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STOCHASTIC DIFFERENTIAL PORTFOLIO GAMES

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Abstract

We study stochastic dynamic investment games in continuous time between two investors (players) who have available two different, but possibly correlated, investment opportunities. There is a single payoff function which depends on both investors' wealth processes. One player chooses a dynamic portfolio strategy in order to maximize this expected payoff, while his opponent is simultaneously choosing a dynamic portfolio strategy so as to minimize the same quantity. This leads to a stochastic differential game with controlled drift and variance. For the most part, we consider games with payoffs that depend on the achievement of relative performance goals and/or shortfalls. We provide conditions under which a game with a general payoff function has an achievable value, and give an explicit representation for the value and resulting equilibrium portfolio strategies in that case. It is shown that non-perfect correlation is required to rule out trivial solutions. We then use this general result explicitly to solve a variety of specific games. For example, we solve a probability maximizing game, where each investor is trying to maximize the probability of beating the other's return by a given predetermined percentage. We also consider objectives related to the minimization or maximization of the expected time until one investor's return beats the other investor's return by a given percentage. Our results allow a new interpretation of the market price of risk in a Black–Scholes world. Games with discounting are also discussed, as are games of fixed duration related to utility maximization.

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1. Introduction

This paper treats various versions of stochastic differential games as played between two 'small' investors, call them A and B. (The investors are called small in that their portfolio trading strategies do not affect the market prices of the underlying assets.) The games considered here are zero-sum, in that there is a single payoff function, with one investor trying to maximize this expected payoff, while simultaneously the other investor is trying to minimize the same quantity. There are two correlated risky investment opportunities, only one of which is available to each investor. The players compete by the choice of their individual dynamic portfolio trading strategy in the risky asset available to them and a risk-free asset that is freely available to both. There is complete revelation, or observation, in that A's strategy is instantaneously observed by B (without error) and vice versa.

For the most part, the games we consider have discontinuous payoffs where Investor A wins if his fortune ever exceeds Investor B's fortune by some predetermined amount, and similarly,

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Investor B wins the game if his fortune ever exceeds Investor A's fortune by some (possibly other) predetermined amount. As we show later, we require non-perfect correlation between the investment opportunities so as to rule out trivial solutions to our games. Specifically, if the investment opportunities available to A and B are the same, then in any of our continuous-time stochastic differential games with perfect revelation, any move by Investor A can be immediately reacted to, and perfectly adjusted for, by Investor B, thus heading off any movement in the state variable. Thus, in our setting, the only interesting games are those where there is non-perfect correlation between the investment opportunities, allowing non-perfect adjustment and reaction between the players.

Aside from the intrinsic probabilistic and game-theoretic interest, such a model is applicable in many economic settings. For example, our results have significant bearing on what is sometimes referred to as active portfolio management, where the objective of an individual investor is to beat the performance of a preselected benchmark portfolio (see e.g. Browne (1999)). While the chosen benchmark is most often a wealth process obtained from a known *deterministic* portfolio strategy (e.g. an index, such as the S&P 500), our results would provide a worst case and minimax analysis for how the benchmark would perform in a game-theoretic setting. These results could then be used in turn, for example, to set conservative capital requirements for a given preassigned maximally acceptable probability of underperformance relative to that benchmark.

Another, perhaps more direct, example occurs in many trading firms, where each individual stock, or sector of stocks, is assigned to its own individual trader. Our model is then applicable to an analysis of the performance of these traders when a component of their compensation is determined by the achievement of relative goals, for example a bonus for the 'best' performer (the winner of the game), and/or a penalty, such as termination, for the worst performer (the loser). Similarly, our results, are of interest in a partial analysis of the competition played out between two fund managers, whose funds are invested in different markets and have different characteristics, who achieve rewards based on the relative performance of their funds.

Finally, we also note that our results also allow new interpretations of the *market price of risk* of an asset in a Black–Scholes world, in that we show that the degree of advantage a player has over the other is determined solely by the market price of risk of his investment opportunity.

An outline of the remainder of the paper, as well as a summary of our main results, are as follows. In the next section, we describe the formal model under consideration here. There are two correlated stocks as well as a risk-free asset called a bond. Each investor can invest freely in the risk-free asset but is allowed to invest in only one of the stocks, according to any admissible dynamic portfolio strategy. His opponent can also invest freely in the risk-free asset, but only in the other stock according to any admissible dynamic portfolio strategy. We then describe how the investors compete. The relevant state variable is the ratio of the two investors, and the game terminates when this ratio first exits an interval.

In Section 3, we provide a general result in optimal control for a stochastic differential game with a general payoff function, in the context of our model. Specifically, we characterize conditions under which the value of this game will be the smooth solution to a particular non-linear Dirichlet problem. The equilibrium, or competitively optimal, controls are then given by an explicit expression involving the derivatives of this value function. We then solve these Dirichlet problems explicitly for various specific examples in subsequent sections. The proof of Theorem 3.1 is presented in the final section of the paper.

In Section 4 we consider the *probability maximizing game*, where Investor A is trying to maximize the probability of outperforming Investor B by a given percentage, before Investor B outperforms him by another given percentage. It turns out that a value for this game exists if and only if a specific *measure of advantage* parameter, which is defined here as the ratio of the market price of risk for A's investment opportunity over the market price of risk for B's investment opportunity, takes values in a particular interval. This interval is determined solely by the instantaneous correlation between the investment opportunities. If this condition is met, then we give explicit solutions for the equilibrium portfolio strategies. Among other results, we show that the disadvantaged player has a relatively bolder strategy than the player who holds the advantage, as would be expected from the classical results of Dubins and Savage (1965) for single-player probability maximizing games. For the symmetric case, where no player holds the advantage, the equilibrium strategies reduce to the growth-optimal strategy.

In Section 5 we consider games where the objective is to minimize the expected time to outperform the other player. There are two cases to consider, depending on which player has the advantage. In the symmetric case, the games do not have a finite value. In the non-symmetric case, the equilibrium portfolio strategies are the individual growth-optimal strategies, and a new connection is made with maximizing logarithmic utility.

In Section 6 we consider games with discounting, where the objective of one player is to maximize the discounted reward achieved upon outperforming his opponent. For this game to have a value, we require a greater degree of advantage to exist than was required for the probability maximizing game.

