The **Limits** of Regular Languages

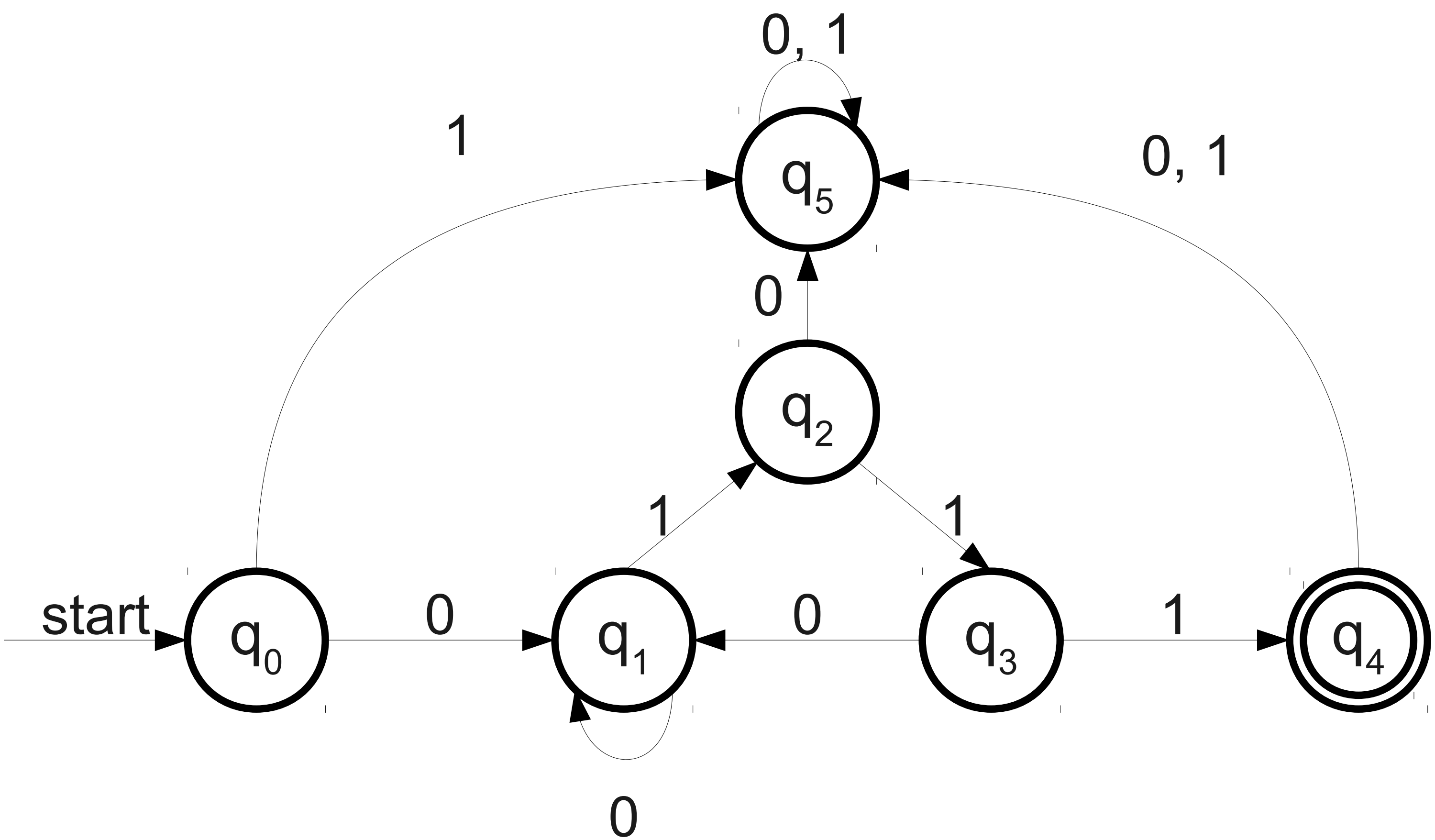
Announcements

- Problem Set 5 due right now.
- Problem Set 6 out, due Friday, November 11 at 2:15PM.
  - Stop by OH with questions!
  - Email **cs103@cs.stanford.edu** with questions!
- Friday Four Square today!

A Counting Argument

- There are more languages than strings (Cantor's theorem; first lecture!)
- There are no more regular languages than strings (can describe regular languages using regular expressions).
- So some languages cannot be regular.
- What are they? What do they look like?

An Important Observation



0 1 1 0 1 1 1

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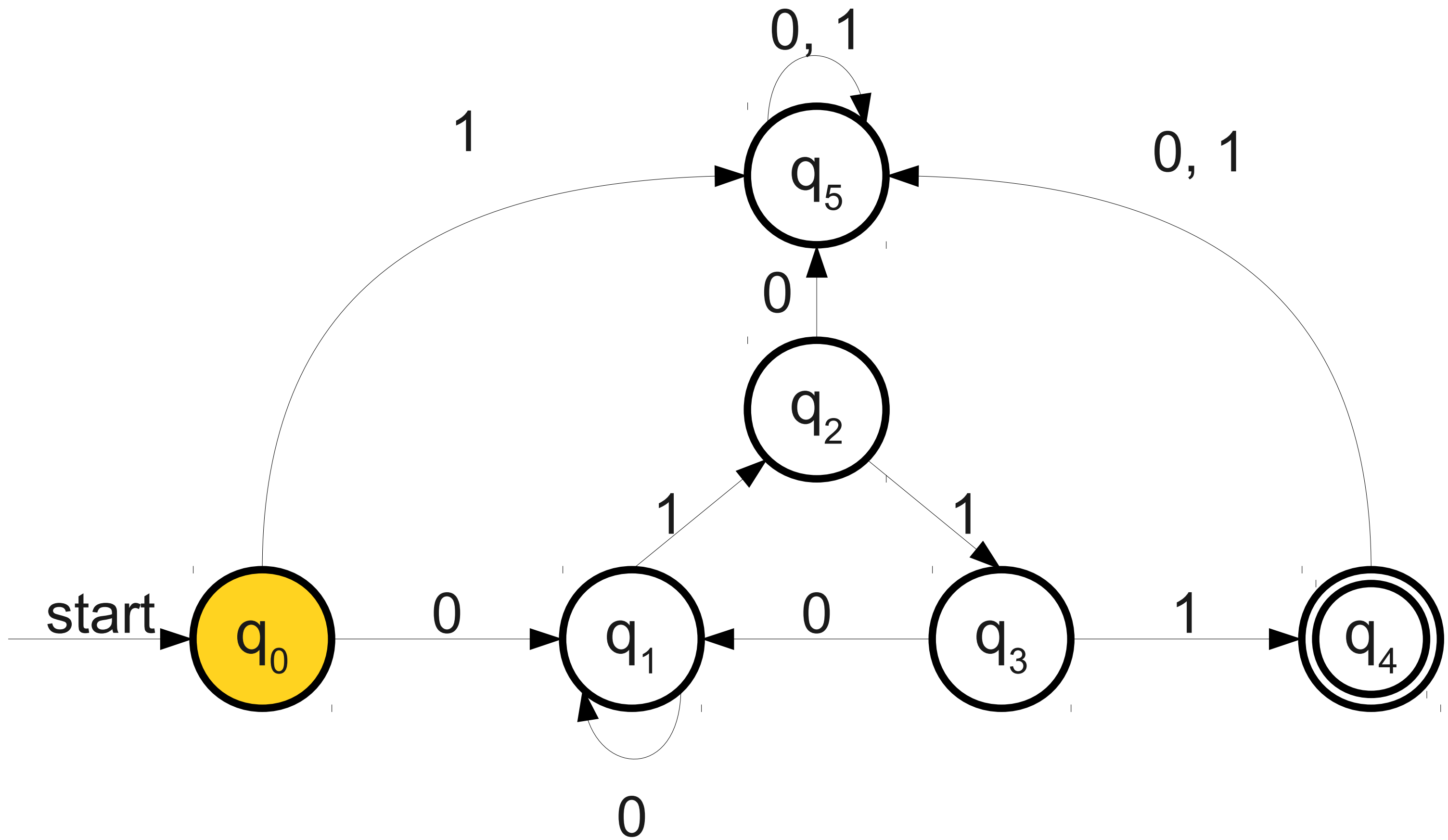

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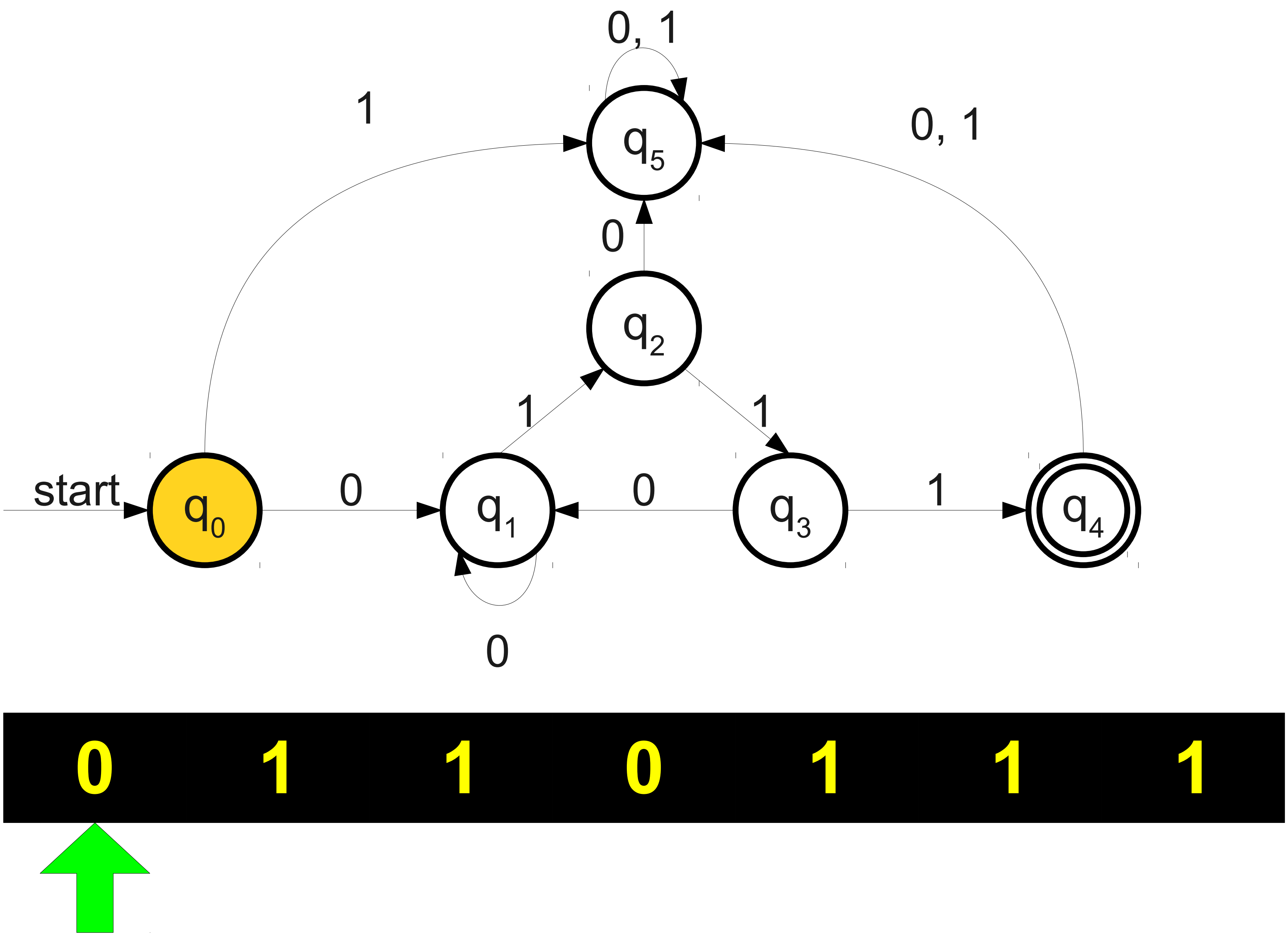

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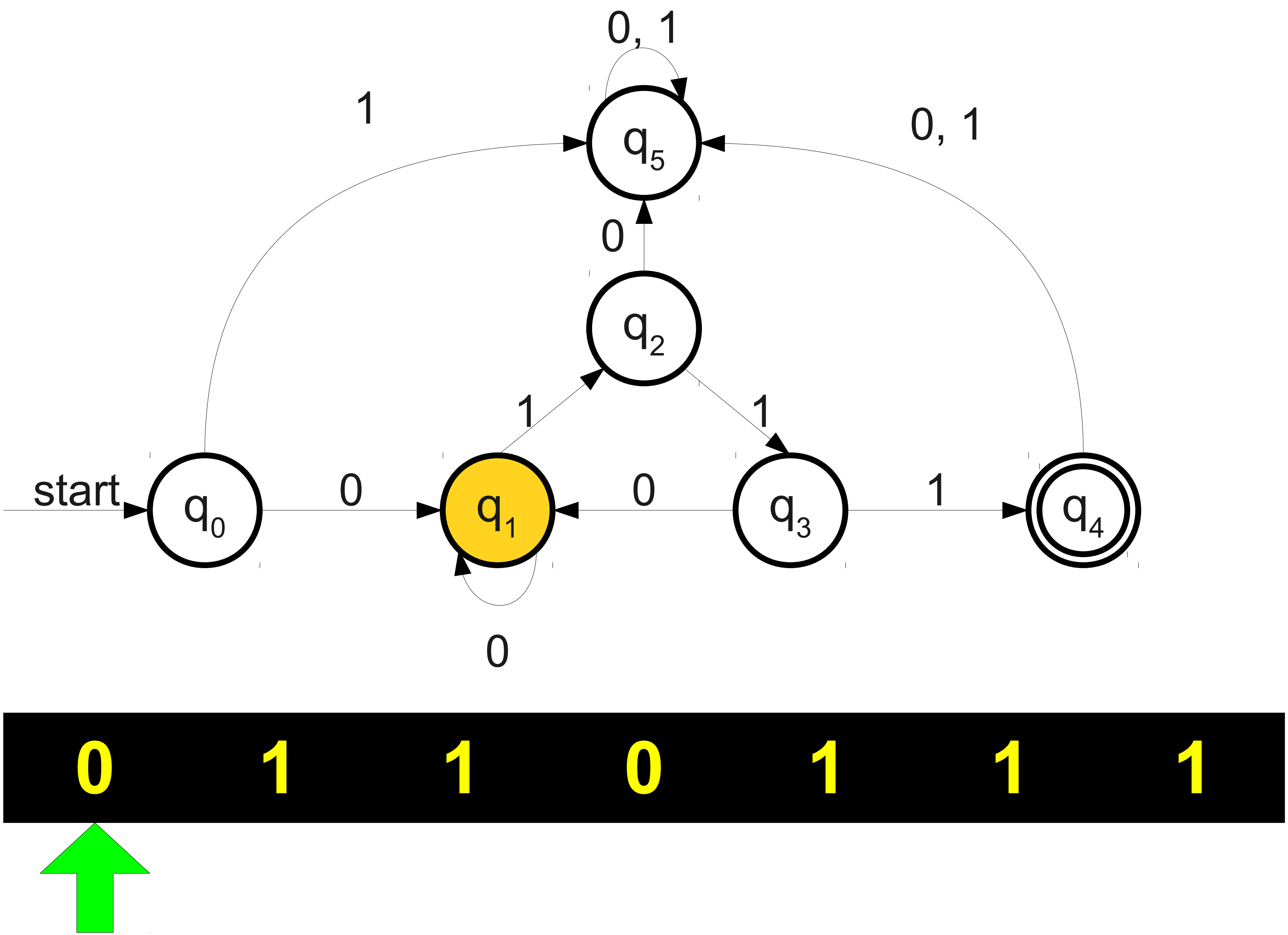

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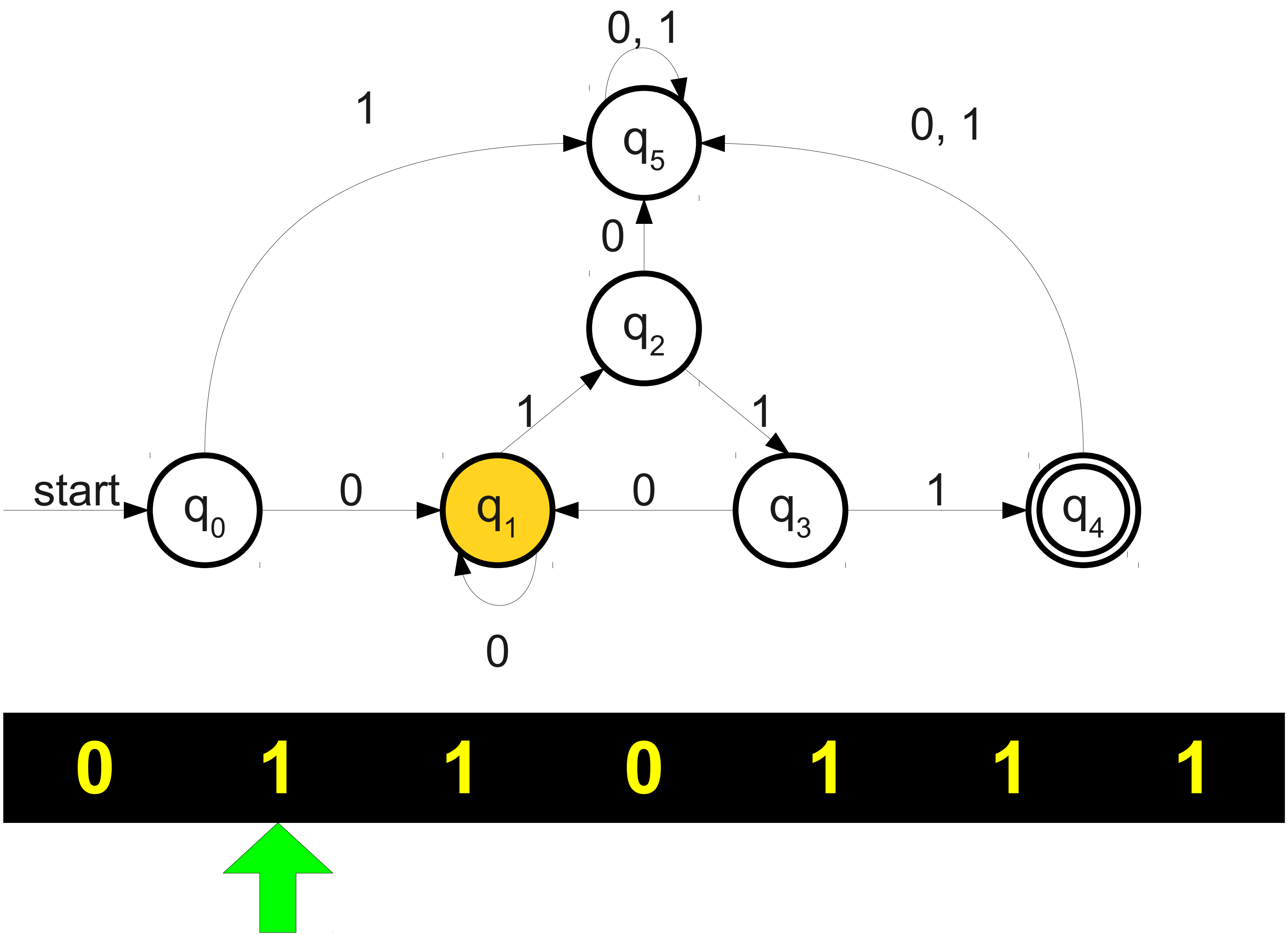

