20 42.08% by James Grosjean and Previn Mankodi

So we know we've got an Ace coming. Now comes the hard part: How much do we bet? Begging the question of how much to bet, how would we *play* the hand? If we bet a large chunk of our bankroll, and then receive a hand such as A4 v. 4, would we double? Expectation-maximizing basic strategy does not take into account how risky this play is. The risk-averse among us would merely hit the hand. After we determine the playing strategy, we can then return to the question of how much to bet.

One way to model risk aversion is to imagine that instead of maximizing the expected return of a gamble, the player maximizes the expected logarithm of his total wealth after the gamble—the so-called Kelly Criterion. Because $\log(0) = -\infty$, busting out is an absolute catastrophe, and a Kelly player will never allow that possibility. With a bit of hand waving, the Kelly-optimal bet size is a fraction f of the player's bankroll, such that $f = E/V$, where E and V are the expectation and variance of the gamble. The derivation of this rule of thumb can be found elsewhere.

Doubling

In the example of our player faced with a double down, we need to compute the decision using backwards induction. First, assuming the player will double, what is the optimal doubling amount (since he can double for less)? Stepping back, we now consider whether hitting or standing is better than optimal doubling.

Let's solve for the optimal doubling amount first. Assume that the player already has a bankroll fraction f riding on the hand, and a normalized bankroll of 1. Then he has to choose the doubling amount $\hat{f}$ by solving this equation, where w, l, and t are the probabilities of winning, losing, and pushing the double down:

$$\begin{array}{ll} \max_{\hat{f}} & w\log(1+f+\hat{f}) + l\log(1-f-\hat{f}) + t\log(1) \\ \text{s.t.} & \hat{f} \in [0, f] \\ & f + \hat{f} \leq 1 \end{array}$$

The constraints say that doubling is up to the amount of the original wager f, and that the total amount of money on the hand $f + \hat{f}$ cannot exceed the bankroll of 1. Since $\log(1) = 0$, the last term drops out of the equation. Taking an unconstrained maximization, the first-order-condition gives us:

$$\begin{aligned} \frac{w}{1+f+\hat{f}} - \frac{l}{1-f-\hat{f}} &= 0 \\ w(1-f-\hat{f}) &= l(1+f+\hat{f}) \\ w(1-f) - w\hat{f} &= l(1+f) + l\hat{f} \\ w - wf - l - lf &= (w+l)\hat{f} \\ \hat{f} &= \frac{w-l}{w+l} - f \end{aligned}$$

This last expression shows that the total amount of money we want riding on the hand, $\hat{f} + f$, will be equal to a constant, $(w-l)/(w+l)$. This constant is familiar. The numerator $w - l$ is just our edge on the total money. The denominator $w+l$ is a scaling factor to take into account the variance. If there were no chance of a tie, then $w + l = 1$, and we will receive only even payoffs on our total money. Then we see that we just want to have a fraction of our bankroll equal to our edge out there.

Recalling the constraints, if the original wager f already exceeds the target amount $(w+l)/(w-l)$, we would not be able to double, because $\hat{f} \geq 0$ is required. Also, if the target amount of money is negative, $(w-l)/(w+l) < 0$, we would not double. This is equivalent to $w - l < 0$, which means that we will not double if doubling is a losing proposition. Finally, because we cannot double for more, if the original wager f is less than half the target amount $f < 0.5(w-l)/(w+l)$, we will have to settle for doubling in full $\hat{f} = f$.

Now, backing up our decision tree, we have to compare optimal doubling to the alternative action A, which would be hitting out for A2-A6, and standing for A7-AT. In this step, we refrain from doubling if:

$$w_A \log(1+f) + l_A \log(1-f) + t_A \log(1) \quad \geq \quad w \log(1+f+\hat{f}) + l \log(1-f-\hat{f}) + t \log(1)$$

After cranking the numbers, we arrive at the charts below. The result is that if f is already a large fraction of the bankroll, the Kelly player will not make any soft doubles.

Look at the entries for A6 v. 4-6. These hands are special because the restriction to take only one card on a double down does not change the play of the cards. Even if we do not double, basic strategy dictates that we will take only one hit card. These hands are strictly an issue of money. The chart shows that for A6 v. 5 in 6-deck S17, we would double in full if the original wager f is 5.33489% of the bankroll or less. The target amount of money $\hat{f} + f$ must be twice this fraction, or about 10.67% of the bankroll. If the original fraction f is above that amount, we would not double at all, because we already have too much money out there. Within the range [0.0533489,0.1066976], we double for less, always so that $\hat{f} + f$ is a constant 10.67% of the bankroll.

For some hands, there is only a critical point for full doubling or not at all (because the double-for-less range got "chopped off" by the alternative action of hitting/standing).

As expected, several "close" doubles, such as A2 v. 5 and A4 v. 4 in 6-deck, are not worth the risk for a Kelly player who has even a small piece of his bankroll on the table.