In Section 7 we consider fixed-duration utility-based games, where both investors obtain utility (or disutility) solely on the basis of their relative wealth, i.e. in terms of their ratio. The value for such games is then given (under appropriate conditions) as the solution to a particular non-linear Cauchy problem, and the saddlepoints, or competitively optimal control functions, are obtained in terms of the derivatives of this value function. An explicit solution is given for the case of power utility.

2. The portfolio model with competition

The model under consideration here consists of three underlying processes: two correlated risky investment opportunities (e.g. stocks, or mutual funds) $S^{(1)}$ and $S^{(2)}$, and a riskless asset B called a bond. The price processes for these assets will be denoted, respectively, by $\{S_t^{(1)}, S_t^{(2)}, B_t,\ t \geq 0\}$. While we allow both investors to invest freely in the risk-free asset, Investor A may trade only in the first stock, $S^{(1)}$, and similarly, Investor B may trade only in the second stock, $S^{(2)}$. While there are only two correlated risky assets in our model, it is without any loss of generality since it is just a simple matter of algebra to generalize our results and analysis to a constant coefficients complete market model (see Duffie (1996)) with n risky stocks driven by n Brownian motions, for any arbitrary $n > 2$. In that case, we would split the n stocks into two groups, say with the first k stocks available to Investor A and the remaining $n-k$ stocks available to Investor B, with A being restricted from trading in B's group and vice versa for B. However, for notational and expositional ease, we consider just the (essentially equivalent) two-asset case.

The probabilistic setting is as follows: we are given a filtered probability space

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathrm{P}),$$

supporting two correlated Brownian motions, $W^{(1)},\ W^{(2)}$, with $\mathrm{E}(W_t^{(1)} W_t^{(2)}) = \rho t$. (Specifically, $\mathcal{F}_t$ is the P-augmentation of the natural filtration $\mathcal{F}_t^W := \sigma\{W_s^{(1)}, W_s^{(2)};\ 0 \leq s \leq t\}$.)

We will assume that the price process for each of the risky stocks follows a geometric Brownian motion, i.e. $S_t^{(i)}$ satisfies the stochastic differential equation

$$dS_t^{(i)} = \mu_i S_t^{(i)}\,dt + \sigma_i S_t^{(i)}\,dW_t^{(i)}, \qquad \text{for} \quad i = 1, 2 \tag{2.1}$$

where μ_i, $i = 1, 2$ are positive constants. The price of the risk-free asset is assumed to evolve according to

$$dB_t = rB_t\,dt \tag{2.2}$$

where $r \geq 0$. To avoid triviality, we assume $\mu_i > r$, for $i = 1, 2$.

For the sequel, let the parameter θ_i denote the risk-adjusted excess return of stock $S^{(i)}$ over the risk-free rate of return, for $i = 1, 2$. Specifically,

$$\theta_i = \frac{\mu_i - r}{\sigma_i}, \qquad \text{for} \quad i = 1, 2. \tag{2.3}$$

The parameter θ_i is also called the *market price of risk* for stock i, for $i = 1, 2$.

Let f_t denote the *proportion* of Investor A's wealth invested in the risky stock $S^{(1)}$ at time t under an investment policy $f = \{f_t,\ t \geq 0\}$, and, similarly, let g_t denote the *proportion* of Investor B's wealth invested in the risky stock $S^{(2)}$ at time t under an investment policy $g = \{g_t,\ t \geq 0\}$. We assume that both $\{f_t,\ t \geq 0\}$ and $\{g_t,\ t \geq 0\}$ are suitable, admissible $\mathcal{F}_t$-adapted control processes, i.e. f_t (or g_t) is a non-anticipative function that satisfies $\mathrm{E}\int_0^T f_t^2\,dt < \infty$ (or $\mathrm{E}\int_0^T g_t^2\,dt < \infty$) for every $T < \infty$.

We place no other restrictions on f or g, for example, we allow f_t (or g_t) ≥ 1, whereby the investor is leveraged and has borrowed to purchase the stock. (We also allow f_t (or g_t) < 0, whereby the investor is selling the stock short; however, for $\mu_i > r$, for $i = 1, 2$, this never happens in any of the problems considered here.)

For the sequel, we will let $\mathcal{G}$ denote the set of admissible controls.

Let X_t^f denote the *wealth* of investor A at time t, if he follows policy $f = \{f_t, t \geq 0\}$, with $X_0 = x$. Since any amount not invested in the risky stock is held in the bond, this process then evolves as

$$\begin{aligned} dX_t^f &= f_t X_t^f \frac{dS_t^{(1)}}{S_t^{(1)}} + X_t^f (1 - f_t) \frac{dB_t}{B_t} \\ &= X_t^f ([r + f_t\sigma_1\theta_1]\,dt + f_t\sigma_1\,dW_t^{(1)}) \end{aligned} \tag{2.4}$$

upon substituting from (2.1) and (2.2) and using the definition (2.3). This is the wealth equation first studied by Merton (1971). Similarly, if we let Y_t^g denote the wealth of investor B under portfolio policy $g = \{g_t,\ t \geq 0\}$, then Y_t^g evolves according to

$$\begin{aligned} dY_t^g &= g_t Y_t^g \frac{dS_t^{(2)}}{S_t^{(2)}} + Y_t^g (1 - g_t) \frac{dB_t}{B_t} \\ &= Y_t^g ([r + g_t\sigma_2\theta_2]\,dt + g_t\sigma_2\,dW_t^{(2)}) \end{aligned} \tag{2.5}$$

where $W_t^{(2)}$ is another (standard) Brownian motion. To allow for complete generality, we allow $W_t^{(2)}$ to be correlated with $W_t^{(1)}$, with correlation coefficient ρ, i.e. $\mathrm{E}(W_t^{(1)} W_t^{(2)}) = \rho t$.