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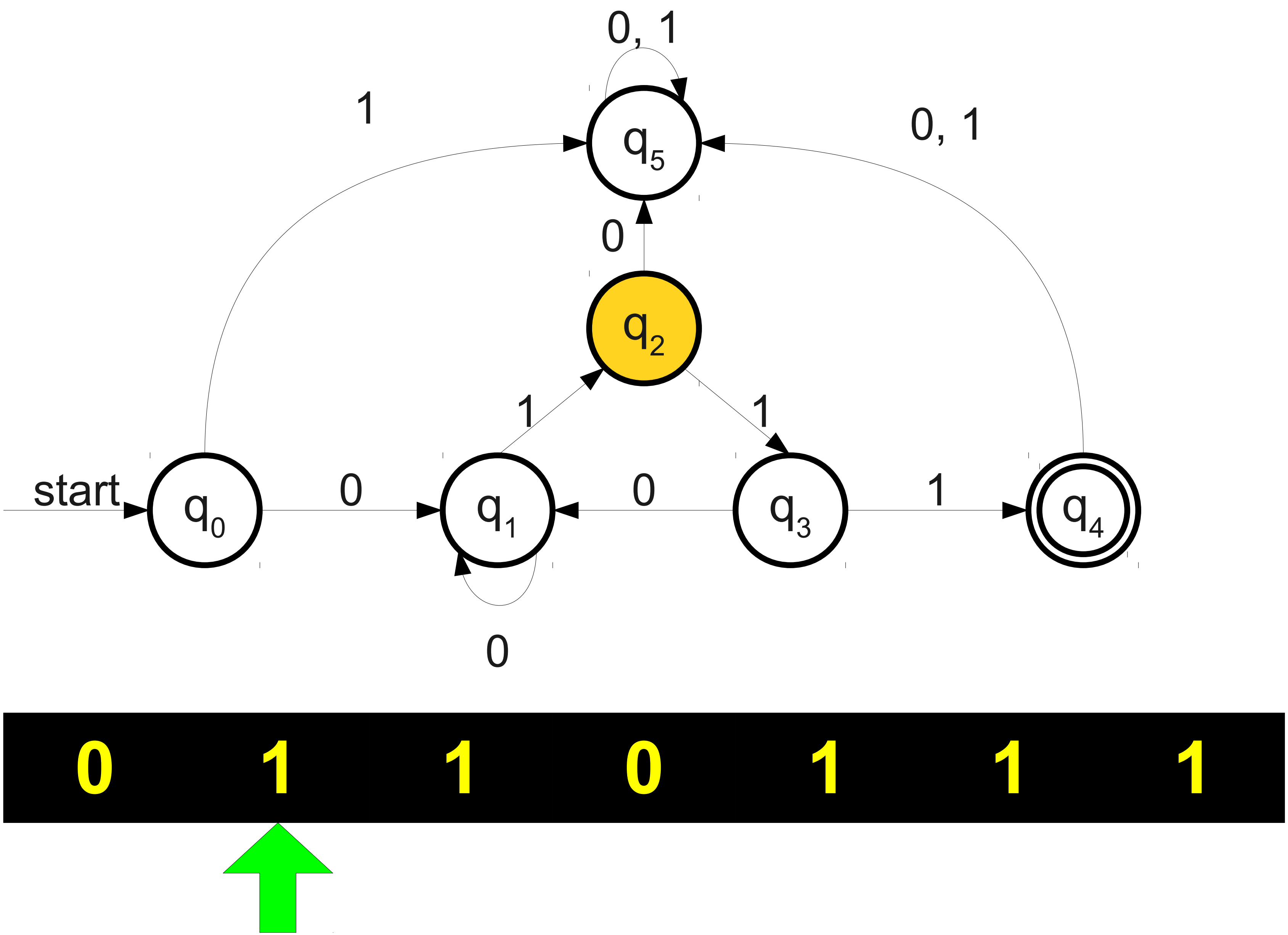

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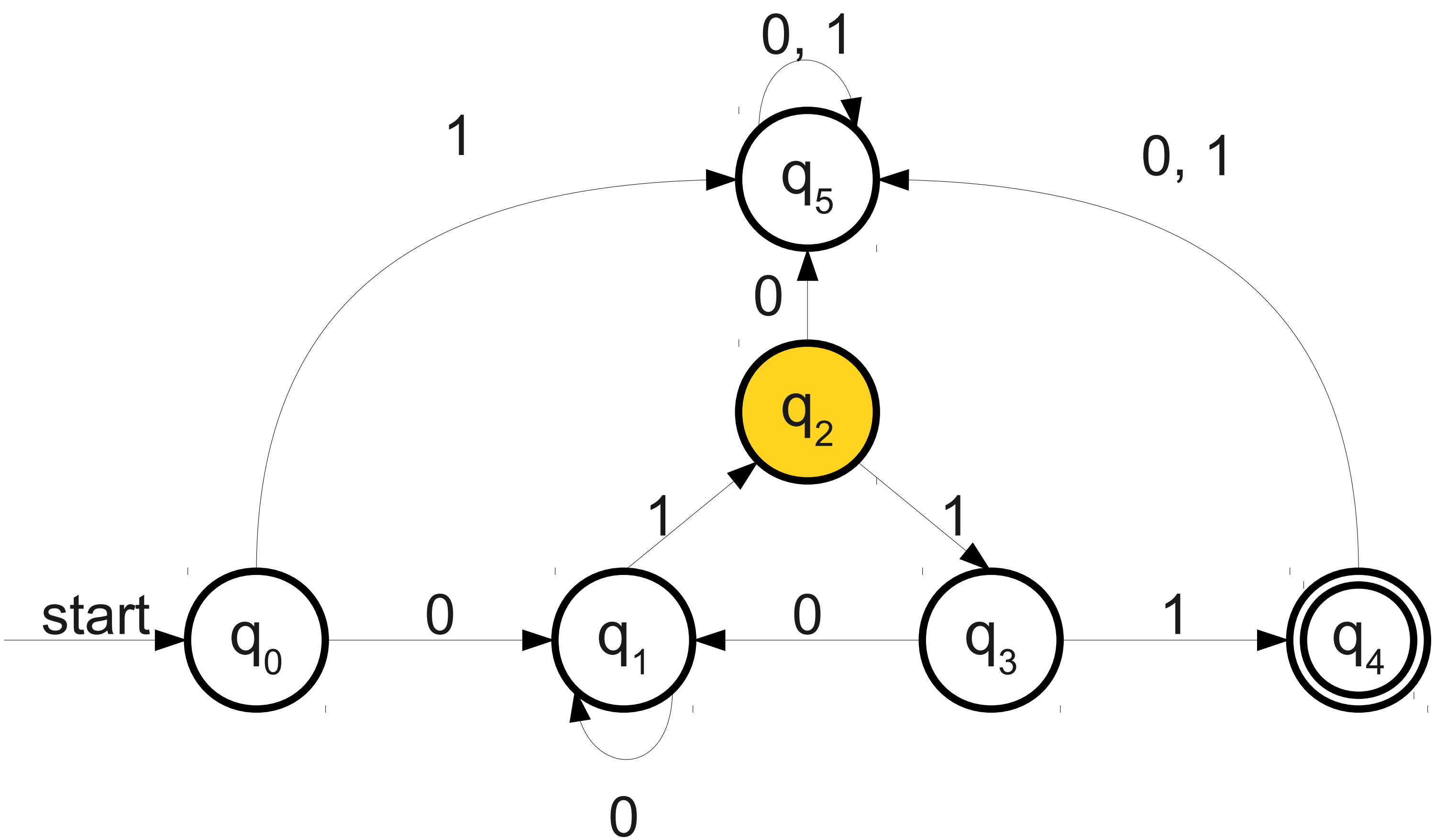

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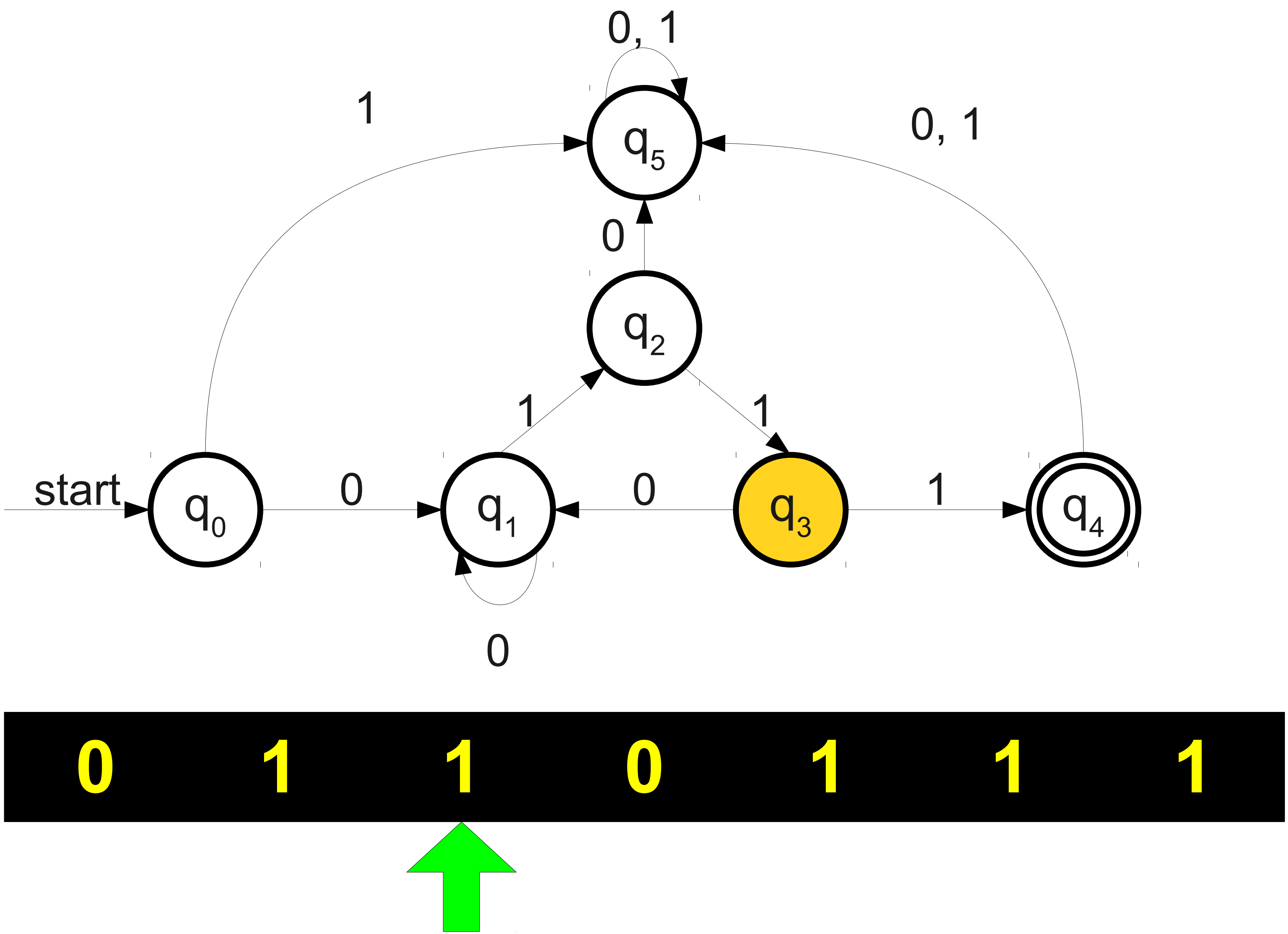

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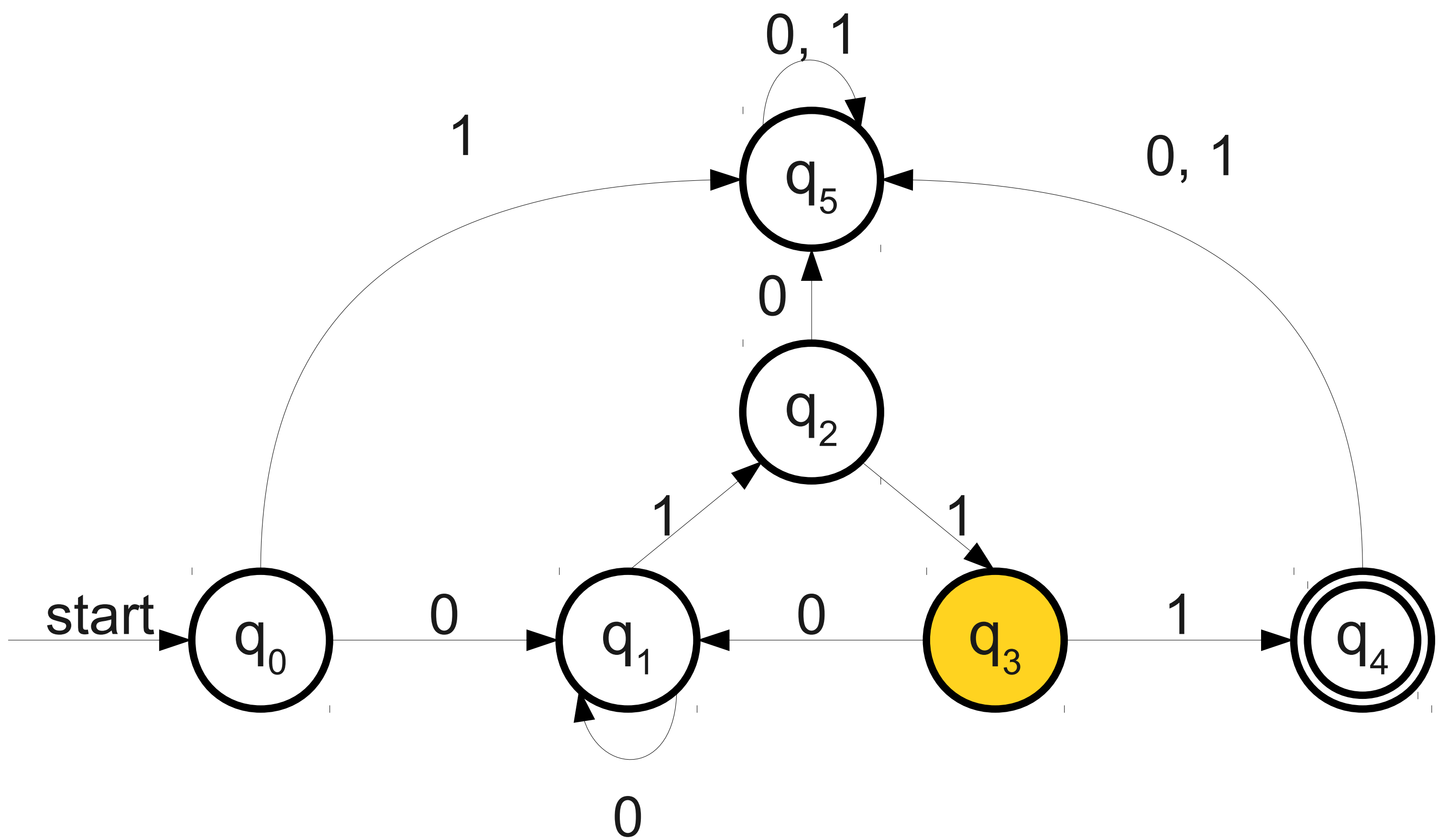