Critical Fractions for Doubling
6-deck

Player Hand	Dealer Upcard 2	3	4	5	6
			S17		
A2	H	H	H	0017547 543296,405963 512647,442721	0176260 557791,395119 524874,430911
A3	H	H	H	0158357 534431,418705 512504,443290	0322337 548336,408110 524762,431575
A4	h	H	0020930 507318,447924 492670,461476	0274664 525728,431377 511621,444767	0450963 539034,420632 523824,432365
A5	h	H	0134761 498905,459542 491884,462571	[0337373,0389394] 517223,442615 510378,445857	[0484669,0633535] 532202,430686 524967,432179
A6	H	[0157437,0239055] 473652,443590 474970,445972	[0335954,0,0671906] 492743,430696 492743,430696	[0533489,1066976] 512878,413984 512878,413984	[0704305,1408608] 523425,394172 523425,394172
A7	S	0200599 510201,359082 505679,416014	0487590 528021,347783 524444,400045	0717093 540269,337141 539261,387218	0729488 586793,306305 560984,369580
			H17		
A2	H	H	H	0023760 543372,405753 512988,442472	0262863 558750,392265 529580,427472
A3	H	H	H	0163685 534584,418425 512845,443030	0396426 550380,404269 529467,427980
A4	h	H	0031595 507869,447112 493486,460847	0279067 525960,431032 511971,444510	0512007 542165,415893 528664,428821
A5	h	H	0143759 499589,458610 492692,461925	[0340552,0393318] 517505,442217 510714,445580	[0527141,0684468] 536006,425268 529544,428536
A6	h	[0154850,0249535] 474307,445095 475838,447250	[0333048,0666094] 493579,431931 493579,431931	[0531858,1063714] 513235,414546 513235,414546	[0677534,1355067] 528228,402155 528228,402155
A7	0022413 488535,375425 488631,430600	0239069 504777,363193 504608,417001	0524303 522781,351753 523400,401002	0733197 537930,338933 538787,387656	0952079 553906,331308 554219,375766
A8	S	S	S	S	0069745 668467,216246 580625,349580

Leading zero and decimal point have been omitted from all entries for space.
If the bankroll fraction f is below the range specified, doubling is for the full fraction.
If the bankroll fraction is above the range, then do not double.
Within the range, the Kelly bettor doubles for a lesser fraction $f^* = \frac{w-l}{w+l} - f$,
where w and l are the probabilities of winning and losing the double down, respectively.
When a single number is shown, not a range, double in full when the betting fraction is less than or equal to the number; otherwise, do not double.
The second line gives win and loss probabilities for basic strategy.
The third line gives win and loss probabilities for doubling.
When not doubling, hit A2-A6 always, stand on A8-AT always, hit A7 v. 9-A, and stand on A7 v. 2-8.
If splitting AA is not an option, double in full v. 6 if $f \leq 0.0019036$ (S17) or $f \leq 0.0116616$ (H17).

Critical Fractions for Doubling
1-deck

Player Hand	Dealer Upcard 2	3	4	5	6
			S17		
A2	H	H	0034074 529862,419649 506901,448352	0373565 555527,396796 531478,425330	0430978 561688,393193 535949,420843
A3	H	H	0127506 524385,433630 507826,453254	0466191 546992,410427 530623,428834	0519305 553157,405994 535389,424460
A4	h	H	0163325 510769,449324 500141,457699	[0454191,0465620] 536052,428195 525331,437838	[0523024,0564434] 541559,421302 529894,429533
A5	h	h	[0162990,0171737] 500628,462652 495675,464380	[0385338,0487725] 523407,441294 517993,443865	[0561437,0777017] 541817,425899 536557,428225
A6	[0036249,0050377] 461169,454072 462698,456037	[0200267,0400533] 477300,441506 479711,442762	[0418056,0836110] 500742,423468 500742,423468	[0755506,1511010] 533329,393313 533329,393313	[0725150,1450298] 525986,392743 525986,392743
A7	S	0157213 517450,360634 508019,413552	0775602 544704,340730 540811,384470	0903408 549046,326751 549416,374869	0876493 575674,313500 559982,367552
A8	S	S	S	S	0001921 686051,203697 585919,344606
			H17		
A2	H	H	0044078 530223,419117 506512,447800	0376581 555635,396641 531680,425183	0502981 563743,389944 540513,417531
A3	H	H	0141393 525139,432680 508801,452374	0469156 547138,410229 530828,428652	0589902 556102,401717 539969,420349
A4	h	H	0177692 511746,448186 501248,456715	[0456288,0468261] 536256,427962 525568,437666	[0569937,0628621] 545826,416184 535189,425664
A5	h	h	[0174110,0186472] 501644,461447 496740,463310	[0387538,0490994] 523590,441047 518183,443634	[0604297,0837785] 545415,420811 540484,423926
A6	[0038541,0077080] 462283,455217 464043,456944	[0201620,0403238] 478109,442344 480689,443425	[0419138,0838273] 501942,424298 501942,424298	[0755122,1510242] 533578,393558 533578,393558	[0707153,1414305] 530975,399393 530975,399393
A7	S	0183753 514174,353278 507510,414134	0806607 540644,343993 540117,385213	0910919 548004,327649 549215,375095	[0975841,1050646] 549844,334678 554340,373295
A8	S	S	S	S	0138572 664778,217813 581224,347964

Leading zero and decimal point have been omitted from all entries for space.
If the bankroll fraction f is below the range specified, doubling is for the full fraction.
If the bankroll fraction is above the range, then do not double.
Within the range, the Kelly bettor doubles for a lesser fraction $f^* = \frac{w-l}{w+l} - f$,
where w and l are the probabilities of winning and losing the double down, respectively.
When a single number is shown, not a range, double in full when the betting fraction is less than or equal to the number; otherwise, do not double.
The second line gives win and loss probabilities for basic strategy.
The third line gives win and loss probabilities for doubling.
When not doubling, hit A2-A6 always, stand on A8-AT always, hit A7 v. 9-A, and stand on A7 v. 2-8.
If splitting AA is not an option, double in full v. 6 if $f \leq 0.0336896$ (S17) or $f \leq 0.0393804$ (H17), and double in full v. 5 if $f \leq 0.0234932$ (S17) or $f \leq 0.0236057$ (H17).

