non-convexity. Since non-convexity is just the negation of convexity, it will be useful to begin by reviewing the justifications for the latter.

I. CONVEXITY

On the importance of the convexity hypothesis, see the entries: EXISTENCE OF GENERAL EQUILIBRIUM, CONVEX PROGRAMMING, CONVEXITY, DUALITY.

The standard convexity hypotheses can be justified in a variety of ways. Three approaches will be reviewed here. The first is relevant mainly (but not exclusively) to consumption, the second to production and the third to both.

(i) Diversification. We assume that the consumption set is always convex. The approach to be considered now has no bearing on this convexity hypothesis.

The classical justification of the convexity of preferences views it as the mathematical expression of a fundamental tendency of economic choice; namely, the propensity to diversify consumption.

Within a traditional cardinalist context (as in, for example, Jevons, Menger and Walras) diversification is the natural consequence of the principle of decreasing marginal utility: successive units of a consumption good yield increasingly smaller amounts of utility. In turn, if decreasing marginal utility is postulated from any origin and for any (simple of composite) commodity what we get, in modern language, is precisely the hypothesis of concavity of the utility function (proof in the differential case: the second derivative matrix of the utility function is negative semi-definite everywhere). Within an ordinalist context, the principle of decreasing marginal utility should be replaced (as was done by Pareto) by the principle of decreasing marginal rate of substitution: keeping utility constant it is increasingly more difficult, i.e. more expensive, to replace units of a consumption good by units of another. Equivalently, indifference hypersurfaces bound convex sets. In other words: preferences are convex.

It should be clear that as an interpretation (but perhaps with less force as a justification), the above applies also to production. Suppose that inputs and outputs are perfectly divisible. Then the convexity hypothesis on the production set simply says that from any initial point at its boundary and for any definition of (simple or composite) input and output commodity, it takes an increasingly large amount of input to produce successive additional units of output.

While the propensity to diversify is plausible enough as a descriptive feature of economic choice (indeed if this were not so much of economics would be seriously out of tune with economic reality) it is by no means a universal principle. A familiar example to illustrate this is the gin and tonic choice situation. One may well like both gin and tonic but hate its mixture (Exercise: do some introspection and come up with a similar example that applies to you, the reader).

The modern theory of choice under risk, i.e., the expected utility theory of von Neumann and Morgenstern (see EXPECTED UTILITY HYPOTHESIS) has provided, in the form of the theory of risk aversion, a powerful reinforcement to the diversification principle. Suppose that preferences over lotteries (with commodity bundles as outcomes and with objective probabilities) are expressible by taking the expectation of a utility function defined on commodity bundles (this is what the Expected Utility Theory yields). Then the concavity hypothesis on this utility function is equivalent, as a matter of the definition of concavity, to the assumption (called risk aversion) that the decision maker would never lose by getting, instead of a risky lottery with commodities or outcomes, the non-risky commodity bundle where the amount of each commodity is precisely the expected amount of that commodity under the given lottery (i.e., the mean of the random variable). To the extent that risk aversion seems more prevalent than its opposite, we thus get additional support for the convexity hypothesis.

(ii) Divisibility and additivity. A production set $Y \subset R^n$ satisfies the *non-increasing returns property* if any feasible technology $y \in Y$ can be scaled down, that is $\alpha y \in Y$ for any $0 \leqslant \alpha \leqslant 1$. The condition can be derived from a more basic requirement, namely, the perfect divisibility of all the inputs used in production. Note: the list of inputs should be exhaustive and inclusive of the non-marketed inputs.

A production set $Y \subset R^n$ satisfies the *additivity property* if $y_1 + y_2 \in Y$ whenever $y_1, y_2 \in Y$, or $Y + Y \subset Y$. The economic interpretation of this condition is straightforward: production activities do not interfere with each other. If activities y_1, y_2 are technically feasible, then it is also feasible, say, to set-up two plants producing, respectively, y_1 and y_2 (if $y_1 = y_2$ then this is what is called *free entry*). Note that for this interpretation to make sense we must again have an exhaustive listing of inputs. In fact, it can be argued that additivity is a test for the exhaustiveness of the listing. In this view a lack of additivity is indicative of an input unaccounted for and available in a fixed amount.

The combination of the two properties above implies that Y is convex. Indeed if y_1, y_2 are feasible and $0 \leqslant \alpha \leqslant 1$ then by non-increasing returns, αy_1, $(1-\alpha)y_2$ are feasible, and therefore, by additivity, $\alpha y_1 + (1-\alpha)y_2$ is also feasible. Although we are not now emphasizing this, we should point out that Y is also a cone, i.e., satisfies the constant returns property: if $y \in Y$ then $\alpha y \in Y$ for an $\alpha \geqslant 0$. To see this note that for an integer $m > \alpha$ we have any $my \in Y$ by additivity and then

$$\alpha y = \frac{\alpha}{m} my \in Y$$

by non-increasing returns.

See Koopmans (1951) for more on this.

(iii) Averaging. In economics we are typically more interested in average than in total magnitudes, e.g., income per capita is a more important concept than total income. It is therefore of great significance that, as we shall now see, the mean behaviour of a collection of economic agents tends to be more regular, more convex-like, than its individual behaviour.

For definiteness the remarks of this subsection will be made in terms of producers. They apply as well to the aggregation of consumers' upper sets (a construct of key importance in welfare economics) or to the aggregation of individual demand correspondences.