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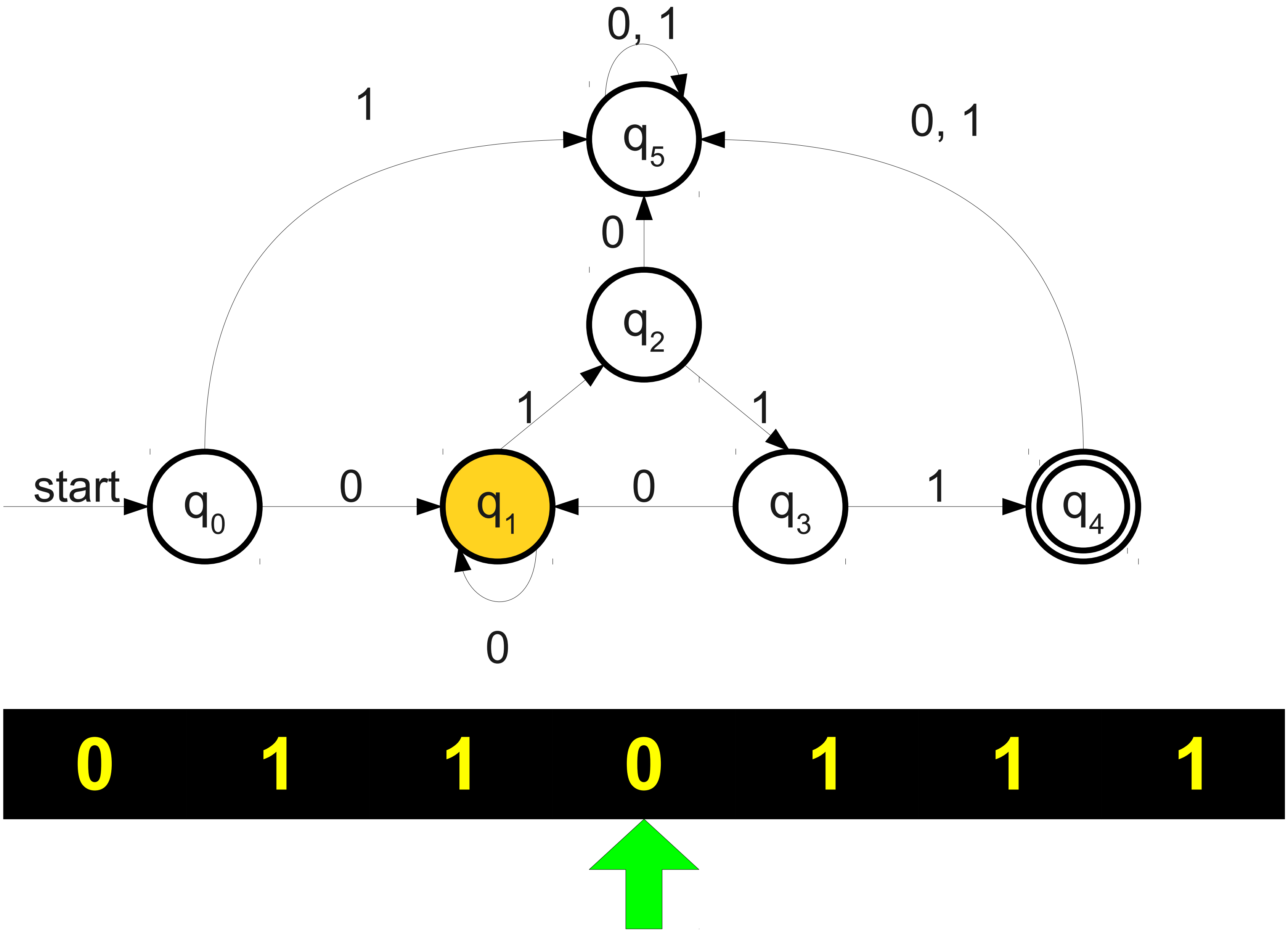

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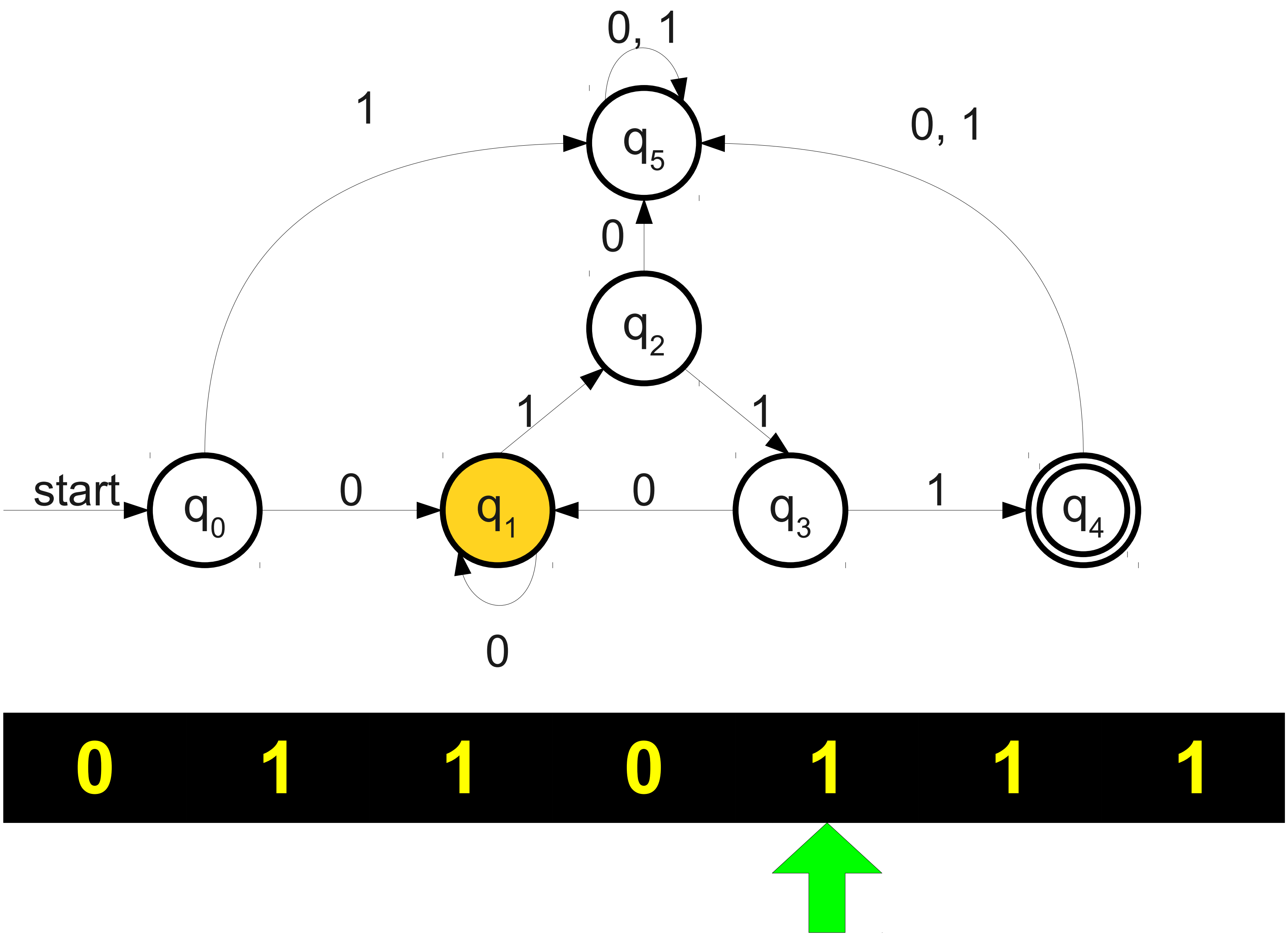

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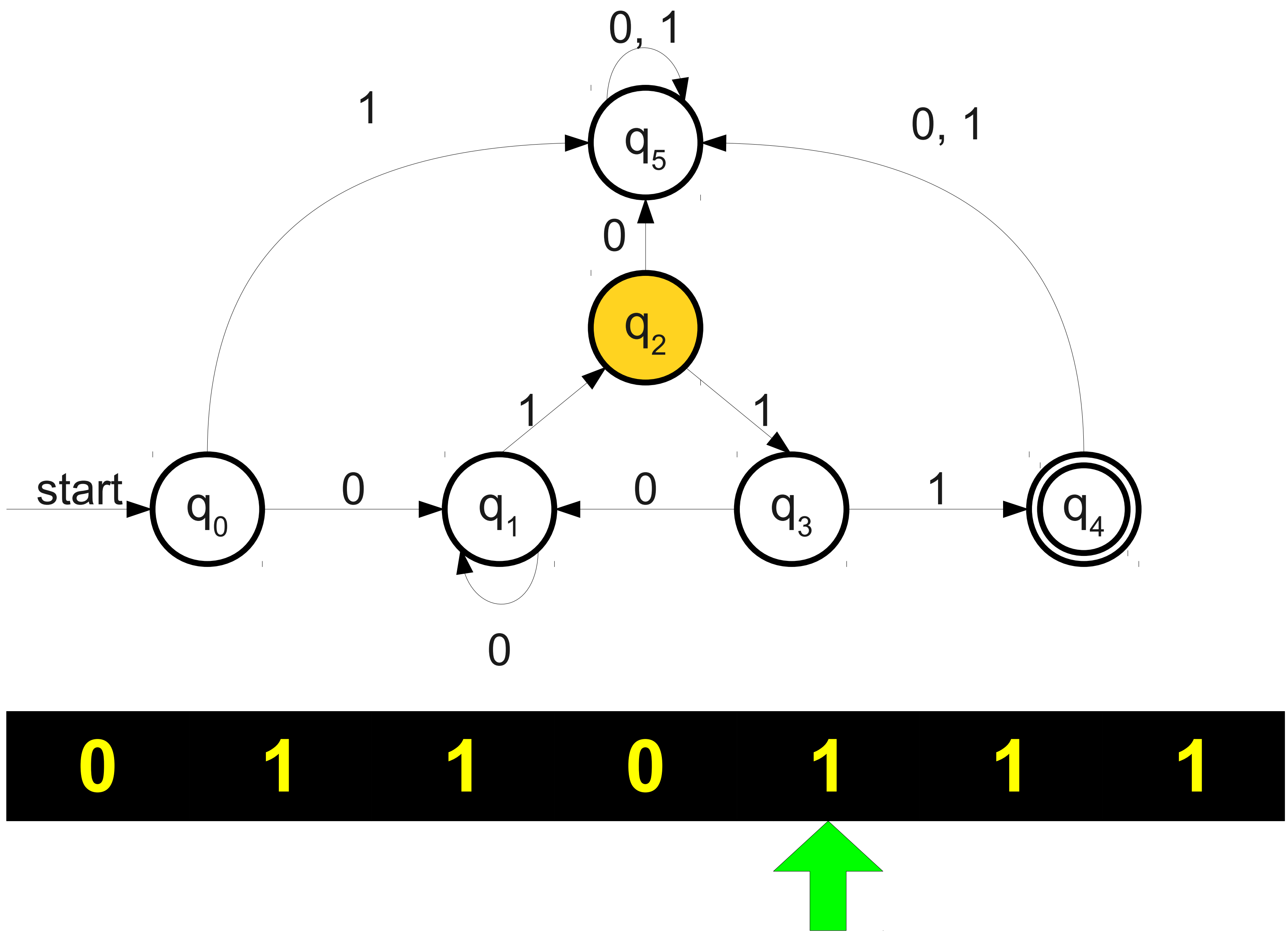

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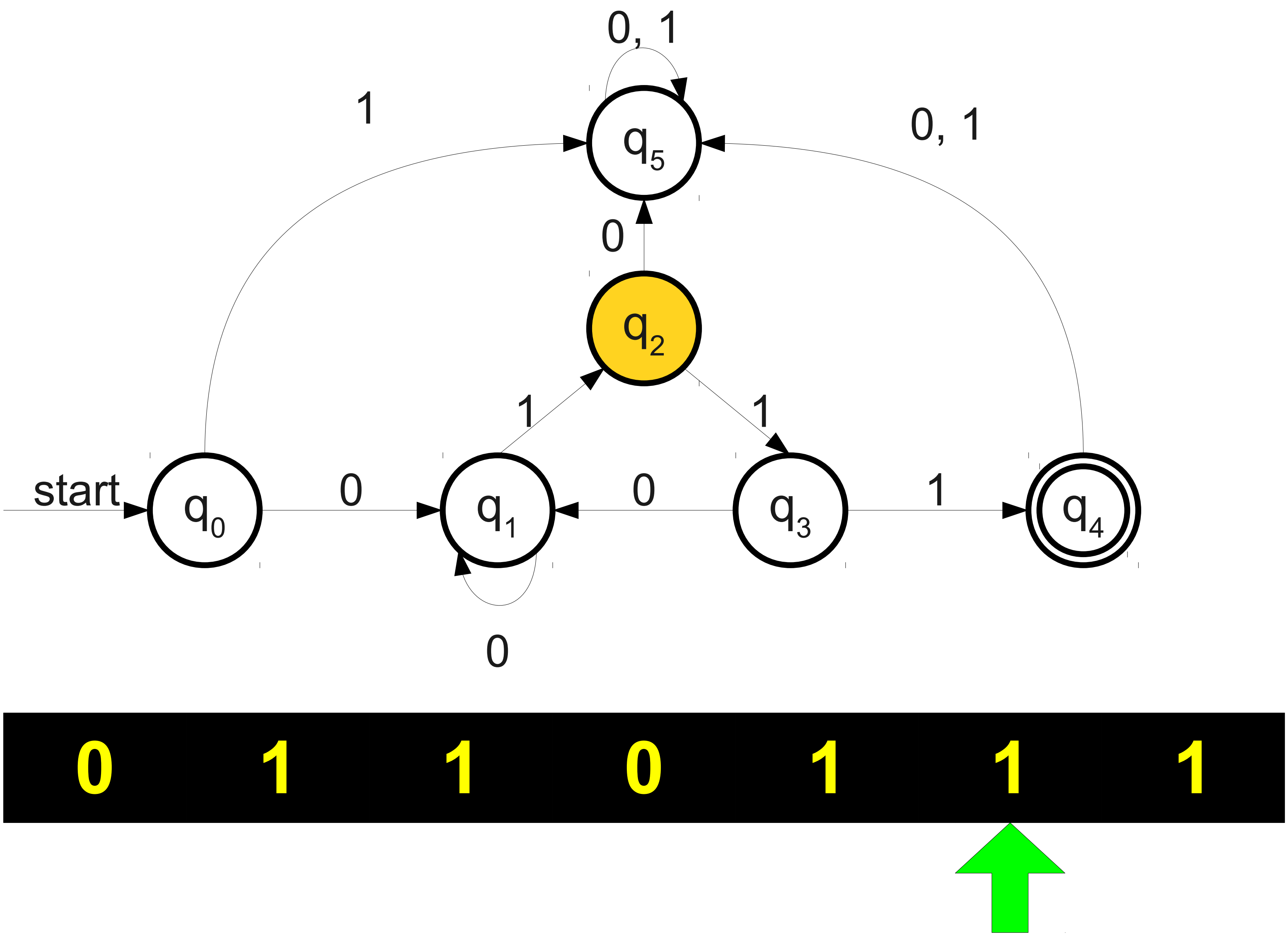

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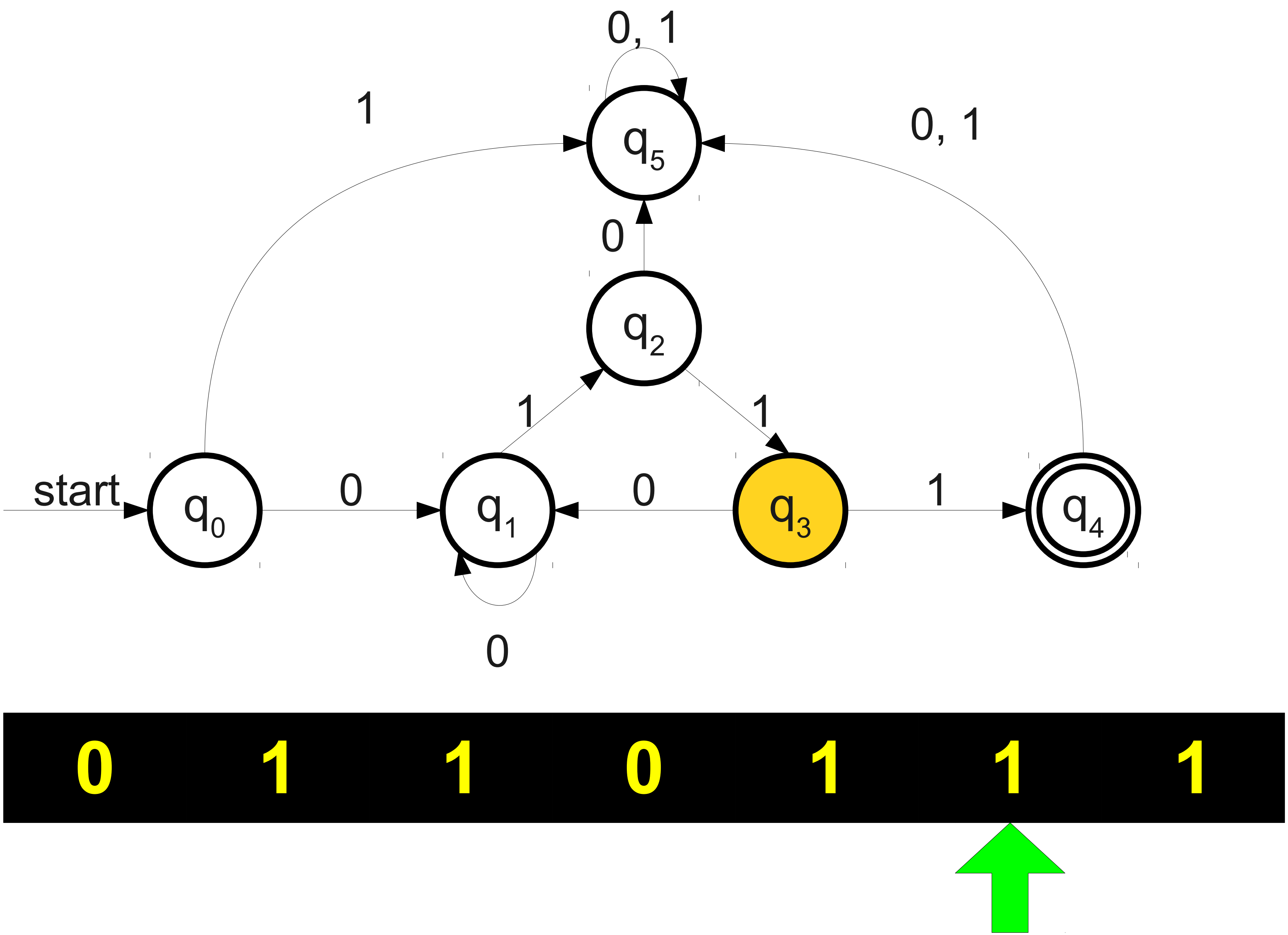

