

ters away, in a manner analogous to the behavior of optical interference fringes.

Appleton's experimental proof that the Kennelly-Heaviside, or *E*, layer really existed, a scientific accomplishment of the highest order, was honored by his election as fellow of the Royal Society (1927), a knighthood (1941), and the Nobel Prize in physics (1947), "for his investigation of the physics of the upper atmosphere, especially for the discovery of the so-called Appleton layers." The last refers to the later discoveries of a second (*F*) layer at more than twice the height of the *E* layer and a third (*D*) layer below the *E* layer. Besides discovering these layers, Appleton and his co-workers showed that the sky wave generally was elliptically polarized, and calculated the reflection coefficients and electron densities of the layers and their diurnal and seasonal variations. His work may also be considered to be of prime technological significance, not only in regard to radio transmission but also as a milestone in the development of radar; the determination of the height of the *E* layer was the first distance measurement made by radio, a technique that was closely followed by Robert Alexander Watson-Watt, the British radar pioneer, who had collaborated with Appleton in atmospheric research and had many subsequent professional contacts with him. The rest of Appleton's life was spent in research flowing from his own discoveries, an endeavor in which he continued to maintain a degree of involvement that was astonishing in view of the many other duties thrust upon him.

Following a three-year tenure as Jacksonian professor of natural philosophy at the University of Cambridge, Appleton was appointed secretary of the government's Department of Scientific and Industrial Research. In 1949 he was made principal and vice-chancellor of the University of Edinburgh, where he remained until his death. Appleton was a great international figure: his first paper on the ionosphere was published in a Dutch journal (in Dutch); he served as vice-president of the (U.S.) Institute of Radio Engineers in 1932; he was president of the International Scientific Radio Union (URSI) from 1934 to 1952; and he was instrumental in organizing the first International Geophysical Year in 1957, a year of maximum sunspot activity. After moving to Edinburgh, he founded the *Journal of Atmospheric Research* (affectionately known as "Appleton's Journal") and served as its editor-in-chief for the rest of his life.

Appleton married Jessie Longson in 1915; they had two daughters, Marjery and Rosalind. A month before his death, Appleton, a widower since 1964, married Mrs. Helen F. Allison, who had been his private secretary for thirteen years.

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CHARLES SÜSSKIND

AQUINAS, SAINT THOMAS (b. Roccasecca, near Monte Cassino, Italy, ca. 1225; d. Fossanuova, near Maenza, Italy, 7 March 1274), not a scientist in the modern sense, but a philosopher and theologian whose synthesis of Christian revelation with Aristotelian science has influenced all areas of knowledge—including modern science, especially in its early development.

Thomas, the youngest of nine children, was born in the castle of the Aquino family. His father, Landolfo, and his older brothers served the Holy Roman Emperor, Frederick II, then warring against the papacy; his mother, Teodora of Chieti, was a Lombard. The family's political situation was precarious, and in 1231 Thomas was placed in the abbey of Monte Cassino for his elementary education. When the abbey was occupied by Frederick's troops in 1239, Thomas was sent to finish his studies at the recently founded University of Naples; his teachers there were Master Martin in grammar and logic and Peter of Ireland in natural science.

Thomas entered the Dominican order at Naples in 1244, against his family's wishes, and was sent to Paris, and then to Cologne, for further studies (1245-1252). The Dominicans at the time were in the forefront of intellectual life: in natural science, groups of friars were synthesizing the heritage of Greece and Rome, which soon appeared in the encyclopedias of Thomas of Cantimpré and Vincent of Beauvais. Albertus Magnus, earlier recruited into the order by Jordan of Saxony, was himself paraphrasing in Latin all of the works of Aristotle that had just been brought to the West, thus rendering them intelligible to the younger friars. Studying under Albert, possibly at Paris and certainly at Cologne (1248-1252), Thomas was soon abreast of the most advanced scholarship of his time, including the major Greek, Arab, and Latin sources that were to revivify the intellectual life of the Middle Ages.

Sent to Paris "to read the *Sentences*" at the priory of Saint-Jacques in 1252, Thomas quickly demonstrated his proficiency as a theologian. There was, however, growing jealousy and antipathy toward the friars (both Dominican and Franciscan) among the

secular masters at the University of Paris, and in 1256 the intervention of Pope Alexander IV was required before Thomas and the Franciscan Bonaventure were accepted as masters at the university. During or before this, his first Paris professorship (1256-1259), Thomas composed his commentary on the *Sentences*, some smaller treatises—including the highly original *De ente et essentia* ("On Being and Essence")—and the disputed question *On Truth*; he also began work on the *Summa contra gentiles*, of special importance for its evaluation of Arab thought.

