Analysis of Common-Collector Colpitts Oscillator

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Introduction

Murphy's rule when paraphrased for oscillators reads [1], "Amplifiers will oscillate but oscillators won't." As we all know, Mr Murphy rules and Mr Newton doesn't so it's sensible to understand well when we can expect oscillators to really oscillate and at what frequency. To be honest, all the different conditions one can use to check for sustained oscillations are different versions of one solid criterion. To apply that criterion we first derive a transfer function between any initial condition based voltage ($Li(0)$ for inductors) or current ($Cv(0)$ for capacitors) and the output voltage. Then to check for oscillations we see if there exists a frequency ω_0 such that the transfer function has poles at $\pm\jmath\omega_0$, i.e., the denominator of the transfer function has roots at $\pm\jmath\omega_0$.

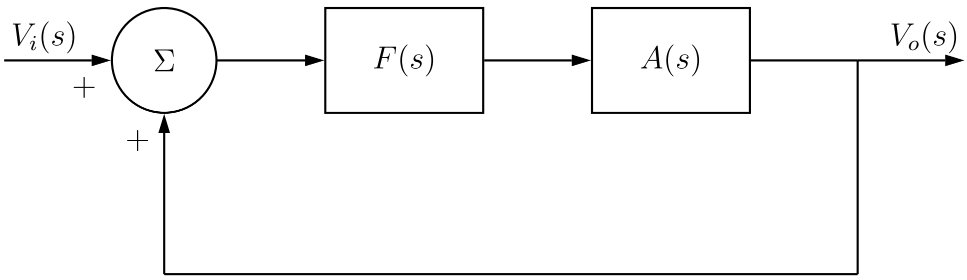

Figure 1: Feedback System

The above idea can be illustrated with the help of the feedback system shown in Figure 1. The transfer function

$$\frac{V_o(s)}{V_i(s)} = \frac{F(s)A(s)}{1 - F(s)A(s)}$$

has a root at $\pm\jmath\omega_0$ if $1 = F(\jmath\omega_0)A(\jmath\omega_0)$ or if $1 - F(\jmath\omega_0)A(\jmath\omega_0) = 0$. Obviously the two conditions are equivalent; our first attempt is to see if $\exists\ \omega_0\ s.t.\ F(\jmath\omega_0)A(\jmath\omega_0) = 1$, but at times it's difficult to separate a feedback circuit in terms of neat blocks like $F(S)$ and $A(S)$ and the feedback. In the latter case we directly check for the roots of $1 - F(s)A(s)$ (which we can obtain by a complete analysis, as shown later) and see if they are on the $\jmath\omega$-axis.

In this note the above mentioned three versions of the criterion for sustained oscillations are demonstrated. In the first version a complete time domain analysis is performed and we see that the output voltage is a sinusoid. In the second version we break the feedback loop at a point in the circuit; set up an input voltage source at that point and then see the output voltage at the break-point. If the gain is unity for the input sinusoid at frequency ω_0 then the circuit will oscillate at that frequency. In this analysis one has to be sure that when the loop closes it doesn't load the network, i.e., the gain doesn't change. If it does load then the analysis in [2] can be used to get an ideal amplifier from the non-ideal elements. Finally in the third version the complete transfer function is derived and then conditions are set such that there are poles on the $\jmath\omega$-axis.

Ideal LC Oscillator—Time Domain Analysis

An ideal LC oscillator is shown in Figure 2.

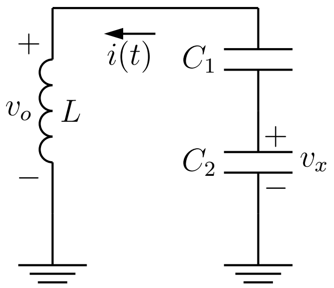

Figure 2: Idealised Colpitts Oscillator Analysis

Writing the KVL around the loop in Figure 2,

$$\begin{aligned} L\frac{di}{dt} + \frac{1}{C_1}\int_0^t idt + \frac{1}{C_2}\int_0^t idt &= 0 \\ L\frac{d^2 i}{dt^2} + \frac{i}{C_1} + \frac{i}{C_2} &= 0 \\ LC\frac{d^2 i}{dt^2} + i &= 0 \end{aligned} \tag{1}$$

