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SURVIVAL AND GROWTH WITH A LIABILITY: OPTIMAL PORTFOLIO STRATEGIES IN CONTINUOUS TIME

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We study the optimal behavior of an investor who is forced to withdraw funds continuously at a fixed rate per unit time (e.g., to pay for a liability, to consume, or to pay dividends). The investor is allowed to invest in any or all of a given number of risky stocks, whose prices follow geometric Brownian motion, as well as in a riskless asset which has a constant rate of return. The fact that the withdrawal is continuously enforced, regardless of the wealth level, ensures that there is a region where there is a positive probability of ruin. In the complementary region ruin can be avoided with certainty. Call the former region the danger-zone and the latter region the safe-region. We first consider the problem of maximizing the probability that the safe-region is reached before bankruptcy, which we call the survival problem. While we show, among other results, that an optimal policy does not exist for this problem, we are able to construct explicit ϵ-optimal policies, for any $\epsilon > 0$. In the safe-region, where ultimate survival is assured, we turn our attention to growth. Among other results, we find the *optimal growth policy* for the investor, i.e., the policy which reaches another (higher valued) goal as quickly as possible. Other variants of both the survival problem as well as the growth problem are also discussed. Our results for the latter are intimately related to the theory of Constant Proportions Portfolio Insurance.

1. Introduction. The problem considered here is to solve for the optimal investment decision of an investor who must withdraw funds (e.g., to pay for some liability or to consume) continuously at a given rate per unit time. Income can be obtained only from investment in any of $n + 1$ assets: n risky stocks, and a bond with a deterministic constant return. The objectives considered here relate solely to what can be termed "goal problems," in that we assume the investor is interested in reaching some given values of wealth (called goals) with as high a probability as possible and/or as quickly as possible.

The fact that the investor must continuously withdraw funds at a fixed rate introduces a new difficulty that was not present in the previous studies of objectives related to reaching goals quickly (cf. Heath et al. 1987). Specifically the forced withdrawals ensure that at certain levels of wealth, there is a positive probability of going bankrupt, and thus the investor is forced to invest in the risky stocks to avoid ruin. In this paper we address the two basic problems faced by such an investor: how to survive, and how to grow. The *survival* problem turns out to be somewhat tricky, in that we prove that no fully optimal policy exists. Nevertheless, we are able to construct ϵ-optimal policies, for any $\epsilon > 0$. The *growth* problem is answered completely, once the survival aspect is clarified.

While this model is directly applicable to the workings of certain economic enterprises, such as a pension fund manager with fixed expenses that must be paid continuously (regardless of the level of wealth in the fund), our results are also related to investment strategies that are referred to as *Constant Proportion Portfolio*

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Insurance (CPPI). In fact a related model was used as the economic justification of CPPI in Black and Perold (1992), where both the theory and application of such strategies is described. In Black and Perold (1992), optimal strategies were obtained for the objective of maximizing utility of consumption, for a very specific utility function, subject to a minimum consumption constraint. However, the analysis and policies of Black and Perold (1992) are relevant only when initial wealth is in a particular region (specifically, when initial wealth is above the "floor"), wherein for that policy, there is no possibility of bankruptcy. Black and Perold (1992) did not address the fact that for the model described there, ruin, or bankruptcy, is a very real possibility when initial wealth is in the complementary region (below the "floor").

Here we focus on the objectives of survival and growth, which are intrinsic objective criteria that are independent of any specific individual utility function. As such, our results for both aspects of the problem will therefore complement the results of Black and Perold (1992) (as well as the more recent related work of Dybvig 1995). Firstly, the survival problem has not been addressed before for this model (although see Majumdar and Radner 1991 and Roy 1995), and secondly, the optimal growth policies we obtain provides another objective justification for the use of the CPPI policies prescribed in Black and Perold (1992), since for this problem we get similar policies as those obtained there.

The remainder of the paper is organized as follows: In the next section, we will describe the model in greater detail, and prove a general theorem in stochastic control from which all our subsequent results will follow. To facilitate the exposition, we will at first consider the case where there is only one risky stock and where the withdrawal rate is constant per unit time. It turns out that the state space (for wealth) can be divided into two regions, which we will call the "danger-zone" and the "safe-region." In the latter region, the investor need never face the possibility of ruin, and so we can concentrate purely on the growth aspects of the investor. (The aforementioned studies of Black and Perold 1992 and Dybvig 1995 considered only this region in their analyses of the maximization of utility from consumption problem.) In the former region, ruin, or bankruptcy, is a possibility (hence the term "danger-zone") and therefore we first concentrate on passing from the danger-zone into the safe-region. This is the *survival problem* and it is completely analyzed in §3. In particular, two problems are considered, *maximizing the probability of reaching the "safe-region" before going bankrupt*, and *minimizing the discounted penalty that must be paid upon reaching bankruptcy*. It is the former problem that does not admit an optimal policy, although we are able to explicitly construct an ϵ-optimal policy. The latter problem does admit an optimal policy, which we find explicitly. The structure of both policies are quite similar, in that they both essentially invest a (different) proportional amount of the distance to the safe-region. In §4 we consider the *growth problem* in the safe-region. We define growth as reaching a given (high) level of wealth as quickly as possible. Two related problems are then solved completely. First, we find the policy that *minimizes the expected time to the* (good) *goal*, and then we find the policy that *maximizes the expected discounted reward of getting to the goal*. Our resulting optimal growth policies turn out to be quite similar to the CPPI policies obtained for a different problem by Black and Perold (1992), in that they invest a (different) proportional amount of the distance from the danger-zone. Extensions to the multiple asset case as well as the case of a wealth-dependent withdrawal rate are discussed in §§5 and 6.

All the control problems considered in this paper are special cases of a particular general control problem that is solved in Theorem 2.1 in §2 below. For this problem we use the Hamilton-Jacobi-Bellman (HJB) equations of stochastic control (see, e.g., Fleming and Rishel 1975, or Krylov 1980) to obtain a candidate optimal policy in

terms of a candidate value function and this value function is then in turn given as the solution to a particular nonlinear Dirichlet problem. These candidate values are then verified and rigorously proved to be optimal by the *martingale optimality principle* (see §V.15 in Rogers and Williams 1987, or §2 in Davis and Norman 1990). The resulting nonlinear differential equations are then solved in turn for each of the problems considered below, yielding the optimal solutions in explicit form.

2. The model and continuous-time stochastic control. Without loss of generality, we assume that there is only one risky stock available for investment (e.g., a mutual fund), whose price at time t will be denoted by P_t. (Extension to the multidimensional case (for a *complete market*) is quite straightforward, and since the excess notation required adds little to the understanding, we will simply outline how to obtain the results for the multidimensional case in a later section.) As is quite standard (see, e.g., Merton 1971, 1990, Davis and Norman 1990, Black and Perold 1992, Grossman and Zhou 1993, Pliska 1986), we will assume that the *price* process of the risky stock follows a *geometric Brownian motion*, i.e., P_t satisfies the stochastic differential equation

$$dP_t = \mu P_t\, dt + \sigma P_t\, dW_t \tag{1}$$

where μ and σ are positive constants and $\{W_t\colon t \geq 0\}$ is a standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, P)$, where $\{\mathcal{F}_t\}$ is the P-augmentation of the natural filtration $\mathcal{F}_t^W := \sigma\{W_s;\ 0 \leq s \leq t\}$. (Thus the instantaneous *return* on the risky stock, dP_t/P_t, is a *linear* Brownian motion.)

The other investment opportunity is a bond, whose price at time t is denoted by B_t. We will assume that

$$dB_t = rB_t\, dt \tag{2}$$

where $r > 0$. To avoid triviality, we assume $\mu > r$.

We assume, for now, that the investor must withdraw funds continuously at a constant rate, say $c > 0$ per unit time, regardless of the level of wealth. (This would be applicable for example if the investor faces a constant liability to which \$$c$ must be paid continuously.) In a later section we generalize this to a case where the withdrawals are wealth-dependent.

Let f_t denote the *total amount of money invested in the risky stock at time t under an investment policy f*. An investment policy f is *admissible* if $\{f_t,\ t \geq 0\}$ is a measurable, $\{\mathcal{F}_t\}$-adapted process for which $\int_0^T f_t^2\, dt < \infty$, a.s., for every $T < \infty$. Let $\mathcal{G}$ denote the set of admissible policies.

For each admissible control process $f \in \mathcal{G}$, let $\{X_t^f,\ t \geq 0\}$ denote the associated wealth process, i.e., X_t^f is the wealth of the investor at time t, if he follows policy f. Since any amount not invested in the risky stock is held in the bond, this process then evolves as

$$\begin{aligned} dX_t^f &= f_t \frac{dP_t}{P_t} + \left(X_t^f - f_t\right)\frac{dB_t}{B_t} - c\, dt \\ &= \left[rX_t^f + f_t(\mu - r) - c\right] dt + f_t \sigma\, dW_t \end{aligned} \tag{3}$$

upon substituting from (1) and (2).

Thus, for Markov control processes f, and functions $\Psi(t, x) \in \mathscr{C}^{1,2}$, the generator of the wealth process is

$$\mathscr{A}^f\Psi(t,x) = \Psi_t + \left[f_t(\mu - r) + rx - c\right]\Psi_x + \frac{1}{2}f_t^2\sigma^2\Psi_{xx}. \tag{4}$$

We will put no constraints on the control f_t (other than admissibility). In particular, we will allow $f_t < 0$, as well as $f_t > X_t^f$. In the first instance, the company is selling the stock short, while in the second instance it is borrowing money to invest long in the stock. (While we do allow shortselling, it turns out that none of our optimal policies will ever in fact do this.)

What will differentiate our model and results from previous work are the objectives considered and the fact that here the withdrawal rate c is constant, and not a decision variable.

The usual portfolio and asset allocation problems considered in the financial economics literature deal with an investor whose wealth also evolves according to a stochastic differential equation as in (3), where instead of being constant, c is now a control variable as well, i.e., the consumption function $c_t = c(X_t^f)$. For a specific utility function $u(\cdot)$, the investor's objective is then to maximize the expected utility of consumption and terminal wealth over some finite horizon, i.e., for $T > 0$, and some "bequest function" $\Psi(\cdot)$, the investor wishes to solve

$$\sup_{f,c} E_x\left(\int_0^T e^{-\lambda t}u(c_t)\,dt + e^{-\lambda T}\Psi\left(X_T^f\right)\right), \tag{5}$$

for some discount factor $\lambda \geq 0$. Alternatively, the investor may wish to solve the discounted infinite horizon problem

$$\sup_{f,c} E_x\int_0^\infty e^{-\lambda t}u(c_t)\,dt. \tag{6}$$

In both of these cases, since the *process* $\{c_t\}$ is usually assumed to be completely controllable, it is clear that for certain utility functions at least, ruin need never occur, since we may simply stop consuming at some level. Alternatively, as is the case when the utility function is of the form $u(c) = c^{1-R}/(1-R)$ for some $R < 1$, or $u(c) = \ln(c)$, the resulting optimal policy takes both investment f_t and consumption c_t to be proportional to wealth, i.e., $f_t = \pi_1(t)X_t$, $c_t = \pi_2(t)X_t$, which in turns makes the optimal wealth process into a geometric Brownian motion, and thus the origin becomes an inaccessible barrier. Classical accounts of such (and more sophisticated) problems are discussed in Merton (1971, 1990) and Davis and Norman (1990) among others.