Consider first the limit situation where there is literally a continuum of firms. Every firm $t \in [0, 1]$ has a production set $Y_t - R^n_t \subset Y_t$ (free disposal). The dependence of Y_t fulfills the technical condition of measurability. Assume further, and the technical condition of measurability. Assume further, and this is very important, that the Y_t are uniformly bounded above, i.e., there is $z \geqslant 0$ such that $y_t \leqslant z$ for any $y_t \in Y_t$ and t. Note that the Y_t need not be convex.

The mean (per firm) production set Y is defined in the obvious ways as the collection of mean vectors $\int_0^1 y(t)\,dt$ obtained by letting $y(t)$ take values in Y_t. It is denoted by $Y = \int_0^1 Y_t\,dt$. It is then a simple consequence of Lyapunov's theorem on the range of a vector measure (see LYAPUNOV THEOREM) that Y is convex. Thus even if the individual supply correspondences are not convex valued the aggregate one will be.

The common sense of this result is illustrated in Figure 1. In it we have a continuum of identical firms. The mean production set is then the convex hull of the common technology.

The limit theory (due to Aumann, 1964, and Vind, 1964) is elegant and conclusive but often one is more interested in obtaining bounds for given, finite situations. In fact, the convexifying effects of averaging were first noted in this context by Farrell (1959) and Rothenberg (1966) and systematically studied by Starr (1969). The key mathematical theorem used by the latter, the Shapley–Folkman theorem (see

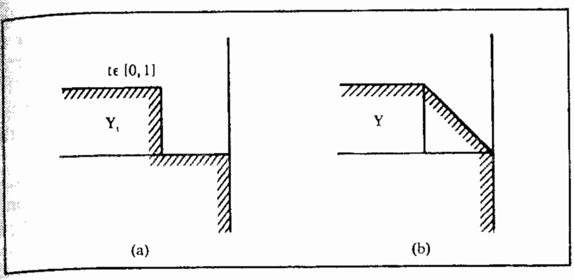

Figure 1

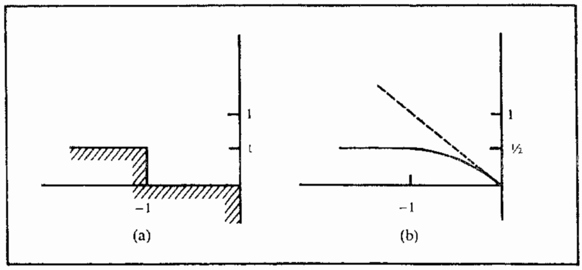

Figure 3

the entry under that heading), was prompted by the economic application.

The Shapley–Folkman Theorem allows us to assert that every vector in the convex hull of the sum of a finite number of production sets $Y_j \subset R^n, j = 1, \ldots, m$ can be obtained as a sum of vectors from the individual convex hulls with at most n of the individual vectors not belonging to the individual sets themselves. Suppose now that the Y_j are uniformly bounded above. It follows that there is a uniform (on j) bound r on the diameters of balls which are contained in the convex hull of Y_j but do not intersect Y_j itself. This r constitutes a measure of the degree of nonconvexity of the family of individual production sets. The Shapley–Folkman theorem implies then that any ball which is contained in the convex hull of $\Sigma_{j=1}^{m} Y_j$ but does not intersect $\Sigma_{j=1}^{m} Y_t$ itself must have diameter at most lr. Hence, the degree of non-convexity of $\Sigma_{j=1}^{m} Y_j$ is bounded independently of the number of firms. So, if m is large the mean production set

$$Y = \frac{1}{m} \sum_{j=1}^{m} Y_j$$

is almost convex. This is illustrated in Figure 2. In many cases of economic interest, e.g., if each production set has a smooth boundary, it is possible to do even better: the degree of nonconvexity may actually go to zero. See Mas-Colell (1985) for more on this.

The averaging theory presented so far is entirely modern. The classics, who lacked the concept of supply correspondence, had no inkling of it. They had, however, a very clear conception of the regularizing effects of aggregation. As an example among many we quote from Walras (1954, p. 95, emphasis in the original):

> There is nothing to indicate that the individual demand curves are ... *continuous*, in other words that an infinitesimally small increase in p_a produces an infinitesimally small decrease in d_a. On the contrary, these functions are often discontinuous. In the case of oats, for example, surely our first holder of wheat will not reduce his demand gradually as the price rises, but he will do it in some intermittent way every time he decides to keep one horse less in his stable. His demand curve will, in reality, take the form of a step curve ... All the other individual demand curves will take the same general form. And yet, the aggregate demand curve can, for all practical purposes, be considered as continuous by virtue of the so-called *law of large numbers*. In fact, whenever a very small increase in price takes place, at least one of the holders of wheat, *out of a large number of them*, will then reach the point of being compelled to keep one horse less, and thus a very small diminution in the total demand for oats will result.