From 1259 to 1268 he was back in Italy, first at Anagni and at Orvieto, where he was associated with the papal courts of Alexander IV and Urban IV, respectively; then at Rome (1265-1267), where he taught at the Dominican priory of Santa Sabina and began his famous *Summa theologiae*; and finally at Viterbo, where he served at the court of Clement IV.

Then, in 1268 or 1269, possibly because of disputes at the University of Paris over the Aristotelianism which he and Albert had introduced, Thomas returned to Paris for a somewhat unusual second professorship (1269-1272). Here he combated both the traditional Augustinian orthodoxy being fostered by such Franciscans as Bonaventure and John Peckham and the heterodox Aristotelianism of Siger of Brabant and his associates, who are usually referred to as Latin Averroists. One of the key issues in the dispute was Thomas' teaching that the world's creation in time cannot be demonstrated by reason alone, since there is no philosophical repugnance in a created universe's having existed from eternity—a thesis with important ramifications for later medieval concepts of infinity.

The condemnation, in 1270, of certain Averroist theses by Étienne Tempier, bishop of Paris, is regarded by some scholars as directed, at least implicitly, against Aquinas' teaching. Of the later condemnation, in 1277, there can be no doubt that two propositions concern matters taught by Thomas, including his thesis on the unicity of the substantial form in man, which bears on the problem of the presence of elements in compounds. Such controversies drew a series of polemical treatises from Aquinas' pen, including *De aeternitate mundi contra murmurantes* ("On the Eternity of the Universe, Against the 'Murmurers'") [i.e., the traditional Augustinians] and *De unitate intellectus contra Averroistas* ("On the Unity of the Intellect, Against the Averroists"). The intellectual ferment also stimulated him to further efforts at philosophical and theological synthesis. During these years he elaborated most of his detailed commentaries on Aristotle and worked steadily on the *Summa theologiae*.

After his second Paris professorship was concluded in 1272, Thomas returned to Italy, this time to Naples,

to erect a Dominican *studium* near the university there. He lectured, directed disputations, and continued writing; but the pace of his work slowed noticeably, partly because of failing health. He suspended all writing activity late in 1273 and died a few months later, while en route to the second Council of Lyons. He was canonized on 18 July 1323 and subsequently was approved by the Roman Catholic Church as its most representative teacher.

Today the name of Thomas is so associated with Catholic orthodoxy that one tends to forget that he was an innovator. In an atmosphere dominated by faith, especially at the University of Paris, he took the leadership in championing the cause of reason. Almost single-handedly he turned the theologians of that university to a study of the pagan Aristotle, to the use of what was then a rigorous scientific method, learned from investigating the world of nature, for probing the mysteries of revelation. Opposing the popular teaching that all knowledge comes by divine illumination, he allowed that man, by sense observation and through the use of unaided reason, could arrive at truth and certitude.

It would be a mistake, of course, to urge that Aquinas' main concern was with the physical universe. Rather, he was preoccupied with questions about God, the angels, and man; first and foremost he was a metaphysician and a theologian. Yet there can be no doubt that, like Aristotle, his approach to metaphysical problems was through the physical sciences. Like St. Paul, he firmly believed that the invisible things of God are seen through his visible creation, provided it is rightly understood (Romans 1:20). So convinced of this was he that in his later life he turned from his unfinished *Summa theologiae* to comment on all the physical works of Aristotle. He probably completed his exposition of *De caelo et mundo*, one of his best works as a commentator, at Naples (1272-1273) and ceased commenting on *De generatione et corruptione* and the *Meteorologica* only shortly before his death.

Furthermore, for a man not usually recognized as a scientist, he made noteworthy contributions to medieval science. These can best be indicated by summarizing his more significant teachings relating to the medieval counterparts of physics, astronomy, chemistry, and the life sciences.

In the high scholastic period, foundations were laid for later medieval discussions that adumbrated the distinction in modern mechanics between kinematics and dynamics. The kinematical content of Thomas' teaching is meager, although he did hold that velocity is a mode of continuous quantity and thus is capable of intensification in the same manner as qualities,

thereby allowing for the type of comparison between qualitative change and local motion later made by Nicole Oresme.