Splitting

As with doubling, splitting may not appeal to the Kelly player who does not want to put more money at risk. Without the ability to split for less, the player would be forced to violate conventional basic strategy if his initial wager were 50% or more of his bankroll. At a wager of "only" 25% of his bankroll, the Kelly player would be unable to split to four hands.

Ever the bane of blackjack research, splitting is more complicated because our maximization requires us to compare four separate options—hitting, and splitting to two, three, or four hands. The expectation for splitting to two hands would be:

$$p \log(1-f) + w_2 \log(1+2f) + w_1 \log(1+f) + l_2 \log(1-2f) + l_1 \log(1-f) + t \log(1)$$

We have introduced probabilities for winning and losing two bets (denoted by the 2 subscript), in addition to the probabilities for winning and losing one bet, tying, and losing to a dealer blackjack (p). The problem gets messier, because these probabilities are not generally known. Most combinatorial analyzers (including ours) are originally designed to produce only the expectation for pair splitting, not the exact payoff distribution. Most analyzers do not enumerate all possibilities of the multi-hand split. They use Griffin's method involving symmetries, card removal, and weighted sums of individual expectations. This approach will not produce variances or payoff distributions. A simulator could produce these statistics, but the accuracy would be insufficient for our needs.

Our approach was to rewrite the combinatorial analyzer to play out the full multi-way split. This can be done in the case of Aces, even under RSA, because of the convenient restriction to one card per split Ace.

Armed with the payoff probabilities for splitting to two, three, or four hands, we then compute the optimal insurance. We do the same for the fourth option, hitting out the AA. Then we select the best option. For once, we won't bore you with the output, except for the optimal insurance amount at the end of this article.

In deciding how far to keep splitting, we have computed a "pre-committed strategy." We could also call this a zero-memory or basic strategy. Our decision is based only on the initial wager size and upcard, not intermediate results on the earlier hands of the split. For instance, our Kelly basic strategist would never split to four hands if his initial wager were 25% of his bankroll, regardless of the upcard, because he would then have 100% of his bankroll at risk, thus allowing for catastrophic loss. In practice, if the player received a Ten on his first Ace, that 25% of the bankroll would no longer be at risk, and he could then consider additional splits. Such complicated decision-making borders on card counting, and we do not consider such possibilities in our basic-strategy analysis.

As expected, we split all the way if the initial wager is small, but at high wagering fractions, we would forgo splitting to hit the AA. In the case of AA v. A, the chart gives the critical fractions for the recomputed bankroll after the insurance bet is lost.

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Critical Fractions for Splitting										
Game Rules	Dealer Upcard									
	2	3	4	5	6	7	8	9	T	A
NoRSA										
6-deck S17	3786	3903	4033	4113	4227	3519	3108	2795	2795	2007
6-deck H17	3784	3901	4023	4109	4173	3519	3108	2795	2795	2296
1-deck S17	4190	4265	4382	4467	4484	4028	3550	3272	2777	3117
1-deck H17	4188	4263	4377	4466	4455	4028	3550	3272	2777	3256
RSA										
6-deck S17	2469	2471	2473	2475	2483	2496	2493	2478	2456	2477
	3240	3246	3251	3257	3276	3312	3262	3094	3031	2692
	3786	3903	4033	4113	4227	3519	–	–	–	–
6-deck H17	2467	2470	2472	2475	2475	2496	2493	2478	2456	2457
	3236	3242	3248	3255	3258	3312	3262	3094	3031	2760
	3784	3901	4023	4109	4173	3519	–	–	–	–
1-deck S17	2494	2495	2495	2496	2495	2499	2499	2495	2477	
	3297	3300	3301	3308	3304	3327	3318	3284	2985	3221
	4190	4265	4382	4467	4484	4028	3550	–	–	–
1-deck H17	2494	2495	2495	2496	2495	2499	2499	2495	2477	
	3297	3300	3301	3308	3301	3327	3318	3284	2985	3270
	4188	4263	4377	4466	4455	4028	3550	–	–	–

Leading zero and decimal point have been omitted for space.

For NoRSA games, split if bet is at or below the critical fraction shown. Example: Split AA v. 4 in 6-deck S17 game if initial wager is 40.33% or less of bankroll.

For RSA games, split to four hands if initial bet is at or below the critical fraction shown on the first line, to three hands if bet is at or below the second number shown, to two hands if bet is at or below the third number; otherwise hit. If a dash is shown, then hit. Example: With a bet of 24.95% or less in a 1-deck game against a dealer 9, split all the way up to four hands. If the bet is 28%, split up to three hands. With 33%, do not split at all; just hit.

Hitting and Standing

In the previous sections we saw that the additional double-down or split money on the table increases our risk. Therefore, Kelly analysis required us to weigh that risk against the increased expectation of doubling or splitting. Now let's backtrack to consider that even hitting and standing decisions can be affected by risk. The reason is that there are three possible outcomes in a blackjack hand—winning, losing, *and pushing.* Two choices could have equal expectations, but if one choice has a higher tie probability, it will have correspondingly lower variance, and would be preferred by a risk-averse player. If we bust out (or stand on a stiff total), we cannot push.

