage device. See [56] for details concerning such languages and machines.

The following result was observed by Anisimov [7]. Its proof is an easy exercise (see [81]).

Theorem 7.9 *A group is regular if and only if it is finite.*

In a series of papers, Muller and Schupp [81], [82] investigate some remarkable connections between groups, pushdown automata, the theory of ends and second-order logic. One of their results [81] is a characterization of context–free groups which involves the notion af accessibility. Subsequently Dunwoody [32] has proved that all finitely presented groups are accessible. So combining these results, one has the following:

Theorem 7.10 *A finitely generated group is context–free if and only if it is virtually free.*

For additional information about rewriting systems and complexity issues for groups the reader may wish to consult the survey article [68].

8 Computability of homological invariants

This section is concerned with decision theoretic aspects of the homological invariants of finitely presented groups. Suppose that G is a group and M a left $\mathbf{Z}G$-module where $\mathbf{Z}G$ denotes the integral group ring of G. If

$$\mathbf{F} : \ldots \to F_3 \to F_2 \to F_1 \to F_0 \to \mathbf{Z} \to 0$$

is a free resolution of $\mathbf{Z}$ by free left $\mathbf{Z}G$-modules then the homology groups of G with coefficients in M are given by $H_*(G, M) = H_*(M \otimes_{\mathbf{Z}G} \mathbf{F})$. Similarly, if M is a left $\mathbf{Z}G$-module, then the cohomology groups with coefficients in M are given by $H^*(G, M) = H^*(Hom_{\mathbf{Z}G}(\mathbf{F}, M))$.

If G is a finitely presented group, then we ask to what extent the sequences of abelian groups $H_*(G, M)$ and $H^*(G, M)$ can be effectively determined?

The most familiar homology group of a group G is $H_1(G, \mathbf{Z}) = G/[G, G]$, the abelianization of G. If G is finitely presented then $G/[G, G]$ is a finitely generated abelian group, and an additive finite presentation for $G/[G, G]$ can be easily written down from a presentation for G. There is then a familiar algorithm for expressing uniquely $G/[G, G]$ as a direct sum of cyclic groups (Smith normal form). This solves the isomorphism problem for $H_1(G, \mathbf{Z})$ and so this group can be effectively determined. We record this as follows:

Proposition 8.1 *The homology group $H_1(G, \mathbf{Z})$ can be effectively determined from a finite presentation for G.*

Next consider the case G is a finite group which may be assumed to be given by a multiplication table presentation. The the additive group of the integral group ring $\mathbf{Z}G$ is a free abelian group of finite rank. Hence a free resolution $\mathbf{F}$ as above can be found in which each F_i is a finitely generated free abelian group (for instance the bar resolution). Now if M is a finitely presented $\mathbf{Z}G$-module, it is also finitely generated as an abelian group and its structure can be completely determined. It is easy to see that all of the maps necessary to compute the homology groups up to any dimension can be effectively computed. Hence each of the finitely generated abelian groups $H_n(G, M)$ and $H^n(G, M)$ can be effectively determined in this case.

Proposition 8.2 *If G is a finite group and M is a finitely presented $\mathbf{Z}G$-module, then each of the homology groups $H_n(G, M)$ and $H^n(G, M)$ can be effectively determined.*

Of course in practice one is not so much interested in computing these groups individually as in establishing general properties of the sequences $H_*(G, M)$ and $H^*(G, M)$ for a group or a collection of groups. The above gives little information about these general questions.

Finally consider the case in which G is a polycyclic-by-finite group. As indicated above, large parts of commutative algebra can be carried out effectively, in particular Hilbert's basis theorem is effective. The analogous results for polycyclic-by-finite groups can likewise be shown to be effective (see [11]). Using this theory one can show the following:

Theorem 8.3 ([11]) *Let G be a polycyclic-by-finite group and let M be a finitely presented $\mathbf{Z}G$-module. Then each of the homology groups $H_n(G, M)$ is a finitely generated abelian group. Moreover, there is a recursive procedure which yields for each $n \geq 0$ a finite presentation of $H_n(G, M)$. The procedure is uniform in the given data.*

In general there is little hope of effectively computing the homology groups $H_n(G, M)$ for M an arbitrary finitely presented $\mathbf{Z}G$-module even when the group G is reasonably nice. Suppose for instance that G is a free group on finitely many generators and let Q be a quotient group of G. Then $\mathbf{Z}Q$ is a cyclic $\mathbf{Z}G$-module. But Q might have unsolvable word problem, in which case the word problem for $\mathbf{Z}Q$ as a $\mathbf{Z}G$-module is unsolvable. Similarly, one can not in general decide whether Q is the trivial group, so one cannot decide whether $\mathbf{Z}Q$ is isomorphic to $\mathbf{Z}$ as a $\mathbf{Z}G$-module.

In view of these considerations it is convenient to restrict one's attention to the case M is a trivial $\mathbf{Z}G$-module and to the case $M = \mathbf{Z}$ in particular. For simplicity we use the abbreviations $H_nG = H_n(G) = H_n(G, \mathbf{Z})$.

Since H_1G can be effectively computed, it is natural to consider H_2G. Now if $G = F/R$ where F is a free group and R is a normal subgroup, then Hopf's formula for H_2G is

$$H_2(G, \mathbf{Z}) = ker\{R/[F, R] \to F/[F, F]\} = (R \cap [F, F])/[F, R].$$

The abelian group $R/[F, R]$ is generated by the images of a set of defining relations for G so if G is given by a finite presentation then H_2G is finitely generated on a set of generators no larger than the number of relations of G. Despite this, the groups H_2G can not be effectively determined.