2.1. Competition

While there are many possible competitive objectives, here we are mainly interested in games with payoffs related to the achievement of relative performance goals and shortfalls. Specifically, for numbers l, u with $lY_0 < X_0 < uY_0$, we say, in terms of objectives for Investor A, that (*upper*) performance goal u is reached if $X_t^f = uY_t^g$, for some $t > 0$ and that (*lower*) performance shortfall level l occurs if $X_t^f = lY_t^g$ for some $t > 0$. In general A wins if performance goal u is reached before performance shortfall level l is reached, while B wins if the converse happens. (Analogous objectives can obviously be stated in terms of Investor B with goal and shortfall reversed.) Some of the specific games we consider in the sequel, stated here from the point of view of Investor A, are: (i) maximizing the probability that performance goal u is reached before shortfall l occurs (equivalently, maximizing the probability that A wins); (ii) minimizing the expected time until the performance goal u is reached; (iii) maximizing the expected time until shortfall l is reached; (iv) maximizing the expected discounted reward obtained upon achieving goal u; (v) minimizing the expected discounted penalty paid upon falling to shortfall level l. In each case, Investor B's objective is the converse. For all these games, the *ratio* of the two wealth processes is a sufficient statistic. In a later section, we also consider a fixed-duration utility-based version of the game where the ratio is also the pertinent state variable.

Since X_t^f is a diffusion process controlled by Investor A, and Y_t^g is another diffusion process controlled by Investor B, the ratio process, $Z^{f,g}$, where $Z_t^{f,g} := X_t^f / Y_t^g$, is a jointly controlled diffusion process. Specifically, a direct application of Itô's formula gives

Proposition 2.1. *For the wealth processes X_t^f, Y_t^g defined by (2.4) and (2.5), let $Z_t^{f,g}$ be defined by $Z_t^f := X_t^f / Y_t^g$. Then*

$$\mathrm{d}Z_t^{f,g} = Z_t^{f,g}(m(f_t, g_t)\,\mathrm{d}t + f_t\sigma_1\,\mathrm{d}W_t^{(1)} - g_t\sigma_2\,\mathrm{d}W_t^{(2)}), \tag{2.6}$$

where the function $m(f, g)$ is defined by

$$m(f, g) \equiv m(f, g : \sigma_1, \sigma_2, \theta_1, \theta_2, \rho) = f\sigma_1\theta_1 - g\sigma_2\theta_2 + g^2\sigma_2^2 - \rho\sigma_1\sigma_2 fg \tag{2.7}$$

and where the parameters θ_i, $i = 1, 2$ are defined in (2.3).

Alternatively, in integral form we have

$$Z_t^{f,g} = Z_0 \exp\left\{\int_0^t [m(f_s, g_s) - \tfrac{1}{2}v^2(f_s, g_s)]\,\mathrm{d}s + \int_0^t f_s\sigma_1\,\mathrm{d}W_s^{(1)} - \int_0^t g_s\sigma_2\,\mathrm{d}W_s^{(2)}\right\} \tag{2.8}$$

where the function $v^2(f, g)$ is defined by

$$v^2(f, g) \equiv v^2(f, g : \sigma_1, \sigma_2, \rho) = f^2\sigma_1^2 + g^2\sigma_2^2 - 2fg\sigma_1\sigma_2\rho. \tag{2.9}$$

A consequence of this is that for Markovian control processes $f_t = f(Z_t^{f,g})$ and $g_t = g(Z_t^{f,g})$ (also referred to as *pure strategies*, see e.g. Friedman (1976)), the ratio process $Z^{f,g}$ of (2.6) is a controlled Markov process whose generator, for arbitrary functions $\varphi(t, z) \in \mathcal{C}^{1,2}$, is given by

$$\mathcal{A}^{f,g}\varphi(t, z) = \varphi_t + m(f_t, g_t)z\varphi_z + \tfrac{1}{2}v^2(f_t, g_t)z^2\varphi_{zz}. \tag{2.10}$$

In the next section we provide a general theorem in stochastic optimal control for differential games associated with the process $\{Z_t^{f,g}, t \geq 0\}$ of (2.6) that covers all the games described above as special cases. In a later section we consider the problem of maximizing the expected discounted utility of the ratio. More general results on stochastic differential games where the diffusion component of the process, as well as the drift, is controllable by both players are discussed in e.g. Fleming and Souganides (1989).

3. Value and equilibrium in a stochastic differential game

For the process $Z^{f,g}$ of (2.6), let

$$\tau_x^{f,g} := \inf\{t > 0 : Z_t^{f,g} = x\} \tag{3.1}$$

denote the first hitting time to the point x under the specific policies $f = \{f_t, t \geq 0\}$ and $g = \{g_t, t \geq 0\}$. For given numbers l, u, with $l < Z_0 < u$, let $\tau^{f,g} := \min\{\tau_l^{f,g}, \tau_u^{f,g}\}$ denote the first escape time from the interval (l, u), under the policies f, g.

For a given non-negative function $\lambda(z) \geq 0$, a given real bounded continuous function $c(z)$, and a function $h(z)$ given for $z = l$, $z = u$, with $h(u) < \infty$, let $\nu^{f,g}(z)$ be the expected *payoff function* under the policy pair f, g, defined by

$$\nu^{f,g}(z) = \mathrm{E}_z\left(\int_0^{\tau^{f,g}} c(Z_t^{f,g}) \exp\left\{-\int_0^t \lambda(Z_s^{f,g})\,\mathrm{d}s\right\} \mathrm{d}t + h(Z_{\tau^{f,g}}^{f,g}) \exp\left\{-\int_0^{\tau^{f,g}} \lambda(Z_s^{f,g})\,\mathrm{d}s\right\}\right). \tag{3.2}$$

(Here and in the sequel, we use the notations $\mathrm{P}_z(\cdot)$ and $\mathrm{E}_z(\cdot)$ as shorthand for $\mathrm{P}(\cdot \mid Z_0 = z)$ and $\mathrm{E}(\cdot \mid Z_0 = z)$.)

The two investors compete in the following form: Investor A would like to choose a control function f in order to *maximize* $\nu^{f,g}(z)$, while simultaneously Investor B is trying to choose a control function g in order to *minimize* $\nu^{f,g}(z)$. We consider here only games with perfect revelation, or perfect observation, so that the players' choices are instantaneously revealed to their opponents. The game, or competition, terminates when the ratio process $Z^{f,g}$ first exits the interval (l, u).

Let

$$\underline{\nu}(z) = \sup_{f \in \mathcal{G}} \inf_{g \in \mathcal{G}} \nu^{f,g}(z) \quad \text{and} \quad \overline{\nu}(z) = \inf_{g \in \mathcal{G}} \sup_{f \in \mathcal{G}} \nu^{f,g}(z)$$

denote the *lower* and *upper* values of the game, respectively.