Standing would be better than hitting if:

$$\begin{aligned} w_S \log(1+f) + l_S \log(1-f) + t_S \log(1) &\geq w_H \log(1+f) + l_H \log(1-f) + t_H \log(1) \\ w_S \log(1+f) + l_S \log(1-f) &\geq w_H \log(1+f) + l_H \log(1-f) \end{aligned}$$

Conveniently, the push term drops out because $\log(1) = 0$. Scouring the numbers, we find two interesting cases: A6 v. 7, and A7 v. A. Basic strategy for the former case is to hit out until a hard 17 or higher is reached. By doing so, we improve our expectation by about 15%, but our variance is increased by reducing the push probability:

A6 v. 7					
Player Action	Probability of			Expectation	Variance
	Winning	Losing	Pushing		
6-deck: Stand above Critical Fraction=0.9961947					
Stand	0.2626345847	0.3664601662	0.3709052491	-0.1038255814	0.6183149996
Hit	0.4436218369	0.3889154072	0.1674627559	0.0547064297	0.8295444507
1-deck: Stand above Critical Fraction=0.9782317					
Stand	0.2637118122	0.3533510149	0.3829371730	-0.0896392027	0.6090276403
Hit	0.4453779090	0.3857323998	0.1688896913	0.0596455092	0.8275527220

As it turns out, if, for some inexplicable reason, we had wagered a monstrous percentage of our bankroll, and then received this hand, Kelly analysis would tell us to stand on the hand. We'd have an excellent chance of pushing immediately with a Ten in the hole, and still other hitting combinations that could generate the zero-variance push outcome. The critical fraction at which a Kelly player would stand instead of hit is shown in the table. Why anyone would have bet nearly 100% of the bankroll is a mystery, but, you know, stuff happens.

For the hand A7 v. A, the theory might actually have practical effects:

A7 v. A					
Player Action	Probability of			Expectation	Variance
	Winning	Losing	Pushing		
6-deck S17 (BS=h): Stand above Critical Fraction=0.1943554					
Stand	0.3568569808	0.4571351856	0.1860078336	-0.1002782048	0.8039364480
Hit	0.3845466601	0.4798912462	0.1355620936	-0.0953445861	0.8553473163
6-deck H17 (BS=h)					
Stand	0.2858067365	0.5111724669	0.2030207967	-0.2253657304	0.7461894909
Hit	0.3519598262	0.5124149114	0.1356252624	-0.1604550852	0.8386289033
1-deck S17 (BS=s)					
Stand	0.3644252855	0.4654398872	0.1701348273	-0.1010146016	0.8196612230
Hit	0.3807731952	0.4891587997	0.1300680051	-0.1083856045	0.8581845557
1-deck H17 (BS=h)					
Stand	0.2969145159	0.5208236628	0.1822618213	-0.2239091470	0.7676028726
Hit	0.3490178802	0.5224049937	0.1285771261	-0.1733871135	0.8413597828

In a 6-deck S17 game, the basic strategy of hitting is Kelly-inferior to standing if the wager is greater than or equal to 19.43554% of the bankroll, which, skipping ahead, would be the case for a Kelly player who knew he had an Ace coming before he had to place his wager.

Surrender

With a soft hand, basic strategy provides an expectation far higher than the -50% return from surrender; however, surrender is a zero-variance option. An Ace as first card warrants a large bet, but if the player does not receive a natural, he can be in trouble, and the Kelly player may then consider surrendering. The mathematical condition is to surrender if:

$$\begin{aligned}\log(1-\tfrac{1}{2}f) &\geq w_{\text{A}}\log(1+f)+l_{\text{A}}\log(1-f)+t_{\text{A}}\log(1)\\ &= w_{\text{H}}\log(1+f)+l_{\text{H}}\log(1-f)\end{aligned}$$

The win, loss, and tie probabilities are for the best Alternative action. That could include count-based deviations in play, or even Kelly-based deviations as given in the previous section.

For any hand with a chance of losing (i.e., a non-natural), there will be some critical fraction of the bankroll above which the player will surrender. To see this, imagine a player with his entire bankroll riding on the hand. The right-hand side of the equation is $w\log(2) + l\log(0) + t\log(1) = w\log(2) + l\log(0) = -\infty$, because $\log(0) = -\infty$ (and $l > 0$). To avoid any chance of that bad outcome, the Kelly player would surrender to lock in a utility payoff of $\log(1-\frac{1}{2}) = \log(1/2) = \log(1) - \log(2) = -\log(2)$.

The calculations show that the "earliest" a Kelly player would surrender would be if he held A5 v. A in a 1-deck H17 game with 57.22587% or more of his bankroll riding. The bankroll fraction at risk must be quite large before we would surrender a soft hand, but, for completeness, the chart is below.

Notice that with A9, the player would sooner surrender against 2-6 than against 7-A. The reason is that a player who holds a 20 is at greater risk from the 2-6 up, because the dealer will have a chance to hit and possibly produce a 21, whereas the big upcards have a good chance of being already pat with totals of 17-19, which pose no threat against our hero's 20.