Optimal investment decisions with *constraints on consumption* have also been considered in the literature previously. Most relevant to our model is the literature on *constant proportion portfolio insurance* (CPPI), as introduced in Black and Perold (1992), where the resulting policy is to invest a *constant proportion of the excess of wealth over a given constant floor*. (As its name suggests, *portfolio insurance* can be loosely considered any trading and investment strategy that ensures that the value of a portfolio never decrease below some limit. Alternative approaches to portfolio insurance using options and other techniques are described in e.g., Luskin 1988.) Black and Perold (1992) introduced this policy as the solution to the discounted infinite horizon problem of (6) *subject to the constraint that* $c_t \geq c_{\min}$, where $c_{\min}$ is

some given constant. The specific utility function considered there was

$$u(c) = \begin{cases} c^{1-R}/(1-R) & \text{for } c \geq c^*, \\ K_1 - K_2 c & \text{for } c \leq c^*, \end{cases}$$

where c^* is a given constant, $R \leq 1$ and K_1, K_2 are constants chosen to ensure $u(\cdot)$ continuous throughout. While others (e.g., Dybvig 1995) have raised some technical questions about the analysis in Black and Perold (1992), more relevant to our point of view is the fact that this model (and the resulting optimal policy) allows for the possibility of *ruin*, or bankruptcy, if wealth is initially below the given floor. This possibility was never addressed in Black and Perold (1992).

In this paper we do not concentrate on the usual utility maximization problems of (5) and (6). Rather, here we are concerned with the *objective* problems of survival and growth. In particular, we first study the problem of how the investor (whose wealth evolves according to (3)) should invest to maximize the probability that the investor survives forever (which turns out to be related to *maximizing the probability of achieving a given fixed fortune before going bankrupt*), as well as the problem of how the investor should invest so as *to minimize the time until a given level of wealth has been achieved.* The former problem is called the *survival problem*, and is discussed in §3. The latter is called the *growth problem* and is the content of §4. Related problems have been studied in general under the label of "goal problems" in the works of Pestien and Sudderth (1985, 1988), Heath et al. (1987) and Orey et al. (1987). The survival problem for some specific related models were studied in Browne (1995) and Majumdar and Radner (1991). The former treated an "incomplete market" model, where the withdrawals are not fixed but rather follow a stochastic process, and the latter treated a model with forced constant consumption but without the possibility of investing in a risk free asset.

Recently, in order to provide a consumption based economic justification for the interesting portfolio strategies introduced in Grossman and Zhou (1993) (where the optimal policy invests a constant proportion of wealth over a *stochastic* floor), Dybvig (1995) considered the consumption-investment problem of (6) with the constraint that *consumption never decrease*, i.e., that $c_t > c_s$, for all $t \geq s$, with $c_0 > 0$. Thus consumption is forced in his model as well. He considered utility functions of the form $u(c) = c^{1-R}/(1-R)$ and $u(c) = \ln(c)$. However he only considered the problem in the *feasible region*, where initial wealth X_0, satisfies $X_0 > c_0/r$, and so for which ruin need not occur. Dybvig (1995) did not consider the case when $X_0 > c_0/r$, and hence where ruin is possibility, and so our results on this problem in §3 will complement his analysis as well. Since in this paper our objectives deals solely with the achievement of particular goals associated with *wealth*, it is clear that if there is a constraint on consumption as in the models of Black and Perold (1992) and Dybvig (1995), we should always set consumption at the minimum level, which in both cases is a constant ($c_{\min}$ in Black and Perold 1992 and c_0 in Dybvig 1995). This is consistent with the model we analyze here, where we will (at least at first) take consumption as a fixed constant c per unit time. This implies that at least for some values of wealth, the origin is accessible, and thus ruin is in fact a possibility.

In the next section we consider the problem of how to invest in order to survive. However, before we study that problem, we need a preliminary result from control theory that will provide the basis of all our future results.

2.1. Optimal control. The problems of survival and growth considered in this paper are all special cases of (Dirichlet-type) optimal control problems of the

following form: For each admissible control process $\{f_t,\ t \geq 0\}$, let

$$\tau_z^f := \inf\{t > 0\colon X_t^f = z\}$$

denote the first hitting time to the point z of the associated wealth process $\{X_t^f\}$ of (3), under policy f. For given numbers (l, u) with $l < X_0 < u$, let $\tau^f := \min\{\tau_l^f, \tau_u^f\}$ denote the first escape time from the interval (l, u).

For a given nonnegative continuous function $\lambda(x) \geq 0$, a given real bounded continuous function $g(x)$, and a function $h(x)$ given for $x = l, x = u$, let $v^f(x)$ be defined by

$$(7)\quad v^f(x) = E_x\left(\int_0^{\tau^f} g(X_t^f)\exp\left\{-\int_0^t \lambda(X_s^f)\,ds\right\}dt + h(X_{\tau^f}^f)\exp\left\{-\int_0^{\tau^f}\lambda(X_s^f)\,ds\right\}\right)$$

with

$$v(x) = \sup_{f\in\mathscr{G}} v^f(x) \quad \text{and} \quad f_v^*(x) = \arg\sup_{f\in\mathscr{G}} v^f(x).$$

We note at the outset that we are only interested in controls (and initial values x) for which $v^f(x) < \infty$.

As a matter of notation, we note first that here, and throughout the remainder of the paper, the parameter γ will be defined by

$$(8)\qquad \gamma := \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2.$$

THEOREM 2.1. *Suppose that $w(x)$: $(l, u) \to (-\infty, \infty)$ is a $\mathscr{C}^2$ function that is the concave increasing (i.e., $w_x > 0$ and $w_{xx} < 0$) solution to the nonlinear Dirichlet problem*

$$(9)\qquad (rx - c)w_x(x) - \gamma\frac{w_x^2(x)}{w_{xx}(x)} + g(x) - \lambda(x)w(x) = 0, \quad \text{for } l < x < u,$$

with

$$(10)\qquad w(l) = h(l) \quad \text{and} \quad w(u) = h(u),$$

and satisfies the conditions:

(i) $w_x^2(x)/w_{xx}(x)$ *is bounded for all* x *in* (l, u);

(ii) *there exists an integrable random variable* Y *such that for all* $t \geq 0, w(X_t^f) \geq Y$;

(iii) $w_x(x)/w_{xx}(x)$ *is locally Lipschitz continuous.*

Then $w(x)$ is the optimal value function, i.e., $w(x) = v(x)$, and moreover the optimal control, f_v^*, can then be written as

$$(11)\qquad f_v^*(x) = -\left(\frac{\mu - r}{\sigma^2}\right)\frac{w_x(x)}{w_{xx}(x)}, \quad \text{for } l < x < u.$$

PROOF. The appropriate HJB optimality equation of dynamic programming for maximizing $v^f(x)$ of (7) over control policies f_t, to be solved for a function v is $\sup_{f\in\mathscr{G}}\{\mathscr{A}^f v + g - \lambda v\} = 0$, subject to the Dirichlet boundary conditions $v(l) = h(l)$ and $v(u) = h(u)$ (cf. Theorem 1.4.5 of Krylov 1980). Since $v(x)$ is independent of

time, the generator of (4) shows that this is equivalent to

$$\sup_{f\in\mathscr{F}}\left\{(f(\mu-r)+rx-c)v_x+\frac{1}{2}f^2\sigma^2v_{xx}+g-\lambda v\right\}=0. \tag{12}$$

Assuming now that (12) admits a classical solution with $v_x>0$ and $v_{xx}<0$ (see, e.g., Fleming and Soner 1993), we may then use standard calculus to optimize with respect to f in (12) to obtain the maximizer $f_v^*=-((\mu-r)/\sigma^2)v_x/v_{xx}$ (compare with (11)). When this $f_v^*(x)$ is then substituted back into (12) and the resulting equation is simplified, we obtain the nonlinear Dirichlet problem of (9) (with $v=w$).

It remains only to verify that the policy f_v^* is indeed optimal. The aforementioned theorem in Krylov (1980) does not apply here, since in particular the degeneracy condition (Krylov 1980, page 23) is not met. We will use instead the martingale optimality principle, which entails finding an appropriate functional which is a uniformly integrable martingale under the (candidate) optimal policy, but a *supermartingale* under any other admissible policy, with respect to the filtration $\mathscr{F}_t$ (see Rogers and Williams 1987, Davis and Norman 1990).

To that end, let $\Lambda^f(s,t):=\int_s^t\lambda(X_v^f)\,dv$, and define the process

$$M(t,X_t^f):=e^{-\Lambda^f(0,t)}w(X_t^f)+\int_0^t e^{-\Lambda^f(0,s)}g(X_s^f)\,ds,\quad\text{for } 0\le t\le\tau^f, \tag{13}$$

where w is the concave increasing solution to (9).

Optimality of f_v^* of (11) is then a direct consequence of the following lemma.