In dynamics, he inaugurated some new directions in the study of causality affecting gravitational and projectile motions. Aquinas would probably look askance at the tendency of present-day historians of science to identify Aristotle's motive powers and resistances with forces and to represent Aristotle's teaching with precise dynamical equations. His own exegesis of the relevant Aristotelian texts, as opposed to that of Averroës and Avempace, discounts any demonstrative intent on Aristotle's part and interprets his statements as dialectical efforts to confute his atomist opponents.

Thus, on the disputed question whether motion through a vacuum would take place instantaneously, Thomas did not follow Aristotle literally. He insisted that if, by an impossibility, a vacuum were to exist, motion through it would still take time—that the temporal character of the motion does not arise uniquely from external resistance but, rather, from the proportion of the mover to the moved (which prevents the movements of the heavens from being instantaneous, although they are not impeded by resistance) and also from the continuity of the distance being traversed. The latter reason, particularly, provoked speculation among fourteenth-century thinkers; some, such as Oresme, saw no necessary connection between spatial continuity and velocity limitation and were led on this account to seek some type of resistance internal to the moving body—thereby foreshadowing the modern concept of inertia.

Thomas' analysis of gravitation is basically Aristotelian, yet it differs in significant respects from that of other commentators. Like all medieval thinkers, he regarded gravitation as the natural motion of a heavy body to its proper place. For Thomas, however, nature was a relational concept, and thus he disagreed with those who defined it as a *vis insita* or as something absolute; it is a principle of motion, either actively or passively, depending on the particular motion that results.

Aquinas held that the body's gravity is the proximate cause of its falling, but only in the manner of a "passive principle." He rejected Averroës' teaching that the medium through which the body falls plays an essential role in its motion and that there is an active source of such motion within the body, whether this be its gravity or its substantial form. In this respect, Aquinas was also at variance with the later teaching of Walter Burley and the Paris terminists, all of whom saw the cause of falling as some type of active force within the body itself, thereby fore-

shadowing animist theories of gravitation such as subsequently proposed by William Gilbert. Again, Thomas disagreed with Bonaventure and Roger Bacon, who regarded place as exerting some type of repulsive or attractive influence on the falling body; for Thomas, there was no repulsion involved, and the attractive aspect of place was sufficiently accounted for by its being the end, or final cause, of the body's movement. Here Thomas implicitly rejected the absolute space and attractive forces later proposed by the Newtonians; his own analysis, it has been remarked, shows more affinity with the ideas behind Einstein's theory of general relativity, although the two are so remote in thought context as to defy any attempt at detailed comparison.

On the subject of *impetus*, authors are divided as to Thomas' teaching. Certainly he has no treatment of the concept to match that found later in Francis de Marcia, Jean Buridan, and Oresme, nor does he use it to explain any details of projectile or gravitational motion. In his later writings, particularly the commentaries on the *Physics* and the *De caelo*, Thomas clearly defends the original Aristotelian teaching on the proximate cause of projectile motion. In some earlier writings, on the other hand, he speaks of a *virtus* in the projectile, and, in one text of the *Physics* commentary, discussing the case of a ball that bounces back from a wall, he mentions that the *impetus* is given not by the wall but by the thrower. Later Thomists, such as Joannes Capreolus and Domingo de Soto, had no difficulty in assimilating a fully developed *impetus* theory to Thomas' teaching, evidently regarding the Aristotelian element in his expositions as reflecting his role as a commentator more than his personal views.

Aquinas took up the problems of the magnet, of tidal variations, and of other "occult" phenomena in a letter entitled *De occultis operationibus naturae* ("On the Occult Workings of Nature"), whose very title shows his preoccupation with reducing all of these phenomena to natural, as opposed to supramundane, causes. Significantly, Thomas' analysis of magnetism was known to Gilbert and was praised by him.

Commenting as he did on the *De caelo* and also, in his theological writings, on the cosmogony detailed in Genesis, Thomas could not help but evaluate the astronomical theories of his contemporaries. He contributed nothing new by way of observational data, nor did he evolve any new theories of the heavens, but his work has an importance nonetheless, if only to show the care with which he assessed the current state of astronomical science. His view of the structure of the universe was basically Aristotelian; he knew of two theories to account for the phenomena of the

heavens, both geocentric in the broad sense: that of Eudoxus, Callippus, and Aristotle, and that of Ptolemy. Aquinas generally employed the Eudoxian terminology; he mentions the Ptolemaic system at least eleven times, and five of these are in his late commentary on the *De caelo*. In most of his references to the Eudoxian or Ptolemaic systems, he refrains from expressing any preference; clearly, he was aware of the hypothetical character of both. At least once, commenting on Ptolemy's cumbersome theory of eccentrics and epicycles, he voices the expectation that this theory will one day be superseded by a simpler explanation.