Player Hand	Dealer Upcard 2	3	4	5	6	7	8	9	T	A
Critical Fractions for Surrender										
6-deck S17										
AA	8834719	8951672	9015446	9161472	9269095	9267865	9000439	8419995	7887137	8272033!
A2	8547635	8717581	8870044	9040007	9149894	9058737	8744772	8146064	7530600	7951710!
A3	8367988	8554798	8730249	8918074	9034132	8814391	8499697	7796614	7144340	7598594!
A4	8179076	8393310	8581672	8787789	8913057	8557567	8150610	7388350	6699910	7177204!
A5	7997996	8227043	8438326	8661958	8810762	8230362	7784181	6947180	6191728	6713269!
A6	8319689	8517446	8702955	8905754	9089044	8951316	7833857	7061948	6466775	6724348!
A7	9207344	9315210	9413634	9491164	9685497	9921559	9683076	7576718	7111768	7872466!
A8	9903496	9918841	9930791	9945535	9971564	9996726	9998248	9964858	8856587	9800906
A9	9999709	9999763	9999830	9999892	9999972	999999*	999999*	999999*	999999*	9999999
6-deck H17										
AA	8825216	8943968	9017781	9162299	9280286	9267865	9000439	8419995	7887137	7903695
A2	8549518	8719073	8874470	9041790	9172022	9058737	8744772	8146064	7530600	7551213
A3	8373292	8558899	8737303	8920759	9067794	8814391	8499697	7796614	7144340	7168709
A4	8187704	8400688	8591234	8791444	8959069	8557567	8150610	7388350	6699910	6717087
A5	8010330	8236933	8450189	8666499	8866915	8230362	7784181	6947180	6191728	6215356
A6	8303488	8504192	8693743	8901865	9037406	8951316	7833857	7061948	6466775	6058230
A7	9160759	9278454	9381795	9478276	9538468	9921559	9683076	7576718	7111768	7008300
A8	9895718	9912897	9925611	9943597	9950066	9996726	9998248	9964858	8856587	9613810
A9	9999659	9999726	9999804	9999884	9999914	999999*	999999*	999999*	999999*	9999959
1-deck S17										
AA	8885414	9028169	9075266	9247560	9314275	9234037	8990694	8403047	8053708	8215611
A2	8503380	8687860	8890258	9133468	9170891	8985742	8665337	8304054	7665297	7880686
A3	8331764	8510431	8764176	9013229	9060452	8701304	8620992	7911972	7288520	7544383
A4	8093004	8340272	8578704	8842099	8913512	8530115	8093907	7349128	6720986	6942655
A5	7908087	8133673	8412133	8691947	8875859	8087381	7645191	6765651	5937558	6260803
A6	8363464	8550526	8791771	9115611	9105132	8983311	7893971	7215689	6569932	6467560
A7	9254179	9380873	9476778	9556455	9645592	9921834	9743145	7665875	7156911	7797187
A8	9919261	9931057	9925206	9957786	9967060	9996694	9998279	9973105	8882698	9795581
A9	9999858	9999730	9999770	9999910	9999955	999999*	999999*	999999*	999999*	999999*
1-deck H17										
AA	8880788	9025113	9078192	9247799	9325316	9234037	8990694	8403047	8053708	7847257
A2	8508598	8691039	8904587	9134800	9197359	8985742	8665337	8304054	7665297	7472496
A3	8341570	8514901	8774666	9015094	9098501	8701304	8620992	7911972	7288520	7117454
A4	8104473	8350909	8592713	8844609	8965045	8530115	8093907	7349128	6720986	6489187
A5	7925613	8145418	8428031	8694740	8926621	8087381	7645191	6765651	5937558	5722587
A6	8355068	8544690	8787523	9114275	9064699	8983311	7893971	7215689	6569932	5921748
A7	9220525	9359241	9453076	9550723	9516078	9921834	9743145	7665875	7156911	6819910
A8	9913820	9927776	9920610	9956977	9948374	9996694	9998279	9973105	8882698	9601565
A9	9999838	9999701	9999744	9999907	9999891	999999*	999999*	999999*	999999*	9999974

Leading zero and decimal point have been omitted for space.
* These indices are greater than 0.9999999.
! At these bet levels, the Kelly player stands on soft 18 against an Ace in 6-deck S17.
Example: With A4 v. 6 in 6-deck S17 game, surrender if betting fraction is greater than or equal to 0.8913057, or 89.13057% of the bankroll.
Do not surrender AT (natural).
When surrender is indicated, alternative is hitting or standing, not doubling.

Insurance

Contrary to what numerous authors have argued, insurance is not a side-bet independent of your main hand. Though usually negative-expectation, the insurance wager will indeed reduce overall variance on the player's "good" hands. For instance, buying insurance when holding a natural will produce a zero-variance payoff of 1 unit, hence the expression "even money."

To determine the Kelly-optimal amount i of insurance when holding a natural with a wagered fraction f of the bankroll, against a dealer's Ace up with probability p of a natural:

$$\max_i [p \log(1+2i) + (1-p)\log(1+\frac{3}{2}f - i)] \text{ s.t. } i \in [0, f/2]$$

An unconstrained maximization produces:

$$\begin{aligned}
\frac{2p}{1+2i} &= \frac{1-p}{1+\frac{3}{2}f-i} \\
2p(1+\frac{3}{2}f-i) &= (1+2i)(1-p) \\
2p+3pf-2pi &= 1-p+2i-2pi \\
2p+3pf &= 1-p+2i \\
3p+3pf-1 &= 2i \\
i &= \frac{1}{2}[3pf+3p-1]
\end{aligned}$$

When the dealer has a natural, the payoff will be $2i = 3pf + 3p - 1$, which, as a fraction of the wager f would be $(3pf+3p-1)/f = 3p + \frac{3p-1}{f}$, which agrees with Griffin's formula in *The Theory of Blackjack*.

