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Establishment of an ordinary generating function and a Christoffel-Darboux type first-order differential equation for the heat equation related Boubaker-Turki polynomials

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Abstract

In this study, we try to find a generating function for the Boubaker-Turki polynomials (or the modified Boubaker polynomials). Since their first definition as an applied physics study, the Boubaker polynomials have been dealt with as non-orthogonal sequences that don't obey to any known Legendre-Laguerre type characteristic differential equation. This generating function is given parallel to a Christoffel-Darboux type first-order differential equation as a guide to further studies.

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1 Introduction

The Boubaker polynomials were proposed in a physics study that yielded a thermal model of a pyrolysis spray device [1]. They represent a mathematical tool for solving the heat transfer equation inside a given domain. The resolution process yielded a formula that led to a sequence of polynomial functions [1] with particular proprieties. These functions, which allowed the formulation of a concrete solution to the heat equation, were officially exposed to the local mathematics community as a new polynomial class [2]. The Boubaker polynomials expansion method was used in the model of blood vessels presented by works of O. Bamidele Awojoyogbe *et al.* in the field of organic tissues modeling [3]. A recent work presented by S. Slama *et al.*[4] presented also a numerical model of the spatial time-dependant evolution of A3 melting point in steel material during a particular sequence of resistance spot welding [4]. While investigating eventual differential equations to these polynomials, the Boubaker-Turki polynomials (modified Boubaker polynomials) were proposed as an improved form that was doted with a characteristic differential equation [5].

2. Historical appearance of the Boubaker-Turki polynomials

2.1 The Boubaker polynomials

The Boubaker polynomials emerged from an attempt to yield a solution to heat equation. In fact, in a calculation step during resolution process [1], an intermediate calculus sequence raised an interesting recursive formula leading to a class of polynomial functions that performs differently with common classes.

The heat equation [1] inside glass layer medium (g) and deposited layer (s) was expressed by (1):

$$\begin{cases} \dfrac{\partial^2 T_g(z,t)}{\partial z^2} = \dfrac{1}{D_g}\dfrac{\partial T_g(z,t)}{\partial t} - \dfrac{1}{k_g}.(P_b - P_s) \\ \dfrac{\partial^2 T_s(z,t)}{\partial z^2} = \dfrac{1}{D_s}\dfrac{\partial T_s(z,t)}{\partial t} - \dfrac{1}{k_s}.P_s \end{cases} \tag{1}$$

where T_g is the absolute temperature inside glass medium, T_s is the absolute temperature inside deposited layer, D_g and D_s are respectively the glass medium and the deposited layer thermal diffusivities, P_b and P_s are the powers transmitted respectively from bulk to glass and from glass to layer, and finally k_g and k_s are respectively the glass medium and the deposited layer thermal conductivities.

According to bulk size and thermal supply, lower heat conduction toward glass layer ($z = -H$) could be considered as issued from an infinite source under constant temperature T_b. Boundary conditions concerned mainly temperature distribution continuity at median plane ($z = -H$) and glass-layer contact plane ($z = 0$).

After proposing a general expression (2) for temperature distribution [1] inside the glass sample:

$$T_n(z,t) = \frac{1}{N} e^{-\frac{A}{\frac{H}{z}+1}} \sum_{m=0}^{\infty} \xi_m . J_m(t) \quad \text{for}: -H < z < 0 \tag{2}$$

where J_{m} is the m-th order first kind Bessel function, N is a fixed integer parameter, A and ξ_{m} are constants to be found; the application of Boundary conditions, and truncation of the infinite sum down to the integer order N lead to the system (3) :

$$\begin{cases} Q_1(z)\xi_0 = \xi_1 \\ Q_1(z)\xi_1 = -2\xi_0 + \xi_2 \\ Q_1(z)\xi_m = \xi_{m-1} + \xi_{m+1} \quad \text{for}: 1 < m < N \\ \ldots \\ Q_1(z)\xi_{N-1} = \xi_{N-2} + \xi_N \\ \xi_{N+1}(z) = 0 \end{cases} \tag{3}$$

Finally, coefficients ξ_{m} are calculated for $z = 0$, and for the given parameters values.