Theorem 8.4 (Gordon [40]) *There is no algorithm to determine of an arbitrary finitely presented group G whether or not $H_2G = 0$.*

As we shall see, this result follows easily from the sorts of constructions used to prove the Adian-Rabin Theorem. However, it should be pointed out that the property $H_2G = 0$ is definitely not a Markov property of G and so the above result is not an instance of the Adian-Rabin Theorem. This follows from the following result:

Theorem 8.5 ([16]) *Every group G which admits a recursively enumerable presentation can be embedded in a finitely presented acyclic group Q; thus by definition $H_nQ = 0$ for $n > 0$.*

While Gordon's result is not implied by the Adian-Rabin Theorem, it does follow easily from any of the constructions used to prove it. In fact the argument shows a certain class of homological properties are not recognizable. To describe these we introduce the following definition.

Definition 8.1 *An abstract property P of finitely presented groups is said to be a* homological Markov property *if there are two finitely presented groups G_+ and G_- such that*

1. *G_+ has the property P; and*
2. *if Y is a finitely presented group such that $H_nG_- \subseteq H_nY$ for $n > 1$ then Y does not have property P.*

These groups G_+ and G_- will be called the positive *and* negative witnesses *for the homological Markov property P respectively.*

Note that the property $H_2G = 0$ is an example of a homological Markov property. In terms of this definition, the arguments for the Adian-Rabin Theorem show the following result which includes Gordon's result.

Theorem 8.6 *If P be a homological Markov property of finitely presented groups, then P is not recursively recognizable.*

Proof: We apply the Technical Lemma used in the proof of Adian-Rabin Theorem. Let Q be a finitely presented acyclic group with unsolvable word problem. Take $K = Q * G_-$ and for any word w of Q construct L_w as in the Technical Lemma. Finally put $\pi_w = L_w * G_+$. Then if $w \neq_Q 1$ it follows from the Mayer-Viettoris sequence for homology of amalgamated free products that $H_nG_- \subseteq H_nL_w \subseteq H_n(gp(\pi_w))$ for $n > 1$. So in this case $gp(\pi_w) \notin P$. On the other hand, if $w =_Q 1$ then $L_w \cong 1$ and so $gp(\pi_w) \cong G_+$ and hence

$gp(\pi_w) \in P$. Since the word problem for Q is unsolvable, it follows that Q is not recursively recognizable. This completes the proof.

To describe abelian groups on a possibly infinite set of generators we use the notation $< X \mid R >_{ab}$ where $X = \{x_1, x_2, \ldots\}$ is a set of generators and $R = \{r_1, r_2, \ldots\}$ is a set of words on X. The abelian group A presented by $< X \mid R >_{ab}$ is then the quotient of the free abelian group on X by the subgroup generated by the words in R. If X is a recursively enumerable set of symbols and R a recursively enumerable set of words in those symbols we say that $< X \mid R >_{ab}$ is an *r.e. abelian group presentation*.

In [16] Baumslag, Dyer and Miller investigated the possibilities for the whole integral homology sequence $H_n G$ for a finitely presented group. Despite the fact that one knows very little about a finitely presented group from its presentation, the sequence of integral homology groups $H_n G$ turns out to be a sequence of recursively presentable abelian groups.

Theorem 8.7 ([16]) *If G is a recursively presented group, then the integral homology sequence $H_n G$ can be described by a recursively enumerable sequence of r.e. abelian group presentations. Moreover, if G is finitely presented, the first two terms of this sequence are finitely generated.*

Whether or not a complete converse to this statement holds has not yet been resolved. However Baumslag, Dyer and Miller [16] have shown that a wide variety of r.e. sequences of recursively presentable abelian groups can be realized as the integral homology sequence of a finitely presented group. To state their results another definition is needed. An r.e. abelian group presentation $< X \mid R >_{ab}$ is called *untangled* if R is a basis of the subgroup it generates, and otherwise *tangled*. Since subgroups of free abelian groups are free they have bases. But a given r.e. abelian group presentation may be tangled and indeed may not be effectively untangled. The situation is summarized by the following result.

Lemma 8.8 ([16]) *Let $< X \mid R >_{ab}$ be an r.e. abelian group presentation of the abelian group A.*

1. *if A is a torsion-free abelian group there is a recursive procedure which transforms $< X \mid R >_{ab}$ into an untangled r.e. abelian group presentation $< Y \mid S >_{ab}$ of A.*
2. *if the word problem for $< X \mid R >_{ab}$ is recursively solvable there is a recursive procedure which transforms $< X \mid R >_{ab}$ into an untangled r.e. abelian group presentation $< Y \mid S >_{ab}$ of A.*

However, there exist abelian groups A having an r.e. abelian group presentation but having no untangled r.e. abelian group presentations at all.

The main result on realizing a sequence of abelian groups as the integral homology of a finitely presented group is as follows:

Theorem 8.9 *Let $A_1, A_2, A_3, \ldots$ be a sequence of abelian groups in which the first two terms are finitely generated. If the A_i's are given by an r.e. sequence of r.e. abelian group presentations each of which is untangled, then there exists a finitely presented group G whose integral homology sequence is the given sequence, that is $H_nG = A_n$ for $n > 0$.*

If one is interested in constructing a finitely presented group with a specified H_nG for a particular n, the restriction to untangled presentations is not necessary. However, the constructions used to build such a group lose control of the homology in adjacent dimensions.

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