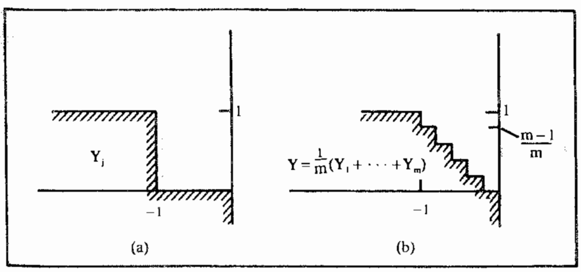

Figure 2

What this says is that if there is enough variation on firms' individual production sets, then, whatever the price system most firms will maximize profits at a single production vector. Therefore, in the limit, supply jumps are smoothed out and aggregation will yield a supply *function*, i.e., it is as if mean supply was generated from a strictly convex production set. Consider the following example. For every $t \in [0, 1]$ the production set it $Y_t = \{(0, 0), (-1, t)\} - R_+^2$, i.e., one unit of input produces t units of output. The corresponding supply correspondence is

$$f_t(p) = \begin{cases} (0, 0) & \text{for} \quad p_1/p_2 > t \\ \{(0, 0), (-1, t)\} & \text{for} \quad p_1/p_2 = t \\ (-1, t) & \text{for} \quad p_1/p_2 < t \end{cases}$$

Hence (normalizing to $p_2 = 1$) mean supply is given by the function $F(p_1) = \int f_t(p_1)\, dt = (p_1 - 1, \frac{1}{2}(1 - p_1^2))$ for $p_1 \leqslant 1$ and $F(p) = (0, 0)$ for $p_1 > 1$. Figure 3 describes the dispersed family of individual production sets and the corresponding (strictly convex) mean production set.

Prompted by a suggestion of Debreu (1972) this 'smoothing by aggregation' problem, which as we have seen can be viewed as an alternative line of attack to the analysis of the convexifying effects of aggregation, has been extensively studied in the last decade. We refer to the excellent survey monograph by Trockel (1984). A conclusion of the research reported in it is that as long as we are interested in the continuity of mean supply and demand then the smoothing intuition can be substantiated by using natural (and weak) concepts of dispersion. Establishing differentiability, however, turns out to be quite a different matter. The theory becomes delicate and powerful mathematical techniques have to be invoked.

II. CAUSES OF NON-CONVEXITIES

We shall concentrate on the production side. As for consumption, recall the gin and tonic example, or the possibility of risk-loving preferences, or the indivisibilities of many consumption goods. Nonetheless, many of these

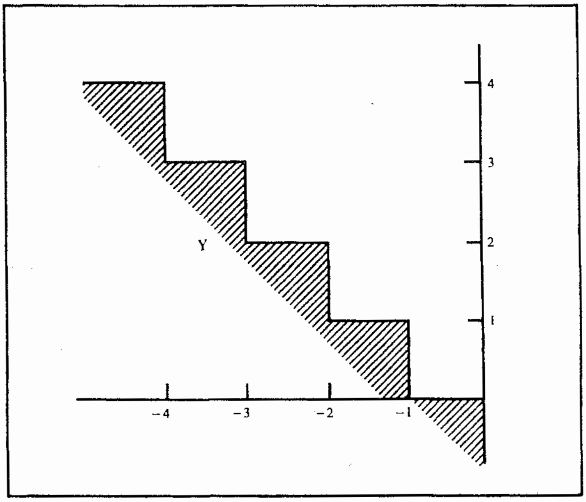

Figure 4

non-convexities, although individually significant, are small from the aggregate point of view and they may well be averaged out in the manner just mentioned. Of course, this can also happen for many production non-convexities. Thus our interest from now on will be on production non-convexities which matter economy-wide.

In I.(ii) we saw that non-increasing returns and additivity jointly yield convexity. As indicated there the violation of either of those two properties can always be formally traced to, respectively, the indivisibility or the fixity of some input. However, it will be useful now to be rather more concrete.

We begin by retaining additivity and examining violations of the non-increasing returns property. Four common instances are:

(a) There is a single input and a single output. The nature of the output, or the input, is such that it can only be produced, or used, in lumps of a fixed size; see Figure 4.

(b) The familiar technology set with set-up cost represented in Figure 5(a) (or, in a smoothed out variation, in Figure 5(b)). Here the production set is a reduced technology giving the total output optimally obtainable from some total cost or labour input. The non-convexity reflects sizeable indivisibilities in some of the physical inputs required in the production process.

(c) The cause of the increasing returns need not be the indivisibility of a physical input. They could also originate in learning and organizational advantages in the internal structure of production. A classical example is Adam Smith's idea of labour productivity being determined, through specialization and the division of labour, by the extent of the market. Smith's idea can be viewed as a brilliant trick to obtain increasing returns on a scale significantly higher than the individual labourer for a world where labour is the only input and where, therefore, there is no capital good whose indivisibility could be appealed to. In Smith's the indivisibilities are present, so to speak, at the level of the performance of individual tasks by individual labourers. Hence, the fewer tasks the latter perform the more productive they will be; see Vassilakis (1986).

(d) Marshallian external economies provide another interesting example. Suppose that the output of an industry is a good proxy for a public positive input (e.g. quality of labour force) to the industry itself. Then the production set of the industry may well be as in Figure 5(b) (with free entry this will be the typical shape). We point out that an indivisibility interpretation, while not impossible, would be here rather constrained.

The four previous examples are compatible with additivity of production sets. It is an interesting fact that if increasing returns prevail then the preservation of additivity does not mitigate the non-convexities. Rather the contrary, it only helps to spread them around. For example, if an output can be produced by means of two elementary technologies each of them using a different single input but both of them exhibiting increasing returns then the isoquants of the production function will be as in Figure 6(a) (see, e.g., Debreu and Koopmans, 1982). Figure 6(b) represents the situation for a finite number of nonlinear elementary activities. Thus we see that a necessary condition for a convex isoquant (an hypothesis very often made in theoretical work) is the availability of an infinite number of elementary activities. Similarly, Figure 6(c) represents the production possibility set for two outputs producible (each of them separately) from a single input with increasing returns. Again, it is non-convex. What all this tells us is that while a fully convex world can be supported by a very parsimonious set of microeconomic hypotheses a conveniently 'semiconvex' world is not so easy to justify.