LEMMA 2.2. *For any admissible policy f, and $M(t,\cdot)$ as defined in* (13), *we have*

$$E\big(M(t\wedge\tau^f,X^f_{t\wedge\tau^f})\big)\le M(0,X_0)=w(x), \tag{14}$$

with equality holding if and only if $f=f_v^*$, where f_v^* is the policy given in (11). *Moreover, under policy f_v^*, the process $\{M(t\wedge\tau^f,X^*_{t\wedge\tau^f})\}$ is a uniformly integrable martingale.*

PROOF. Applying Ito's formula to $M(t,X_t^f)$ of (13) using (3) shows that for $0\le s\le t\le\tau^f$

$$M(t,X_t^f)=M(s,X_s^f)+\int_s^t e^{-\Lambda^f(s,v)}Q(f_v;X_v^f)\,dv+\int_s^t e^{-\Lambda^f(s,v)}\sigma f_v w_x(X_v^f)\,dW_v \tag{15}$$

where $Q(z;y)$ denotes the quadratic (in z) defined by

$$\begin{aligned}Q(z;y):=z^2\left[\frac{1}{2}\sigma^2w_{xx}(y)\right]+z[(\mu-r)w_x(y)]\\+(ry-c)w_x(y)+g(y)-\lambda(y)w(y).\end{aligned}$$

Recognize now that since $Q_{zz}(z;y)=\sigma^2w_{xx}(y)<0$, we always have $Q(z;y)\le 0$, and the maximum is achieved at the value

$$z^*(y):=-\left(\frac{\mu-r}{\sigma^2}\right)\frac{w_x(y)}{w_{xx}(y)}$$

with corresponding maximal value

$$Q(z^*;y)=(ry-c)w_x(y)-\gamma\frac{w_x(y)^2}{w_{xx}(y)}+g(y)-\lambda(y)w(y)\equiv 0$$

where the final equality follows from (9). Therefore the second term in the r.h.s. of (15) is always less than or equal to 0. Moreover (15) shows that we have

$$\begin{aligned}
&\int_0^{t\wedge\tau^f} e^{-\Lambda^f(0,v)}\sigma f_v w_x\big(X_v^f\big)\,dW_v \\
&\quad = M\big(t\wedge\tau^f, X^f_{t\wedge\tau^f}\big) - w(x) - \int_0^{t\wedge\tau^f} e^{-\Lambda^f(0,v)} Q\big(f_v; X_v^f\big)\,dv \\
&\quad \geq M\big(t\wedge\tau^f, X^f_{t\wedge\tau^f}\big) - w(x).
\end{aligned}$$

Thus, by (ii) we see that the stochastic integral term in (15) is a local martingale that is in fact a *supermartingale*. Hence, taking expectations on (15), with $s = 0$, therefore shows that

$$\begin{aligned}
(16)\quad E\big(M\big(t\wedge\tau^f, X^f_{t\wedge\tau^f}\big)\big) &\leq w(x) + E\left(\int_0^{t\wedge\tau^f} e^{-\Lambda^f(0,v)} Q\big(f_v; X_v^f\big)\,dv\right) \\
&\leq w(x) + E\left(\int_0^{t\wedge\tau^f} e^{-\Lambda^f(0,v)}\left[\sup_{f_v} Q\big(f_v; X_v^f\big)\right] dv\right) \\
&= w(x)
\end{aligned}$$

with the equality in (16) being achieved at the policy f_v^*.

Thus we have established (14).

Note that under the policy f_v^* of (11), the wealth process X^* satisfies the stochastic differential equation

$$(17)\qquad dX_t^* = \left[\left(rX_t^* - c - 2\gamma\frac{w_x(X_t^*)}{w_{xx}(X_t^*)}\right)dt - \sqrt{2\gamma}\,\frac{w_x(X_t^*)}{w_{xx}(X_t^*)}dW_t\right] I_{\{t\leq\tau^*\}}$$

where $\tau^* := \tau^{f_v^*}$. By (iii) this equation admits a unique strong solution (Karatzas and Shreve 1988, Theorem 5.2.5).

Furthermore note that under the (optimal) policy, f_v^*, we have, for all $0 \leq s \leq t \leq \tau^*$,

$$(18)\qquad M(t, X_t^*) = M(s, X_s^*) - \sqrt{2\gamma}\int_s^t \exp\left\{-\int_s^v \lambda\big(X_\rho^*\big)\,d\rho\right\}\frac{w_x^2(X_v^*)}{w_{xx}(X_v^*)}dW_v$$

which by (i) above is seen to be a uniformly integrable martingale. This completes the proof of the theorem. □

We now return to the survival problem.

3. Maximizing survival. We consider in this section two objectives related to maximizing the survival of the investor. First we consider the problem of *minimizing the probability of ruin* which is related to the problem of *maximizing the probability of reaching a particular given upper level of wealth before a given lower level*. We will show that an optimal strategy for this latter problem does not exist, although exploiting the solution to a related solvable problem will allow us to explicitly construct ϵ-optimal ones. Next we consider the related objective of *minimizing the expected discounted penalty of ruin*, which is equivalent to *minimizing the expected discounted time to*

bankruptcy. This problem does admit an optimal solution and we find it explicitly. The structure of the (optimal) survival policies obtained in this section are similar in that they all invest a fixed fraction of the positive distance of wealth to a particular goal.

3.1. Minimizing the probability of ruin. The evolutionary equation (3) exhibits clearly that under policy f, the wealth process is a diffusion with drift function m and diffusion coefficient function v given respectively by

$$m(f,x,t) = f_t(\mu - r) + rx - c, \qquad v(f,x,t) = f_t^2\sigma^2. \tag{19}$$

Thus for any admissible control $f < \infty$ there is a region (in X space) where there is a positive probability of *bankruptcy*. This is due to the fact that while the variance of the wealth process is completely controllable, as is apparent from (19), the *drift* is not completely controllable due to the fact that $c > 0$, and hence the drift can be negative at certain wealth levels. This feature differentiates this model from those usually studied in the investment literature (e.g., Merton 1971, 1990, Pliska 1986, Davis and Norman 1990), with Majumdar and Radner (1991) being a notable exception. (For results on an "incomplete market" model where the variance, as well as the drift, is also not completely controllable, see Browne 1995.) Specifically, let a denote the bankruptcy level or point, with corresponding "bankruptcy time" (or *ruin time*) τ_a^f, where $0 \le a < X_0$. One survival objective is then to choose an investment policy which minimizes the probability of ruin, i.e., one which minimizes $P(\tau_a^f < \infty)$, or equivalently, maximizes $P(\tau_a^f = \infty)$ (see, e.g., Majumdar and Radner 1991, Browne 1995, Roy 1995).

Clearly this objective is meaningless for $X_t^f \ge c/r$. To see this directly, consider the case where the wealth level is $x > c/r$. We may then choose a policy which puts all wealth into the bond, and then under this policy the probability of bankruptcy is 0. Specifically, if we take $f = 0$ for $x > c/r$, (3) shows that the wealth will then follow the deterministic differential equation $dX_t = (rX_t - c)\,dt$, $X_0 = x > c/r$, which exhibits exponential growth and for which $P(\tau_{x-\epsilon} = \infty) = 1$, for all $\epsilon > 0$. Thus *the survival problem is interesting and relevant only in the region* $a < x < c/r$, which we will call the "danger-zone." This is of course due to the fact that $c/r = c\int_0^\infty e^{-rt}\,dt$ is the amount that is needed to be invested in the perpetual bond to pay off the forced withdrawals forever. Since the investor need never face the possibility of ruin for $x > c/r$, we will call the region $(c/r, \infty)$ the "safe-region."

Our objective in this section therefore is to determine a strategy that *maximizes the probability of hitting the safe-region or "safe point," c/r, prior to the "bankruptcy point," a, when initial wealth is in the danger-zone*, i.e., $a < x < c/r$. As noted above, we will show that an optimal policy for this problem *does not exist*, necessitating the construction of an ϵ-optimal strategy.

A somewhat related survival problem with constant withdrawals was studied in Majumdar and Radner (1991) in a different setting, although without a risk-free investment, and hence without a safe-region. Moreover, their results are *not applicable* to our case since here $\inf_f v^2(f,x,t) = 0$, which violates the conditions of the model in Majumdar and Radner (1991). As we shall see, it is in fact precisely this fact that negates the existence of an optimal policy for our problem. A related survival model which allows for investment in a risk-free asset, but where the "withdrawals" are assumed to follow another (possibly dependent). Brownian motion with drift, was treated in Browne (1995). Since the Brownian motion is unbounded, there was no safe-region in Browne (1995) either. A discrete-time model with constant withdrawals that does allow for a risk-free investment was treated in Roy (1995), but with no borrowing allowed and bounded support for the return on the risky asset.

To show explicitly why no policy obtains optimality for the model treated here, and how we may construct ϵ-optimal strategies, we will first consider the following problem: for any point b in the danger-zone, i.e., with $a < x < b < c/r$, we will find the optimal policy to maximize the probability of hitting b before a. For b strictly less than c/r, an optimal policy *does exist* for this problem, and we will identify it in the following theorem. To that end let

$$V(x: a, b) = \sup_{f \in \mathcal{F}} P_x(\tau_a^f > \tau_b^f), \quad \text{and let} \quad f_V^* = \arg\sup_{f \in \mathcal{F}} P_x(\tau_a^f > \tau_b^f).$$

THEOREM 3.1. *The optimal policy is to invest, at each wealth level $a < x < b$, the state dependent amount*

$$f_V^*(x) = \frac{2r}{\mu - r}\left(\frac{c}{r} - x\right). \tag{20}$$

The optimal value function is

$$V(x: a, b) = \frac{(c - ra)^{\gamma/r+1} - (c - rx)^{\gamma/r+1}}{(c - ra)^{\gamma/r+1} - (c - rb)^{\gamma/r+1}}, \quad \text{for } a \le x \le b, \tag{21}$$

where γ is defined by (8).

REMARK 3.1. Note that the policy of (20) invests less as the wealth gets closer to the goal b. In fact, it invests a *constant proportion of the distance to the "safe point" c/r, regardless of the value of the goal b, and the bankruptcy point a*. It is interesting to observe that while here this constant proportion is *independent of the underlying diffusion parameter* σ^2, this does not hold when there are multiple risky stocks in which to invest in (see §5 below). The constant proportion is greater (less) than 1 as $\mu/r <$ ($>$) 3. Thus it is interesting to observe that as the wealth gets closer to the bankruptcy point, a, the optimal policy does *not* "panic" and start investing an enormous amount, rather the optimal policy stays calm and invests at most $f_V^*(a) = 2(c - ra)/(\mu - r)$. The investor does get increasingly more cautious as his wealth gets closer to the goal b. (This behavior should be compared with the "timid" vs. "bold" play in the discrete-time problems considered in the classic book of Dubins and Savage 1965. See also Majumdar and Radner 1991 and Roy 1995.)

Observe further that the investor is borrowing money to invest in the stock only when $x < 2c/(\mu + r)$ but not when $2c/(\mu + r) < x < c/r$. This can be seen by observing directly that in the former case $f_V^*(x) > x$ and in the latter case $f_V^*(x) < x$. (The fact that $2c/(\mu + r) < c/r$ follows from the assumption that $\mu > r$.)

PROOF. While we could prove Theorem 3.1 from a more general theorem in Pestien and Sudderth (1985) (see also Pestien and Sudderth 1988) which we will discuss later (see Remark 3.4 below), recognize that this is simply a special case of the control problem solved in Theorem 2.1 for $l = a, u = b$ with $\lambda = 0$, $g = 0$ and $h(b) = 1$, $h(a) = 0$. As such the nonlinear Dirichlet problem of (9) for the optimal value function V becomes in this case

$$(rx - c)V_x - \gamma\frac{V_x^2}{V_{xx}} = 0, \quad \text{for } a < x < b \tag{22}$$

subject to the (Dirichlet) boundary conditions $V(a) = 0, V(b) = 1$.