For larger values of N, and when $z = 0$, a sequence of polynomial functions $B_{\mathrm{m}}(X)$ was defined according to the structure of the relation (3), and resumed in the relation (4):

$$\begin{cases} B_0(X) = 1 \\ B_1(X) = X \\ B_2(X) = X^2 + 2 \\ B_m(X) = X.B_{m-1}(X) - B_{m-2}(X) \quad \text{for}: \mathrm{m} > 2 \end{cases} \tag{4}$$

Using the recursive relations (4) and results from the previous study [1] , the authors established, an explicit monomial form (eq. 5 and 6) and a recursive definition of the Boubaker polynomials coefficients (7). Demonstrations of equations (5-6) are available in appendices of previous studies [2-4].

$$B_n(X)=1.X^n-(n-4).X^{n-2}+\sum_{p=2}^{\xi(n)}\left[\frac{(n-4p)}{p!}\prod_{j=p+1}^{2p-1}(n-j)\right].(-1)^p.X^{n-2p}\,;\quad \text{with } \xi(n)=\frac{2n+((-1)^n-1)}{4} \tag{5}$$

$$B_n(X)=\sum_{p=0}^{\xi(n)}\left[\frac{(n-4p)}{(n-p)}C_{n-p}^{p}\right].(-1)^p.X^{n-2p} \tag{6}$$

where :

$$\xi(n)=\left\lfloor\frac{n}{2}\right\rfloor=\frac{2n+((-1)^n-1)}{4}$$

(The symbol : ⌊ ⌋ designates the floor function, introduced by Iverson in 1962)

$$\begin{cases} B_n(X)=\sum_{j=0}^{\xi(n)}\left[b_{n,j}X^{n-2j}\right];\ \xi(n)=\frac{2n+((-1)^n-1)}{4} \\ b_{n,0}=1; \qquad b_{n,1}=-(n-4); \\ b_{n,j+1}=\frac{(n-2j)(n-2j-1)}{(j+1)(n-j-1)}\times\frac{(n-4j-4)}{(n-4j)}\times b_{n,j} \\ b_{n,\frac{2n+((-1)^n-1)}{4}}=\begin{cases}(-1)^{\frac{n}{2}}\times 2 & \text{if n even}\\ (-1)^{\frac{n+1}{2}}(n-2) & \text{if n odd}\end{cases} \end{cases} \tag{7}$$

According to the relations (4-7), the first Boubaker polynomials are given by (8) :

$$B_0(\mathrm{X}) = 1; \qquad B_1(\mathrm{X}) = X; \qquad B_2(\mathrm{X}) = X^2 + 2;$$
$$B_3(\mathrm{X}) = X^3 + X; \qquad B_4(\mathrm{X}) = X^4 - 2;$$
$$B_5(\mathrm{X}) = X^5 - X^3 - 3X; \qquad B_6(\mathrm{X}) = X^6 - 2X^4 - 3X^2 + 2; \qquad (8)$$
$$B_7(\mathrm{X}) = X^7 - 3X^5 - 2X^3 + 5X; \; B_8(\mathrm{X}) = X^8 - 4X^6 + 8X^2 - 2;$$
.....

2.2. The Modified Boubaker polynomials

Later, the authors proposed, through a specialized study [5], a new version of these polynomials. As opposed to the earlier defined polynomials, the modified Boubaker polynomials, defined by (9):

$$\tilde{B}_n(X) = 2^n.X^n - 2^{n-2}(n-4).X^{n-2} + \sum_{p=2}^{\xi(n)}\left[\frac{(n-4p)}{p!}\prod_{j=p+1}^{2p-1}(n-j)\right].2^{n-2p}(-1)^p.X^{n-2p}\,;\xi(n) = \frac{2n+((-1)^n-1)}{4} \qquad (9)$$

are solutions to a second order characteristic, but non proper equation (10):

$$16.(1-X^2)\tilde{B}_n''(X) - 4.X\tilde{B}_n'(X) + n^2\tilde{B}_n(X) = 32.(n-1)T_{n-2}(X); \quad for \ \mathrm{n} > 2 \qquad (10)$$

where $T_n(X)$, for $n > 2$, are the Chebyshev [6,7] first order polynomials.