Let us now retain non-increasing returns but drop additivity. It is very easy to see how the (negative) interference of two activities can cause non-convexities. The theory of external dis-economies provides classical examples (see Baumol and Oates, 1975, or Starret, 1972). Suppose that any of two activities (producing, respectively, laundry and smoke producing output) uses labour under constant returns. Then any

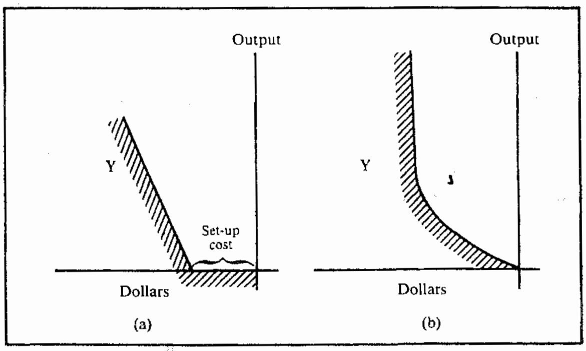

Figure 5

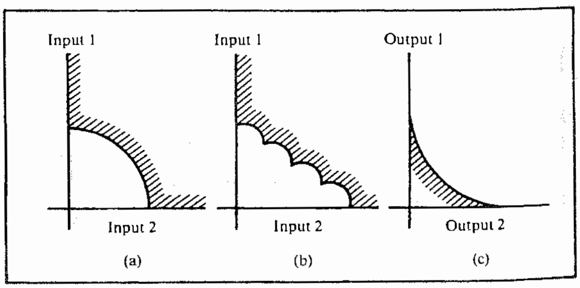

Figure 6

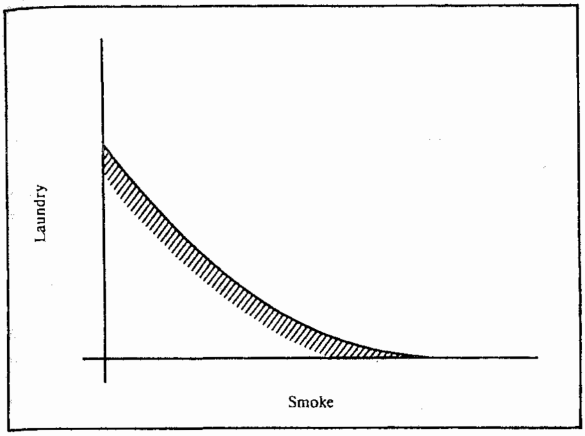

Figure 7

degree of interference will generate a non-convex production possibility set. See Figure 7.

Observe that while Figure 7 and Figure 6(c) are identical, the underlying reasons for the non-convexity are very different. Here the technology is of constant returns but additivity breaks down while there the technology is of increasing returns and it is additivity that makes the convexity unavoidable. We may also note that both external economies and external diseconomies are sources of non-convexities. But again the reasons are not the same in the two cases.

III. THE NON-CONVEXITY PROBLEM

Significant non-convexities create great difficulties both for equilibrium and for welfare theory. We comment on them in turn.

It is obvious, in the first place, that the existence of Walrasian price-taking equilibria is not to be expected. For example, in Figure 5, only the no production outcome can be sustained by prices. Technically, the convex valuedness and continuity (more precisely: upper hemicontinuity) of supply, required for existence proofs, will fail.

In itself, the above would not be very destructive. It is not clear after all that in a world with large non-convexities the conditions for perfect competition would be met. Walrasian equilibria may not be therefore the most sensible solution concept to look at. The point is, however, that delicate existence problems are present in any of the many, arguably more appropriate, solution concepts proposed in the literature (some will be reviewed in the next subsections). There is a way to see that the difficulty is intrinsic to the non-convex physical environment. Consider a collection $Y_1, \ldots, Y_m \subset R^l$ of production sets and define the feasible set

$$F = \left\{ (y_1, \ldots, y_m) \in \prod_{j=1,\ldots,m} Y_j : \sum_j y_j \geq 0 \right\}.$$

If every Y_j is convex, the F is convex, i.e., it has a simple structure. However, if the Y_j may not be convex then, even if they are otherwise quite nice (e.g., they have smooth boundary and satisfy free disposal), the set F may be far from simple, it may even be formed by several disconnected pieces (e.g., one piece could be the no production point, another a high production region that, so to speak, becomes feasible only due to substantial increasing returns). Directly or indirectly the complexity of the set F bears on the likelihood of existence for any solution concept we may consider.

To obtain, through an equilibrium or an explicitly optimizing process, economic outcomes with good welfare properties (say, Pareto optimality) is also no mean feat in a non-convex world. So much so that most equilibrium approaches simply do not get it. See Calsamiglia (1977) for an impossibility theorem which, in essence, asserts that any decentralized equilibrium notion which guarantees optimality with non-convexities must include as one of its steps the solution of an infinite dimensional programming problem.

The previous remarks should perhaps come as no surprise. The global maximum of an arbitrary function is not characterized by any sort of local conditions. Without some type of structural restriction finding it is a programming problem of intractable complexity. A restriction that proves useful is to limit the permissible non-convexities to those that arise from the indivisibility of explicit inputs or outputs (as in Figure 4). Then the methods of integer programming can be appealed to. Although those are still complex when compared to convex or linear programming (also, Figure 4 is misleading as to the higher dimensional possibilities), there is nonetheless an extensive body of technical literature and the field is undergoing rapid progress (e.g. Scarf, 1981, 1984). In particular, Scarf (1984) shows that for integer programming problems there is a way to associate to every feasible point a finite system of neighbourhoods in such a way that to test for global optimality it suffices to test every neighbourhood set.