If $\underline{\nu}(z) = \overline{\nu}(z)$ for every z, then the *value of the game* is given by $\nu(z) := \underline{\nu}(z) = \overline{\nu}(z)$. This value can be attained if a Nash equilibrium, equivalently a saddlepoint for the payoff $\nu^{f,g}(z)$, exists, i.e. if there exist two strategies, $f^* = \{f_t^*, t \geq 0\}$ and $g^* = \{g_t^*, t \geq 0\}$ such that for all $z \in (l, u)$, and all other admissible f and g

$$\nu^{f,g^*}(z) \leq \nu^{f^*,g^*}(z) \leq \nu^{f^*,g}(z). \tag{3.3}$$

If (3.3) holds, then $\nu(z) = \nu^{f^*,g^*}(z)$ (see for example Elliott (1976); Fleming and Souganides (1989); Maitra and Sudderth (1996)). The saddlepoint strategies f^*, g^* are referred to as the equilibrium, or competitively optimal, strategies.

In the following theorem, we provide an explicit evaluation of the value of the game as the appropriate solution to a particular non-linear Dirichlet problem, as well as an evaluation of the competitively optimal strategies f_t^* and g_t^*, under suitable conditions. To enable the reader to proceed directly to the specific examples and applications in the subsequent sections, the proof of this theorem is presented in the final section of the paper.

To state results more compactly, let us first introduce some notation and definitions:

1. For an arbitrary function $\psi(z) \in \mathcal{C}^2$, let Γ denote the differential operator defined by

$$\Gamma\psi(z) := (1-\rho^2)[\psi_z(z) + z\psi_{zz}(z)]^2 - \psi_z(z)^2. \tag{3.4}$$

2. For the sequel, we will say that an increasing strictly concave function $\psi(z) \in \mathcal{C}^2$ (so $\psi_{zz} < 0$) is *sufficiently fast-increasing* on an interval (a, b) if the following condition holds:

$$2\psi_z(z) + z\psi_{zz}(z) > 0, \qquad \text{for all } a < z < b. \tag{3.5}$$

   (Observe that the fast-increasing condition (3.5) is equivalent to requiring that the Arrow–Pratt measure of relative risk-aversion for ψ, defined as $-z\psi_{zz}/\psi_z$, is less than 2.)

3. For the sequel, the parameter κ will denote the ratio of the market prices of risk for the two risky assets. Specifically, for θ_i as defined in (2.3) for $i = 1, 2$, define the parameter κ by

$$\kappa := \kappa(\theta_1, \theta_2) = \frac{\theta_1}{\theta_2}. \tag{3.6}$$

   We will see later that the parameter κ is a measure of the degree of *advantage* one player has over the other. Investor A is said to have the advantage if $\kappa > 1$ and Investor B has the advantage if $\kappa < 1$. In the symmetric case the two are neutral.

Theorem 3.1. *Suppose that $\Psi(z) : (l, u) \mapsto \Re$ is a $\mathcal{C}^2$ strictly concave, sufficiently fast-increasing (as in (3.5)) solution to the non-linear Dirichlet problem for $l \le z \le u$:*

$$\frac{z\Psi_z(z)^2}{2\Gamma\Psi(z)}\theta_2^2[(1-\kappa^2)\Psi_z(z) - (1+\kappa^2-2\rho\kappa)(\Psi_z(z) + z\Psi_{zz}(z))] + c(z) - \lambda(z)\Psi(z) = 0 \tag{3.7}$$

with

$$\Psi(l) = h(l) \quad \text{and} \quad \Psi(u) = h(u). \tag{3.8}$$

Also suppose that for all admissible policies f and g, $\mathrm{E}_z(\tau^{f,g}) < \infty$ for $l < z < u$, and that $\Psi(z)$ satisfies the following conditions:

(i) *for all admissible policies f and g, and for all $t \ge 0$, the following moment condition holds*

$$\int_0^t \mathrm{E}([Z_s^{f,g}\Psi_z(Z_s^{f,g})]^2[f_s^2 + g_s^2])\,\mathrm{d}s < \infty; \tag{3.9}$$

(ii) *the function $z\Psi_z(z)H(z)$ is bounded on (l, u), where*

$$H(z) := \Psi_z(z)[\Psi_z(z) + z|\Psi_{zz}(z)|]/|\Gamma\Psi(z)|;$$

(iii) *the function $zH(z)[1 + H(z)]$ is Lipschitz continuous on (l, u).*

Then $\Psi(z)$ is the value of the game described earlier, i.e. $\Psi(z) = \nu(z) \equiv \nu^{f^*,g^*}(z)$, and moreover this value is achieved at the saddlepoint control functions, or competitively-optimal portfolio strategies, $f_\nu^*(z)$ and $g_\nu^*(z)$, given by

$$f_\nu^*(z) = \frac{\theta_1}{\sigma_1}\left(\frac{\Psi_z(z)}{\Gamma\Psi(z)}\right)\left[\left(\frac{\rho}{\kappa} - 1\right)(\Psi_z(z) + z\Psi_{zz}(z)) - \Psi_z(z)\right] \tag{3.10}$$

$$g_\nu^*(z) = \frac{\theta_2}{\sigma_2}\left(\frac{\Psi_z(z)}{\Gamma\Psi(z)}\right)[(1 - \rho\kappa)(\Psi_z(z) + z\Psi_{zz}(z)) - \Psi_z(z)]. \tag{3.11}$$

Remark 3.1. The technical conditions (i), (ii) and (iii) above play a role at various points in the proof of Theorem 3.1, which is presented in the final section. Specifically, if condition (i) holds, then a stochastic integral term, which in general is only a continuous local martingale, is a martingale. If condition (ii) holds, then a particular martingale is uniformly integrable (and hence the martingale stopping theorem is valid), and condition (iii) ensures that the stochastic differential equation for the competitively optimal ratio process $Z^{*,*}$, obtained when the controls of (3.10) and (3.11) are placed back into (2.6), admits a strong solution. (If (iii) holds, then $zf_\nu^*(z)$, $zg_\nu^*(z)$ and $zm(f_\nu^*, g_\nu^*)$ are all Lipschitz continuous.)