Off the top of a d-deck game, the probability that the dealer also will have blackjack (conditional on having an Ace up) is $p = \frac{16d-1}{52d-3}$:

Kelly Criterion Insurance Off-the-Top Player Natural v. Ace Up			
Number of Decks	Dealer Natural Probability	Formula for Insurance i	Critical Fraction for Some Insurance
1	15/49	$(45/98)f - (4/98)$	$4/45 \approx 0.08889$
2	31/101	$(93/202)f - (8/202)$	$8/93 \approx 0.08602$
6	95/309	$(285/618)f - (24/618)$	$24/285 \approx 0.08421$
8	127/413	$(381/826)f - (32/826)$	$32/381 \approx 0.08399$

Notice that when insurance is a negative bet, we will never buy full insurance:

$$\begin{aligned}
p &< \frac{1}{3} \\
3p &< 1 \\
i &= (1/2)[3pf+3p-1] \\
&< (1/2)3pf \\
&< (1/2)f
\end{aligned}$$

The last column of the above table shows that above some critical fraction, we will buy partial insurance ($i > 0$). Since this critical fraction is less than 9% of the bankroll, the Kelly player will be buying some insurance if his wager had been based on knowledge of the imminent Ace.

Now suppose we are considering insuring a hand other than a natural. If p is the probability that the dealer has a natural, w, l, and t are the win, loss, and push ("tie") probabilities against a non-natural ($p + w + l + t = 1$), then we select the insurance amount i:

$$\begin{aligned}
\max_i \quad & [p\log(1-f+2i) + w\log(1+f-i) + l\log(1-f-i) + t\log(1-i)] \\
\text{s.t.} \quad & i \in [0, f/2] \\
& f + i \leq 1
\end{aligned}$$

Notice the additional constraint that $f + i \leq 1$. The player needs to have chips to insure. His initial wager and additional insurance wager cannot exceed his bankroll. We did not have this constraint in the case of a player blackjack, because we assume the player can "mark" the hand. That is, he takes a short-term marker to complete the hand. This is a common practice in old-school Vegas casinos in the case of double downs where the casino wishes to save the time of having an empty-pocketed player fetch additional money to complete the hand. In the case of a player blackjack, the casino incurs no credit risk. A player can take "even money" without having additional chips to buy insurance, so we assume that "partial money" would be acceptable also.

An unconstrained maximization of the insurance problem produces:

$$\begin{aligned}
\frac{2p}{1-f+2i} &= \frac{w}{1+f-i} + \frac{l}{1-f-i} + \frac{t}{1-i} \\
\frac{2p}{1-f+2i} &= \frac{w}{1+f-i} + \frac{l}{1-f-i} + \frac{1-p-w-l}{1-i}
\end{aligned}$$

Unfortunately, the solution for i, as given by the last equation, will be the root of a cubic polynomial. The formula would be too ugly to produce here, and we will have to resort to numerical tactics. Off-the-top, the probability of a dealer natural would be $p = 16d/(52d-3)$ for a d-deck game. The rest of the numbers come from a brute-force combinatorial analysis.

For the hand AA v. A, we present only the optimal insurance amount for the optimal initial wager, instead of any arbitrary wager, in the summary at the end of this article.

Kelly Criterion Insurance, Off-the-Top
6-deck

Bet Size f	Prob-ability	Player Hand A2	A3	A4	A5	A6	A7	A8	A9	AT
S17*										
	p	96/309 ≈ 0.31067961								0.3074
	w	0.2860	0.2749	0.2631	0.2514	0.2364	0.2651	0.3771	0.5079	0.6926
	l	0.3270	0.3409	0.3556	0.3701	0.3625	0.3308	0.1843	0.0532	0
	t	0.0764	0.0735	0.0706	0.0679	0.0904	0.0934	0.1280	0.1283	0
6%		0	0	0	0	0	0	0	0	0
10%		0	0	0	0	0	0	0.68	1.97	0.73
15%		0.87	0.71	0.54	0.37	0.34	0.73	2.53	4.55	3.03
20%		2.01	1.80	1.57	1.36	1.33	1.86	4.22	7.04	5.34
25%		2.99	2.74	2.46	2.20	2.19	2.84	5.76	9.42	7.65
30%		3.83	3.53	3.22	2.92	2.91	3.69	7.12	11.67	9.95
35%		4.50	4.17	3.82	3.49	3.50	4.39	8.27	13.77	12.26
40%		5.02	4.66	4.29	3.94	3.96	4.94	9.22	15.66	14.56
45%		5.39	5.01	4.62	4.25	4.29	5.34	9.93	17.28	16.87
50%		5.60	5.21	4.81	4.43	4.49	5.59	10.38	18.57	19.17
55%		5.65	5.26	4.86	4.48	4.56	5.69	10.57	19.40	21.48
60%		5.56	5.18	4.78	4.41	4.50	5.63	10.48	19.67	23.79
65%		5.32	4.95	4.58	4.23	4.33	5.43	10.10	19.26	26.09
70%		4.94	4.60	4.25	3.93	4.03	5.07	9.43	18.11	28.40
75%		4.42	4.12	3.81	3.52	3.62	4.56	8.47	16.25	30.70
80%		3.77	3.51	3.25	3.00	3.10	3.92	7.25	13.76	33.01
85%		3.00	2.79	2.58	2.39	2.47	3.13	5.77	10.79	35.32
90%		2.11	1.96	1.82	1.68	1.74	2.21	4.05	7.44	37.62
95%		1.10	1.03	0.95	0.88	0.92	1.17	2.12	3.82	39.93
99%		0.23	0.21	0.20	0.18	0.19	0.24	0.44	0.78	41.77
H17										
	w	0.2717	0.2611	0.2500	0.2385	0.2332	0.2426	0.3398	0.4830	0.6926
	l	0.3409	0.3543	0.3686	0.3827	0.3858	0.3532	0.2091	0.0656	0
	t	0.0768	0.0739	0.0707	0.0681	0.0703	0.0935	0.1404	0.1407	0
6%		0	0	0	0	0	0	0	0	0
10%		0	0	0	0	0	0	0.40	1.79	0.73
15%		0.69	0.54	0.37	0.21	0.16	0.45	2.10	4.26	3.03
20%		1.77	1.57	1.36	1.15	1.09	1.46	3.65	6.63	5.34
25%		2.71	2.47	2.21	1.96	1.89	2.35	5.05	8.89	7.65
30%		3.50	3.22	2.93	2.64	2.56	3.09	6.28	11.01	9.95
35%		4.14	3.83	3.51	3.19	3.10	3.70	7.32	12.96	12.26
40%		4.64	4.30	3.95	3.61	3.52	4.18	8.16	14.69	14.56
45%		4.98	4.63	4.27	3.91	3.82	4.52	8.79	16.16	16.87
50%		5.19	4.82	4.45	4.09	3.99	4.72	9.20	17.29	19.17
55%		5.24	4.88	4.51	4.15	4.05	4.79	9.36	18.00	21.48
60%		5.16	4.80	4.44	4.09	4.00	4.74	9.28	18.19	23.79
65%		4.94	4.60	4.25	3.92	3.84	4.55	8.95	17.79	26.09
70%		4.59	4.27	3.95	3.65	3.57	4.24	8.36	16.74	28.40
75%		4.11	3.83	3.54	3.27	3.20	3.81	7.52	15.06	30.70
80%		3.51	3.27	3.02	2.79	2.74	3.26	6.45	12.81	33.01
85%		2.79	2.60	2.41	2.22	2.18	2.60	5.14	10.08	35.32
90%		1.96	1.83	1.69	1.56	1.54	1.83	3.62	6.99	37.62
95%		1.03	0.96	0.89	0.82	0.81	0.96	1.90	3.60	39.93
99%		0.21	0.20	0.18	0.17	0.17	0.20	0.39	0.74	41.77