IV. EXTERNALITIES

An approach which to a large extent salvages the equilibrium part of Walrasian theory is based on the observation that if all non-convexities in aggregate technologies are external to the single production unit then the decision problem of the individual firm is conventionally looking and, therefore, price taking behaviour is not doomed from the start. The existence of a price taking equilibrium has in fact been proved in considerable generality (see, e.g., Shafer and Sonnenschein, 1976; the problem alluded to in section III remains but it can be handled by means of survival hypotheses).

Recently this externality approach has been successfully exploited for the study, by means of dynamic competitive methods, of increasing returns effects in the process of capital accumulation and growth (see Romer, 1986).

Because of the presence of externalities the above type of price taking equilibria will typically fail to be Pareto optimal. The other side of the coin is that if external effects are internalized or, simply, priced out, then any Walrasian equilibrium will automatically be Pareto optimal but, because of the non-convexities, it is now the existence of equilibria which will be in serious difficulty (see Starret, 1972).

V. IMPERFECT COMPETITION

If increasing returns prevail then either the economic equilibrium is very inefficient or individual firms will end up being large. If so, they will be endowed with market power which suggests imperfect competition theory as a proper analytical framework. Interestingly, to this conceptual argument a technical one can be added. The nonlinearity of profit functions will increase the likelihood that firms' optimal productions react continuously to market parameters. This is illustrated in Figure 8 for an output-setting monopolist facing a linear demand function (and maximizing profits in terms of

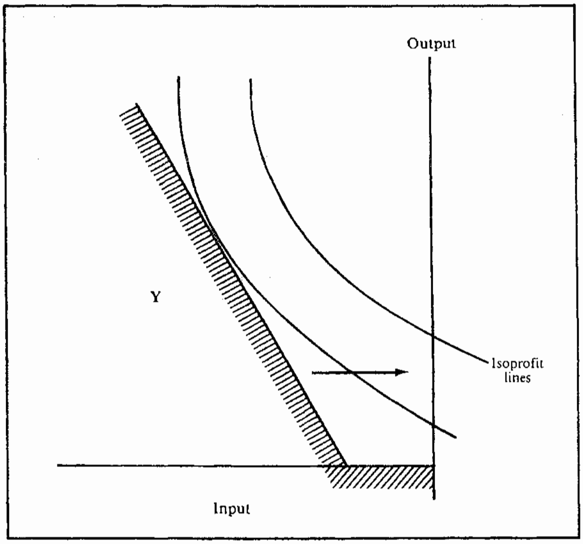

Figure 8

input). It follows that an existence theory for imperfectly competitive equilibria with increasing returns may be available. This is indeed so. It has been developed both for the perceived and the objective demand approach to imperfect competition. The perceived demand case is somewhat easier since the hypotheses of no joint production plus linearity of perceived demand will automatically imply the concavity of profit functions; see Arrow and Hahn (1971), Silvestre (1978) and the survey article by Hart (1985). Altogether, imperfect competition is one of the most promising approaches to increasing returns.

Let us consider a particularly simple example (see Fraysse-Moreau, 1981, and Dasgupta-Ushio, 1981). A certain good can be produced with zero marginal cost but there is a (non-sunk) set-up cost of c. There is free entry and the inverse demand function is $p = 1 - (1/N)Q$, where N is a market size parameter. Such a market will always have a Cournot quantity-setting equilibrium with free entry. The number of active firms will be $\sqrt{N/4c} - 1$ (more precisely, the integer closest from above to this number) while the production per active firm is $2\sqrt{cN}$ and the equilibrium price is $2\sqrt{c/N}$. It is instructive to evaluate the welfare loss. Adopting total surplus as a welfare measure the full optimum would have a single firm producing N at zero price for a total welfare of $(N/2) - c$. In the imperfectly competitive equilibria total welfare would be (approximately)

$$\frac{N}{2} - 3c - \frac{\sqrt{c}}{2}\sqrt{N}.$$

Hence the welfare loss is $2c + (\sqrt{c}/2)\sqrt{N}$. This is of order $\sqrt{N}$, a non-negligible number if N is large (although $\sqrt{N}/N \to 0$ as $N \to \infty$). Is this loss due to the unbounded increasing returns or to the imperfect competition? One way to answer this is to compare it with the situation which is in every way identical except that individual firms have a capacity limit k. Then the welfare loss at the imperfect competition equilibrium can be computed to be of order $k^2/2N$, which is a small number if N is large. Hence increasing returns seem to make quite a difference. Alternatively one could say that the unlimited increasing returns model is inherently much less competitive than the case with bounded non-convexities which, for N large, is almost Walrasian.

VI. WELFARE THEORY

A Pareto optimal allocation in a non-convex environment satisfies the same first-order necessary conditions as in the convexity case. There must be a price system such that at every production (resp. at every consumption) the price hyperplane must be 'tangent' to the corresponding production set (resp. indifference surfaces). Here tangent means that the firm (resp. the consumer) satisfies the first order necessary conditions for profit (resp. utility) maximization. This is the classical marginal cost pricing principle, so called because for a technology characterized by a single output and a single input it leads to the equality of output price to marginal cost. (Warning: With more than one input cost maximization is not a necessary condition for optimality.) A modern and rigorous analysis of this theory is contained in Bonnisseau and Cornet (1986b). Surprisingly, by using the mathematical techniques of non-smooth optimization it is possible to relax considerably the differentiability hypotheses.