The astronomical data reported by Aquinas, according to an extensive analysis by Thomas Litt, were those of a well-informed thirteenth-century writer; he errs in one or two particulars, but on matters of little theoretical consequence. His treatise on comets, included in a work by Lynn Thorndike, is one of the most balanced in the high Middle Ages, rejecting fanciful explanations and pointing out how little is actually known about these occurrences.

In his more philosophical views, however, Thomas was not so fortunate. He believed in the existence of spheres that transport the heavenly bodies and, with his contemporaries, regarded such bodies as incorruptible. He was convinced also of the existence of an empyrean heaven, the dwelling place of the blessed and known only through revelation, but nonetheless included in the corporeal universe. He accorded an extensive causality to the heavenly bodies, while excepting from this all actions that are properly human (i.e., that arise from man's intelligence and deliberate will) and completely fortuitous events, so as to discourage any naive credence in the astrologers of his day.

Aquinas has no treatment of alchemy to match that of his teacher Albertus Magnus, but he does discuss one topic that had important bearing on later views of the structure of matter, that of the presence of elements in compounds. Earlier thinkers, attempting to puzzle out Aristotle's cryptic texts, favored one of two explanations of elemental presence offered by Avicenna and Averroës, respectively. Dissatisfied with both, Aquinas formulated a third position, which soon became the most popular among the Schoolmen. He taught that the elements do not remain actually in the compound, but that their qualities give rise to "intermediate qualities" that participate somewhat in each extreme; these intermediate qualities are in turn proper dispositions for a new substantial form, that of the compound, which is generated through the alteration that takes place. Since the elemental qualities remain "in some way" in the compound, one can

say also that the substantial forms of the elements are present there, too—not actually, but virtually.

The subsequent influence of Thomas' teaching has been traced in considerable detail by Anneliese Maier, who characterized it as inaugurating a modern direction that dominated treatments of the problem in later Scholasticism (*Studien* III, pp. 89-140). Duns Scotus and his school, particularly, became enamored of the theory and attempted a consistent development of its ramifications. Nominalists such as William of Ockham and Gregory of Rimini took it up, too, as did such Paris terminists as Buridan, Oresme, Albert of Saxony, and Marsilius of Inghen. The basic explanation continued to be taught through the sixteenth century and, coupled with Aristotelian teaching on *minima naturalia*, became the major alternative to a simplistic atomist view of the structure of matter before the advent of modern chemistry.

In biology and psychology, Aquinas followed Aristotle, Galen, and the medieval Arab tradition; his work is noteworthy more for its philosophical consistency than for its scientific detail. He wrote commentaries on Aristotle's *De anima*, *De sensu et sensato*, and *De memoria et reminiscentia*, all based on the texts of William of Moerbeke, his fellow Dominican. Also, ca. 1270, he composed a letter to a Master Philippus, who seems to have been a physician and professor at Bologna and Naples, on the motion of the heart (*De motu cordis*), explaining how the principle "Whatever is moved, is moved by another" is saved in this phenomenon. Like his contemporaries, he believed in spontaneous generation and countenanced a qualified type of evolution in the initial formation of creatures. Catholic thinkers, on the basis of his philosophy, have been more open to evolutionary theories than have fundamentalists, who follow a strict, literal interpretation of the text of Genesis.

Thomas was a mild man, objective and impersonal in his writing, more cautious than most in giving credence to reported facts. He showed neither the irascible temperament of Roger Bacon, nor the subtle questioning of Duns Scotus, nor the pious mysticism of Bonaventure. Calm and methodical in his approach, proceeding logically, step by step, he offered proof where it could be adduced, appealing to experience, observation, analysis, and (last of all) authority. He appreciated the importance of textual criticism, and possibly was one of the instigators of Moerbeke's many Latin translations of scientific treatises from the original Greek. He had a penetrating intellect and a strong religious faith, both of which led him to seek a complete integration of all knowledge, divine as well as human. Working with the science of his time, he succeeded admirably in this attempt, thus providing

a striking example for all who were to be similarly motivated in the ages to come.