Remark 3.2. The parameters θ_i/σ_i, $i = 1, 2$, in the optimal control functions, $f_\nu^*(z)$ and $g_\nu^*(z)$ of (3.10) and (3.11), are the *individual* growth-optimal portfolio strategies for the respective investors. Specifically, Investor A will choose $f(z) = \theta_1/\sigma_1$ for all z, if he is interested in any or all of the following individual objectives, without any regard to actions by Investor B: maximizing logarithmic utility of wealth at a fixed terminal time; minimizing the expected time to reach any arbitrary fixed (obviously higher than initial) level of wealth; maximizing the *growth rate* of wealth, defined by $\sup_f\{\liminf_{t\to\infty}\log(X_t^f)/t\}$. (See e.g. Merton (1990), Chapter 6 and Browne (1998) for reviews and further optimality results.) The obvious analogous results hold for Investor B.

3.1. The symmetric case

In the (fully) symmetric case, $\mu_1 = \mu_2 = \mu$ and $\sigma_1 = \sigma_2 = \sigma$, and so the market prices of risk for the two stocks are the same, i.e. $\theta_1 = \theta_2 \equiv \theta$ say, and of course $\kappa = 1$. For this case we observe that the determining ODE for the optimal value function of (3.7) reduces to

$$-\theta\frac{z\Psi_z(z)^2}{\Gamma\Psi(z)}(1-\rho)[\Psi_z(z) + z\Psi_{zz}(z)] + c(z) - \lambda(z)\Psi(z) = 0, \tag{3.12}$$

and the associated equilibrium controls of (3.10) and (3.11) reduce to

$$f_\nu^*(z) = \frac{\theta}{\sigma}\left(\frac{\Psi_z(z)}{\Gamma\Psi(z)}\right)[(\rho - 1)(\Psi_z(z) + z\Psi_{zz}(z)) - \Psi_z(z)] \tag{3.13}$$

$$g_\nu^*(z) = \frac{\theta}{\sigma}\left(\frac{\Psi_z(z)}{\Gamma\Psi(z)}\right)[(1 - \rho)(\Psi_z(z) + z\Psi_{zz}(z)) - \Psi_z(z)]. \tag{3.14}$$

Observe that the only difference between the players' strategies in (3.13) and (3.14) is in the treatment of the instantaneous correlation ρ.

3.2. The complete, symmetric case

The 'complete' case occurs when $\rho^2 = 1$, in that there is then only *one* Brownian motion in the model. Without any loss of generality, let us consider only the case $\rho = 1$. For the symmetric version of this case it is seen that the control functions of (3.13) and (3.14) reduce further to the growth-optimal proportion $f_\nu^*(z) \equiv g_\nu^*(z) = \theta/\sigma$, regardless of the particulars of the objective of the game and the value of $\Psi(z)$. However, when both players choose this policy, the functions $m(\cdot,\cdot)$ of (2.7) and $v^2(\cdot,\cdot)$ of (2.9) both reduce to zero, i.e. in this case we have

$$m\left(\frac{\theta}{\sigma}, \frac{\theta}{\sigma} : \sigma, \sigma, \theta, \theta, 1\right) \equiv v^2\left(\frac{\theta}{\sigma}, \frac{\theta}{\sigma} : \sigma, \sigma, 1\right) \equiv 0$$

and as such we see from (2.6) that for the resulting ratio process we have $\mathrm{d}Z_t = 0$ for all t. As such, the state never changes, as any movement by a player will be immediately negated by his opponent. (This is never optimal if $\rho^2 < 1$.) The ODE of (3.12) reduces to the degenerate $\Psi(z) = c(z)/\lambda(z)$, which need not be the value to the game.

This degeneracy should be contrasted with the discrete-time complete case treated by Bell and Cover (1980), where a *randomized* version of the growth-optimal strategy is shown to be game-theoretic optimal for maximizing the probability of beating an opponent in a *single* play. Such a result obviously *cannot* hold in a continuous-time stochastic differential game with full revelation, since any randomization by a player will be immediately revealed to the other player, who can immediately (and exactly) adjust.

4. The probability maximizing game

In this section, we consider the game where for two given numbers $l < 1 < u$, the objective of Investor A is to maximize the probability that he will outperform Investor B by $u-1\%$ before Investor B can outperform him by $1/l - 1\%$. Similarly, Investor B wants to maximize the probability that he will outperform Investor A by $1/l - 1\%$ before Investor A can outperform him by $u - 1\%$. Put more simply: Investor A wants to maximize the probability of reaching u while Investor B is trying to maximize the probability of reaching l. Single-player games with related objectives have been studied previously in Pestien and Sudderth (1985, 1988), Mazumdar and Radner (1991) and Browne (1995, 1997, 1999).

Let $V(z)$ denote the value for this game—should it indeed exist: i.e.

$$V(z) = \sup_f \inf_g \mathrm{P}_z(\tau_l^{f,g} > \tau_u^{f,g}) = \inf_g \sup_f \mathrm{P}_z(\tau_l^{f,g} > \tau_u^{f,g}). \tag{4.1}$$

Theorem 3.1 applies to the probability maximizing game by taking $\lambda = c = 0$ in (3.7), and setting $h(l) = 0$ and $h(u) = 1$. Specifically, by Theorem 3.1, we find after simplification that $V(z)$ must be the fast-increasing (in the sense of (3.5)) concave solution to

$$(1-\kappa^2)\Psi_z(z) - (1+\kappa^2 - 2\rho\kappa)(\Psi_z(z) + z\Psi_z(z)) = 0, \qquad \text{for } l < z < u \tag{4.2}$$

with $V(l) = 0$ and $V(u) = 1$.

The solution to the non-linear Dirichlet problem of (4.2), subject to the boundary conditions $\Psi(l) = 0$, $\Psi(u) = 1$, is seen to be $\Psi(z) = (z^\gamma - l^\gamma)/(u^\gamma - l^\gamma)$, where the parameter γ is defined by

$$\gamma = \gamma(\kappa, \rho) := \frac{1-\kappa^2}{1+\kappa^2 - 2\rho\kappa}. \tag{4.3}$$

Observe that for $\rho^2 < 1$, the denominator of (4.3) is positive for all κ. As such, the sign of γ depends on the sign of the numerator. Specifically, $\gamma < 0$ if A has the advantage (i.e. if $\theta_1 > \theta_2$), while $\gamma > 0$ if B has the advantage.