A glance at part A in Figure 9 suffices to see that the first order necessary conditions are not sufficient for optimality. For (local) sufficiency one has to check second order conditions. Roughly speaking if preferences are convex the second order conditions require that the curvature of the indifference surface by larger than the curvature of the production set, e.g., as in points B and C in Figure 9. Note that point B is only a local optimum.

It is possible to obtain necessary and sufficient conditions for Pareto optimality by appealing to some form of non-linear prices. Observe, for example, that in Figure 9 one may separate the production set and the indifference surface at point C by a non-linear 'price' surface (dotted line) relative to which the firm maximizes 'profits' and the consumer utility. Note that no such non-linear prices exist for point B; see Brown and Heal (1978). Non-linear prices belong to an inherently infinite dimensional price space. Hence the

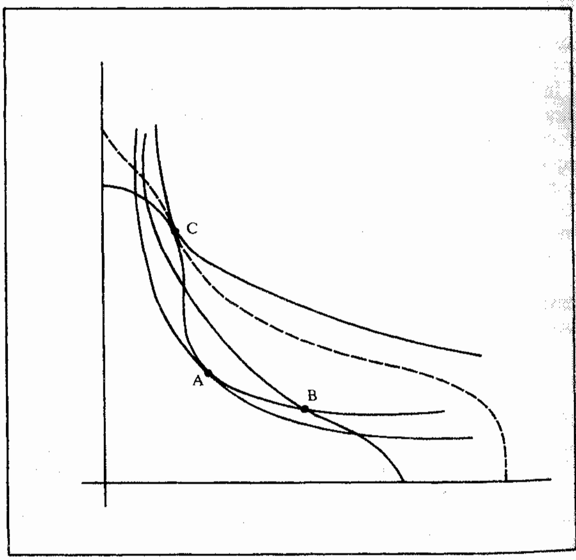

Figure 9

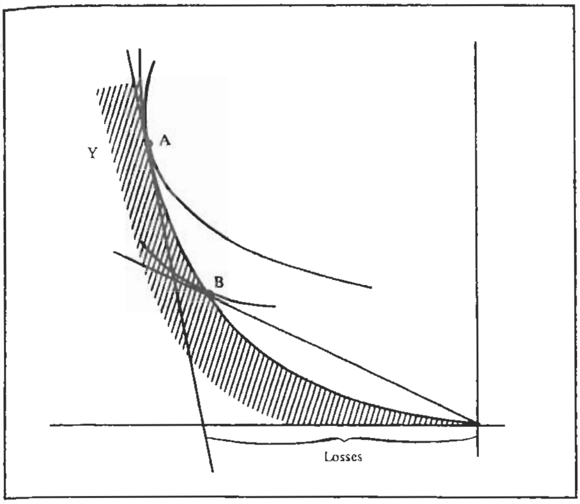

Figure 10

impossibility of reaching a global optimum by using them is not in conflict with the theorem of Calsamiglia mentioned in Section III. For iterative procedures leading to a local optimum see Heal (1973).

Typically if the productions at an optimum are evaluated at the corresponding optimality prices the firms with significant non-convexities will be making losses (marginal cost will be lower than average cost); see point A in Figure 10. The accounting identities will be taken care of by the lump sum transfers inherent to an optimum (in other words, losses will be covered by receipts from non distortionary taxes). But suppose this is politically infeasible i.e., prices and productions must be such that total profits are non-negative, although they can be limited to be non-positive. Then if we retain the hypothesis that consumers maximize utility given prices (suppose that preferences are convex) Pareto optimality will typically not be reachable. In the one output–one input case, the requirement that profits be zero (i.e., that average and marginal cost by the same) determines the outcome; see point B in Figure 10. Not so in the multiproduct case. The 'regulatory constraint' of zero profit is compatible with a range of choice of prices and production. This leads to a classical second best problem studied by Boiteux (see Guesnerie, 1981, for a modern point of view).

VII. OTHER EQUILIBRIUM APPROACHES

Imperfect competition is not the only equilibrium approach compatible (to some extent) with non-convexities. A variety of others, more influenced by a planning outlook, have been proposed. Among them are:

(a) Generalized marginal cost pricing equilibrium where firms are assumed to follow the principles described in the previous section, consumers are price takers and distribution rules (including tax subsidies) are given. See Guesnerie (1975), Mantel (1979), Beato (1982) and the recent synthesis by Bonnisseau and Cornet (1986a).

(b) Models where, in contrast to (a), firms do act as profit maximizing price takers but where prices are supplemented by quantity constraints, e.g. perceptions of possible sales. A good example is Dehez and Drèze (1986).

(c) A more abstract approach has been taken by, among others, Dierker, Guesnerie and Neuefeind (1985), Kamiya (1986), Vohra (1986), and Bonnisseau and Cornet (1986a). Their idea is to analyse the equilibria of systems where firms' behaviour is described by pricing rules (given *a priori*), which specify the prices acceptable at different production decisions; (a) and (b) are included but so are other rules, e.g., average cost pricing.