where $C = \frac{C_1 C_2}{C_1 + C_2}$. The general solution to the differential equation (1) can be written as,

$$i(t) = k_1 e^{\frac{\jmath}{\sqrt{LC}}t} + k_2 e^{\frac{-\jmath}{\sqrt{LC}}t}, \tag{2}$$

constants k_1 and k_2 depend on initial conditions. Let us assume that the circuit has the initial conditions $i(0) = i_0$ and $\frac{di(0)}{dt} = 0$. With these initial conditions the general solution is,

$$i(t) = \frac{i_0}{2}\left(e^{\frac{\jmath}{\sqrt{LC}}t} + e^{\frac{-\jmath}{\sqrt{LC}}t}\right) = i_0 \cos\frac{1}{\sqrt{LC}}t. \tag{3}$$

Now one can see that the voltages v_x and v_o in Figure 2 can be written as,

$$v_x(t) = -\frac{1}{C_2}\int_0^t idt = -\frac{1}{C_2}\frac{i_0}{\frac{1}{\sqrt{LC}}}\sin\frac{1}{\sqrt{LC}}t \tag{4}$$

$$v_o(t) = L\frac{di}{dt} = -Li_0\frac{1}{\sqrt{LC}}\sin\frac{1}{\sqrt{LC}}t \tag{5}$$

The ratio of the maximum values of v_o and v_x can be written as:

$$\text{gain} = \frac{Li_0\frac{1}{\sqrt{LC}}}{\frac{1}{C_2}\frac{i_0}{\frac{1}{\sqrt{LC}}}} = \frac{LC_2}{LC} = \frac{C_1 + C_2}{C_1}. \tag{6}$$

The gain in equation (6) is greater than one. The above analysis shows that in an ideal LC circuit in Figure 2 there will be sustained oscillations at the frequency $\frac{1}{\sqrt{LC}}$ and that the ratio of voltages v_o and v_x is greater than one. This fact (gain greater than 1) is not so important here but it will be important to demonstrate that the common-collector Colpitts oscillator will sustain oscillations even when the cc-amplifier doesn't act as an ideal element. Note that this circuit will not have sustained oscillations if any of the three elements is not ideal, i.e., if they have a finite resistance. As one knows there doesn't exist an ideal inductor or capacitor so to obtain sustained oscillation an amplifier block is used with LC circuits as discussed next.

Common-Collector Colpitts Oscillator—Breaking the Loop

Small-signal equivalent circuit of a common-collector Colpitts oscillator [3, Section 4-1] is shown in Figure 3.

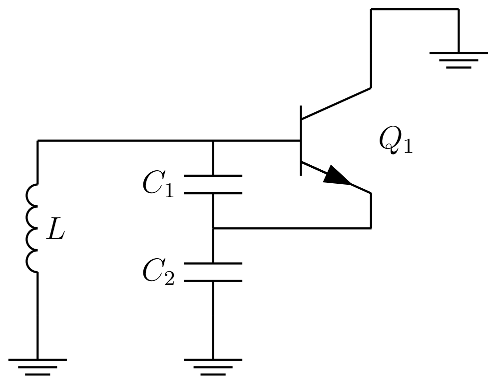

Figure 3: Common-collector Colpitts oscillator (small-signal equivalent)

A simple analysis can be performed on this circuit to see if it will work as an oscillator and if yes then at what frequency. In a feedback system if the phase shift around the loop is 0 or 2π and the gain is unity then it will have sustained oscillations. In practice the gain needs to be larger than unity (why?). To do this analysis the feedback loop in Figure 3 is broken at the base of the transistor as shown in Figure 4 and a relationship between v_i and v_o is dervied to investigate if the conditions for oscillation are met or not.