This definition allowed an establishment of a quasi-polynomial expression (11) of the Boubaker-Turki polynomials [5]:

$$\tilde{B}_n(X) = \left\langle X + \sqrt{X^2-1}\right\rangle^n\left[8X^2 - 3 - 8X\sqrt{X^2-1}\right] + \left\langle X - \sqrt{X^2-1}\right\rangle^n\left[8X^2 - 3 + 8X\sqrt{X^2-1}\right] \qquad (11)$$

or by setting the simplified forms (12) :

$$\xi = \left\langle X + \sqrt{X^2 - 1} \right\rangle \quad \text{and} \quad \xi' = \xi^{-1} = \left\langle X - \sqrt{X^2 - 1} \right\rangle \tag{12}$$

with the properties expressed by (13):

$$\xi + \xi' = 2X \quad \text{and} \quad \xi.\xi' = 1 \tag{13}$$

we obtain the simplified analytic relation, (14) :

$$\tilde{B}_n(X) = (\xi)^n \left[8X\xi' - 3 \right] + \left(\xi' \right)^n \left[8X\xi - 3 \right] \tag{14}$$

3. Investigations of the Boubaker-Turki polynomials

3.1 Second kind Chebyshev polynomial expansion of the Boubaker-Turki polynomials

The main purpose of this section is to solve the system (15):

$$\tilde{B}_n(X) = \sum_{j=0}^{n} P_j(X).U_{n-j}(X) + \sum_{j=0}^{n} Q_j(X).T_{n-j}(X) \tag{15}$$

where T_n and U_n are respectively the Chebyshev[6,7] polynomials of the first and second kind, P_j and Q_j are unknown polynomials of degree *j*.
In fact, the early defined Boubaker polynomials could not be expressed in function of the Chebyshev polynomials since they obeyed to different recursive relations. As opposed to this, the Boubaker-Turki polynomials presented some similarities with the Dickson polynomials that are easily expressed in function of Chebyshev ones.

Using the first terms of the evoked polynomials we obtained the empirical solution (16-17):

$$\begin{cases} P_0(X)=0 \\ P_1(X)=4x \\ P_j(X)=0;\ j>1. \end{cases} \quad \text{and} \quad \begin{cases} Q_0(X)=2 \\ Q_j(X)=0;\ j>0 \end{cases} \tag{16-17}$$

We conjectured hence that we have the relation:

$$\tilde{B}_n(X)=4X.U_{n-1}(X)-2T_n(X) \tag{18}$$

We set an analytical demonstration to this relation (see Appendix) using the expressions (19-20) given by Abramowitz et al. [8] :

$$T_n(X)=\frac{n}{2}\sum_{p=0}^{\xi(n)}(-1)^p.2^{n-2p}\frac{(n-p-1)!}{p!(n-2p)!}.(X)^{n-2p} \tag{19}$$

$$U_n(X)=\sum_{p=0}^{\xi(n)}(-1)^p.2^{n-2p}\frac{(n-p)!}{p!(n-2p)!}.(X)^{n-2p} \tag{20}$$

where:

$$\xi(n)=\left\lfloor\frac{n}{2}\right\rfloor=\frac{2n+\left[(-1)^n-1\right]}{4}$$

3.2 Establishment of an ordinary generating function for the defined Boubaker-Turki polynomials

An ordinary generating function *f(X,t)* for a polynomial sequence $P_n(X)$ is the function that verifies (21):

$$f(X,t)=\sum_{n=0}^{\infty}P_n(X)t^n \tag{21}$$

Thanks to the established relation (§ 3.1) and to the known[9] generating functions (22) and (23):

$$\sum_{n=0}^{\infty} T_n(X)t^n = \frac{1-t.X}{1-2X.t+t^2} \tag{22}$$

$$\sum_{n=0}^{\infty} U_n(X)t^n = \frac{1}{1-2X.t+t^2} \tag{23}$$

we can calculate (24):