The general solution to the second-order nonlinear ordinary differential equation of (22) is $K_1 - K_2(c - rx)^{\gamma/r+1}$, where K_1, K_2 are arbitrary constants which will be determined from the boundary conditions. The boundary condition $V(a) = 0$ determines that $K_1 = K_2(c - ra)^{\gamma/r+1}$, and the boundary condition $V(b) = 1$ then determines K_2, which then leads directly to the function $V(x)$ given in (21). It is clear that this function V is in $\mathscr{C}^2$ and does in fact satisfy $V_x > 0$ and $V_{xx} < 0$, and moreover satisfies conditions (i), (ii) and (iii) of Theorem 2.1 on the interval (a, b). (Condition (ii) is trivially met since V is bounded on (a, b).) As such V is indeed the optimal value function and the associated optimal control function f_V^* of (20) is then obtained by substituting the function V of (21) for w in (11). □

Note that under policy f_V^*, the wealth process, say X_t^*, satisfies the stochastic differential equation

$$dX_t^* = (c - rX_t^*)\,dt + \frac{2\sigma}{\mu - r}(c - rX_t^*)\,dW_t, \quad \text{for } t \le T^*, \tag{23}$$

where $T^* = \min\{\tau_a^*, \tau_b^*\}$, and $\tau_z^* = \inf\{t > 0\colon X_t^* = z\}$. This is obtained by placing the control (20) into the evolutionary equation (3). Equation (23) defines a linear stochastic differential equation, i.e., X^* is a time-homogeneous diffusion on (a, b) with drift function $\mu_*(x) = c - rx$, and diffusion coefficient function ${\sigma^2}_*(x) = ((2\sigma/(\mu - r))(c - rx))^2 \equiv (2/\gamma)(c - rx)^2$. As such its *scale* function is defined by

(24)

$$S^*(x) = \int^x \exp\left\{-\int^y \frac{2\mu_*(u)}{\sigma_*^2(u)}du\right\} dy \equiv -(\gamma + r)^{-1}(c - rx)^{\gamma/r+1}, \quad \text{for } a \le x \le b$$

where $\gamma = \frac{1}{2}((\mu - r)/\sigma)^2$. For this process therefore,

$$P_x(\tau_a^* > \tau_b^*) = \frac{S^*(x) - S^*(a)}{S^*(b) - S^*(a)} \equiv \frac{(c - ra)^{\gamma/r+1} - (c - rx)^{\gamma/r+1}}{(c - ra)^{\gamma/r+1} - (c - rb)^{\gamma/r+1}},$$

which of course agrees with (21). Thus the process $\{S^*(X_t^*)\}$ is a diffusion in *natural scale*, and is therefore a (uniformly integrable) martingale with respect to the filtration $\mathscr{F}_t$ (as is the optimal value function), i.e., $E(S^*(X_t^*)|\mathscr{F}_s) = S^*(X_s^*)$ for $0 \le s \le t \le \tau^*$, where $\tau^* := \min\{\tau_a^*, \tau_b^*\}$. Note further that the scale function $S^*(x)$ of (24), is increasing in x (although *negative*) for $0 \le x < c/r$.

3.1.1. Inaccessibility of the safe-region under f_V^* and ϵ-optimal strategies. While we have found a policy that maximizes the probability of reaching any $b < c/r$ before any $a < b$, it is important to realize that if we extend b to c/r, then this policy *will never achieve the safe point c/r with positive probability in finite time*. We can of course extend the function displayed in (21) to the point c/r to get

$$V(x\colon a, c/r) = P_x(\tau_a^* > \tau_{c/r}^*) \tag{25}$$

$$= \frac{S^*(x) - S^*(a)}{S^*(c/r) - S^*(a)} \equiv 1 - \left(\frac{c - rx}{c - ra}\right)^{\gamma/r+1} < \infty, \quad \text{for } a < x \le c/r$$

which shows than in fact c/r is an *attracting barrier* for the process X^*. However, it is an *unattainable* barrier. (See §15.6 in Karlin and Taylor (1981) for a discussion of the

boundary classification terminology used here.) To verify this, first recall that if we let $s^*(x) = dS^*(x)/dx$ denote the *scale density* of the diffusion X^*, then its *speed density* is given by

$$m^*(x) = \left(\sigma_*^2(x)s^*(x)\right)^{-1} \equiv \left(\frac{2}{\gamma}(c - rx)^2(c - rx)^{\gamma/r}\right)^{-1} \tag{26}$$

$$= \frac{\gamma}{2}(c - rx)^{-(\gamma/r+2)}$$

and it is well known then that

$$E_x\left(\min\{\tau_a^*, \tau_{c/r}^*\}\right) < \infty \quad \text{if and only if} \quad \int_x^{c/r}[S^*(c/r) - S^*(y)]m^*(y)\,dy < \infty.$$

However, it can be seen from (24) and (26) that the latter quantity is

$$\int_x^{c/r}[S^*(c/r) - S^*(y)]m^*(y)\,dy = \frac{\gamma}{2(\gamma + r)}\int_x^{c/r}\frac{1}{c - ry}dy = \infty,$$

and thus we see that while f_V^* minimizes the probability of hitting the ruin point a, and so is in fact optimal for the problem of $\min_f P_x(\tau_a^f = \infty)$, it does so in a way which makes the upper goal, c/r, unattainable in finite expected time. In fact, under f_V^* we have $\tau_{c/r}^* = \infty$ a.s., and thus *no optimal policy exists* for the problem of maximizing the probability of reaching the safe-region prior to bankruptcy!

Intuitively, what's going on is that as the wealth gets closer to the boundary of the safe-region, c/r, the investor gets increasingly more cautious so as not to forfeit his chances of getting there. This of course entails investing less and less, but in continuous-time, where the wealth is infinitely divisible, this just means eventually investing (close to) nothing. However while this in turn does in effect shut off the drift of the resulting wealth process (see (23)), it also shuts off the variance, and some positive variance is needed to cross over the c/r-barrier from the danger-zone into the safe-region. This is not supplied by the policy described above, which essentially tells the investor that the best he can hope to do (i.e., with maximal probability) is to try to get pulled into an asymptote that is drifting toward c/r.

In terms of our Theorem 2.1, it is clear that V is no longer concave increasing for $x > c/r$ (i.e., for $x > c/r$, we have $V_x(x) > 0$ and $V_{xx}(x) < 0$), and thus Theorem 2.1 is not valid for any $u > c/r$.

REMARK 3.3. This difficulty disappears if $r = 0$, since if there is no risk-free investment, the investor always faces a positive probability of ruin and the only way to survive is to always invest in the risky stock. To see this, note that letting $a = 0$ and taking limits as $r \downarrow 0$ (so that the "safe point" goes to infinity, i.e., when $r = 0$, there is always a positive probability of bankruptcy) shows that the value function, $V(x: 0, c/r)$, then goes to an exponential, i.e., as $r \downarrow 0$,

$$V(x: 0, c/r) \to 1 - \exp\left\{-\frac{\mu}{2\sigma^2 c}x\right\} \tag{27}$$

and for this case the (unconstrained) optimal control to minimize the probability of ruin is to always invest the *fixed constant* $2c/\mu$. (This model then becomes a

degenerate special case of Browne 1995.) In this case the optimal wealth process follows a *linear Brownian motion* with drift c and diffusion coefficient $2c\sigma/\mu$, for which the probability of ruin is the exponential (27). Ferguson (1965) conjectured that an ordinary investor (in discrete-time and space) can asymptotically *minimize the probability of ruin* by *maximizing the exponential utility of terminal wealth, for some risk aversion parameter.* It is interesting to observe that for this model the conjecture turns out to be true. To verify this, one would have to solve the finite-horizon utility maximization problem for the utility function $u(x) = \delta - \eta \exp\{-2cx/\mu\}$, with arbitrary $\eta > 0$ and δ. Since this problem is then essentially a special case of the (Cauchy) problem considered in §3 of Browne (1995), we refer the reader there for further details. If we impose the constraint that the investor is not allowed to borrow, then it can be shown (Browne 1995, Theorem 3) that the optimal control in this case is $f^* = \max\{x, 2c/\mu\}$, whereby the investor must invest all his wealth in the risky stock when wealth is below the critical level $2c/\mu$. In this case the value function is no longer concave below $2c/\mu$. Such extremal behavior (or "bold" play, ala Dubins and Savage 1965) and nonconcavity of the value function below a threshold is also a feature of the optimal policies in the related survival models studied in Majumdar and Radner (1991) and Roy (1995), where borrowing is not allowed.

REMARK 3.4. This inaccessibility and the resulting nonexistence of an optimal policy can be best understood in the context of the more general "goal" problem: Consider a controlled diffusion $\{Y_t^f\}$ on the interval (a, b) satisfying

$$dY_t^f = m(f, y)\, dt + v(f, y)\, dW_t,$$

with the objective of determining a control to maximize the probability of hitting b before a. Let $\Psi(x)$ denote the optimal value function for this problem, i.e., $\Psi(x) = \sup_{f \in \mathcal{F}} P_x(\tau_a^f > \tau_b^f)$ with optimal control $\psi(x) = \arg\sup_{f \in \mathcal{F}} P_x(\tau_a^f > \tau_b^f)$. This problem was first studied by Pestien and Sudderth (1985, 1988), who showed—using a different formulation—that

$$\psi(x) = \arg\sup_f \left\{ \frac{m(f, x)}{v^2(f, x)} \right\}, \tag{28}$$

and indeed our $f_V^*(x)$ of (20) can be obtained from maximizing m/v^2 for m and v^2 in (19). However as noted in Pestien and Sudderth (1985, 1988), this is the case only when $\inf_x v_*^2(f, x) > 0$, where $\psi = m_*/v^2{}_*$.

These results can be obtained from somewhat simpler methods (albeit with some lesser generality) then those used in Pestien and Sudderth (1985, 1988) as follows: $\Psi(x)$ must satisfy the HJB equation

$$\sup_f \left\{ m(f, x)\Psi_x + \frac{1}{2} v^2(f, x)\Psi_{xx} \right\} \tag{29}$$

$$\equiv \sup_f \left\{ \left[\frac{m(f, x)}{v^2(f, x)} \Psi_x + \frac{1}{2}\Psi_{xx} \right] \cdot v^2(f, x) \right\} = 0,$$

subject to the Dirichlet boundary conditions $\Psi(a) = 0, \Psi(b) = 1$.

If Ψ is a *classical solution* to the HJB equation (29), then we must have $\Psi_x > 0$ and $\Psi_{xx} < 0$. Therefore as long as $v^2(f, x) > 0$, it is clear from (29) that the maximum of (29) occurs at the maximum of m/v^2, which by (28) is denoted by ψ. If we now let $\rho(x) = \sup_f\{m(f, x)/v^2(f, x)\}$, i.e., $\rho(x) = m(\psi(x), x)/v^2(\psi(x), x)$, then the solution to (29) subject to the Dirichlet conditions is simply $\Psi(x) = \int_a^x s(z)\, dz / \int_a^b s(z)\, dz$,

where $s(z) = \exp\{-2\int^z p(y)\,dy\}$, with which our value function (21) of course agrees, for $b < c/r$.