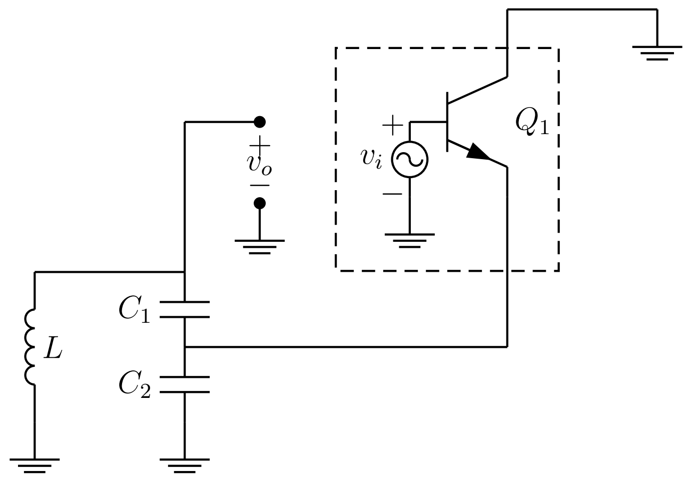

Figure 4: Open-loop CC Colpitts oscillator (small-signal equivalent)

The analysis is considerably simplified if the common-collector transistor in the dashed box in Figure 4 is replaced by its Thevenin equivalent and we assume that the input impedance of the cc-amplifier is infinity. The infinite input impedance assumption means that when the break point is connected back it doesn't load the feedback network, i.e., the gains calculated by breaking the loop don't change when the loop is connected.

A quick analysis of the common-collector hybrid-π model shows that the Thevenin voltage $V_{TH} = v_i$ and the Thevenin resistance is $R_{TH} = r = \dfrac{1}{g_m}$. After replacing the transistor with its Thevenin equivalent, the circuit to analyse now is given in Figure 5.

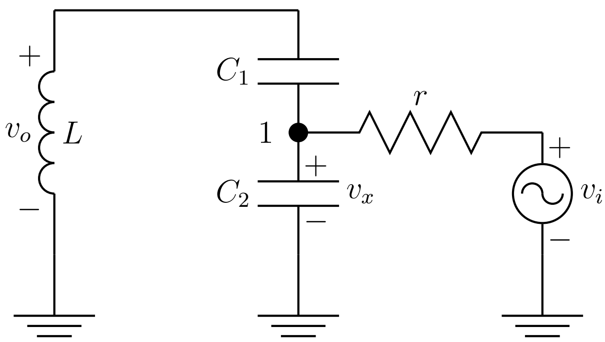

Figure 5: Open-Loop Colpitts Oscillator

Writing KCL at the node 1 in Figure 5 we get,

$$\frac{V_x(s)-V_i(s)}{r}+V_x(s)C_2s+\frac{V_x(s)C_1s}{1+LC_1s^2}=0, \tag{7}$$

now simplifying equation (7) we get,

$$\frac{V_x(s)}{V_i(s)}=\frac{1+LC_1s^2}{1+LC_1s^2+rs(LC_1C_2s^2+C_1+C_2)}. \tag{8}$$

Also note that,

$$V_o(s)=\frac{V_x(s)LC_1s^2}{1+LC_1s^2}, \tag{9}$$

putting together equations (8) and (9) we can write,

$$\frac{V_o(s)}{V_i(s)}=\frac{LC_1s^2}{1+LC_1s^2+rs(LC_1C_2s^2+C_1+C_2)} \tag{10}$$

The frequency ω at which the term in side the bracket in the denominator of equation (10) is equal to zero is called the resonant frequency. This is because the gain or the transfer function $\frac{V_o(s)}{V_i(s)}$ at the resonant frequency should be a real number and this can be true only if all the co-effecients of the odd powers of $s=\jmath\omega$ are zero. The resonant frequency is then given as:

$$-LC_1C_2\omega^2+C_1+C_2=0\Rightarrow\omega=\frac{1}{\sqrt{L\frac{C_1C2}{C_1+C_2}}}.$$

Substituting the value of the resonant frequency in the equation (10) we get,

$$\begin{aligned}\frac{V_o(\jmath\omega)}{V_i(\jmath\omega)} &= \frac{-LC_1\frac{C_1+C_2}{LC_1C_2}}{1-LC_1\frac{C_1+C_2}{LC_1C_2}}\\ &= \frac{C_1+C_2}{C_1}\end{aligned} \tag{11}$$

This shows that the phase-shift from v_i to v_o at the resonant frequency is 0 or 2π and the gain is greater than unity. These two conditions are sufficient to sustain oscillations at the resonant frequency $\omega=\frac{1}{\sqrt{L\frac{C_1C2}{C_1+C_2}}}$.