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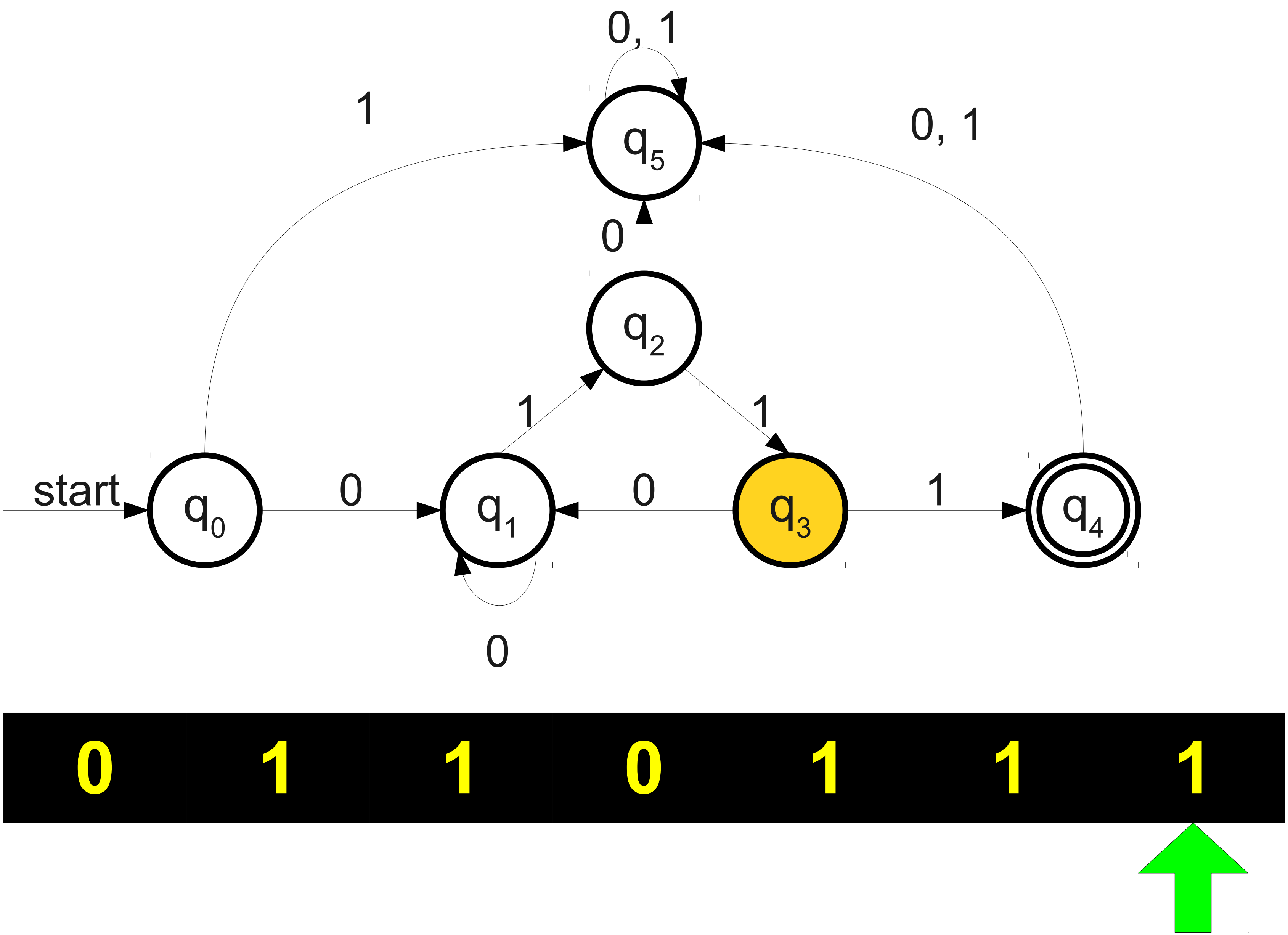

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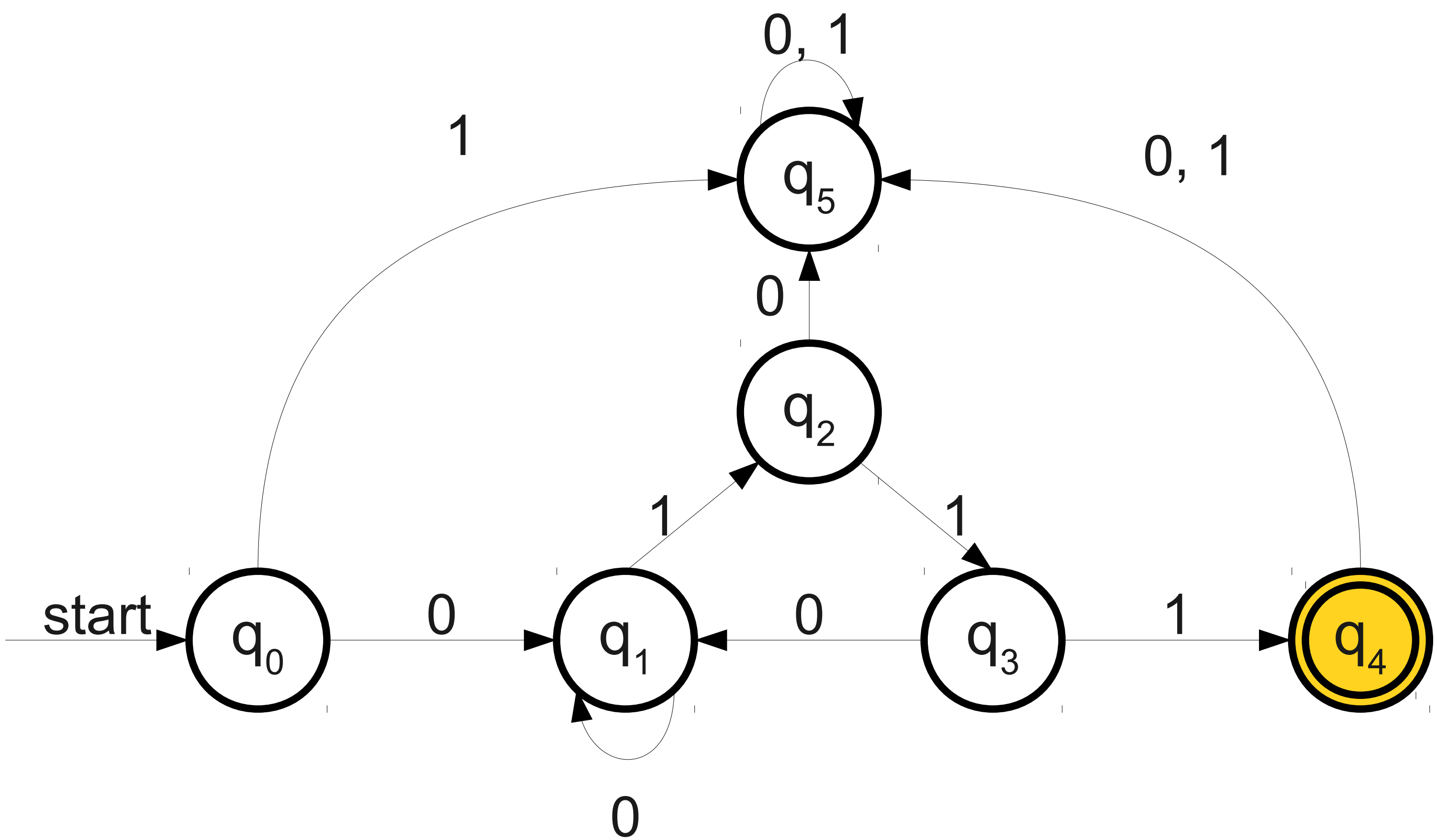

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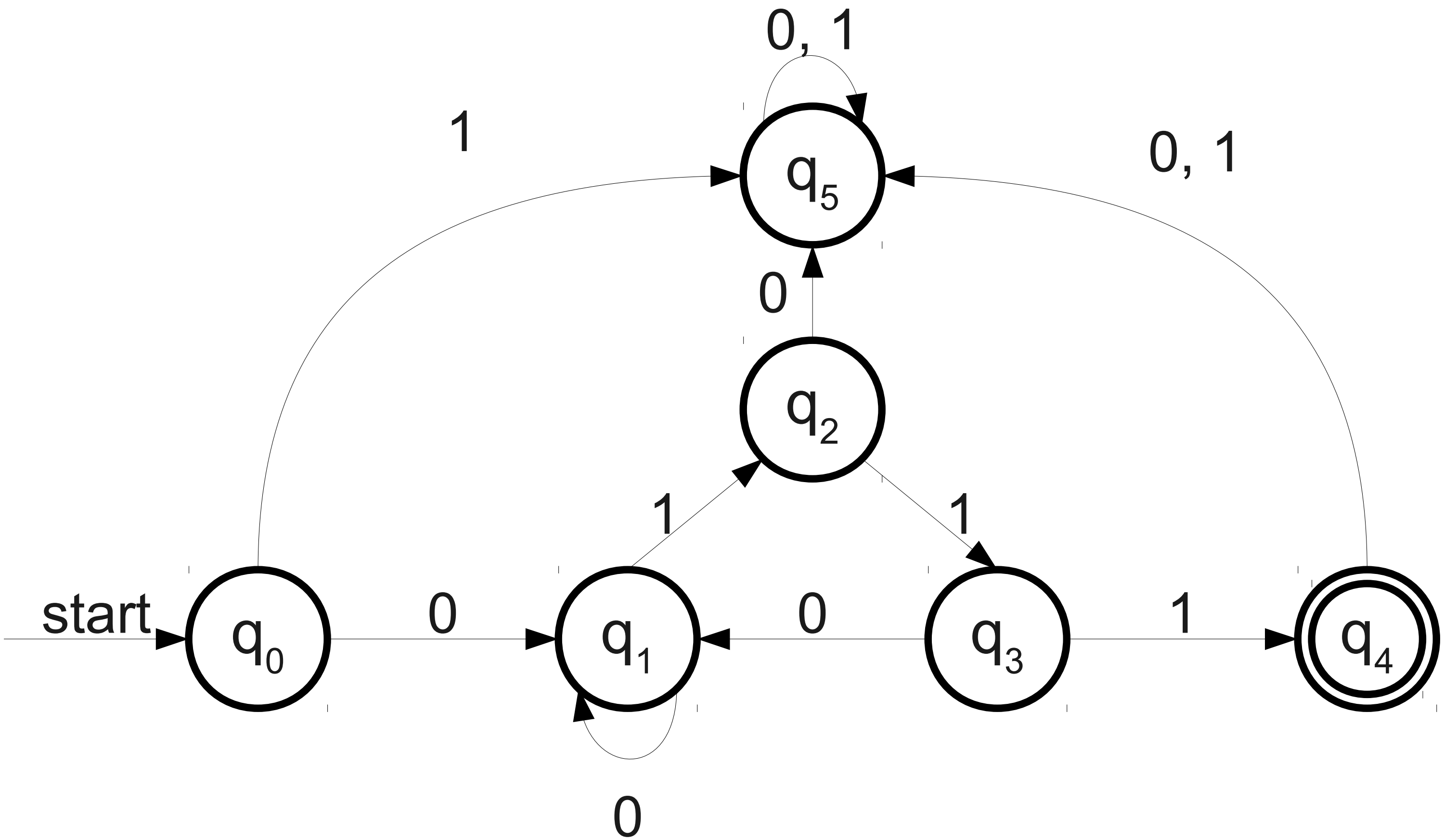

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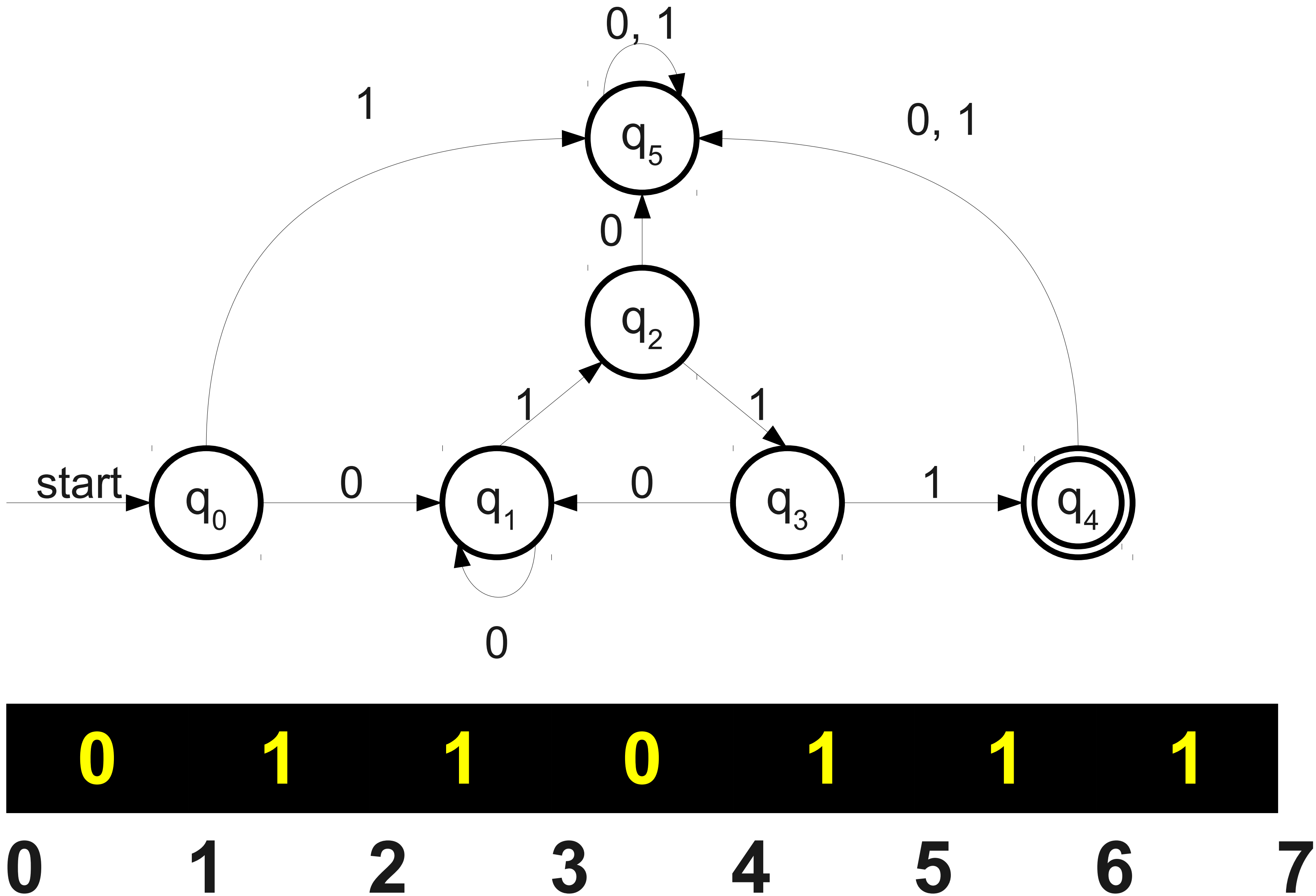

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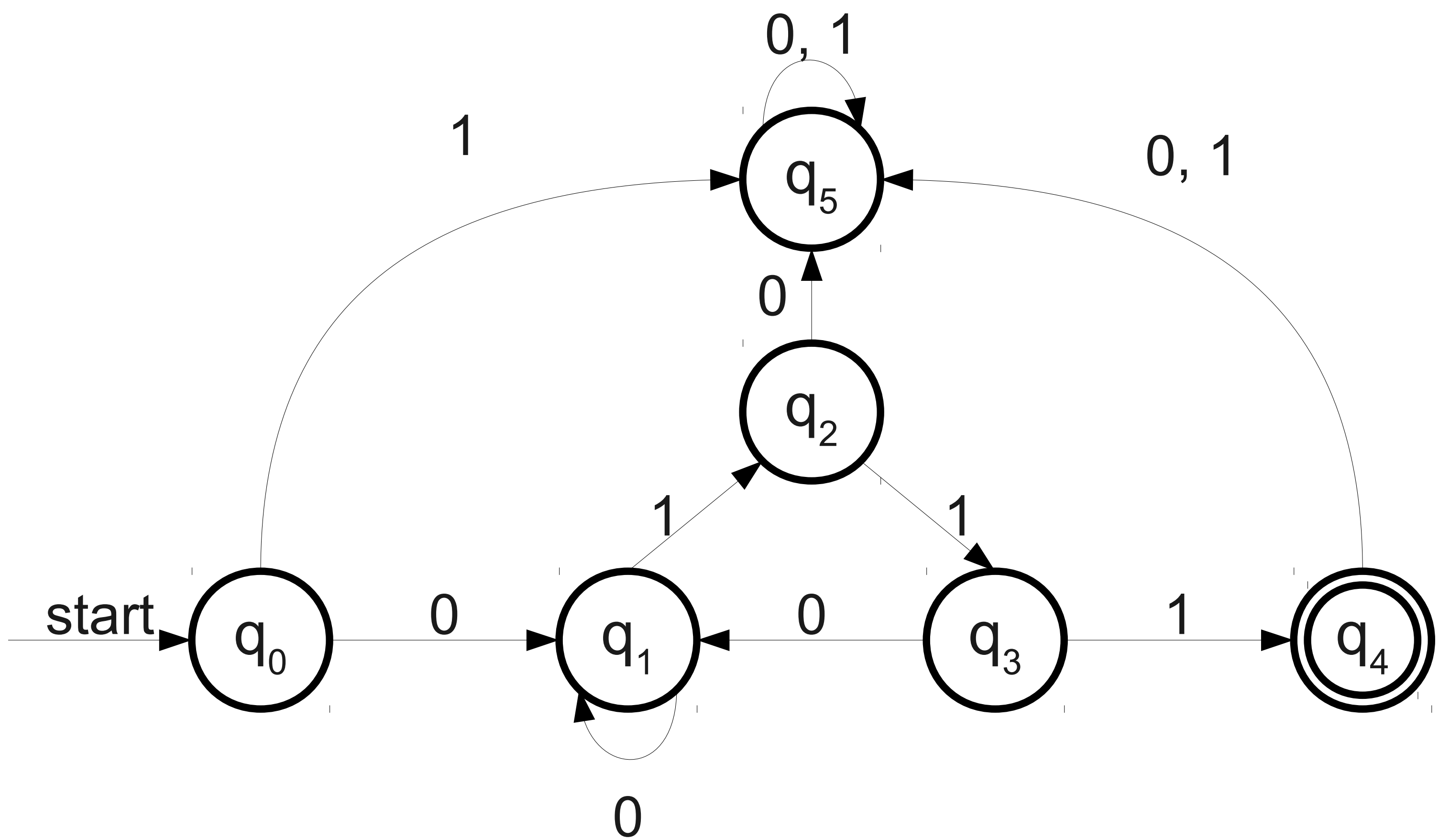