$$\sum_{n=1}^{\infty} \tilde{B}_n(X).t^n = 4X\sum_{n=1}^{\infty} U_{n-1}(X)t^n - 2\sum_{n=1}^{\infty} T_n(X)t^n = 4X\sum_{n=0}^{\infty} U_n(X)t^{n+1} - 2\sum_{n=1}^{\infty} T_n(X)t^n \tag{24}$$

we can write:

$$\sum_{n=0}^{\infty} \tilde{B}_n(X).t^n - \tilde{B}_0(X) = 4Xt\frac{1}{1-2X.t+t^2} - 2\left(\frac{1-t.X}{1-2X.t+t^2} - T_0(X)\right) \tag{25}$$

and then :

$$\sum_{n=0}^{\infty} \tilde{B}_n(X).t^n = 4Xt\frac{1}{1-2X.t+t^2} - 2\left(\frac{1-t.X}{1-2X.t+t^2} - 1\right) + 1 = \frac{1+3t^2}{1-2X.t+t^2} \tag{26}$$

This yields finally the ordinary generating function (27) :

$$\boxed{f_B(X,t) = \frac{1+3t^2}{1-2X.t+t^2} = \sum_{n=0}^{\infty} \tilde{B}_n(X).t^n} \tag{27}$$

For values of t close to unity, and $X = 0$, we can verify that this sum has the mean value 2. In fact we have demonstrated that:

$$\tilde{B}_n(0) = 2\cos(\frac{n+2}{2}\pi); n \geq 1 \Rightarrow \tilde{B}_n(0) = \begin{cases} 1; & \text{for} \quad n=0 \\ 0, & \text{for} \quad n=4q+1, 4q+3 \\ 2, & \text{for} \quad n=4q+2 \\ -2, & \text{for} \quad n=4q \end{cases}$$

it is obvious that due to the symmetrical values, 2 and -2 :

$$\lim_{t\to 1}\left\langle \sum_{\mathrm{n}=0}^{\infty} \tilde{B}_n(X).t^{\mathrm{n}} \right\rangle\Bigg|_{X=0} = 1 + \lim_{t\to 1}\left\langle \sum_{\mathrm{n}=0}^{\infty} \pm 2.t^{\mathrm{n}} \right\rangle = 2 = \lim_{t\to 1}\left[\frac{1+3t^2}{1-2X.t+t^2}\right]\Bigg|_{X=0}$$

A similar result can be demonstrated for t = -1.

3.3 A Christoffel-Darboux type first-order differential equation for the Boubaker-Turki polynomials

The main relation (4), the equations (9) and (11) allow writing (28):

$$\tilde{B}_{j+1}(X)\times\tilde{B}_j(Y) - \tilde{B}_j(X)\times\tilde{B}_{j+1}(Y) = 2(X-Y)\tilde{B}_j(X)\tilde{B}_j(Y) - \left[\tilde{B}_{j-1}(X)\times\tilde{B}_j(Y) - \tilde{B}_j(X)\times\tilde{B}_{j-1}(Y)\right] \qquad (28)$$

which gives (29):

$$\tilde{B}_j(X)\tilde{B}_j(Y) = \frac{\theta_j - \theta_{j-1}}{2(X-Y)} \qquad (29)$$

where (30):

$$\theta_j = \tilde{B}_{j+1}(X)\times\tilde{B}_j(Y) - \tilde{B}_j(X)\times\tilde{B}_{j+1}(Y) \qquad (30)$$

By adding the expressions (30) for j=0 to n, we obtain the equation (31):

$$\sum_{j=0}^{n}\tilde{B}_j(X)\tilde{B}_j(Y)=\frac{\tilde{B}_{n+1}(X)\times\tilde{B}_n(Y)-\tilde{B}_n(X)\times\tilde{B}_{n+1}(Y)}{2(X-Y)} \quad (31)$$

Finally, by imposing X→ Y, we obtain Christoffel-Darboux[10] type first-order differential equation (32)

$$\sum_{j=0}^{n}\tilde{B}_j^2=\frac{\tilde{B}'_{n+1}(X)\times\tilde{B}_n(X)-\tilde{B}'_n(X)\times\tilde{B}_{n+1}(X)}{2} \quad (32)$$

5. Conclusion

This work presents an ordinary generating function for the established Boubaker-Turki polynomials [1-5, 11-13], as a guide to establish a second order characteristic differential equation to these polynomials. The yielded Christoffel-Darboux type first-order differential equation seems to be an important supply for further investigations on properties that may lead to a characteristic homogenous second order equation.