Observe further that for the solution found above we have $\Psi_z > 0$, regardless of the sign or magnitude of γ, while $\Psi_{zz} < 0$ only for $\gamma < 1$. Moreover, the required fast-increasing condition of (3.5), $2\Psi_z + z\Psi_{zz} > 0$, holds only for the case where $-1 < \gamma$. Thus, we see that we require $-1 < \gamma < 1$ for the game to have a value. It follows from (4.3) that this requirement is equivalent to the following two requirements on the parameters ρ and κ:

$$\rho < \kappa \quad \text{and} \quad \rho < \frac{1}{\kappa}. \tag{4.4}$$

Since we assumed that $\theta_i > 0$ for $i = 1, 2$, it follows that $\kappa > 0$ and hence these conditions are trivially satisfied if $\rho \le 0$. Otherwise they are equivalent to

$$\rho < \kappa < \frac{1}{\rho}. \tag{4.5}$$

Assuming that (4.4) holds, it is straightforward to verify that conditions (i), (ii) and (iii) of Theorem 3.1 hold (in particular, $\Psi(z)$ is bounded) and, as such, it is seen by Theorem 3.1 that the *value of the game*, $V(z)$, is indeed given by

$$V(z) := V(z; \gamma, u, l) = \frac{z^\gamma - l^\gamma}{u^\gamma - l^\gamma}, \qquad \text{for} \quad l < z < u \tag{4.6}$$

where γ is defined in (4.3). Therefore, since we now have the value of the game, $V(z)$, in explicit form, we can now use (3.10) and (3.11) of Theorem 3.1 to obtain the equilibrium, or competitively optimal, portfolio strategies. Specifically, by substituting $V(z)$ of (4.6) for $\Psi(z)$ in (3.10) and (3.11) and then simplifying (and using the definition of γ from (4.3)), we obtain the following.

Theorem 4.1. *Suppose that (4.4) holds, then for $l < z < u$ and γ as defined in (4.3), the value of the probability maximizing game of (4.1) is given by $V(z)$ of (4.6), and the associated optimal portfolio policies are given by*

$$f_V^*(z) = \frac{\theta_1}{\sigma_1} C \tag{4.7}$$

$$g_V^*(z) = \frac{\theta_2}{\sigma_2} \kappa^2 C, \tag{4.8}$$

where C is the positive constant given by

$$C := C(\kappa, \rho, \gamma) = \frac{(\rho/\kappa - 1)\gamma - 1}{(1 - \rho^2)\gamma^2 - 1}. \tag{4.9}$$

Observe that the portfolio strategies of (4.7) and (4.8) are *constant proportion* portfolio strategies: regardless of the level of wealth of the individual investor, or the level of wealth of his competitor (or their ratio), the proportion of wealth invested in the risky asset (available to that investor) is held constant, with the remainder in the risk-free asset. Moreover, for each investor, the constant is independent of the levels l and u. (See Browne (1998) for further optimality properties of constant proportion portfolio strategies.)

To see that these constants are positive, and so both players take a positive position in their respective stock, we need only show that $C > 0$. The denominator of C is always negative (since $\gamma^2 < 1$), while the sign of the numerator of C depends on the sign of the quadratic $Q_1(\kappa; \rho)$, where

$$Q_1(\kappa; \rho) = \rho\kappa^2 - 2\kappa + \rho, \tag{4.10}$$

since the numerator of C can be written as $Q_1(\kappa; \rho)/\kappa$.

For $\rho < 0$, $Q_1(\kappa; \rho)$ is trivially negative, and so $C > 0$. For $\rho > 0$, the two roots to the equation $Q_1(\kappa) = 0$ are given by

$$\kappa^- = \frac{1}{\rho}(1 - \sqrt{1 - \rho^2}) \quad \text{and} \quad \kappa^+ = \frac{1}{\rho}(1 + \sqrt{1 - \rho^2}),$$

with $Q_1(\kappa) < 0$ for $\kappa^- < \kappa < \kappa^+$. Since we required $\kappa < 1/\rho$, it is clear that we are only interested in the smaller root, κ^-, and so for $\kappa^- < \kappa < 1/\rho$, it follows that $Q_1(\kappa) < 0$. Moreover, a simple computation will show that $\kappa^- < \rho$, for $\rho > 0$, and since we in fact required $\kappa > \rho$, we finally see that for all relevant κ, we have $Q_1(\kappa) < 0$, giving $C > 0$.

Remark 4.1. The value function of (4.6) shows one manner in which the parameter κ is a measure of advantage. Specifically, consider the probability maximizing game with $l = 1/u$ and $Z_0 = 1$. Then it is natural to say that the player who has the higher probability of winning is the one with the advantage. Some direct manipulations will show that $V(1 : \gamma, u, 1/u) > \frac{1}{2}$ if and only if $\gamma < 0$, i.e. if and only if $\kappa > 1$. That is, Investor A has the advantage (a greater probability of winning) if his investment opportunity has the higher market price of risk.

Remark 4.2. Observe that the only structural difference in the investment policies of (3.10) and (3.11) is in the treatment of the measure of advantage parameter κ. Specifically, we see from (3.10) and (3.11) that if A has the advantage, then the *relative* investment of B is greater, with the converse holding if B has the advantage. Thus a relatively 'bolder' strategy must be followed by the disadvantaged player, in particular on the order of the square of the measure of advantage parameter κ.

It is interesting to note that the determination of which player invests the larger *absolute* fraction of his wealth turns out to depend only on the instantaneous returns $\mu_i, i = 1, 2$ and not the volatility parameters $\sigma_i, i = 1, 2$. Specifically, after simplifying we observe that

$$\frac{f_V^*}{g_V^*} = \frac{\sigma_2\theta_2}{\sigma_1\theta_1} \equiv \frac{\mu_2 - r}{\mu_1 - r}$$

implying that the player with the *lower* instantaneous return must invest *more* in his stock, in order to overcome the advantage of the other player. As can be seen, the volatility parameters, σ_1 and σ_2, do not play a role in determining which player invests a larger fraction of wealth.