However, it is also clear from (29) that for $v^2(f, x) = 0$, the HJB equation *need not hold*, and therefore, no policy is in general optimal when this is the case, which is precisely what is happening here for $b = c/r$ (see also Example 4.1 in Pestien and Sudderth 1988).

For more details on the general problem from a different perspective, we refer the reader to the fundamental papers of Pestien and Sudderth (1985, 1988). We now return to the problem of determining a 'good' strategy for crossing the c/r barrier.

An ϵ-optimal strategy. As we have just seen, the inaccessibility of c/r is due to the fact that f_V^* dictates an investment policy that causes the drift and variance of the resulting wealth process to go to zero as the c/r barrier is approached from below. A practical way around this difficulty is to modify f_V^* as follows:

Let f_δ^* denote the (suboptimal) policy which agrees with f_V^* below the point $c/r - \delta$, and then above it invests κ in the risky stock until the c/r barrier is crossed, i.e.,

$$f_\delta^*(x) = \begin{cases} f_V^*(x) & \text{for } x \le c/r - \delta, \\ \kappa & \text{for } x > c/r - \delta. \end{cases}$$

Now $V(x_0, a, c/r)$ as given in (25) is an *upper bound* on the probability of escaping the interval $(a, c/r)$ into the safe-region starting from an initial wealth level $x_0 < c/r$ (see Krylov 1980, page 5). Without loss of generality, we may take $a = 0$ here. Thus for any $\epsilon > 0$, and initial wealth $x_0 < c/r$, the best we can do is find a policy which gives

$$V(x_0: 0, c/r) - \epsilon = 1 - \left(1 - \frac{rx_0}{c}\right)^{\gamma/r+1} - \epsilon \tag{30}$$

as its value. Therefore for any given ϵ, and initial wealth $x_0 < c/r$, we need to find $\delta = \delta(x_0, \epsilon)$ and $\kappa = \kappa(x_0, \epsilon)$ which will achieve the value (30). To keep the drift and diffusion parameters continuous, we must take $\kappa = (2r/(\mu - r))\delta(x_0, \epsilon)$, and so $f_\delta(x) = (2r/(\mu - r))\max\{c/r - x, \delta\}$, which then gives a corresponding wealth process X^δ which has the (continuous) drift function $\mu_\delta(x)$ and diffusion function $\sigma_\delta^2(x)$ given by $\mu_\delta(x) = \max\{c - rx, 2r\delta + rx - c\}$, and $\sigma_\delta^2(x) = \max\{2(c - rx)^2/\gamma, 2r^2\delta^2/\gamma\}$.

The scale *density* for this new process, defined by

$$s_\delta(y) = \exp\left\{-\int^y \frac{2\mu_\delta(z)}{\sigma_\delta^2(z)}dz\right\}$$

can then be written as

$$s_\delta(y) = \begin{cases} (c - ry)^{\gamma/r} & \text{for } y \le c/r - \delta, \\ (r\delta)^{\gamma/r}\, e^{\gamma/(2r)}\sqrt{2\pi}\,\phi\left(\dfrac{y + 2\delta - c/r}{\sqrt{r\delta^2/\gamma}}\right) & \text{for } y \ge c/r - \delta, \end{cases}$$

where ϕ denotes the standard normal p.d.f.

The probability of reaching the safe-region from initial wealth $x_0 < c/r$ under this policy is therefore

$$V_\delta(x_0: 0, c/r) = \frac{\int_0^{x_0} s_\delta(y)\,dy}{\int_0^{c/r} s_\delta(y)\,dy} \equiv V(x_0: 0, c/r)\left(1 + \delta^{\gamma/r+1} H(\gamma, r, c)\right)^{-1}$$

where V is as in (25) and H is given by

$$H(\gamma, r, c) = (1 + \gamma/r)(r/c)^{\gamma/r+1}\, e^{\gamma/(2r)}\sqrt{2\pi r/\gamma}\left[\Phi\left(2\sqrt{\gamma/r}\right) - \Phi\left(\sqrt{\gamma/r}\right)\right],$$

where Φ denotes the standard normal c.d.f. Setting $V_\delta = V - \epsilon$, and then solving for δ therefore gives

$$\delta(x_0, \epsilon) = \left(\frac{\epsilon}{H(\gamma, r, c)\left[V(x_0: 0, c/r) - \epsilon\right]}\right)^{r/(\gamma+r)}. \tag{31}$$

Therefore, for the particular $\delta(x_0, \epsilon)$ given in (31), the policy f_δ^* is within ϵ of optimality. Since we chose $a = 0$ here purely for notational convenience, we summarize this in the following theorem for the case with an arbitrary bankruptcy point a, with $0 \le a < x_0$.

THEOREM 3.2. *The policy f_δ^*, given by*

(32)

$$f_\delta^*(x) = \begin{cases} f_V^*(x) & \text{for } a < x \le c/r - \delta, \\ \dfrac{2r}{\mu - r}\left[\epsilon/\left(H(\gamma, r, c)\left[V(x_0: a, c/r) - \epsilon\right]\right)\right]^{r/(\gamma+r)} & \text{for } x \ge c/r - \delta, \end{cases}$$

is an ϵ-optimal policy for maximizing the probability of crossing the c/r barrier before hitting the point a, starting from an initial wealth level x_0, where $a < x_0 < c/r$, and $V(\cdot: a, c/r)$ is the function given by (25).

3.2. Minimizing discounted penalty of bankruptcy. Suppose now that instead of minimizing the *probability* of ruin, we are instead interested in choosing a policy that *maximizes the time until bankruptcy*, in some sense. Obviously, this problem is nontrivial only in the danger-zone $a < x < c/r$, which is the case considered here. Maximizing the *expected* time until bankruptcy is a trivial problem, since there are any number of policies under which the expected time to bankruptcy is in fact infinite. In particular the ϵ-optimal policy described above gives a positive probability of reaching the c/r barrier, and since the safe-region ($x \ge c/r$) is absorbing, it therefore gives an infinite expected time to ruin. Thus we need to look at other criteria. Here we will consider the objective of minimizing the expected *discounted* time until bankruptcy (a related problem without forced withdrawals was treated in Dutta 1994 in a different framework, and in an incomplete market in Browne 1995). In particular, suppose there is a large penalty, say M, that must be paid if and when the ruin point a is hit. If there is a (constant) discount rate $\lambda > 0$, then the amount due upon hitting this point is therefore $Me^{-\lambda\tau_a^f}$, and we would like to find a policy that minimizes the expected value of this penalty. Clearly, this policy is equivalent to the policy that *minimizes* $E_x(e^{-\lambda\tau_a^f})$.

To that end, let $F(x) = \inf_{f \in \mathcal{F}} E_x(e^{-\lambda\tau_a^f})$, and let f_F^* denote the associated optimal policy, i.e., $f_F^* = \arg\inf_{f \in \mathcal{F}} E_x(e^{-\lambda\tau_a^f})$. For reasons that will become clear

soon, define the constants η^+ and D by

$$\eta^+ = \eta^+(\lambda) = \frac{1}{2r}\left[(r+\gamma+\lambda) + \sqrt{D}\right], \tag{33}$$

$$D = D(\lambda) = (\gamma+\lambda-r)^2 + 4r\gamma, \tag{34}$$

where γ is defined by (8). The optimal policy and optimal value function for this problem is then given in the following theorem.

THEOREM 3.3. *The optimal control is*

$$f_F^*(x) = \frac{\mu - r}{\sigma^2(\eta^+ - 1)}\left(\frac{c}{r} - x\right), \quad \text{for } a < x < c/r, \tag{35}$$

and the optimal value function is

$$F(x) = \left(\frac{c-rx}{c-ra}\right)^{\eta^+}, \quad \text{for } a \le x \le c/r. \tag{36}$$

REMARK 3.5. Note that $\eta^+ > 1$, and that $F(a) = 1$, $F(c/r) = 0$, and $F(x)$ is monotonically decreasing on the interval $(a, c/r)$, as is the optimal policy f_F^*, which once again invests a *constant proportion of the distance to the goal.* Observe too that we are therefore once again faced with the problem that under this policy, the safe-point c/r is inaccessible. However, in this case it is indeed the unique optimal policy. The condition $F(a) = 1$ is by construction, but the fact that $F(c/r) = 0$ is determined by the optimality equation itself, i.e., optimality determines that the c/r barrier is inaccessible. The intuition behind this is that this policy—although it never allows the fortune to cross the c/r-barrier—does indeed minimize the expected *discounted* time until ruin. The best one can do in this case is to get trapped in an asymptote approaching c/r, which this policy tries to do. Any additional investment near the c/r-barrier (such as in the ϵ-optimal strategy of the previous problem) allows a greater possibility of hitting a, thus increasing the value of $E_x(e^{-\lambda\tau_a})$.

REMARK 3.6. As a consistency check, note too that when we substitute the control f_F^* of (35) back into the evolutionary equation (3), we obtain a wealth process, say X_t^λ, that satisfies the stochastic differential equation

$$dX_t^\lambda = \left[\frac{2\gamma}{(\eta^+-1)r} - 1\right](c - rX_t^\lambda)\,dt + \frac{\mu - r}{\sigma(\eta^+-1)r}(c - rX_t^\lambda)\,dW_t, \quad \text{for } t < T^\lambda, \tag{37}$$

where $T^\lambda = \min\{\tau_a^\lambda, \tau_{c/r}^\lambda\}$, where $\tau_z^\lambda = \inf\{t > 0\colon X_t^\lambda = z\}$.

For this process, it is well known that the Laplace transform of τ_a^λ evaluated at the point λ, say $L(x\colon \lambda) = E_x(e^{-\lambda\tau_a^\lambda})$, is the unique solution of the Dirichlet problem

$$\left[\frac{2\gamma}{(\eta^+-1)r} - 1\right](c - rx)L_x + \frac{1}{2}\left(\frac{\mu - r}{\sigma(\eta^+-1)r}(c-rx)\right)^2 L_{xx} - \lambda L = 0,$$

with $L(a\colon \lambda) = 1$ and $L(c/r\colon \lambda) = 0$. It can be easily checked that we do in fact have $F(x) \equiv L(x\colon \lambda)$. It should be noted that τ_a^λ is a defective random variable, with $E_x(\tau_a^\lambda) = \infty$, as can be seen from the fact that

$$L(x\colon 0) = \left(\frac{c-rx}{c-ra}\right)^{\gamma/r+1} \equiv 1 - V(x\colon a, c/r),$$

where $V(x: \cdot, \cdot)$ is the function defined by (21). This of course is due to the fact that c/r essentially acts as an absorbing barrier, and it can be hit with positive probability (albeit in infinite time). Specifically, as $\lambda \downarrow 0$, it is clear that $\eta^+(\lambda) \to \eta^+(0) \equiv \gamma/r + 1$, and thus $F(x)$ converges (uniformly in x) to the probability that the bankruptcy point a is hit before the safe point c/r, which implies that the control $f_F^*(x)$ converges (uniformly in x) to the control $f_V^*(x)$ of (20), i.e. as $\lambda \downarrow 0$:

$$F(x) \to 1 - V(x: a, c/r) \quad \text{and} \quad f_F^*(x) \to \frac{2}{\mu - r}(c - rx) \equiv f_V^*(x).$$

Note also that for $\lambda > 0$, we have $f_V^* > f_U^*$, which is of course consistent with the fact that a "bolder" strategy maximizes the probability of survival, while a "timid" strategy maximizes expected playing time (for subfair games).