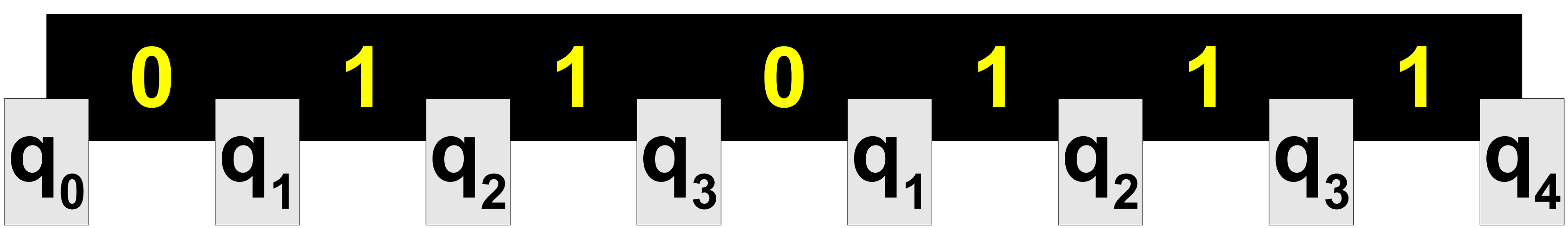

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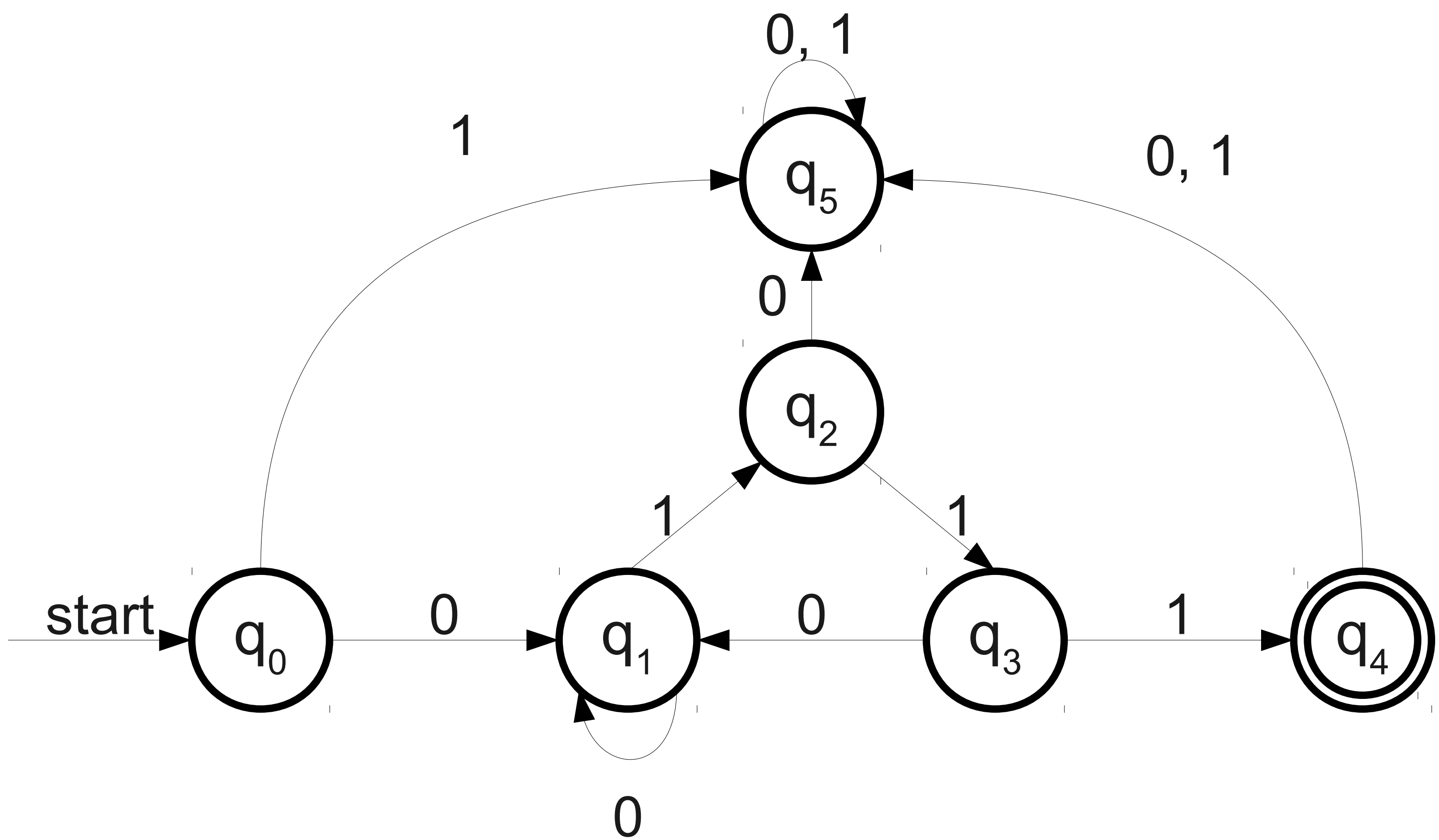

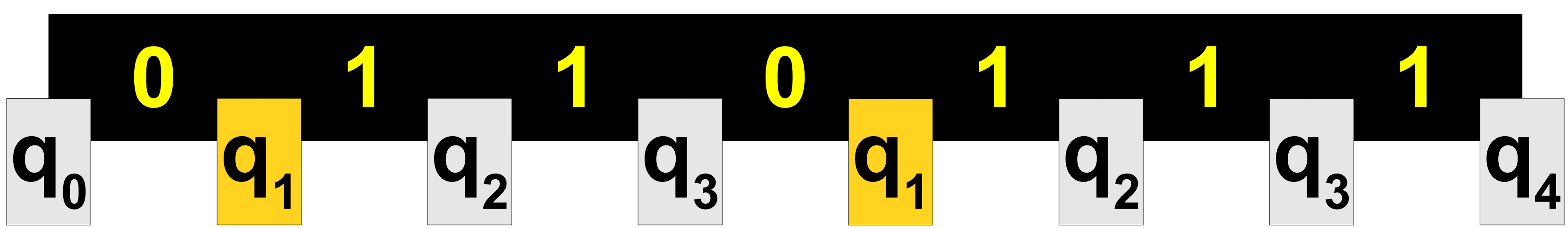

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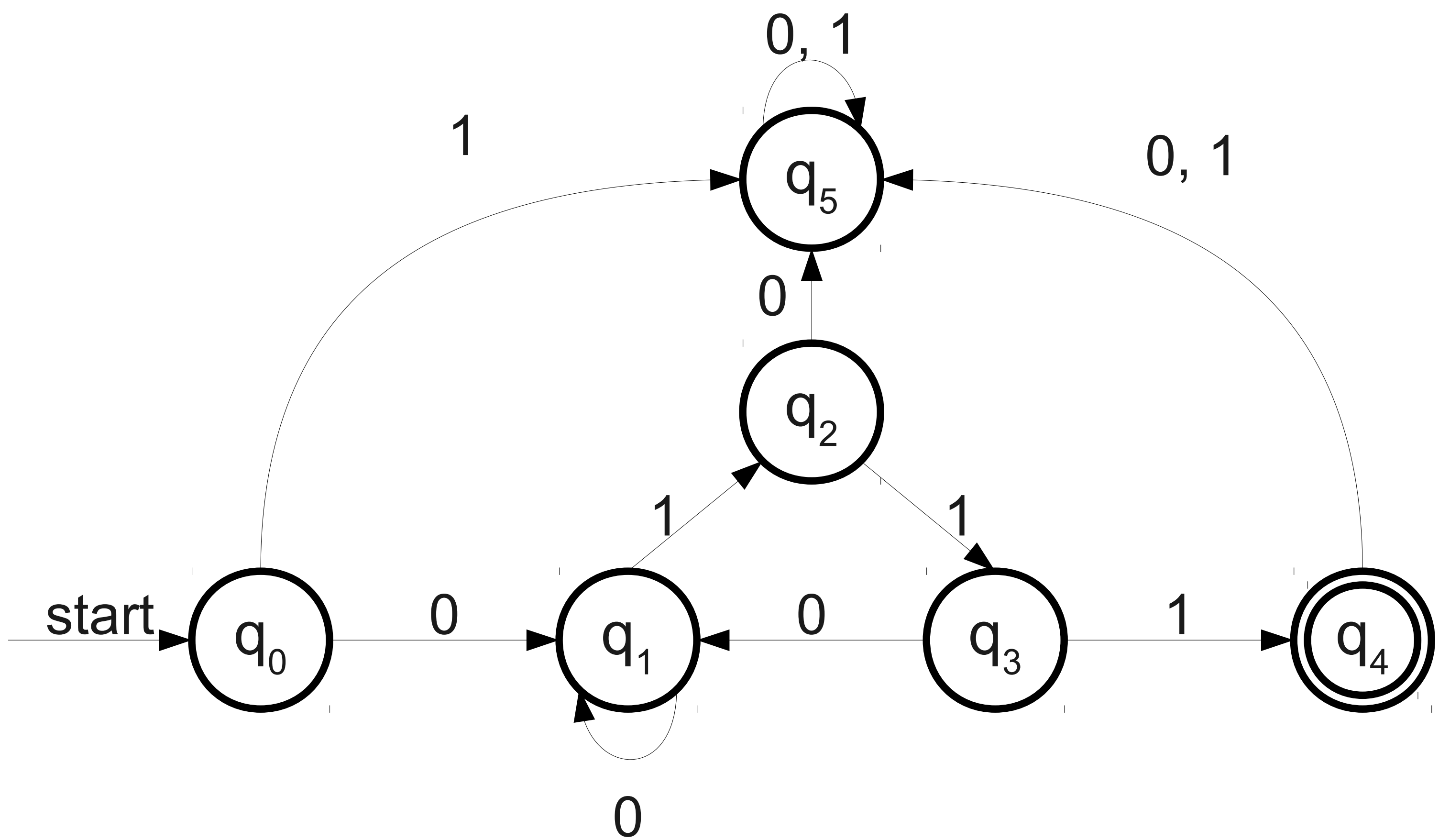

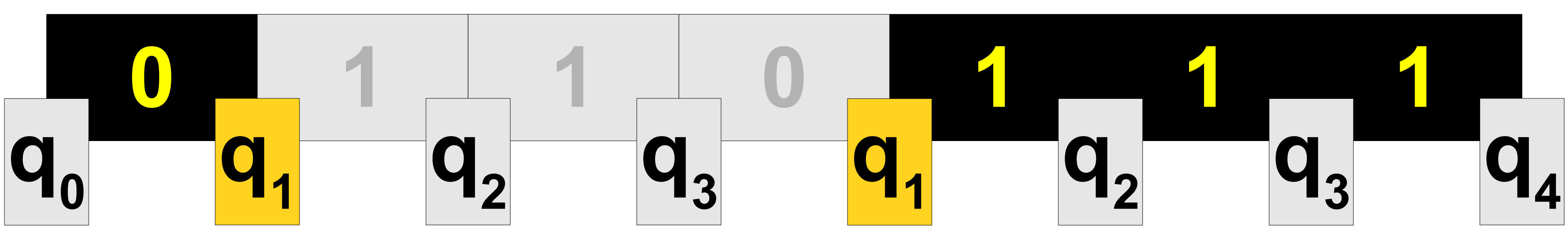

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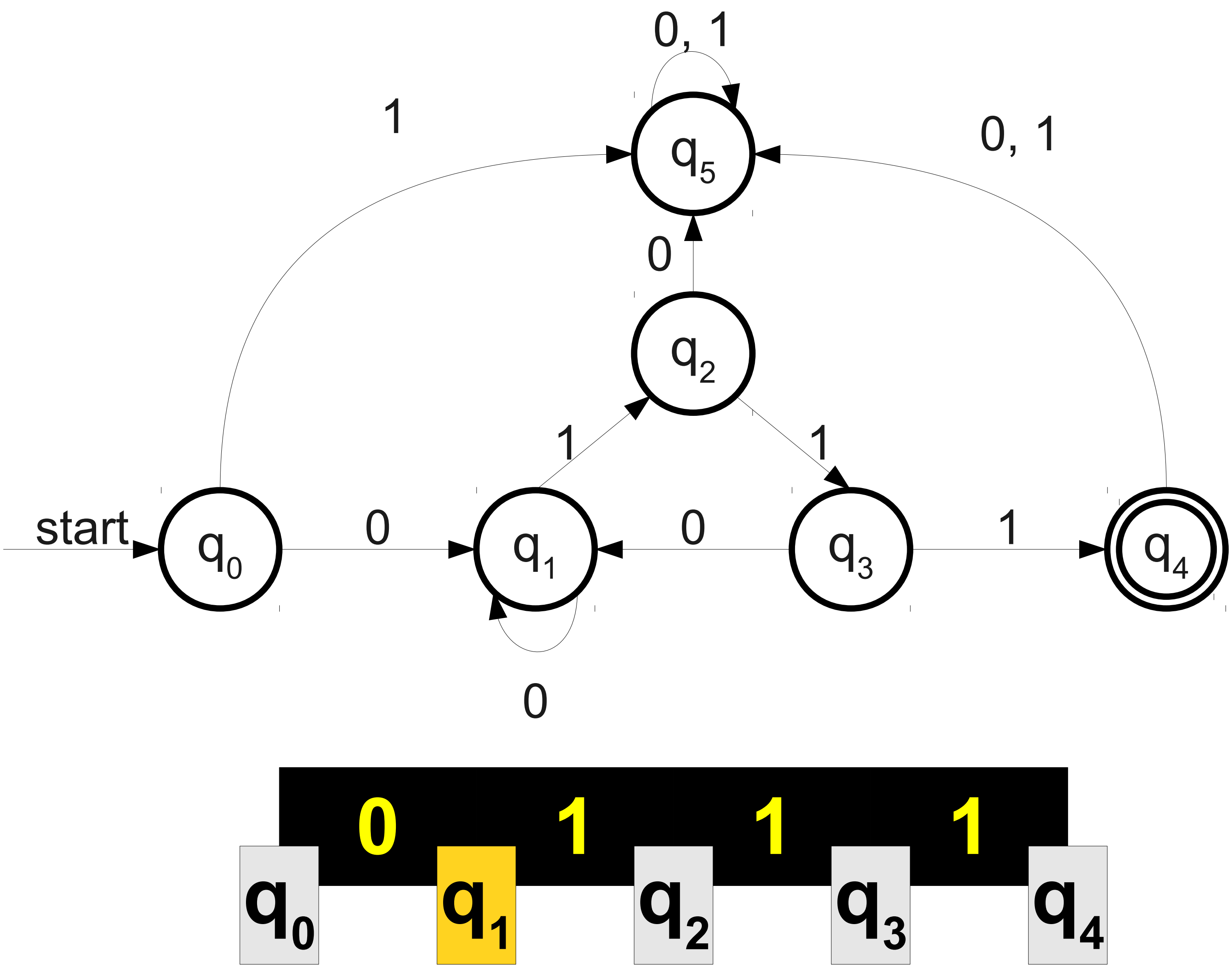

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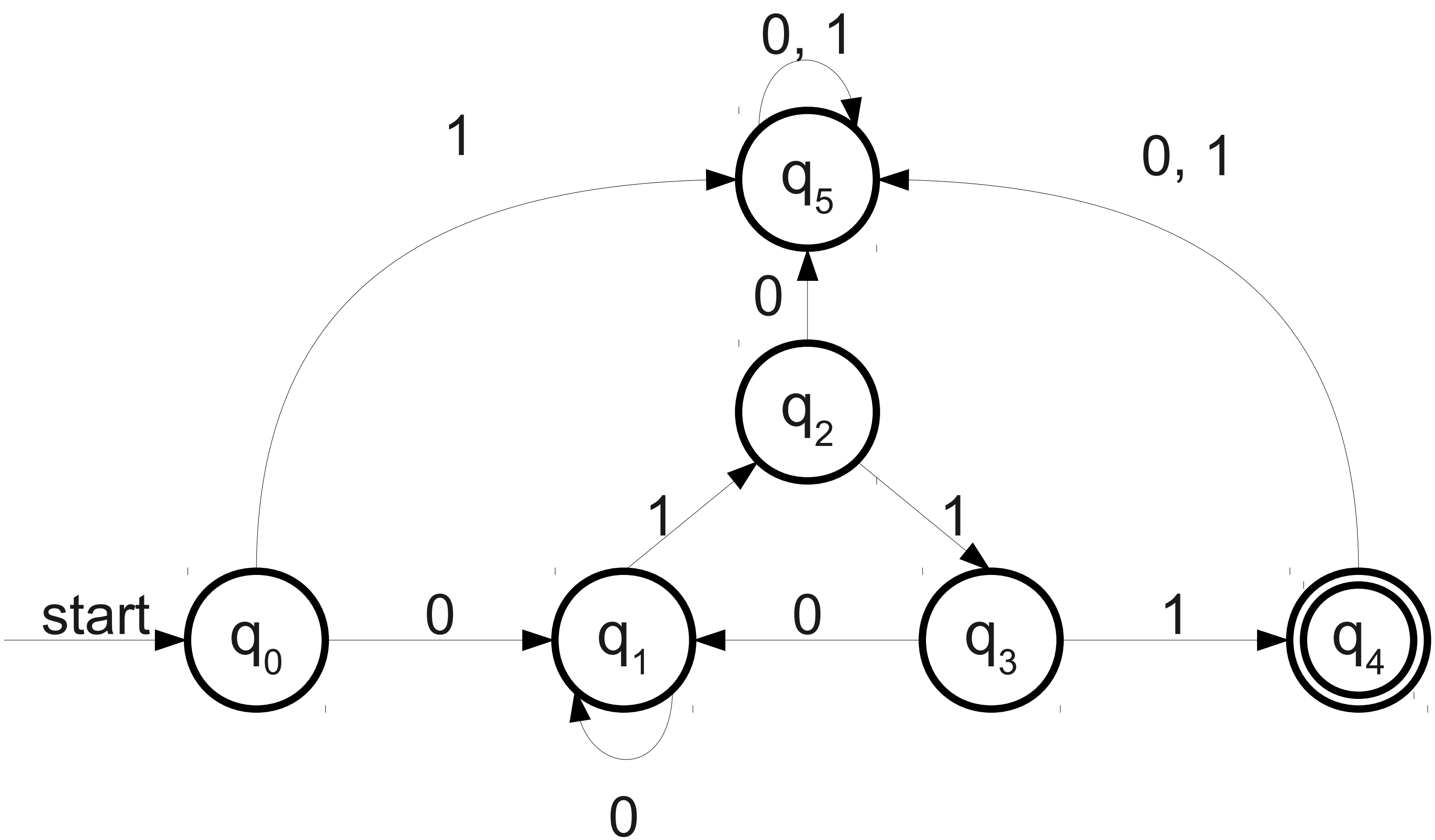