As could be expected, none of the above approaches yields equilibria with good first best properties (or, for that matter, second best ones; but this has been less studied). This is true even for the notion of marginal cost pricing equilibrium, which is directly inspired by welfare considerations (see Guesnerie, 1975; Beato and Mas-Colell, 1985). There is, however, an exception: if there is a single production set (i.e., the entire production sector is under a single management) and the curvature of the indifference surfaces is larger than the curvature of the production surface then the marginal cost pricing equilibrium will be Pareto optimal (see Quinzii, 1986).

(d) An approach based on (non-linear) Lindhalian prices is pursued in Mas-Colell and Silvestre (1986). The equilibrium is always Pareto optimum (in the one output–one input case it picks the Pareto optima compatible with average cost pricing) but with non-convexities it may not exist (curvature conditions will guarantee existence).

VIII. SUSTAINABILITY

As it is well known there is a close relationship in a convex world between the notion of Walrasian equilibrium and the cooperative game theory concept of the core (see CORES). With significant non-convexities Walrasian equilibria can easily fail to exist. This is not so clear for the core. In fact the basic intuition of increasing returns seems to suggest that it is difficult for small coalitions to improve their positions by themselves, thus making the core a prime candidate for the analysis of increasing returns economies.

Let $Y \subset R^l$ be a production technology freely available to any agent in the economy. A final allocation of goods is in the core if there is no coalition of agents that can guarantee each of its members a preferred outcome by using only their endowments and the technology Y. Note that a core allocation is automatically Pareto optimal. A more general approach would let coalitions have their own technologies; these are the so-called coalition production economies (see, e.g., Oddou, 1976). By constructing coalition specific inputs it is possible to view them as a limiting case of the common technology framework.

In the above setting the core has been studied by Scarf (1986). It turns out that the 'basic intuition' described above is not easily substantiated. Indeed, if Y is not a convex cone then it is always possible to find a collection of agents yielding an empty core. This is disappointing. There are, however, some special cases for which the core will be non-empty.

(a) There is one output, one input, the technology exhibits decreasing average cost, and consumers own no output.

(b) Consumers derive no utility from input goods and the technology satisfies the property of *distributivity*. By using prices the latter can be described thus: for any efficient production y there is a price system p such that $0 = p \cdot y \geqslant p \cdot z$ for any $z \in Y$ such that $z^- \leqslant y^-$, i.e., z should use at most as much input as y.

(c) A particular case of distributive production sets is when there is a single input, average cost decreases radially and the set of output productions attainable from any fixed input is convex (see Sharkey, 1979). Recall that this property is not additive. Neither is distributivity.

(d) As with marginal cost pricing relative curvature conditions can also be applied to guarantee a non-empty core (see Quinzii, 1986).

There is an intimate connection between the core approach and the sustainability problem in the theory of natural monopoly (see Sharkey, 1979; Baumol, Panzar and Willig, 1975). Suppose our production set Y is additive. This is often described as a natural monopoly situation on the ground that the combined productions of two firms can always be taken care of at least as efficiently by a single firm. The sustainability problem consists in designing a production and compensating (i.e., pricing) system which is immune to (necessarily inefficient) entry. By viewing an entrant as the coalition of its customers the link to core theory becomes clear and it helps to explain the 'paradox' of the existence of unsustainable natural monopolies (i.e., the additivity of Y is far from guaranteeing the non-emptiness of the core).

In the theory of natural monopoly a particularly important role is played by the hypothesis that a Walrasian equilibrium exists (e.g., in the one-input, one-output case this says that the demand forthcoming at the minimum average cost is an exact multiple of a minimum efficiency scale). Of course, this implies that the core is non-empty and a sustainable arrangement exists. But more is true. Under weak conditions the Walrasian equilibrium is the only point in the core (a related result, emphasizing the possibility of big players more than non-convexities, is in Shitovitz, 1973). Finally, we note that there is a close link between this result and many non-cooperative models of competition 'à la Bertrand'. Indeed, it is often possible to understand the latter as core models in which there are restrictions on which coalitions can form (e.g. they include only one firm) and on the way they can split gains (e.g. only through a uniform price system). The theme that under conditions of free entry (i.e., additivity of the aggregate production set) the existence of a Walrasian equilibrium will imply the non-emptiness and efficiency of the set of non-cooperative equilibria is also common in the latter theory (see Baumol, Panzer and Willig 1982, Grossmann 1981, or Mas-Colell 1985).

A. MAS-COLELL

See also CONSUMPTION SET; CONVEX PROGRAMMING; CONVEXITY; CORES; DUALITY; EXISTENCE OF GENERAL EQUILIBRIUM; EXTERNALITY; GENERAL EQUILIBRIUM; INCREASING RETURNS; LYAPUNOV THEOREM; PLANNING; SHAPLEY-FOLKMAN THEOREM.

BIBLIOGRAPHY

Arrow, K. and Hahn, F. 1971. *General Competitive Analysis*. San Francisco: Holden-Day.

Baumol, W. and Oates, W. 1975. *The Theory of Environmental Policy*. Engelwood Cliffs, NJ: Prentice-Hall.

Baumol, W., Panzar, J. and Willig, R. 1982. *Contestable Markets and the Theory of Industrial Structure*. San Diego: Harcourt, Brace, Jovanovich.

Beato, P. 1982. The existence of marginal cost pricing with increasing returns. *Quarterly Journal of Economics* 97, 669–89.

Beato, P. and Mas-Colell, A. 1985. On marginal cost pricing with given tax-subsidy rule. *Journal of Economic Theory* 37(2), 356–65.

Bonnisseau, J.-M. and Cornet, B. 1986a. Existence of equilibria when firms follow bounded losses pricing rules. *Journal of Mathematical Economics*, forthcoming.