PROOF. Theorem 2.1 is again relevant, however since Theorem 2.1 deals with the maximization problem, recognize that $F = -\sup_f\{-E_x(e^{-\lambda\tau_a^f})\}$. We can now apply Theorem 2.1 to $\tilde{F} := -F$ with $\lambda(x) = \lambda$, $g = 0$, $h(a) = -1$. Reverting back to F, we then see that the nonlinear Dirichlet problem of (9) for F becomes then:

$$(rx - c)F_x - \gamma\frac{F_x^2}{F_{xx}} - \lambda F = 0, \quad \text{for } a < x < c/r, \tag{38}$$

subject to the Dirichlet boundary condition $F(a) = 1$, where $\gamma = \frac{1}{2}((\mu - r)/\sigma)^2$. Observe of course that we now require $F_x < 0$ and $F_{xx} > 0$.

The nonlinear second-order ordinary differential equation in (38) admits the two solutions $C(c - rx)^{\eta^+}$, and $K(c - rx)^{\eta^-}$, where C and K are constants to be determined from the boundary condition, and where η^+, η^- are the roots to the quadratic equation $\tilde{Q}(\eta) = 0$, where

$$\tilde{Q}(\eta) = \eta^2 r - \eta(\gamma + \lambda + r) + \lambda. \tag{39}$$

To determine which (if any) of these two solutions are appropriate we need to examine these roots in greater detail. The discriminant of (39) is the constant D of (34) which is clearly positive, and thus the two roots are real and, for $\lambda > 0$, distinct. In particular

$$\eta^+ = \frac{1}{2r}\left[(r + \gamma + \lambda) + \sqrt{D}\right] \quad \text{and} \quad \eta^- = \frac{1}{2r}\left[(r + \gamma + \lambda) - \sqrt{D}\right]. \tag{40}$$

Since $\eta^+\eta^- = \lambda/r > 0$, both roots are of the same sign, and since $\eta^+ > 0$, they are both positive. The boundary condition $F(a) = 1$ determines the constants C, K as $C = (c - ra)^{-\eta^+}$ and $K = (c - ra)^{-\eta^-}$ and so clearly $C > 0$ and $K > 0$, and therefore, $F_x < 0$ for both solutions. However it is easy to check the roots in (40) to see that $\eta^+ > 1$, while $\eta^- < 1$, and so $F_{xx} > 0$ only for the root η^+. Thus we find that the (unique) solution to the (38) that satisfies $F(a) = 1$ and $F_x < 0$ $F_{xx} > 0$ is given by the function $F(x)$ defined in (36). Moreover, it is a simple matter to check that conditions (i), (ii) and (iii) of Theorem 2.1 are indeed met for F (F is bounded on (a, b)), and so we may conclude that F is optimal. The associated optimal control function, f_F^* of (35), is then obtained by placing F (or $\tilde{F} = -F$) into (11). □

REMARK 3.7. An alternative proof of Theorem 3.4 can be constructed by modifying the arguments in Orey et al. (1987), who treat the converse problem of maximizing discounted time to a goal, to deal with the minimization problem treated here.

The evolutionary equation (3) would have to be reparameterized by taking $f_t = \pi_t \cdot (c/r - X_t^f)$, for admissible control processes π, and then applying the results of Orey et al. (1987) to the further transformed process $Y_t^\pi = \ln[(c - rX_t^\pi)/(c - rb)]$.

Note that since the wealth process, say X_t^λ, under the policy f_F^*, satisfies the stochastic differential equation (37), we can apply Ito's formula to the function $F(\cdot)$ given by (36) to show, after simplification, that

(41)

$$dF(X_t^\lambda) = F(X_t^\lambda)\left[\frac{(\eta^+)^2 r - \eta^+(r+\gamma)}{\eta^+ - 1}dt - \sqrt{2\gamma}\,\frac{\eta^+}{\eta^+ - 1}dW_t\right], \quad \text{for } 0 < t < T^\lambda.$$

The quadratic of (39) then shows that $(\eta^+)^2 r - \eta^+(r+\gamma) = \lambda(\eta^+ - 1)$, and thus substituting this into the r.h.s. of (41), and then solving the resulting (linear) stochastic differential equation gives

$$F(X_t^\lambda) = F(X_0)\exp\left\{\left(\lambda - \gamma\left(\frac{\eta^+}{\eta^+ - 1}\right)^2\right)t - \sqrt{2\gamma}\,\frac{\eta^+}{\eta^+ - 1}W_t\right\}, \quad \text{for } t \le T^\lambda,$$

which shows that the value function $F(\cdot)$ operating on the process X_t^λ is a *geometric Brownian motion* on the interval $(0, 1)$, for $a \le X_t^\lambda \le c/r$.

Unfortunately, as noted above, this policy, while optimal for the stated problem, *will never cross the c/r barrier into the safe-region*, and thus the investor should utilize a policy similar to the ϵ-optimal policy described earlier to get into the safe-region. Since this can be achieved at relatively little cost, we will assume for the sequel that the investor does in fact invest in a way that will allow a positive probability of getting into the safe-region. When (if) the safe-region is achieved, the investor no longer faces the problem of bankruptcy, and should then be concerned with other optimality criteria. We consider two such criteria in the next section.

4. Optimal growth policies, in the safe-region. Suppose now that we have survived, i.e., we have achieved a level $x > c/r$. As noted earlier it is clear that in this region there need never be a possibility of ruin, and therefore the investor who has achieved this safe-region will be interested in criteria other than survival. In particular, we assume here that in this region the investor is interested in *growth*, by which we mean achieving a high level of wealth as quickly as possible. Suppose therefore that there is now some target goal, which we will denote again by b with $b > x$, which the investor wants to get to (e.g., to pay out dividends) as quickly as possible. In this section we consider two related aspects of this problem. First we consider the problem of *minimizing the expected time to the goal*, and then we consider the related problem of *maximizing the expected discounted reward of achieving the goal*. In both cases, the optimal strategies are interesting generalizations of the *Kelly criterion* that has been studied in discrete-time in Kelly (1956), Breiman (1961) and Thorp (1969), and in continuous-time in Pestien and Sudderth (1985) and Heath et al. (1987). (See also Theorem 6.5 in Merton 1990, where it is called the *growth-optimum strategy*. For Bayesian versions of both the discrete and continuous-time Kelly criterion, see Browne and Whitt 1996.) Such policies dictate investing a *constant multiple* of the wealth in the risky stock. Here our policies invest a constant multiple of the *excess wealth* over the boundary c/r, in the risky stock. This will make the c/r boundary inaccessible from above, ensuring that the investor will stay in the safe-region forever, almost surely.

4.1. Minimizing the time to a goal. To formalize this, let $X_0 = x$, but now with $c/r < x < b$. For $\tau_b^f := \inf\{t > 0: X_t^f = b\}$, let

$$G(x) = \inf_{f \in \mathcal{G}} E_x(\tau_b^f), \quad \text{and let} \quad f_G^* = \arg\inf_{f \in \mathcal{G}} E_x(\tau_b^f).$$

THEOREM 4.1. *For the problem of minimizing the expected time to the goal b, the optimal policy is to invest*

$$f_G^*(x) = \frac{\mu - r}{\sigma^2}\left(x - \frac{c}{r}\right), \quad \text{for } c/r < x < b. \tag{42}$$

The optimal value function is

$$G(x) = \frac{1}{r + \gamma}\ln\left(\frac{rb - c}{rx - c}\right), \quad \text{for } c/r < x \leq b. \tag{43}$$

REMARK 4.1. Note that the proportion $(\mu - r)/\sigma^2$ in (42) is the same proportion as in the ordinary continuous-time Kelly criterion (or optimal growth policy) (see Heath et al. 1987, Merton 1990, Browne and Whitt 1996). However in our policy f_G^*, this proportion operates only on *the excess wealth over the boundary* c/r. Under this policy therefore, the lower boundary, c/r is inaccessible. It is quite interesting to observe that this policy is independent of the goal b. This is quite remarkable, since while it was to be expected a priori that the optimal policy should look something like (42) near the point c/r, which ensures that c/r is inaccessible from above, it is not clear why one should expect such behavior to continue throughout even when the wealth is far away from c/r. Nevertheless, it appears that the best one can do is to simply put c/r into the safe asset, and leave it there forever, continuously compounding at rate r. This is the endowment which will finance the withdrawal at the constant rate c forever. (Recall, $c/r = c\int_0^\infty e^{-rt}\,dt$.) Once this is done, the optimal policy then plays the best *ordinary* optimal growth game with the remainder of the wealth, $x - c/r$. This policy is quite similar to the policy prescribed in Proposition 11 of Black and Perold (1992) as a form of CPPI (see also Dybvig 1995). Thus, we have shown that CPPI has another optimality property associated with it, namely that of *optimal growth*.