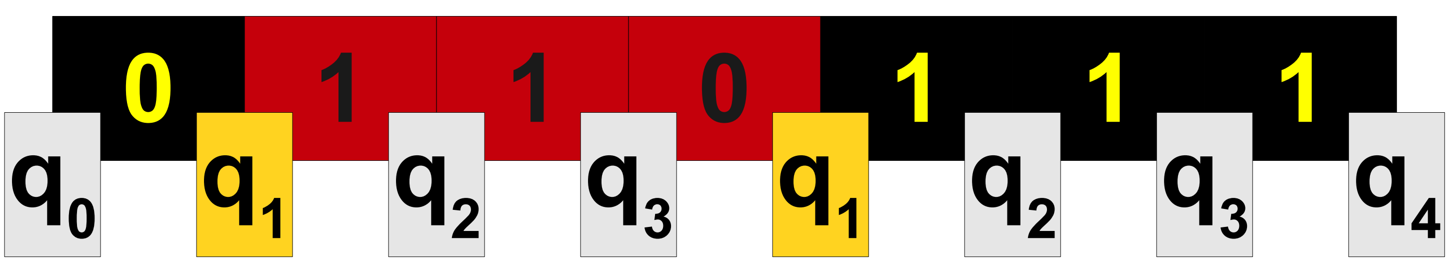

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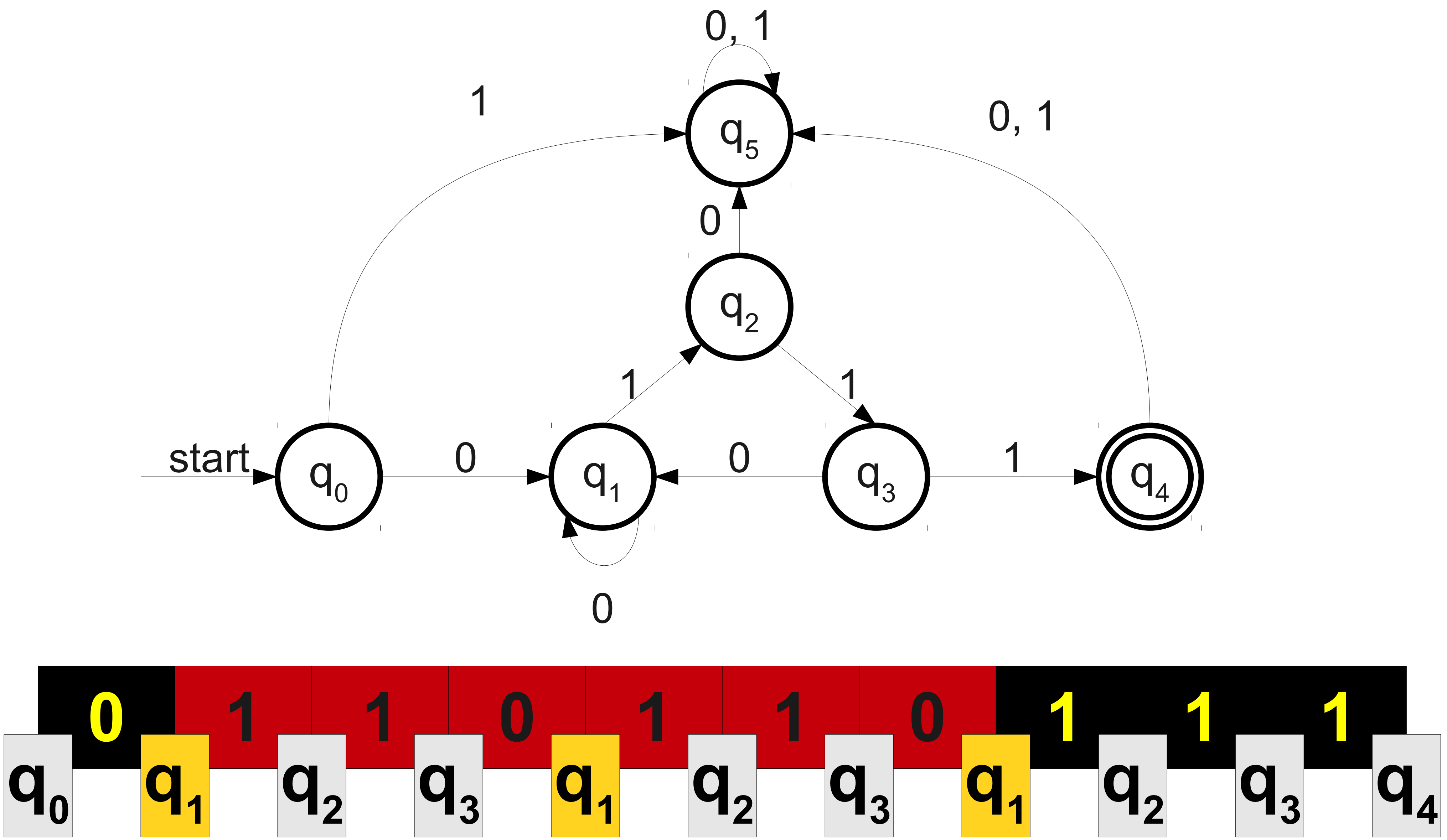

Visiting Multiple States

- Let D be a DFA with n states.
- Any string *w* accepted by D that has length at least n must visit some state twice.
  - Number of states visited is equal to the length of the string plus one.
  - By the **pigeonhole principle**, some state is duplicated.
- The substring of *w* between those revisited states can be removed, duplicated, tripled, quadrupled, etc. without changing the fact that *w* is accepted by D.

Informally

- Let L be a regular language.
- If we have a string w ∈ L that is "sufficiently long," then we can split the string into three pieces and "pump" the middle.
- Write w = xyz.
- Then xy^0z, xy^1z, xy^2z, …, xy^nz, … are all in L.
  - **Notation**: y^n means "n copies of y."

The Weak Pumping Lemma

- The **Weak Pumping Lemma for Regular Languages** states that

For any regular language L,

There exists a positive natural number n such that

For any $w \in L$ with $|w| \geq n$,

There exists strings x, y, z such that

For any natural number i,

$w = xyz$,

$y \neq \varepsilon$

$xy^iz \in L$

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∀ regular language L,

∃ a positive natural number n such that

∀ w ∈ L with |w| ≥ n,

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∀ natura

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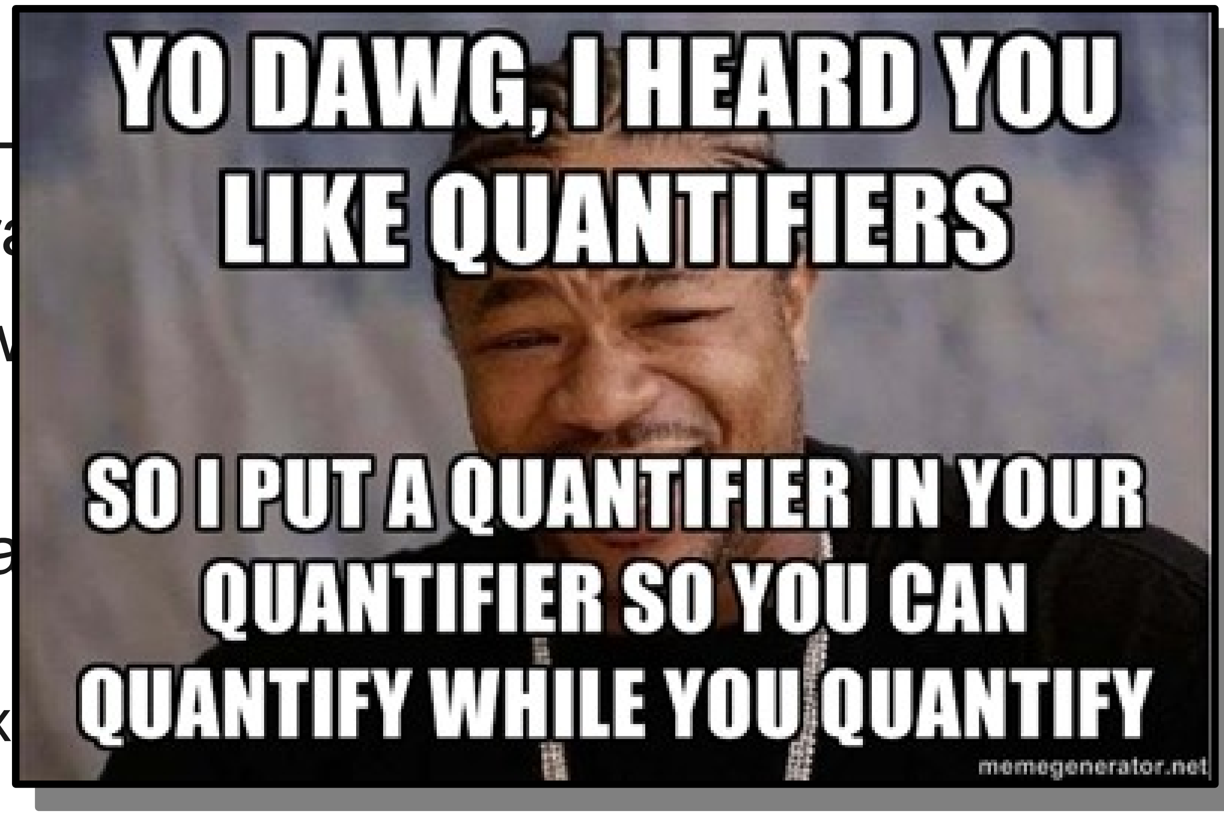

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This number n is sometimes called the <u>pumping length</u>.

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Strings longer than the pumping length must have a special property.

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The Weak Pumping Lemma

- Let Σ = {0, 1} and
  L = { w | w contains 00 as a substring. }
- Any string of length 3 or greater can be split into three pieces, the second of which can be "pumped."

1	0	0	1	0

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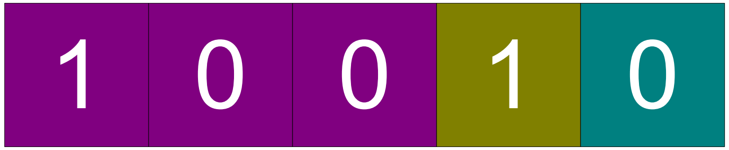

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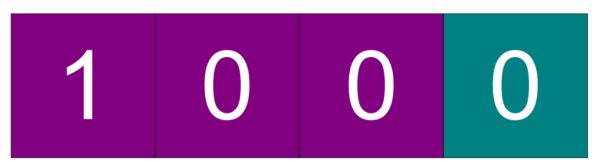

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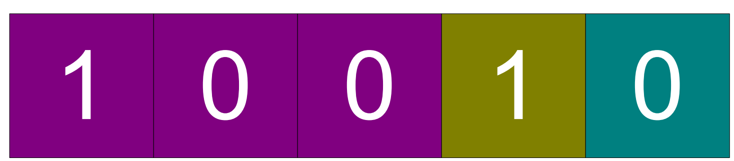

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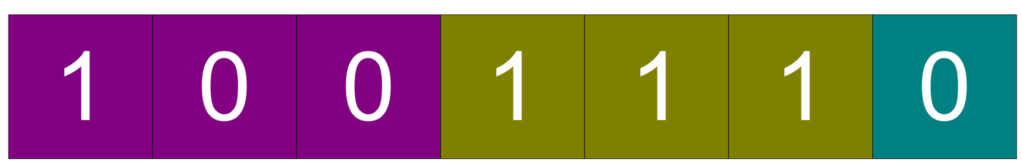

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1	0	0

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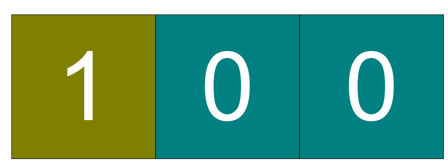

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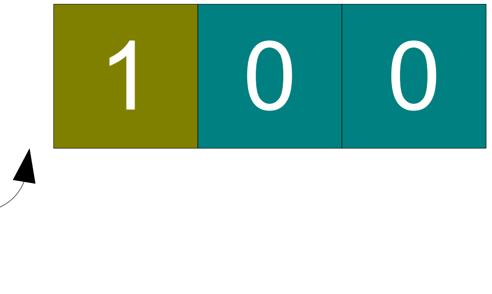

The first piece is just the empty string! This is perfectly fine.

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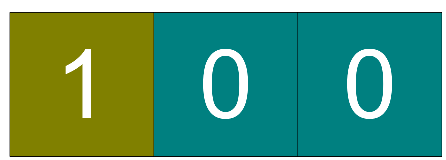

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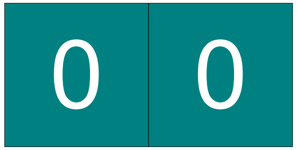

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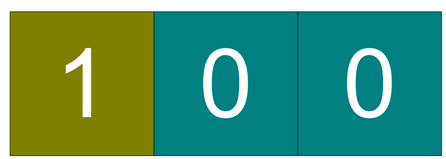

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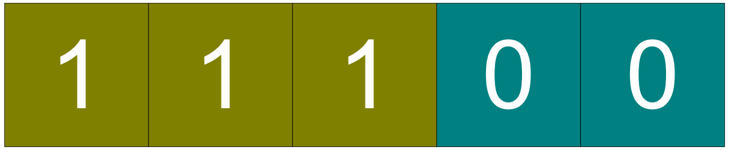

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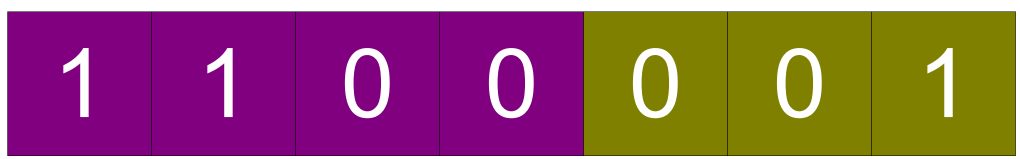

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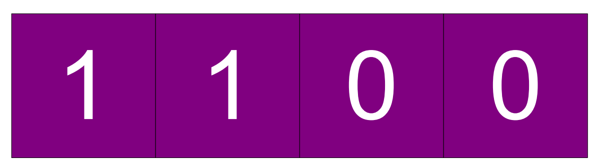

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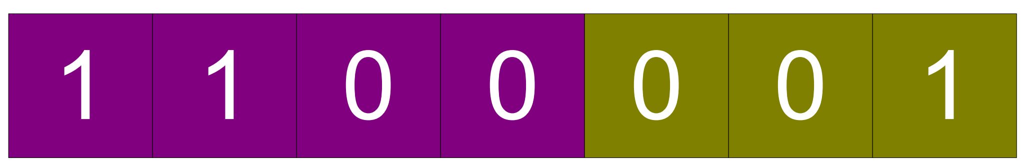

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1	1	0	0	0	0	1	0	0	1	0	0	1

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The weak pumping lemma
holds for finite languages
because the pumping
length can be longer than
the longest string!

Testing Equality

- The **equality problem** is defined as follows:

  Given two strings x and y, report whether x = y.

- Let Σ = {**`0`**, **`1`**, **`?`**}. We can encode the equality problem as a string of the form x**`?`**y.
  - "Is 001 equal to 110?" would be **`001?110`**
  - "Is 11 equal to 11?" would be **`11?11`**
  - "Is 110 equal to 110?" would be **`110?110`**
- Let *EQUAL* = { w?w | w ∈ {0, 1}* }
- **Question**: Is *EQUAL* a regular language?