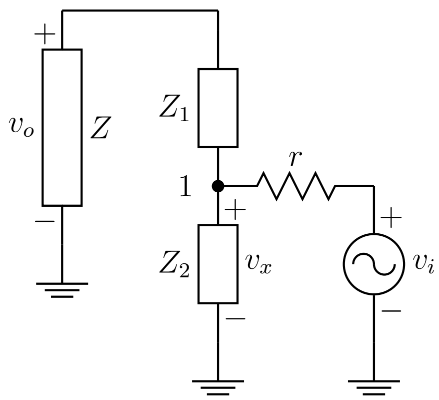

Figure 6: Colpitts Oscillator Analysis with Non-ideal Inductor

A General Analysis

In a general case where the resonant circuit is not a simple combination of L, C_1, and C_2, the above analysis can be repeated by replacing them with general impedances Z_1, Z_2, and Z_3.

Following an analysis similar to the one done for Figure 5 the transfer function for the circuit in Figure 6 is given as,

$$\frac{V_o(s)}{V_i(s)} = \frac{ZZ_2}{(Z + Z_1)Z_2 + r(Z + Z_1 + Z_2)} \tag{12}$$

Let us consider the case when $Z = Ls + R$, i.e., a non-ideal inductor, $Z_1 = \frac{1}{C_1 s}$, and $Z_2 = \frac{1}{C_2 s}$. The transfer function with these values is:

$$\frac{V_o(s)}{V_i(s)} = \frac{(Ls + R)C_1 s}{1 + s(RC_1 + r(C_1 + C_2 + LC_1C_2s^2)) + LC_1s^2 + rRC_1C_2s^2} \tag{13}$$

The conditions for sustained oscillation are satisfied if two real numbers k and ω can be found such that,

$$k = \left.\frac{V_o(s)}{V_i(s)}\right|_{s=\jmath\omega}.$$

The numbers k and ω can be found graphically or by solving the two nonlinear equations (equating the real and imaginary parts); ω is the frequency of oscillation and k is the gain.

Loading on the LC Tank—Complete Transfer Function

In the preceding analysis it has been assumed that the cc-amplifier has an infinite input impedance. In a general analysis the loading due to the finite input impedance of the cc-amplifier also needs to be taken into account. The analysis gets a little more complicated but the idea of breaking the loop and working out the frequency at which the phase shift is 0 or 2π and then testing if the gain at that frequency is greater than unity works well. The analysis in [2, Section 8.5] which shows how to consider the effect of loading on feedback amplifiers can be used.

Here we analyse the entire circuit in one go and see what we get. Please note that testing for the frequency when $A(\jmath\omega)f(\jmath\omega) = 1$ is the same as getting the closed-loop transfer function and checking if any of its poles are on the $\jmath\omega$-axis.

Figure 7 shows the transistor replaced by the hybrid-π model. The voltage source Li_0 is due to the initial current in the inductor. The KVL at the two nodes can be written as:

$$(V_x - V_o)C_1 s + \frac{V_x - V_o}{r_\pi} + V_x C_2 s \quad = \quad g_m(V_o - V_x) \tag{14}$$

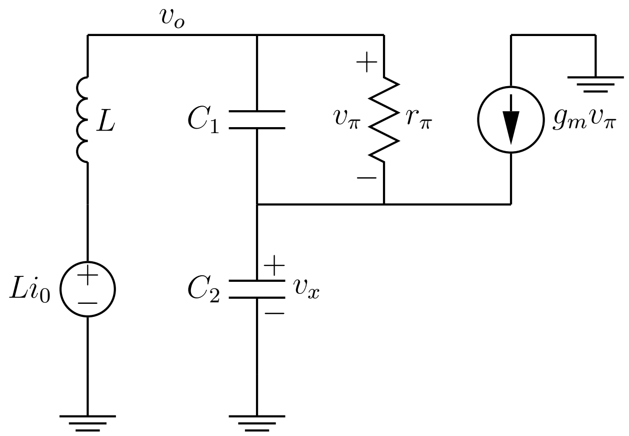

Figure 7: Complete Colpitts Oscillator

$$\frac{V_o - Li_0}{Ls} + \frac{V_o - V_x}{r_\pi} + (V_o - V_x)C_1 s \quad = \quad 0 \tag{15}$$