PROOF. Since here we are minimizing expected time, we could apply Theorem 2.1 to $\tilde{G}(x) = \sup_f\{-E_x(\tau_b^f)\}$, with $g(x) = -1$, $\lambda = 0$, $h(b) = 0$. Recognizing that $G = -\tilde{G}$, it is then seen that in terms of G, Theorem 2.1 now requires $G_x < 0$ and $G_{xx} > 0$, and that the nonlinear Dirichlet problem of (9) specializes to

$$(rx - c)G_x - \gamma\frac{G_x^2}{G_{xx}} + 1 = 0, \quad \text{for } c/r < x < b, \tag{44}$$

subject to the boundary condition $G(b) = 0$. It is readily verified that the function G of (43) satisfies this, and that moreover for this function we have $G_x < 0$ and $G_{xx} > 0$ for all $c/r < x < b$. The control function $f_G^*(x)$ of (42) is obtained by substituting G of (43) for ν in (11). However, note that while it is easy to see that conditions (i) and (iii) of Theorem 2.1 are satisfied by G, it is also clear that G is *unbounded* on $(c/r, b)$, since $G(x) \to \infty$ as $x \downarrow c/r$. Thus it is doubtful that condition (ii) of

Theorem 2.1 holds for this case. Nevertheless, we will show that Theorem 4.1 holds and f_G^* is indeed the optimal policy, however the final proof of this awaits the development in §4.2, and we will complete the proof there after Lemma 4.3. □

Note that when we substitute the control f_G^* of (42) back into the evolutionary equation (3), we obtain an (optimal) wealth process, say X^b, that satisfies

$$(45)\quad dX_t^b = (r + 2\gamma)\Big(X_t^b - \frac{c}{r}\Big)\,dt + \sqrt{2\gamma}\Big(X_t^b - \frac{c}{r}\Big)\,dW_t, \quad \text{for } 0 < t < \tau_b^*,$$

where $\tau_b^* := \inf\{t > 0\colon X_t^b = b\}$, which is again a linear stochastic differential equation. (It is clear from this that c/r is in fact an inaccessible *lower* boundary for X^b.) The solution to (45) is

$$X_t^b = \Big(X_0^b - \frac{c}{r}\Big)\exp\Big\{(r + \gamma)t + \sqrt{2\gamma}W_t\Big\} + \frac{c}{r}, \quad \text{for } 0 \le t < \tau_b^*,$$

from which it follows that

$$(46)\qquad G\big(X_t^b\big) = G(X_0) - t - \frac{\sqrt{2\gamma}}{r + \gamma}W_t, \quad \text{for } 0 \le t < \tau_b^*,$$

i.e., under the (optimal) policy f_G^*, the process $\{G(X_t^b) - G(X_0)\}$, follows a simple Brownian motion on $(0, \infty)$ with a drift coefficient equal to -1. (From this it is easy to recover the value function (43) from (46) by evaluating the expected value of (46) at $t = \tau_b^*$ using the fact that $G(b) = 0$, which then gives $E_x(\tau_b^*) = G(x)$.)

REMARK 4.2. The "minimal time to a goal" problem for the case $c = 0$ was first solved in the fundamental paper of Heath et al. (1987) without direct recourse to HJB methods (see also Schäl 1993). The result in that case is simply (42) with $c = 0$ (see §4 in Heath et al. 1987). Merton (1990, Theorem 6.5), also obtained this policy via another, rather complicated, argument. Since the proof given here holds too for the case $c = 0$, our results also provide an alternative and complementary proof for that case to the ones in Heath et al. (1987) and Merton (1990).

In fact, it is possible to apply the results of Heath et al. (1987) to construct a different proof of Theorem 4.1. First one would need to reparameterize the wealth equation (3) by taking $f_t = \pi_t \cdot (X_t^f - c/r)$, and then applying results of Heath et al. (1987) to the further transformed process $Y_t^\pi = \ln[(rX_t^\pi - c)/(rb - c)]$. However the results in Heath et al. (1987) are specific to the case where the controls must lie on a given constant set that is independent of the current wealth, while the approach here, based on the HJB methods of Theorem 2.1 could be modified to allow for a state dependent opportunity set.

4.2. Maximizing expected discounted reward of achieving the goal. Suppose now that instead of minimizing the expected time to the goal b, we are instead interested in maximizing $E_x(e^{-\lambda\tau_b^f})$, for $c/r < x \le b$. To that end let

$$U(x) = \sup_{f\in\mathcal{G}} E_x\big(e^{-\lambda\tau_b^f}\big), \quad \text{and let} \quad f_U^*(x) = \arg\sup_{f\in\mathcal{G}} E_x\big(e^{-\lambda\tau_b^f}\big).$$

As we show in the following theorem, the optimal policy for this problem also invests a (different) constant proportion of the excess wealth above the c/r barrier, and is hence another version of the CPPI strategy as in Black and Perold (1992).

THEOREM 4.2. *The optimal control is*

$$f_U^*(x) = \frac{\mu - r}{\sigma^2(1 - \eta^-)}\left(x - \frac{c}{r}\right), \quad \text{for } c/r < x < b, \tag{47}$$

and the optimal value function is

$$U(x) = \left(\frac{rx - c}{rb - c}\right)^{\eta^-}, \quad \text{for } c/r \le x \le b, \tag{48}$$

where $\eta^- \equiv \eta^-(\lambda)$ *was defined previously in* (40).

REMARK 4.3. Recall that η^- is the root that satisfies $0 \le \eta^- < 1$ to the quadratic equation $\tilde{Q}(\eta) = 0$, where $\tilde{Q}(\cdot)$ is given in (39). Note that $U(b) = 1, U(c/r) = 0$, with $U(x)$ monotonically increasing on $(c/r, b)$. As was the case earlier in §3, the fact that $U(b) = 1$ is by construction, but it is optimality that causes $U(c/r) = 0$, and hence makes the danger-zone inaccessible from the safe-region.

PROOF. The proof is essentially the same as for Theorem 3.3. Specifically, here Theorem 2.1 applies directly with $u = b$, $\lambda(x) = \lambda > 0$, $g = 0$ and $h(b) = 1$. Thus the nonlinear Dirichlet problem of (9) for this case specializes to

$$(rx - c)U_x - \gamma\frac{U_x^2}{U_{xx}} - \lambda U = 0, \quad \text{for } c/r < x < b, \tag{49}$$

subject to the boundary condition $U(b) = 1$. Since we require $U_x > 0$ and $U_{xx} < 0$, it is clear the solution of interest here involves the smaller root, η^-, to the quadratic $\tilde{Q}(\eta) = 0$ (see (39)), since $\eta^- < 1$. The control function $f_U^*(x)$ of (47) is then obtained by substituting U of (48) for ν into (11). Finally, it is easy to check that U of (48) satisfies conditions (i), (ii) and (iii) of Theorem 2.1, and we may therefore conclude that f_U^* is indeed optimal. □

It is interesting to observe that when we place the control f_U^* back into the evolutionary equation (3), we find that the resulting optimal wealth process, say $\tilde{X}_t^\lambda$, satisfies the stochastic differential equation

(50)

$$d\tilde{X}_t^\lambda = \left[\frac{2\gamma}{(1 - \eta^-)r} + 1\right]\left(r\tilde{X}_t^\lambda - c\right)dt + \frac{\sqrt{2\gamma}}{(1 - \eta^-)r}\left(r\tilde{X}_t^\lambda - c\right)dW_t, \quad \text{for } t < \tau_b^\lambda$$

where $\tau_b^\lambda = \inf\{t > 0\colon \tilde{X}_t^\lambda = b\}$. An application now of Ito's formula to the function $U(\cdot)$ of (48) using (50) (and (39)) gives

$$U\left(\tilde{X}_t^\lambda\right) = U(X_0)\exp\left\{\left(\lambda - \gamma\left(\frac{\eta^-}{1 - \eta^-}\right)^2\right)t + \sqrt{2\gamma}\,\frac{\eta^-}{1 - \eta^-}W_t\right\}, \quad \text{for } t \le \tau_b^\lambda,$$

which shows that the value function $U(\cdot)$ operating on the process $\tilde{X}_t^\lambda$ is a *geometric Brownian motion* on the interval $(0, 1)$, for $c/r < \tilde{X}_t^\lambda < b$.

REMARK 4.4. Orey, Pestien and Sudderth (1987), using different methods, studied some general goal problems with a similar objective as that considered here, and as a particular example study a version of our problem with $r = c = 0$ (Orey Pestien and Sudderth 1987, page 1258). An alternative proof of Theorem 4.3 can therefore be

constructed by using the results of Orey et al. (1987) using the transformation and reparameterization described above in Remark 4.2.

We may now use the results of Theorem 4.2 to complete the proof of Theorem 4.1. However, we first need the following lemma, which is of independent interest since it is applicable to more general processes than those considered here (for related results, see Schäl 1993, §4).

LEMMA 4.3. *Suppose that for every* $\lambda > 0$, *we have*

$$v(x;\lambda) = \inf_f E_x\left(\frac{1 - e^{-\lambda\tau^f}}{\lambda}\right), \quad \textit{with optimal control } f^*(x;\lambda),$$

with $v(x;\lambda) < \infty$, $\lim_{\lambda \downarrow 0} v(x;\lambda) = v(x;0) < \infty$, *and* $\lim_{\lambda\downarrow 0} f^*(x;\lambda) = f(x;0)$.
Then

$$\lim_{\lambda\downarrow 0} \inf_f E_x\left(\frac{1 - e^{-\lambda\tau^f}}{\lambda}\right) = \inf_f E_x\left(\lim_{\lambda\downarrow 0} \frac{1 - e^{-\lambda\tau^f}}{\lambda}\right) \equiv \inf_f E_x(\tau^f), \tag{51}$$

with $\inf_f E_x(\tau^f) = v(x;0)$ *and with optimal control* $f(x;0)$.

PROOF. It is the first equality in (51) that needs to be established since the second is just an identity. To proceed, it is obvious that $\lambda^{-1}[1 - e^{-\lambda\tau^f}] \le \tau^f$ for all $\lambda \ge 0$, and hence $E_x(\lambda^{-1}[1 - e^{-\lambda\tau^f}]) \le E_x(\tau^f)$, as well as $\inf_f E_x(\lambda^{-1}[1 - e^{-\lambda\tau^f}]) \le \inf_f E_x(\tau^f)$. Since the r.h.s. of this inequality is independent of the parameter λ, it follows that we may take limits on λ to get

$$\lim_{\lambda\downarrow 0} \inf_f E_x\left(\lambda^{-1}\left[1 - e^{-\lambda\tau^f}\right]\right) \le \inf_f E_x(\tau^f). \tag{52}$$

For notational convenience now, let f_λ^* denote the policy $f^*(\cdot;\lambda)$, and for any policy f, let $\tau[f] = \tau^f$. Note that under this notation, we may write $v(x;\lambda) = E_x(\lambda^{-1}[1 - e^{-\lambda\tau[f_\lambda^*]}])$.

To go the other way now, suppose that there is an admissible policy, say $\tilde{f}$, such that $\tau[f_\lambda^*] \to^{a.s.} \tau[\tilde{f}]$ as $\lambda \downarrow 0$. Then it is clear that

$$\inf_f E_x(\tau^f) \le E_x\left(\tau\left[\tilde{f}\right]\right) = E_x\left(\lim_{\lambda\downarrow 0} \lambda^{-1}[1 - e^{-\lambda\tau[f_\lambda^*]}]\right).$$

An application of Fatou's lemma then shows that

$$E_x\left(\lim_{\lambda\downarrow 0} \lambda^{-1}[1 - e^{-\lambda\tau[f_\lambda^*]}]\right) \le \lim_{\lambda\downarrow 0} E_x(\lambda^{-1}[1 - e^{-\lambda\tau[f_\lambda^*]}]),$$

and since $E_x(\lambda^{-1}[1 - e^{-\lambda\tau[f_\lambda^*]}]) \equiv \inf_f E_x(\lambda^{-1}[1 - e^{-\lambda\tau^f}])$, we in turn conclude that

$$\inf_f E_x(\tau^f) \le \lim_{\lambda\downarrow 0} \inf_f E_x\left(\lambda^{-1}\left[1 - e^{-\lambda\tau^f}\right]\right). \tag{53}$$

The inequalities (52) and (53) yield (51). □

COMPLETION OF PROOF OF THEOREM 4.1. Observe first that $\eta^-(\lambda) \to 0$ as $\lambda \downarrow 0$, and that therefore $f_U^* \to f_G^*$ as $\lambda \downarrow 0$, where f_U^* and f_G^* are given by (47) and (42).