Table gives the percentage of the bankroll that a Kelly player would wager on insurance. This amount may not exceed $f/2$, nor may it exceed $1 - f$ for A2-A9, where f is the fraction of the bankroll wagered on the hand (left column). Win (w), loss (l), and tie (t) probabilities are also provided ($p + w + l + t = 1$).

* With betting fractions $f \geq 0.1943554$, Kelly player stands on A7 v. A in 6-deck S17, producing $p = 0.3107, w = 0.2460, l = 0.3151, t = 0.1282$.

Example: A player holding A9 in a 6-deck S17 game with 85% of his bankroll on the line would wager an additional 10.79% for insurance. Roughly 31.07% of the time, his A9 will lose to a natural, 50.79% of the time he'll win, 12.83% push, and 5.32% of the time he'll lose to a non-natural (a multi-card 21).

Kelly Criterion Insurance, Off-the-Top
1-deck

Bet Size f	Probability	A2	A3	A4	A5	A6	A7	A8	A9	AT
		Player Hand								
S17										
	p	$16/49 \approx 0.32653061$								0.3061
	w	0.2747	0.2658	0.2498	0.2339	0.2270	0.2454	0.3780	0.5085	0.6939
	l	0.3203	0.3335	0.3535	0.3726	0.3616	0.3135	0.1828	0.0501	0
	t	0.0785	0.0741	0.0702	0.0670	0.0849	0.1146	0.1126	0.1149	0
1%		0	0	0	0	0	0	0	0	0
5%		0.48	0.43	0.34	0.26	0.27	0.43	1.07	1.74	0
10%		1.83	1.73	1.56	1.39	1.42	1.75	3.03	4.44	0.51
15%		3.04	2.88	2.63	2.39	2.44	2.92	4.85	7.05	2.81
20%		4.08	3.88	3.56	3.25	3.31	3.95	6.50	9.58	5.10
25%		4.97	4.72	4.33	3.96	4.05	4.83	7.97	11.98	7.40
30%		5.70	5.41	4.96	4.54	4.65	5.56	9.24	14.25	9.69
35%		6.26	5.94	5.44	4.98	5.12	6.14	10.30	16.33	11.99
40%		6.66	6.31	5.78	5.29	5.44	6.56	11.12	18.19	14.29
45%		6.90	6.53	5.98	5.46	5.64	6.83	11.69	19.74	16.58
50%		6.98	6.60	6.03	5.51	5.70	6.93	11.99	20.91	18.88
55%		6.90	6.52	5.96	5.43	5.64	6.89	12.00	21.56	21.17
60%		6.67	6.30	5.75	5.24	5.45	6.68	11.73	21.57	23.47
65%		6.29	5.93	5.41	4.93	5.14	6.33	11.16	20.84	25.77
70%		5.77	5.44	4.95	4.51	4.71	5.83	10.30	19.36	28.06
75%		5.11	4.81	4.38	3.99	4.18	5.18	9.17	17.17	30.36
80%		4.32	4.07	3.70	3.37	3.53	4.40	7.77	14.41	32.65
85%		3.40	3.20	2.92	2.65	2.79	3.48	6.14	11.21	34.95
90%		2.37	2.24	2.03	1.85	1.95	2.43	4.28	7.69	37.24
95%		1.24	1.17	1.06	0.97	1.02	1.27	2.23	3.93	39.54
99%		0.26	0.24	0.22	0.20	0.21	0.26	0.46	0.80	41.38
H17										
	w	0.2629	0.2541	0.2401	0.2225	0.2269	0.2351	0.3406	0.4834	0.6939
	l	0.3358	0.3480	0.3671	0.3859	0.3813	0.3518	0.2078	0.0624	0
	t	0.0748	0.0714	0.0662	0.0651	0.0653	0.0866	0.1252	0.1276	0
1%		0	0	0	0	0	0	0	0	0
5%		0.42	0.36	0.29	0.20	0.22	0.31	0.92	1.65	0
10%		1.70	1.61	1.45	1.28	1.32	1.51	2.73	4.24	0.51
15%		2.85	2.70	2.47	2.23	2.29	2.56	4.38	6.74	2.81
20%		3.83	3.64	3.35	3.04	3.11	3.47	5.88	9.13	5.10
25%		4.67	4.44	4.08	3.71	3.80	4.24	7.20	11.40	7.40
30%		5.35	5.08	4.67	4.25	4.36	4.87	8.34	13.51	9.69
35%		5.87	5.58	5.13	4.67	4.78	5.35	9.27	15.44	11.99
40%		6.24	5.93	5.44	4.95	5.07	5.70	9.99	17.12	14.29
45%		6.46	6.13	5.62	5.11	5.24	5.90	10.48	18.50	16.58
50%		6.53	6.19	5.67	5.16	5.28	5.97	10.73	19.51	18.88