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(available also online at : **http://planetmath.org/encyclopedia/BoubakerPolynomials.html**)

APPENDIX

In this appendix we demonstrate the conjectured relation (A.1):

$$\tilde{B}_n(X) = 4X.U_{n-1}(X) - 2T_n(X) \tag{A.1}$$

where U_n are the Chebyshev polynomials of the second kind, T_n are the Chebyshev polynomials of the first kind and $\tilde{B}_n(X)$ denotes the Boubaker-Turki polynomials (A.2) :

$$\tilde{B}_n(X) = \sum_{p=0}^{\xi(n)} (-1)^p . 2^{n-2p} \frac{(n-4p)}{p!} \frac{(n-p-1)!}{(n-2p)!} .(X)^{n-2p}; \text{ where: } \xi(n) = \left\lfloor \frac{n}{2} \right\rfloor = \frac{2n + \left[(-1)^n - 1\right]}{4} \tag{A.2}$$

We knew that Chebyshev polynomials of the first and second kind are expressed by (A.3) and (A.4):

$$T_n(X) = \frac{n}{2} \sum_{p=0}^{\xi(n)} (-1)^p . 2^{n-2p} \frac{(n-p-1)!}{p!(n-2p)!} .(X)^{n-2p} \tag{A.3}$$

$$U_n(X) = \sum_{p=0}^{\xi(n)} (-1)^p . 2^{n-2p} \frac{(n-p)!}{p!(n-2p)!} .(X)^{n-2p} \tag{A.4}$$

Let's calculate the expression (A.5):

$$4X.U_{n-1}(X)-2T_n(X)=$$
$$4.X.\sum_{p=0}^{\xi(n-1)}(-1)^p.2^{n-2p-1}\frac{(n-p-1)!}{p!(n-2p-1)!}.(X)^{n-2p-1}-2\left(\frac{n}{2}\sum_{p=0}^{\xi(n)}(-1)^p.2^{n-2p}\frac{(n-p-1)!}{p!(n-2p)!}.(X)^{n-2p}\right) \quad (A.5)$$

The coefficient of the (n-2p)-order term of the sum (A.4) is hence:

$$c_p=(-1)^p 2^{n-2p}\left[\frac{2.(n-p)!}{p!(n-2p)!}-\left(n\frac{(n-p-1)!}{p!(n-2p)!}\right)\right] \quad (A.6)$$

Due to the proprieties of the factorials we obtain (A.7):

$$c_p=(-1)^p 2^{n-2p}\left[\frac{2.(n-1-p)!(n-2p)}{p!(n-2p)!}-\left(n\frac{(n-p-1)!}{p!(n-2p)!}\right)\right]. \quad (A.7)$$

We can notice that :

$$\frac{2.(n-1-p)!(n-2p)-n(n-p-1)!}{p!(n-2p)!}=\frac{(n-1-p)![(2n-4p)-n]}{p!(n-2p)!}=\frac{(n-1-p)![(n-4p]}{p!(n-2p)!} \quad (A.8)$$

Finally we obtain (A.9):

$$c_p=(-1)^p.2^{n-2p}\frac{(n-4p)}{p!}\frac{(n-p-1)!}{(n-2p)!} \quad (A.9)$$

and (A.10):

$$4X.U_{n-1}(X)-2T_n(X)=\sum_{p=0}^{\xi(n)}c_p.(X)^{n-2p}=\sum_{p=0}^{\xi(n)}(-1)^p.2^{n-2p}\frac{(n-4p)}{p!}\frac{(n-p-1)!}{(n-2p)!}(X)^{n-2p}=\tilde{B}_n(X) \quad (A.10)$$