Note further that for any $c/r < x < b$, we have

$$\lim_{\lambda \downarrow 0} \frac{1 - U(x)}{\lambda} = G(x),$$

where U and G are given by (48) and (43). Finally, since $\eta^-(\lambda) \to 0$ as $\lambda \downarrow 0$, we have $X^b_{t \wedge \tau_b} \to^{\text{a.s.}} \tilde{X}^\lambda_{t \wedge \tau^\lambda_b}$ as $\lambda \downarrow 0$, where X^b_t and $\tilde{X}^\lambda_t$ are defined by (45) and (50), from which it is clear that $\tau^\lambda_b \to^{\text{a.s.}} \tau_b$ as $\lambda \downarrow 0$. Therefore, Lemma 4.3 may be applied directly to Theorem 4.2 to deduce Theorem 4.1. □

5. The multiple asset case. As promised earlier, here we show how all of our previous results extend in a very straightforward way to the case with multiple risky stocks. The model here is that of a complete market (as in, e.g., Karatzas and Shreve 1988) where there are n risky assets generated by n independent Brownian motions. The prices of these stocks evolve as

$$dP_i(t) = P_i(t)\left[\mu_i\, dt + \sum_{j=1}^{n} \sigma_{ij}\, dW_t^{(j)}\right], \qquad i = 1, \ldots, n, \tag{54}$$

while the riskless asset, B_t, still evolves as $dB_t = rB_t\, dt$. The wealth of the investor therefore evolves as

$$dX^f_t = \left[rX^f_t - c + \sum_{i=1}^{n} f_i(\mu_i - r)\right] + \sum_{i=1}^{n}\sum_{j=1}^{n} f_i \sigma_{ij}\, dW_t^{(j)}, \tag{55}$$

where now f_i denotes the total amount of money invested in the ith stock.

If we introduce now the matrix $\boldsymbol{\sigma} = (\sigma)_{ij}$, and the (column) vectors $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)^T$, $\mathbf{f} = (f_1, \ldots, f_n)^T$, and then set $\mathbf{A} = \boldsymbol{\sigma}\boldsymbol{\sigma}^T$, we may write the generator of the (one-dimensional) wealth process, for functions $\Psi(x) \in \mathscr{C}^2$ as

$$\mathscr{A}^{\mathbf{f}}\Psi(x) = \left(\mathbf{f}^T(\boldsymbol{\mu} - r\mathbf{1}) + rx - c\right)\Psi_x + \frac{1}{2}\mathbf{f}^T\mathbf{A}\mathbf{f}\Psi_{xx}, \tag{56}$$

where $\mathbf{1}$ denotes a vector of 1's. The assumption of completeness implies that $\mathbf{A}^{-1}$ exists, and thus all our results will go through exactly as before. In particular, if an optimal value function for a specific problem is denoted by $\nu(x)$, the optimal control vector is $\mathbf{f}^*_\nu(x)$ where

$$\mathbf{f}^*_\nu(x) = -\mathbf{A}^{-1}(\boldsymbol{\mu} - r\mathbf{1})\frac{\nu_x}{\nu_{xx}}. \tag{57}$$

The differential equations ((22), (38), (44) and (49)), and hence the value functions ((21), (36), (43) and (48)), all remain the same except for the fact that now the scalar γ is evaluated as

$$\gamma = \frac{1}{2}(\boldsymbol{\mu} - r\mathbf{1})^T\mathbf{A}^{-1}(\boldsymbol{\mu} - r\mathbf{1}). \tag{58}$$

It is interesting to note that for the problem of maximizing the probability of reaching b before a when b is in the danger-zone, considered in §3.1, the optimal policy now *does depend on the variances and covariances* of the risky assets, since

instead of (20), in the multiple asset case we now get

$$\mathbf{f}_V^*(x) = \mathbf{A}^{-1}(\boldsymbol{\mu} - r\mathbf{1})\frac{r}{\gamma}\left(\frac{c}{r} - x\right). \tag{59}$$

The ϵ-optimal policy of §3 needs to be modified, but the extension is straightforward and we leave the details for the reader. For reference, we note further that if we define the vector $\mathbf{K}$ by $\mathbf{K} := \mathbf{A}^{-1}(\boldsymbol{\mu} - r\mathbf{1})$, then the optimal controls (35), (42) and (47) of §§3.2, 4.1 and 4.2 become, respectively

$$\mathbf{f}_F^*(x) = \mathbf{K}(\eta^+ - 1)^{-1}\left(\frac{c}{r} - x\right), \qquad \mathbf{f}_G^*(x) = \mathbf{K}\left(x - \frac{c}{r}\right), \tag{60}$$

$$\mathbf{f}_U^*(x) = \mathbf{K}(1 - \eta^-)^{-1}\left(x - \frac{c}{r}\right).$$

6. Linear withdrawal rate. In this section we show how all of our previous results and analysis for the case of forced withdrawals at the *constant* rate $c > 0$ can be generalized to the case where there is a *wealth-dependent* withdrawal rate, $c(x)$ where

$$c(x) = c + \theta x.$$

Here we will only consider the case where $0 \le \theta < r$. For notational ease, we will consider again only the case with one risky stock. The generalization to the multiple stock case as in the previous section is very straightforward, and so we leave the details for the reader.

For this case the evolutionary equation (3) becomes

$$\begin{aligned} dX_t^f &= f_t\frac{dP_t}{P_t} + \left(X_t^f - f_t\right)\frac{dB_t}{B_t} - \left(c + \theta X_t^f\right)dt \\ &= \left[(r - \theta)X_t^f + f_t(\mu - r) - c\right]dt + f_t\sigma\, dW_t. \end{aligned} \tag{61}$$

If we now define $\tilde{r} := r - \theta > 0$, then for Markov control processes f, and $\Psi \in \mathscr{C}^2$ the generator of the wealth process is

$$\tilde{\mathscr{A}}^f\Psi(x) = \left[f(\mu - r) + \tilde{r}x - c\right]\Psi_x + \frac{1}{2}f^2\sigma^2\Psi_{xx}. \tag{62}$$

The parameter $\tilde{r}$ is simply the *adjusted* (*risk-free*) *compounding rate*. Essentially, nothing really changes except for the fact that the danger-zone is now the region $x < c/\tilde{r}$, and the safe-region is its complement. The differential equations (22), (38), (44) and (49) all remain the same except for the fact that we must replace $rx - c$ with $\tilde{r}x - c$. The parameter γ in all those equations, as well as here, is still defined as in (8), i.e., $\gamma = \frac{1}{2}((\mu - r)/\sigma)^2$, where r is the *standard* interest rate. Thus, the previous analysis will go through with relatively little change, and so we will only point out the essential differences. In particular, the structure of the policies remain the same, in that the optimal *survival* policies of §3 invest a fixed proportion of the distance to the (new) safe-region barrier, $c/\tilde{r}$, while the optimal *growth* policies of §4 invest a fixed proportion of the excess of wealth over the barrier.

6.1. Survival problems in the danger-zone. The analysis of §§3.1 and 3.2 can be repeated almost verbatim. What changes is that now for $a < x < b < c/\tilde{r}$, the value

function $V(x; a, b)$ of (21) becomes instead

$$V(x: a, b) = \frac{(c - \tilde{r}a)^{\gamma/\tilde{r}+1} - (c - \tilde{r}x)^{\gamma/\tilde{r}+1}}{(c - \tilde{r}a)^{\gamma/\tilde{r}+1} - (c - \tilde{r}b)^{\gamma/\tilde{r}+1}}, \quad \text{for } a \le x \le b. \tag{63}$$

Since (11) still holds, the resulting optimal control becomes, instead of (20),

$$f_V^*(x) = \frac{2\tilde{r}}{\mu - r}\left(\frac{c}{\tilde{r}} - x\right). \tag{64}$$

For the discounted problem of §3.2 (as well as for the discounted problem of §4.2), the quadratic $\tilde{Q}(\cdot)$ of (39) changes to $\tilde{Q}(\eta) = \eta^2\tilde{r} - \eta(\gamma + \lambda + \tilde{r}) + \lambda$, and thus the two (real) roots to $\tilde{Q}(\eta) = 0$, denoted by $\tilde{\eta}^+$ and $\tilde{\eta}^-$, become, instead of (40),

$$\tilde{\eta}^{+,-} = \frac{1}{2\tilde{r}}\left[(\tilde{r} + \gamma + \lambda) \pm \sqrt{(\gamma + \lambda + \tilde{r})^2 - 4\tilde{r}\lambda}\right]. \tag{65}$$

It is easy to check that once again, we have $0 \le \tilde{\eta}^- < 1 < \tilde{\eta}^+$, and so the optimal value function (36) and the optimal control function (35) become, respectively

$$F(x) = \left(\frac{c - \tilde{r}x}{c - \tilde{r}a}\right)^{\tilde{\eta}^+}, \qquad f_F^*(x) = \frac{\mu - r}{\sigma^2(\tilde{\eta}^+ - 1)}\left(\frac{c}{\tilde{r}} - x\right). \tag{66}$$

6.2. Growth policies in the safe-region. Once again, the analysis is almost identical to that in §§4.1 and 4.2. The value and the optimal control functions for the minimal expected time to the goal problem, (43) and (42) are replaced respectively by

$$G(x) = \frac{1}{\tilde{r} + \gamma}\ln\left(\frac{\tilde{r}b - c}{\tilde{r}x - c}\right), \qquad f_G^*(x) = \frac{\mu - r}{\sigma^2}\left(x - \frac{c}{\tilde{r}}\right), \quad \text{for } c/\tilde{r} < x < b. \tag{67}$$

Note that the optimal (Kelly) proportion of the excess wealth invested in the stock, $(\mu - r)/\sigma^2$, is unchanged in this case.

Similarly for the discounted problem considered in §4.2, the value function (48) and optimal policy (47) become

$$U(x) = \left(\frac{\tilde{r}x - c}{\tilde{r}b - c}\right)^{\tilde{\eta}^-}, \qquad f_U^*(x) = \frac{\mu - r}{\sigma^2(1 - \tilde{\eta}^-)}\left(x - \frac{c}{\tilde{r}}\right). \tag{68}$$

The case where $\theta > r$, and hence with $\tilde{r} < 0$, introduces new difficulties that will be discussed elsewhere.

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