Remark 4.3. Observe further that since f_V^* and g_V^* are constants, Proposition 2.1 implies that the optimal ratio process, $Z^{*,*}$, is a geometric Brownian motion. Specifically, when we place the optimal controls of (4.7) and (4.8) into the functions $m(f, g)$ of (2.7), and $v^2(f, g)$ of (2.9), we find that they reduce to (using the obvious identity $\theta_1 = \theta_2\kappa$)

$$m\left(\frac{\theta_1 C}{\sigma_1}, \frac{\theta_2\kappa^2 C}{\sigma_2}\right) = C^2\theta_1^2\kappa(\kappa - \rho) \text{ and } v^2\left(\frac{\theta_1 C}{\sigma_1}, \frac{\theta_2\kappa^2 C}{\sigma_2}\right) = C^2\theta_1^2(1 + \kappa^2 - 2\rho\kappa). \tag{4.11}$$

From (2.8), we find that the optimal ratio process is the geometric Brownian motion

$$Z_t^{*,*} = Z_0 \exp\{\tfrac{1}{2}C^2\theta_1^2(\kappa^2 - 1)t + \theta_1 C(W_t^{(1)} - W_t^{(2)})\}. \tag{4.12}$$

Observe that the constant m in (4.11) is positive (since $\kappa > \rho$), regardless of which player has the advantage, i.e. whether $\kappa > 1$ or $\kappa < 1$. However, the sign of $\mathrm{E}\ln(Z_t^{*,*})$ depends on whether $\kappa > 1$ or $\kappa < 1$, with $\mathrm{E}\ln(Z_t^{*,*}) > 0$ if Investor A has the edge, and vice versa if Investor B has the edge.

Remark 4.4. Proposition 2.1 exhibits the fact that for any admissible control functions $f(z)$, $g(z)$, the ratio process $Z^{f,g}$ is a diffusion process with *scale* function given by

$$S^{f,g}(z) = \int^z \exp\left\{-\int^\xi \frac{2}{y}\left[\frac{m(f(y), g(y))}{v^2(f(y), g(y))}\right]\mathrm{d}y\right\}\mathrm{d}\xi, \quad \text{for} \quad l < z < u, \tag{4.13}$$

where $m(f, g)$ and $v^2(f, g)$ are the functions defined in (2.7) and (2.9). As such for these given policies, the probability that Investor A wins the game can be written as

$$\mathrm{P}_z(\tau_u^{f,g} < \tau_l^{f,g}) = \frac{S^{f,g}(z) - S^{f,g}(l)}{S^{f,g}(u) - S^{f,g}(l)}. \tag{4.14}$$

It follows from the single-player results of Pestien and Sudderth (1985, 1988) (see also Browne (1997), Remark 3.4) that for any given control function $g(z)$, Investor A can *maximize* the probability in (4.14) by choosing the control policy that pointwise *maximizes* the ratio $[zm(f, g)]/[z^2v^2(f, g)]$, which is equivalent to the pointwise maximizer of $m(f, g)/v^2(f, g)$. Similarly, for any given control policy $f(z)$, Investor B can *minimize* the probability in (4.14) by choosing g to be the pointwise *minimizer* of the quantity $m(f, g)/v^2(f, g)$. Some computations will now accordingly show that the minimax value of the the function $m(f, g)/v^2(f, g)$ in fact occurs at the policies f_V and g_V of (4.7) and (4.8). See Nilakantan (1993) for some more general results along these lines.

Remark 4.5. The value function of (4.6) can be used to set conservative capital requirements by setting it equal to a given preassigned probability of outperformance, say p, and then inverting for the required initial capital. Specifically, setting $V(z_0) = p$ and then solving for z_0 gives $z_0 = (l^\gamma + p[u^\gamma - l^\gamma])^{1/\gamma}$.

4.1. The symmetric case

For the symmetric case, we have $\theta_1 = \theta_2 = \theta$, $\sigma_1 = \sigma_2 = \sigma$, and $\kappa = 1$. For this case, so long as $\rho^2 < 1$, (4.3) becomes $\gamma(1, \rho) = 0$. As such, by taking limits appropriately in (4.6) we observe that in the symmetric case the optimal value function reduces to

$$\lim_{\gamma\to 0} V(z; \gamma, u, l) = V(z; 0, u, l) = \ln\left(\frac{z}{l}\right) \Big/ \ln\left(\frac{u}{l}\right). \tag{4.15}$$

Moreover, for $\rho^2 < 1$, we see that C of (4.9) reduces to $C(1, \rho, 0) = 1$, and, as such, the competitively optimal controls of (4.7) and (4.8) reduce in the symmetric case to $f_V^* = g_V^* = \theta/\sigma$.

Since the function in (4.15) satisfies the appropriate version of the Dirichlet problem of (3.12), we have the following.

Corollary 4.1. *In the symmetric case, so long as $\rho^2 < 1$, the value of the game is given by (4.15), and the competitively optimal policies for the probability maximizing problem are for each player to play the growth-optimal strategy, θ/σ.*

Observe that while the correlation parameter ρ does not play an explicit role here at all, in either the value function of (4.15) or the game-theoretic controls θ/σ, all of this holds *only* for $\rho^2 < 1$. Specifically, the limit in (4.15) is valid only for $\rho^2 < 1$. This can be seen by observing that from (4.3) we have $\gamma(\kappa, 1) = (1+\kappa)/(1-\kappa)$. As such,

$$\lim_{\rho\to 1}\lim_{\kappa\to 1}\gamma(\kappa,\rho) = 0 \neq \lim_{\kappa\to 1}\lim_{\rho\to 1}\gamma(\kappa,\rho) = \infty.$$

5. Expected time minimizing/maximizing games

In this section we consider games where the objective is the minimization (maximization) of the expected time for one investor to outperform the other by a given percentage. The existence of a value for such games depends on which investor has the advantage, i.e. whether $\kappa > 1$ or $\kappa < 1$. Since the game is symmetric, in that one player's advantage is the other's disadvantage, we need only consider one game. Here we choose to study only the case where Investor A has the advantage (i.e. $\kappa > 1$) and as such is the minimizer (Investor A would be the maximizer if he were at a disadvantage with $\kappa < 1$). Single-player games with minimal/maximal expected time objectives have been studied in Heath *et al.* (1987) and Browne (1997, 1999).