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Académie des Sciences, replacing Fourier, on 7 June 1830.

Arago was at once volatile and warm-hearted in his personal relations. He either forged strong bonds with fellow scientists or engaged in sharp polemics that often were provoked by priority controversies. Among his closest friends were Alexander von Humboldt, with whom he shared a room in Paris from 1809 to 1811, Gay-Lussac, and Malus; and among his relatives, the physicist Alexis Petit and the astronomer Claude L. Mathieu. He had a stormy relationship with Biot, Thomas Young, and Brewster, but it did not blind him to their scientific merits. In both his writings and his public appearances, Arago conveyed a contagious sense of excitement that won him a large following. His personal style, which spilled over into his work habits, was that of a romantic—restless, inquisitive, volatile, and constantly bubbling with enthusiasm and optimism. Married in 1811, Arago had three sons and lived in an apartment at the observatory. In his later years he gradually lost his eyesight, went blind, and was reduced to dictating to his students.

Arago's most important original work in science was carried out before 1830, for his younger brothers, particularly Étienne, drew him into politics following the July Revolution of 1830. He was repeatedly elected deputy for his native department (Pyrénées-Orientales) and for Paris between 1830 and 1852, and sat on the left in the Chamber of Deputies, delivering influential speeches on educational reform, freedom of the press, and the application of scientific knowledge to technological progress, particularly concerning canals, steam engines, railroads, the electric telegraph, and photography. He also was twice named president of the Paris Municipal Council. The peak of Arago's political career came after the February Revolution of 1848, when he was made a member of the provisional government and was named, successively, minister of the navy and the army and president of the Executive Committee. As minister he signed decrees outlawing corporal punishment and improving the rations of sailors on the high seas, and abolishing slavery in the French colonies. His politics were those of a constitutional liberal, passionately concerned with social reform (he helped found *La réforme* in 1843), freedom of association, and education of the lower classes. He was, however, violently opposed to mob rule and to the socialistic programs espoused by Louis Blanc, Alexandre Ledru-Rollin, and Louis Blanc. Arago's effective political career ended following his loss of control over the revolutionaries during the June days of 1848.

Arago's scientific life was dominated by a persistent interest in physical phenomena related to electricity, magnetism, and, above all, to light. His earliest investigations with Biot in 1805 and 1806 continued the work of Borda on the factors affecting the refraction of light passing through the atmosphere of the earth. They helped to verify the formulas given in Laplace's *Mécanique céleste*, which were based on the assumption that the atmosphere is composed of concentric rings of a mixture of oxygen and nitrogen, with density as a function of altitude. Biot and Arago showed experimentally that temperature and pressure were significant variables, whereas humidity and the traces of carbon dioxide in the atmosphere could be disregarded. But when Arago extended his investigations to refraction in liquids and solids—with Petit in 1813 and Fresnel in 1815—he recognized the failure of the current theory of emission and particle attraction to account for the empirical formulas he derived. After his return from the geodetic expedition to extend meridian triangulations from Barcelona to the Balearic Islands, Arago became a vocal critic of the Newtonian emission theory and, by 1816, an ardent supporter of the undulatory theory.

The original source of Arago's interest was Thomas Young's classic paper of 1801 on the color of thin glass plates and the discovery of polarization by Malus in 1808. Arago continued their independent investigations by passing beams of polarized light through a variety of gaseous and crystalline substances at various degrees of incidence to study the light's properties. His results, which suggested the usefulness of the undulatory theory, included the discovery of chromatic polarization by the use of thin mica plates (1811), rediscovered independently by Brewster; the elaboration of the conditions necessary to produce Newton's rings (1811); and the observation of special cases of rotary polarization (1812), which were shortly thereafter made a general law of optics by Biot.

It was this series of disparate experiments that caused Fresnel to write to Arago in 1815 to announce his theory of stellar aberration and the explanation of diffraction phenomena by undulatory principles. Although Fresnel's "discoveries" had retraced the work of Bradley and Thomas Young, Arago urged him to pursue his investigations and agreed to collaborate with the young engineer. Together they published a series of papers advocating the undulatory theory of light, answering one by one the criticisms of the partisans of emission theory, especially Arago's colleagues and former friends, Laplace and Biot. In this collaborative enterprise Fresnel supplied the crucial mathematical analyses and the seminal concept of