Solving the above two equations (14) and (15) simultaneously, we obtain:

$$\frac{V_o}{Li_o} = \frac{1 + g_m r_\pi + s r_\pi (C_1 + C_2)}{1 + g_m r_\pi + s r_\pi (C_1 + C_2) + s^2 C_2 L + s^3 C_1 C_2 r_\pi L} \tag{16}$$

There will be sustained oscillations if the denominator of the above transfer function has roots on the $\jmath\omega$-axis (why?). Substituting $s = \jmath\omega$ in the denominator of the transfer function (16) and equating it to zero we have ($\beta = g_m r_\pi$)

$$1 + \beta - C_2 \omega^2 L + \jmath(\omega r_\pi (C_1 + C_2) - \omega^3 C_1 C_2 r_\pi L) = 0 \tag{17}$$

Equating the imaginary part to zero we get

$$\omega^2 = \frac{C_1 + C_2}{C_1 C_2 L}$$

and with this value of ω the real part of the equation is,

$$1 + \beta = LC_2 \frac{C_1 + C_2}{C_1 C_2 L} = \frac{C_1 + C_2}{C_1} \Rightarrow C_2 = \beta C_1 \tag{18}$$

This implies that provided we choose the capacitors appropriately the oscillator will have a sustained oscillation at $\omega = \sqrt{\frac{(C_1+C_2)}{C_1 C_2 L}}$.

Lossy Inductor

When the inductor in Figure 7 is lossy the transfer function can be obtained by substituting $Ls + R$ for Ls in equation (16). After the substitution the denominator of the transfer function can be written as:

$$1 + \beta + s(r_\pi (C_1 + C_2) + RC_2) + s^2 (LC_2 + C_1 C_2 r_\pi R) + s^3 C_1 C_2 r_\pi L = 0 \tag{19}$$

As before, to test if there will be sustained oscillations with a lossy inductor, equation (19) should have a solution for $s = \jmath\omega$. Unless a full nonlinear analysis is done it's a bit difficult to check this condition but one can perform a simple Routh-Hurwitz test and see what are the conditions for the denominator to have roots in the right-half-plane (not on the $\jmath\omega$-axis though). This condition turns out to be:

$$(LC_2 + C_1 C_2 r_\pi R)(r_\pi (C_1 + C_2) + RC_2) \leq (1 + \beta) C_1 C_2 r_\pi L. \tag{20}$$

If the above inequality were to be an equality then a couple of roots will lie on the $\jmath\omega$ axis. These roots can be obtained by equating the denominator with $s = \jmath\omega$ to zero and solving for ω. The imaginary part of this identity gives:

$$\omega_0^2 = \frac{r_\pi(C_1 + C_2) + RC_2}{C_1 C_2 r_\pi L}, \tag{21}$$

with this value of ω_0 inequality (20) can be written as:

$$Rr_\pi \leq \frac{1+\beta}{\omega_0^2 C_1 C_2} - \frac{L}{C_1}. \tag{22}$$

This gives a condition in which oscillations will start but to test for sustained oscillations a nonlinear analysis needs to be performed. In general with most transistors if the above inequality (20) holds then there will be sustained oscillations.

Mr James Webb's Design

The oscillator circuit can be reduced to the following schematic in Figure 8 for the purposes of our analysis here.

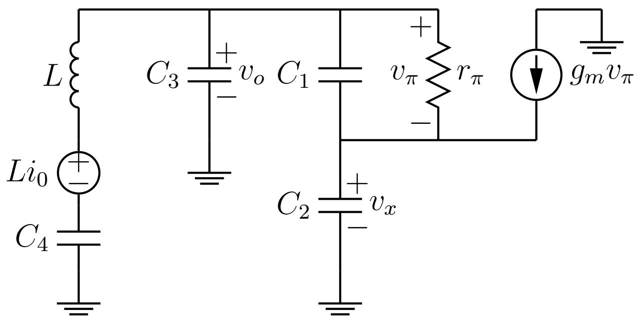

Figure 8: Oscillator Designed by James Webb

The circuit in Figure 8 doesn't show the capacitance C_π explicitly because if C_π is not negligible then it can be just added to C_1. The capacitances C_3 and C_4 represent the equivalent capacitances in that particular section of the oscillator. The KVL at the two nodes in the circuit in Figure 8 can be written as:

$$\begin{aligned} (V_x - V_o)C_1 s + \frac{V_x - V_o}{r_\pi} + V_x C_2 s &= g_m(V_o - V_x) && (23) \\ \frac{V_o - Li_0}{Ls + \frac{1}{C_4 s}} + \frac{V_o}{C_3 s} + \frac{V_o - V_x}{r_\pi} + (V_o - V_x)C_1 s &= 0 && (24) \end{aligned}$$

The above two equations (23) and (24) can be simultaneously solved using the following `maple` script.

```
##    Title:  pllfosc.ma
##    Created: Fri Jun 22 16:58:14 2001
##    Modified: Time-stamp: <2001-06-24 15:35:02 Himanshu>
##    Author:  Himanshu Pota <hrp@wattle.ee.adfa.edu.au>
##
## Description: Oscillation frequency for the VCO designed by James Webb.

#restart; read(‘C:/Hemanshu/courses/Electronics4/colpitts/pllfosc.ma‘);
```

```
interface(echo=2);
eq1:=(Vx-Vo)*C1*s + (Vx-Vo)/rpi+Vx*C2*s=gm*(Vo-Vx);
eq2:=(Vo-Li0)/(L*s+1/(C4*s)) + (Vo-Vx)/rpi + (Vo-Vx)*C1*s + Vo*C3*s=0;
sol:=solve({eq1,eq2},{Vx,Vo}):
assign(sol):
tf:=collect(simplify(normal(Vo/Li0)),s);
dentf:=collect(denom(tf),s);
#dentf:=evalc(subs(s=I*omega,dentf));
#C4:=11e-9; C3:=45e-12; C1:=10e-12; C2:=33e-12; L:=40e-9;
numerdentf:=subs({C4=11e-9, C3=45e-12, C1=10e-12, C2=33e-12,
                  L=40e-9,s=I*omega},dentf);
fosc:=evalf(sqrt(-coeff(numerdentf,omega)/coeff(numerdentf,omega^3))/2/Pi);
tfsimple:=limit(tf,{C3=0,C4=infinity});
```

The ratio of V_o and Li_0 gives the transfer function. For this transfer function to represent an oscillator it should have poles on the $\jmath\omega$-axis. That means for a particular valuse of ω both the real and imaginary parts of the denominator at $s = \jmath\omega$ must go to zero.

Equating the imaginary part of the denominator with $s = \jmath\omega$ to zero, we obtain:

$$\omega^2 = \frac{C_1C_2 + C_1C_3 + C_1C_4 + C_2C_3 + C_2C_4}{L(C_1C_2C_4 + C_1C_3C_4 + C_2C_3C_4)} \tag{25}$$

Equating the real part of the denominator with $s = \jmath\omega$ to zero, we obtain:

$$-\omega^2 LC_4(C_2 + (1+\beta)C_3) + (1+\beta)(C_3 + C_4) + C_2 = 0 \tag{26}$$

The expression of ω from equation (25) needs to be substitued in the above equation (26) to set values for C_1 and C_2.

With the values of the capacitances in one of James Webb's classic design, we have $C_1 = 10\ pF$, $C_2 = 33\ pF$, $C_3 = 45\ pF$, $C_1 = 11\ nF$, and $L = 40\ nH$, which gives:

$$f_{\text{osc}} = 109.90\ \text{MHz}.$$

References

[1] Jonathan Scott. *Analog Electronic Design—Principles and Practice of Creative Design*. Prentice-Hall, Inc., NJ 07632, 1992. ISBN 0-13-033192-9.

[2] Paul R. Gray and Robert G. Meyer. *Analysis and Design of Analog Integrated Circuits*. John Wiley and Sons, Brisbane, 3rd edition, 1993.

[3] Ulrich L. Rohde. *Microwave and Wireless Synthesizers — Theory and Design*. John Wiley & Sons, Inc., Brisbane, 1997.