55%		6.45	6.11	5.59	5.09	5.21	5.90	10.73	20.03	21.17
60%		6.23	5.89	5.39	4.90	5.02	5.71	10.48	19.99	23.47
65%		5.87	5.55	5.07	4.61	4.72	5.39	9.97	19.30	25.77
70%		5.38	5.08	4.64	4.22	4.32	4.94	9.21	17.94	28.06
75%		4.76	4.50	4.10	3.73	3.82	4.38	8.20	15.97	30.36
80%		4.02	3.80	3.47	3.15	3.23	3.71	6.96	13.46	32.65
85%		3.17	2.99	2.73	2.48	2.54	2.93	5.51	10.52	34.95
90%		2.21	2.09	1.90	1.73	1.77	2.04	3.85	7.24	37.24
95%		1.15	1.09	0.99	0.90	0.92	1.07	2.01	3.72	39.54
99%		0.24	0.22	0.20	0.19	0.19	0.22	0.41	0.76	41.38

Table gives the percentage of the bankroll that a Kelly player would wager on insurance. This amount may not exceed $f/2$, nor may it exceed $1 - f$ for A2-A9, where f is the fraction of the bankroll wagered on the hand (left column). Win (w), loss (l), and tie (t) probabilities are also provided ($p + w + l + t = 1$).

Strategy Summary

Since we have now described how to play the Ace for any given wager size, a "simple" grid search can tell us what the Kelly-optimal bet size is. As it turns out, the betting fraction is so high that we split Aces only once; therefore, the RSA rule is rendered irrelevant. Also, because we apply the current, standard rule of delivering only one card to each split Ace, the DAS rule is irrelevant. Even DOA is irrelevant, because we do not make any soft doubles at this betting fraction. The four game combinations remaining are summarized in the final chart. To see that the Kelly player has a lower expectation, but a significantly lower variance, look at the statistics for conventional basic strategy shown in the last section of the table.

Risk-Averse Kelly Player (Logarithmic Utility) Bet and Strategy Summary										
Player	Strategy Dealer Upcard						Insurance Amount (%)			
Hand	2	3-5	6	7-8	9-T	A	1-deck S17	1-deck H17	6-deck S17	6-deck H17
AA	H	P2	P2	H	H	H	7.1920			
AA	P2	P2	P2	H	H	H		6.7170		
AA	H	H	P2	H	H	H			5.5850	
AA	H	H	H	H	H	H				5.1640
A2	H	H	H	H	H	H	6.7819	6.3283	5.1966	4.7999
A3	H	H	H	H	H	H	6.4234	6.0045	4.8264	4.4550
A4	H	H	H	H	H	H	5.8806	5.5110	4.4436	4.0986
A5	H	H	H	H	H	H	5.3749	5.0163	4.0784	3.7532
A6	H	H	H	H	H	H	5.5401	5.1358	4.1059	3.6580
A7	S	S	S	S	H	→	6.6901/S	5.7759/H	5.0994/S	4.3370/H
A8	S	S	S	S	S	S	11.3885	10.1701	9.5032	8.4523
A9	S	S	S	S	S	S	18.8774	17.5889	16.2809	15.3368
AT	S	S	S	S	S	S	15.2422	15.0039	15.3992	15.5228
Initial Bet Size							42.083	41.564	41.813	42.081
Expected Log							0.109287	0.108297	0.101221	0.099911
Expectation E							0.506794	0.506994	0.480314	0.474827
Variance V							1.054115	1.063227	1.041375	1.036903
E/V							0.480777	0.476844	0.461230	0.457928
Basic Strategy DOA RSA										
Expectation E							0.531806	0.530438	0.516840	0.514730
Variance V							1.495439	1.497643	1.461095	1.498066
E/V							0.355619	0.354182	0.353735	0.343596
Insurance and bet size are shown as a percentage of initial bankroll. Player does not double down. Player does not surrender. P2 means to split once. Split Aces receive one card only.										