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Using the Weak Pumping Lemma

$EQUAL = \{ w?w \mid w \in \{0, 1\}^* \}$

0	0	0	?	0	0	0

Using the Weak Pumping Lemma

$$EQUAL = \{ w?w \mid w \in \{0, 1\}^* \}$$

0	0	0	?	0	0	0

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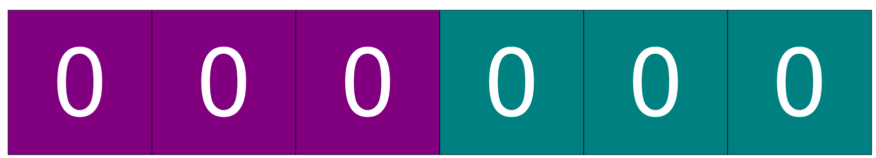

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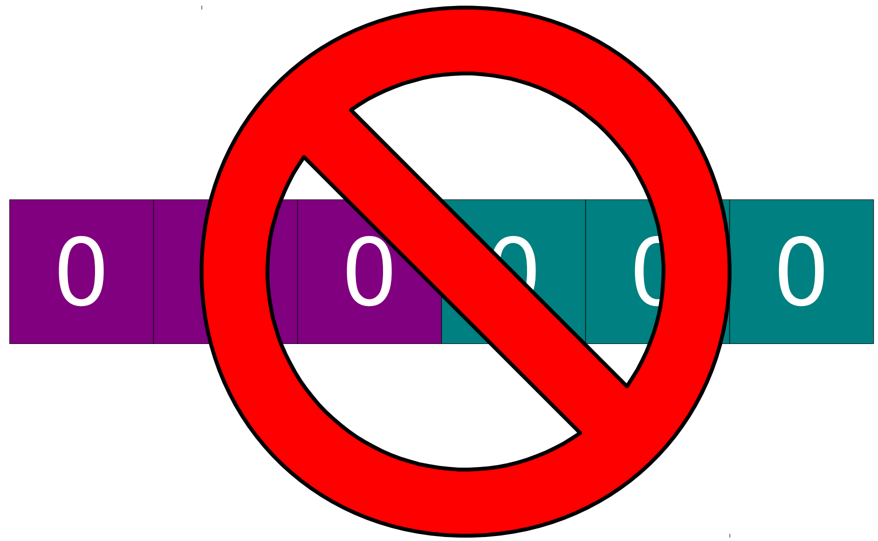

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0	0	0	?	0	0	0

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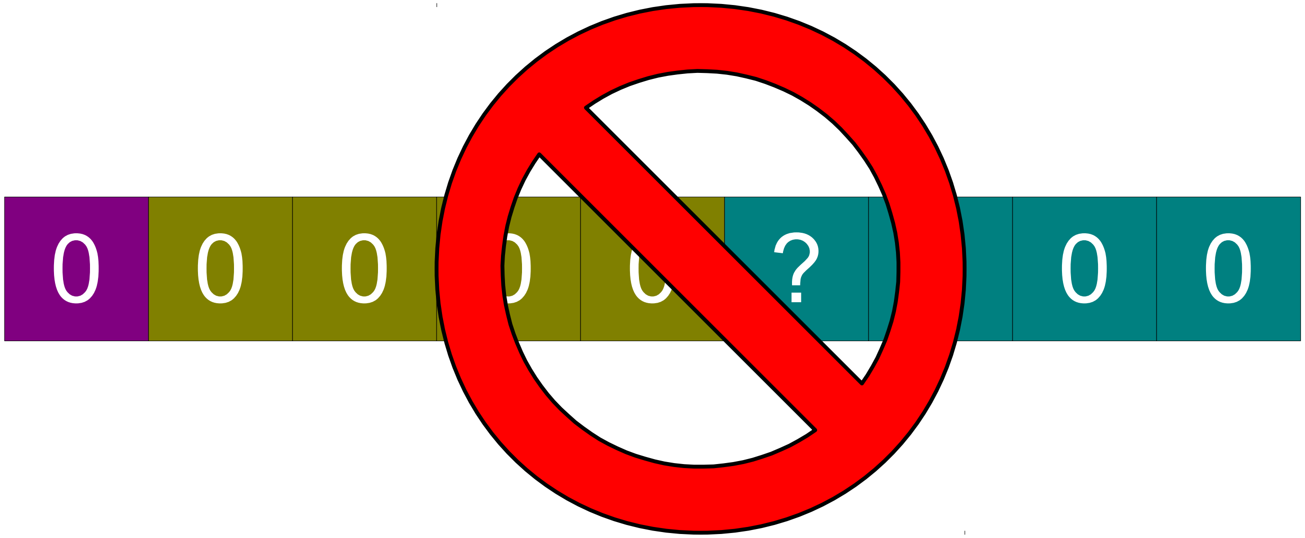

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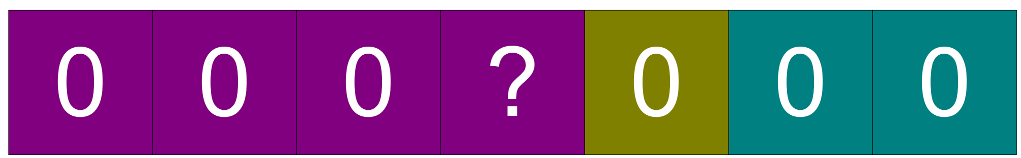

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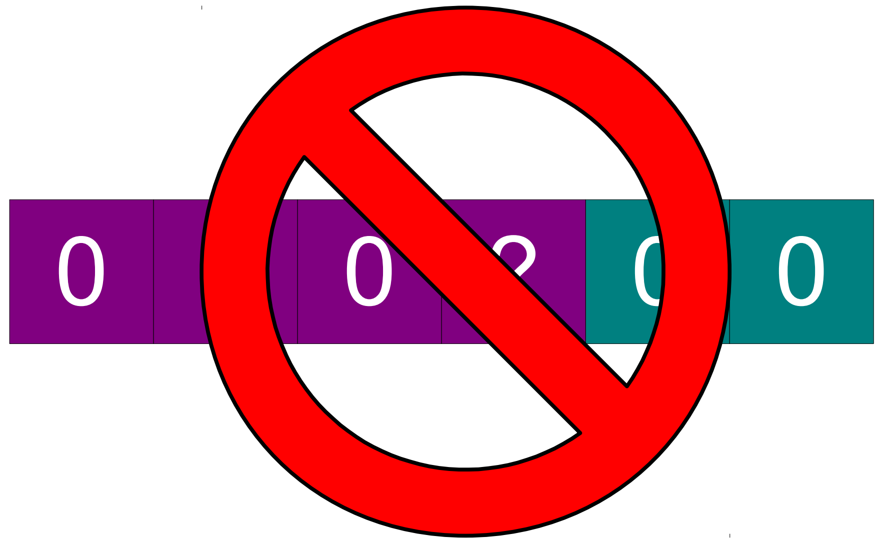

What's Going On?

- The weak pumping lemma says that for "sufficiently long" strings, we should be able to pump some part of the string.
- We can't pump any part containing the ?, because we can't duplicate or remove it.
- We can't pump just one part of the string, because then the strings on opposite sides of the ? wouldn't match.
- **Can we formally show that *EQUAL* is not regular?**

For any regular language L,

  There exists a positive natural number n such that

    For any $w \in L$ with $|w| \geq n$,

      There exists strings x, y, z such that

        For any natural number i,

          $w = xyz$,

          $y \neq \varepsilon$

          $xy^iz \in L$

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The hardest part of most proofs with the pumping lemma is choosing some string that we should be able to pump but cannot. In this case, we already saw a good example, so we'll choose it here.

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At this point, we have some string that we should be able to split into pieces and pump. The rest of the proof shows that no matter what choice we make, this is impossible.

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Nonregular Languages

- The weak pumping lemma describes a property common to all regular languages.
- Any language L which does not have this property **cannot be regular**.
- What other languages can we find that are not regular?

A Canonical Nonregular Language

- Consider the language $L = \{ 0^n1^n \mid n \in \mathbb{N} \}$.

  $L = \{ \varepsilon, 01, 0011, 000111, 00001111, \ldots \}$
- L is a **classic example** of a nonregular language.
- Intuitively: If you have only finitely many states in a DFA, you can't “remember” an arbitrary number of 0s.
- How would we prove that L is nonregular?

The Pumping Lemma as a Game

- The weak pumping lemma can be thought of as a game between **you** and an **adversary**.
- **You win** if you can prove that the pumping lemma fails.
- **The adversary wins** if the adversary can make a choice for which the pumping lemma succeeds.
- The game goes as follows:
  - **The adversary** chooses a pumping length n.
  - **You** choose a string w with $|w| \geq n$ and $w \in L$.
  - **The adversary** breaks it into x, y, and z.
  - **You** choose an i such that $xy^iz \notin L$ (if you can't, you lose!)

The Pumping Lemma Game

ADVERSARY	YOU

The Pumping Lemma Game

ADVERSARY	YOU
Maliciously choose pumping length n.	

The Pumping Lemma Game

ADVERSARY	YOU
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Theorem: $L = \{ 0^n1^n \mid n \in \mathbb{N} \}$ is not regular.

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Case 1: y consists solely of 0s.

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Case 1: y consists solely of 0s. Then $xy^0z = xz = 0^{n - |y|}1^n$, and since $|y| > 0$, $y \notin L$.

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Case 1: y consists solely of 0s. Then $xy^0z = xz = 0^{n - |y|}1^n$, and since $|y| > 0$, $xz \notin L$.

Case 2: y consists solely of 1s. Then $xy^0z = xz = 0^n1^{n - |y|}$, and since $|y| > 0$, $xz \notin L$.

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Proof: By contradiction; assume L is regular. Let n be the pumping length guaranteed by the weak pumping lemma. Consider the string w = 0^n1^n. Then |w| = 2n ≥ n and w ∈ L, so we can write w = xyz such that y ≠ ε and for any natural number i, xy^iz ∈ L. We consider three cases:

Case 1: y consists solely of 0s. Then $xy^0z = xz = 0^{n - |y|}1^n$, and since |y| > 0, xz ∉ L.

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Counting Symbols

- Consider the alphabet $\Sigma = \{ 0, 1 \}$ and the language

  BALANCE = { w | w contains an equal number of 0s and 1s. }
- For example:
  - 01 $\in$ *BALANCE*
  - 110010 $\in$ *BALANCE*
  - 11011 $\notin$ *BALANCE*
- **Question:** Is *BALANCE* a regular language?

BALANCE and the Weak Pumping Lemma

BALANCE = { w | w contains an equal number of 0s and 1s. }

1	1	0	0	0	1

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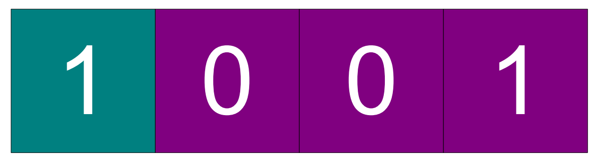

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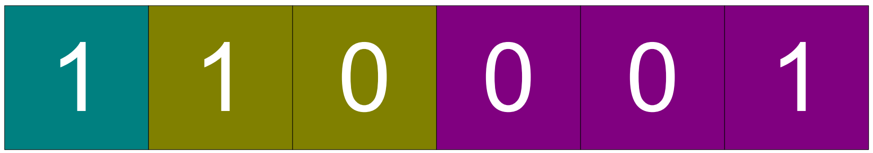

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An Incorrect Proof

Theorem: BALANCE is regular.

Proof: We show that *BALANCE* satisfies the condition of the pumping lemma. Let n = 2 and consider any string w ∈ *BALANCE* such that |w| ≥ 2. Then we can write w = xyz such that x = z = ε and y = w, so y ≠ ε. Then for any natural number i, $xy^iz = w^i$, which has the same number of 0s and 1s. Since *BALANCE* passes the conditions of the weak pumping lemma, *BALANCE* is regular. ■

An Incorrect Proof

Theorem: BALA

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The Weak Pumping Lemma

- The **Weak Pumping Lemma for Regular Languages** states that

For any regular language L,

There exists a positive natural number n such that

For any $w \in L$ with $|w| \geq n$,

There exists strings x, y, z such that

For any natural number i,

$w = xyz$,	w can be broken into three pieces,
$y \neq \varepsilon$	where the middle piece isn't empty,
$xy^iz \in L$	where the middle piece can be replicated zero or more times.

The Weak Pumping Lemma

- The **Weak Pumping Lemma for Regular Languages** states that

For any regular language L,

This says nothing about languages that aren't regular!

There exists a positive natural number n such that

For any w ∈ L with |w| ≥ n,

There exists strings x, y, z such that

For any natural number i,

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$y \neq \varepsilon$	where the middle piece isn't empty,
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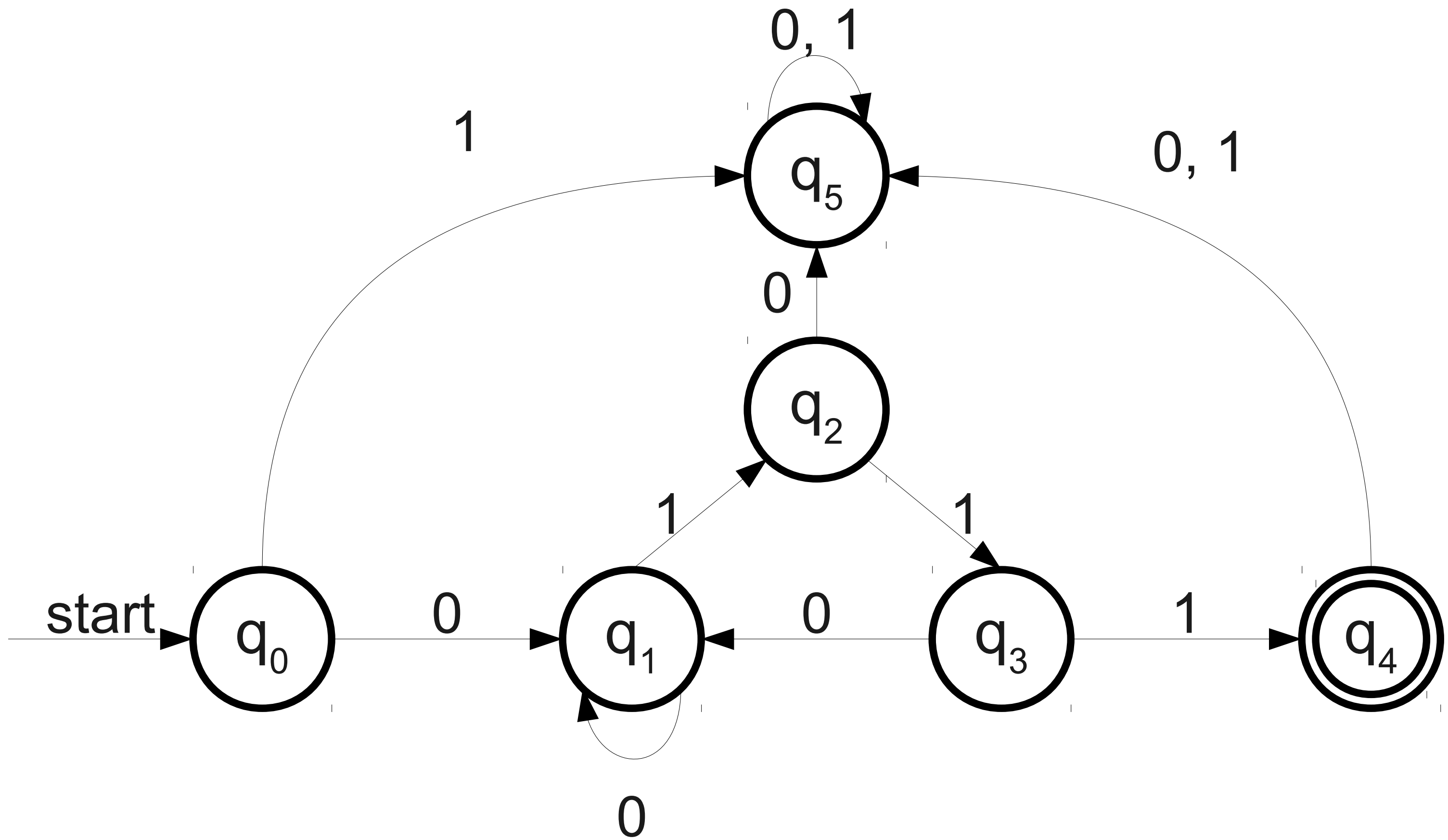
Caution with the Pumping Lemma

- The weak (and full) pumping lemma describe a **necessary** condition of regular languages.
  - L is regular → L passes the pumping lemma
- The weak (and full) pumping lemma is not a **sufficient** condition of regular languages.
  - "L passes the pumping lemma → L is regular" is **not true**.
- If a language fails the pumping lemma, it is definitely **not** regular.
- If a language passes the pumping lemma, we learn nothing about whether it is regular or not.

BALANCE is Not Regular

- The language *BALANCE* can be proven not to be regular using a stronger version of the pumping lemma.
- To see the full pumping lemma, we need to revisit our original insight.

An Important Observation



0 1 1 0 1 1 0 1 1 0 1 1 1

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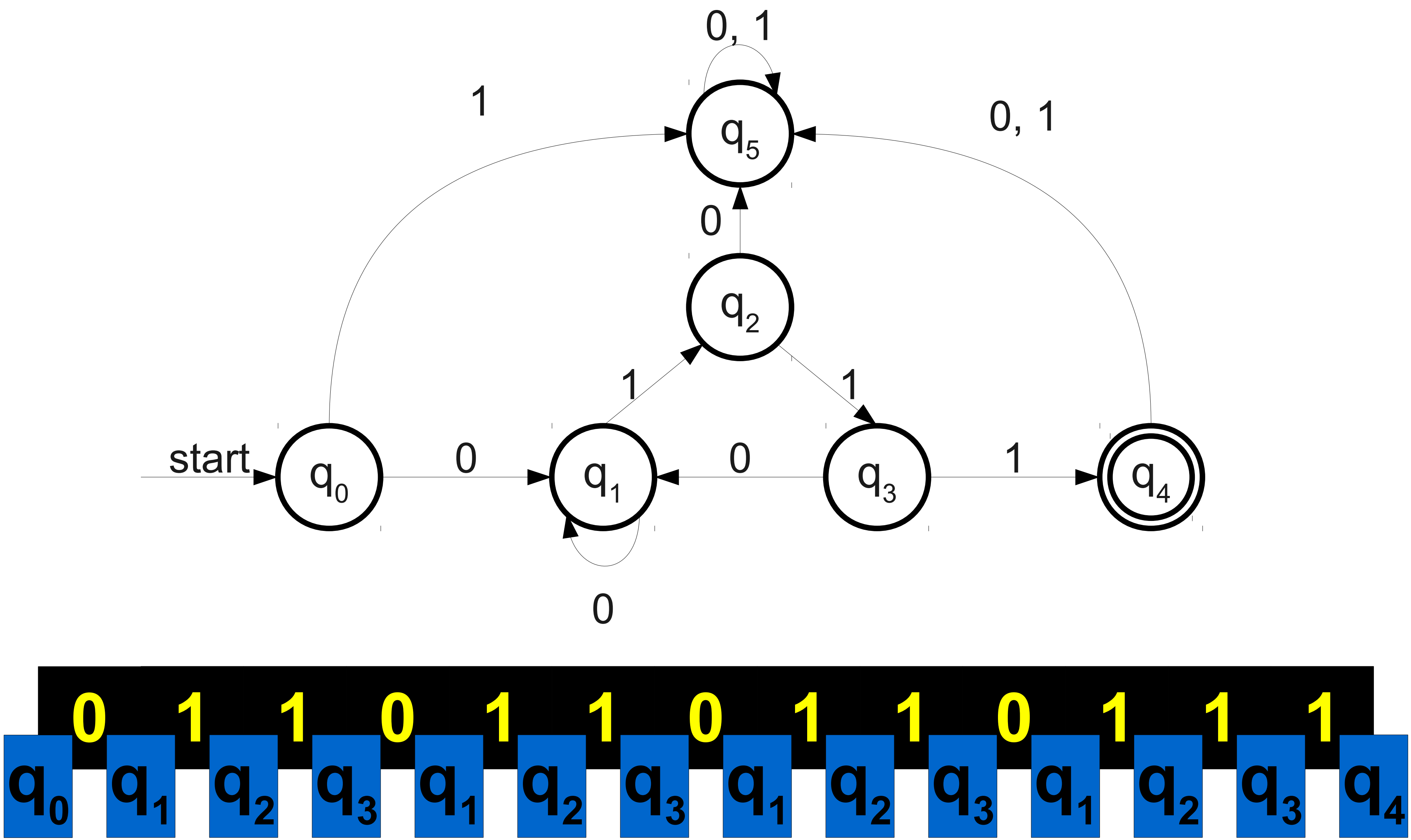