Bonnisseau, J.-M. and Cornet, B. 1986b. Valuation equilibrium and Pareto optimum in non-convex economies. *Journal of Mathematical Economics*, forthcoming.

Brown, D. and Heal, G. 1978. Equity, efficiency and increasing returns. *Review of Economic Studies* 46, 571–85.

Calsamiglia, X. 1977. Decentralized resource allocation and increasing returns. *Journal of Economic Theory* 14, 262–85.

Dasgupta, P. and Ushio, Y. 1981. On the rate of convergence of oligopoly equilibria in large markets. *Economic Letters* 8, 13–17.

Debreu, G. 1959 *Theory of Value*. New York: Wiley.

Debreu, G. and Koopmans, T. 1982. Additively decomposed quasiconvex functions. *Mathematical Programming* 24, 1–38.

Dehez, P. and Drèze, J. 1986. Competitive equilibria with increasing returns. Louvain-la-Neuve: Core Discussion Paper No. 8623.

Dierker, E., Guesnerie, R. and Neuefeind, W. 1985. General equilibrium when some firms follow special pricing rules. *Econometrica* 53, 1369–93.

Farrell, M. 1959. The convexity assumption in the theory of competitive markets. *Journal of Political Economy* 67, 377–91.

Fraysse, J. and Moureau, M. 1981. Cournot equilibrium in large markets under increasing returns. *Economic Letters* 8(3), 217–20.

Grossman, S. 1981. Nash equilibrium and the industrial organization of markets with large fixed costs. *Econometrica*, 1149–72.

Guesnerie, R. 1975. Pareto optimality in non-convex economies, *Econometrica* 43, 1–29.

Guesnerie, R. 1981. *Modéles de l'économie publique*. Paris: Editions du CNRS.

Hart, O. 1985. Imperfect competition in general equilibrium: an overview of recent work. In *Frontiers of Economics*, ed. K. Arrow and S. Honkapohja, Oxford: Blackwell.

Heal, G. 1973. *The Theory of Economic Planning*. Amsterdam: North-Holland.

Kamiya, K. 1986. Existence and uniqueness of equilibria with increasing returns. *Journal of Mathematical Economics*.

Koopmans, T. 1951. Analysis of production as an efficient combination of activities. In *Activity Analysis of Production and Allocation* ed. T. Koopmans, New York: Wiley, 33–97.

Mantel, R. 1979. Equilibrio con rendimientos crecientes a escala. *An. Asoc. Argentina Econ. Polit.* 1, 271–82.

Mas-Colell, A. 1985a. La libre entrada y la eficiencia económica: un análisis de equilibrio parcial. *Revista Española de Economía* 2(1), 135–52.

Mas-Colell, A. 1985b. *The Theory of General Economic Equilibrium: A Differentiable Approach*. Cambridge: Cambridge University Press.

Mas-Colell, A. and Silvestre, J. 1986. Cost share equilibria: A Lindahlian approach. Working Paper, MSRI, Berkeley, CA.

Nikaido, H. 1968. *Convex Structures and Economic Theory*. New York: Academic Press.

Oddou, C. 1976. Theorèmes d'existence et d'équivalence pour des économies avec production. *Econometrica* 44, 265–82.

Quinzii, M. 1986. Rendements croissants et équilibre général. PhD dissertation, Université Paris II.

Rockafeller, T. 1970. *Convex Analysis*. Princeton: Princeton University Press.

Romer, P. 1986. Increasing returns and long-run growth. *Journal of Political Economy* 94(5), 1002–36.

Scarf, H. 1981. Production sets with indivisibilities, Part I: Generalities. *Econometrica* 49, 1–32.

Scarf, H. 1984. Neighborhood systems for production sets with indivisibilities. Cowles Foundation Discussion Paper No. 728.

Scarf, H. 1986. Notes on the core of a productive economy. Ch. 21 in *Contributions to Mathematical Economics*, ed. W. Hildenbrand and A. Mas-Colell, Amsterdam: North-Holland.

Shafer, W. and Sonnenschein, H. 1976. Equilibrium, commodity taxation, and lump sum transfers. *International Economic Review* 17, 601–611.

Sharkey, W. 1979. Existence of the core when there are increasing returns. *Econometrica* 47, 869–76.

Shitovitz, B. 1973. Oligopoly in markets with a continuum of traders. *Econometrica* 41, 467–501.

Silvestre, J. 1978. Increasing returns in general non-competitive analysis. *Econometrica* 46, 397–402.

Starr, R. 1969. Quasi-equilibria in markets with non-convex preferences. *Econometrica* 37, 25–38.

Starrett, D. 1972. Fundamental non-convexities in the theory of externalities. *Journal of Economic Theory* 4, 180–99.

Trockel, W. 1984. *Market Demand: an analysis of large economies with non-convex preferences.* New York: Springer-Verlag.

Vassilakis, S. 1986. Increasing returns and strategic behavior. PhD thesis, Johns Hopkins University.

Vind, K. 1964. Edgeworth-allocations in an exchange economy with many traders. *International Economic Review* 5(2), 165–77.

Vohra, R. 1986. On the existence of equilibrium with increasing returns: a synthesis. *Journal of Mathematical Economics.*

Walras, L. 1874-7. *Eléments d'économie politique pure.* Definitive edn, 1926, trans. by W. Jaffé as *Elements of Pure Economics*, Homewood, Ill.: Irwin, 1954.