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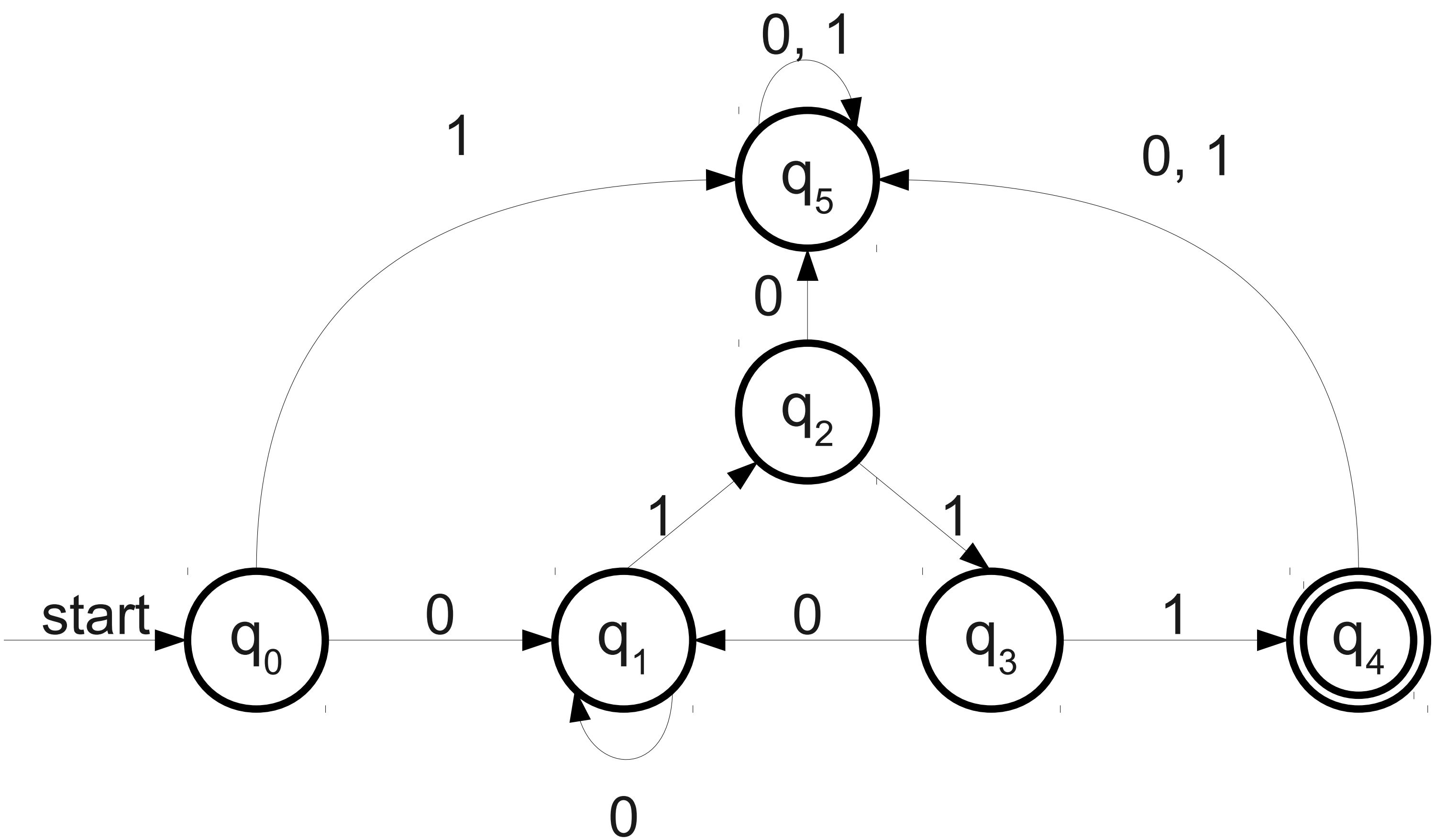

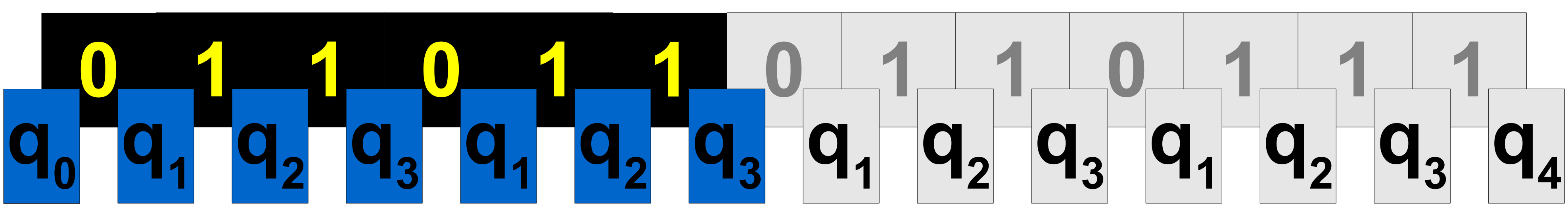

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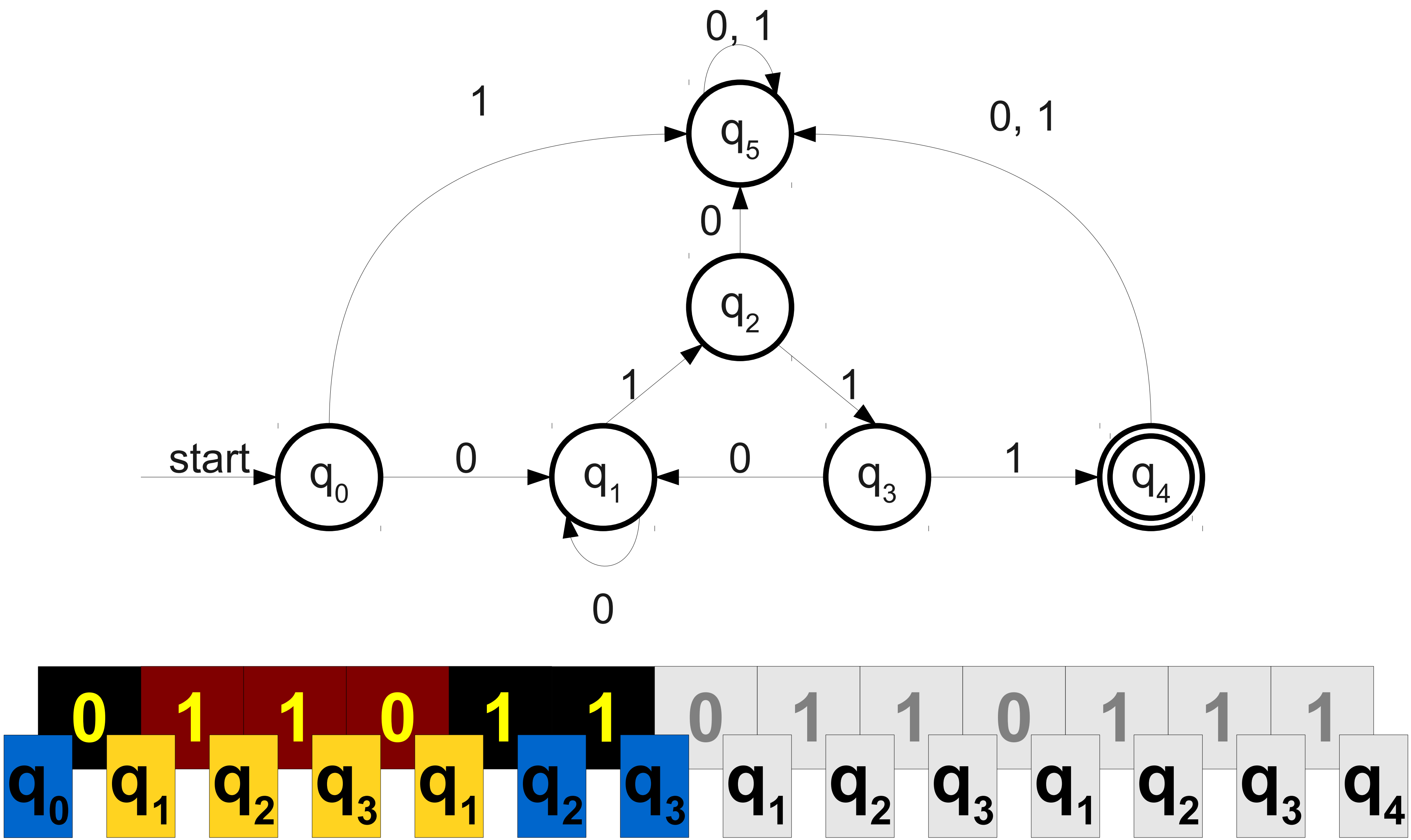

Weak Pumping Lemma Intuition

- Let D be a DFA with n states.
- Any string *w* accepted by D that has length at least n must visit some state twice.
  - Number of states visited is equal to |w| + 1.
  - By the **pigeonhole principle**, some state is duplicated.
- The substring of *w* in-between those revisited states can be removed, duplicated, tripled, quadrupled, etc. without changing the fact that *w* is accepted by D.

Pumping Lemma Intuition

- Let D be a DFA with n states.
- Any string *w* accepted by D that has length at least n must visit some state twice **within its first n characters**.
  - Number of states visited is equal **n + 1**.
  - By the **pigeonhole principle**, some state is duplicated.
- The substring of *w* in-between those revisited states can be removed, duplicated, tripled, quadrupled, etc. without changing the fact that *w* is accepted by D.

The Weak Pumping Lemma

For any regular language L,

There exists a positive natural number n such that

For any $w \in L$ with $|w| \geq n$,

There exists strings x, y, z such that

For any natural number i,

$w = xyz$, w can be broken into three pieces,

$y \neq \varepsilon$ where the middle piece isn't empty,

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The Pumping Lemma

For any regular language L,

There exists a positive natural number n such that

For any $w \in L$ with $|w| \geq n$,

There exists strings x, y, z such that

For any natural number i,

$w = xyz$,	w can be broken into three pieces,
$\|xy\| \leq n$,	where the first two pieces occur at the start of the string,
$y \neq \varepsilon$	where the middle piece isn't empty,
$xy^iz \in L$	where the middle piece can be replicated zero or more times.

Why This Change Matters

- The restriction $|xy| \leq n$ means that we can limit where the string to pump must be.
- If we specifically craft the first n characters of the string to pump, we can force y to have a specific property.
- We can then show that y cannot be pumped arbitrarily many times.

BALANCE and the Pumping Lemma

BALANCE = { w | w contains an equal number of 0s and 1s. }

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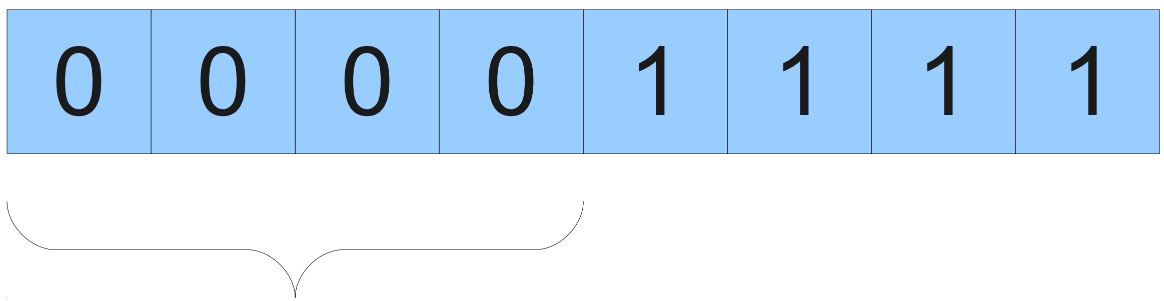

Since |xy| ≤ 4, the string to pump must be somewhere in here.

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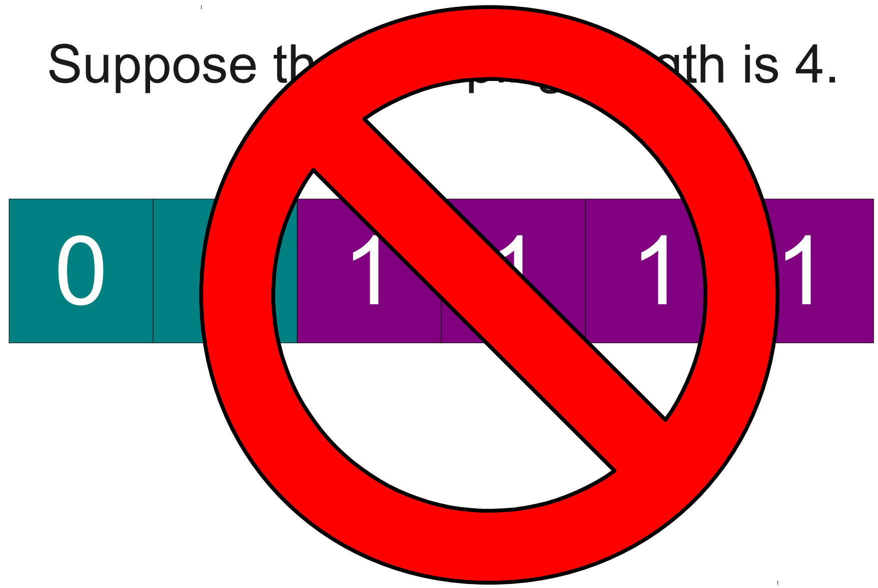

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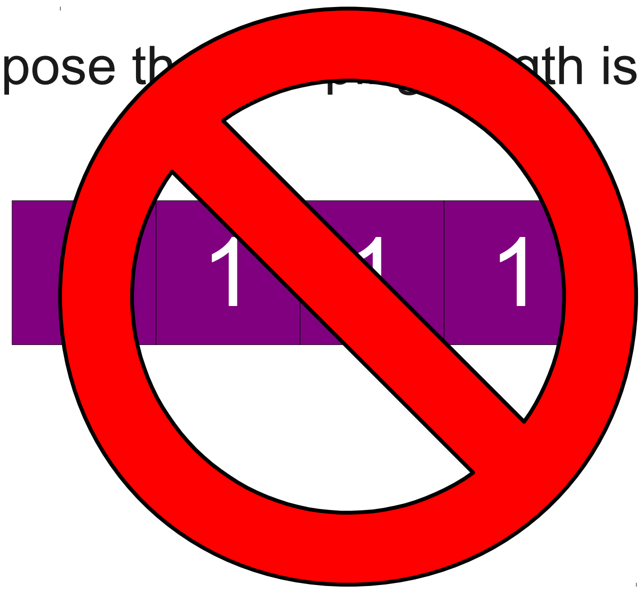

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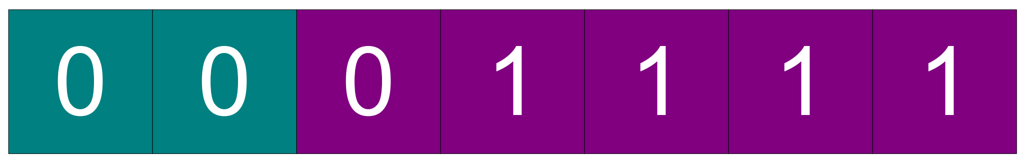

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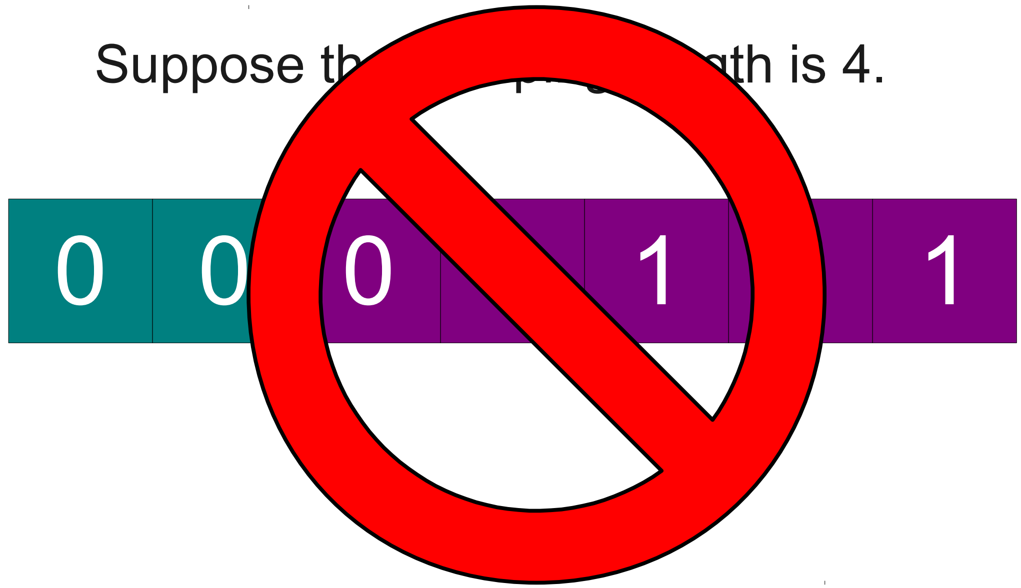

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This is why the pumping lemma is more powerful than the weak pumping lemma. We can force y to be made purely of 0s, rather than some combination of 0s and 1s.

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Summary of the Pumping Lemma

- Using the **pigeonhole principle**, we can prove the **weak pumping lemma** and **pumping lemma**.
- These lemmas describe essential properties of the regular languages.
- Any language that fails to have these properties cannot be regular.

Next Time

- **Beyond Regular Languages**
    - Context-free languages.
    - Context-free grammars.