If Investor A has the advantage, in that $\kappa > 1$, then he is trying to minimize the expected time to the performance goal u, while Investor B, in an effort to stop him, is trying to maximize the same expected time. Let $G^*(z)$ denote the value to this game, should it exist, i.e.

$$G^*(z) = \inf_f \sup_g \mathrm{E}_z(\tau_u^{f,g}) = \sup_g \inf_f \mathrm{E}_z(\tau_u^{f,g}), \qquad \text{for} \quad z < u. \tag{5.1}$$

As we show in the following theorem, the equilibrium portfolio policies turn out to be the individual growth-optimal portfolio policies.

Theorem 5.1. *Let $G^*(z)$ be the value of the game in (5.1) with associated optimal strategies $f^*(z)$ and $g^*(z)$.*
Then, for $\kappa > 1$,

$$G^*(z) = \frac{2}{\theta_2^2(\kappa^2-1)}\ln\left(\frac{u}{z}\right), \qquad \textit{with} \quad f^*(z) = \frac{\theta_1}{\sigma_1}, \quad g^*(z) = \frac{\theta_2}{\sigma_2} \quad \textit{for all } z \le u. \tag{5.2}$$

Proof. While Theorem 3.1 is stated in terms of a maximization objective for Investor A and a minimization objective for Investor B, it can be applied to $G^*(z)$ of (5.1) by taking $c(z) = -1$, $\lambda = 0$ and $h(u) = 0$. Specifically, $G^*(z) = -\tilde{G}(z)$ where

$$\tilde{G}(z) = \sup_f \inf_g \{-\mathrm{E}_z(\tau_u^{f,g})\} = \inf_g \sup_f \{-\mathrm{E}_z(\tau_u^{f,g})\}, \qquad \text{for} \quad z < u.$$

As such, Theorem 3.1 applies directly to $\tilde{G}$, which in turn must be the fast-increasing concave solution to

$$\frac{zG_z(z)^2}{2\Gamma G(z)}\theta_2^2[(1-\kappa^2)G_z(z) - (1+\kappa^2-2\rho\kappa)(G_z(z)+zG_{zz}(z))] - 1 = 0, \qquad \text{for} \quad z < u, \tag{5.3}$$

with $\tilde{G}(u) = 0$. It can be checked that the appropriate solution to (5.3) is indeed given by $\tilde{G}(z) \equiv -G^*(z)$, where $G^*(z)$ is given in (5.2). (Observe that $-G^*(z)$ is sufficiently fast-increasing and concave only for $\kappa > 1$.)

It is easy to see that conditions (i), (ii) and (iii) of Theorem 3.1 hold for the appropriate value functions in the respective cases. In particular, condition (i) holds since for this case $\mathrm{d}G^*(z)/\,\mathrm{d}z = -2[z\theta_2^2(\kappa^2-1)]^{-1}$, and so (3.9) reduces to

$$\left(\frac{2}{\theta_2^2(1-\kappa^2)}\right)^2 \int_0^t ([f(Z_s^{f,g})]^2 + [g(Z_s^{f,g})]^2)\,\mathrm{d}s < \infty,$$

which must hold by the *admissibility* requirement on the policies f and g.

As such, we may conclude that G^* is the value of the game and substitute it into (3.10) and (3.11) to obtain the competitively optimal controls, which in this case reduce to the individual growth-optimal strategies.

5.1. Connections with logarithmic utility

Observe that if we take logarithms in (2.8) and then take expectations, we get

$$\mathrm{E}(\ln(Z_t^{f,g})) = \ln(Z_0) + E\int_0^t [m(f_s, g_s) - \tfrac{1}{2}v^2(f_s, g_s)]\,\mathrm{d}s \tag{5.4}$$

where m and v^2 are the functions given in (2.7) and (2.9). Observe now that for any given g, the argument that achieves the maximum value of $m(f, g) - v^2(f, g)/2$ is θ_1/σ_1 and, similarly, for any given f, the argument that minimizes $m(f, g) - v^2(f, g)/2$ is given by θ_2/σ_2. As such, it is clear that the growth-optimal policies give the minimax value for $m - v^2/2$, given by

$$m\left(\frac{\theta_1}{\sigma_1}, \frac{\theta_2}{\sigma_2}\right) - \tfrac{1}{2}v^2\left(\frac{\theta_1}{\sigma_1}, \frac{\theta_2}{\sigma_2}\right) = \tfrac{1}{2}\theta_2^2(\kappa^2 - 1). \tag{5.5}$$

This in turn implies, by (5.4), that these policies are also the competitively optimal policies for the game where Investor A is trying to maximize the value of $\mathrm{E}\ln(Z_T^{f,g})$, while Investor B is trying to minimize the same quantity, for a fixed terminal time T. This of course is trivially obvious, since, for every t, we have $\ln(Z_t^{f,g}) = \ln(X_t^f) - \ln(Y_t^g)$, and so the minimax occurs when each player maximizes the expected logarithm of his own terminal wealth. Thus we find that *maximizing individual logarithmic utility is also game-theoretically optimal for minimizing (resp. maximizing) the expected time to beat an opponent*. (This generalizes the single-player results of Heath *et al.* (1987), Merton (1990), Chapter 6, and Browne (1997, 1999).) While this observation is now obvious in light of the logarithmic value function of (5.2), this was by no means obvious a priori. Fixed-horizon utility-based games will be discussed in Section 7.

5.2. The symmetric case

The equivalence between the minimal/maximal expected time game and the logarithmic utility game of fixed duration just discussed does *not* carry forth to the symmetric case.

Specifically, the argument above, for the utility-based game of fixed duration (i.e. where for some fixed T, Investor A wants to maximize $\mathrm{E}\ln(Z_T^{f,g})$ while B is trying to minimize the same quantity) is still valid for the *symmetric* case, where $\kappa = 1$. As such we see from (5.5) that the minimax value of $m - v^2/2$ is zero, and so the value of this game is $\mathrm{E}\ln(Z_T^{f,g}) = Z_0$, with saddlepoint, or competitively optimal polices, given by $f = g \equiv \theta/\sigma$.

However, as we see from (5.2) of Theorem 5.1, the goal-based game of (5.1) does *not have a finite value* in the symmetric case where $\kappa = 1$. The reason for this is the fact that in the symmetric case, for the game of (5.1), where the lower goal is 0, the expected time to the upper goal, u, is infinite. This follows directly from elementary properties of geometric Brownian motion, and the fact that the minimax value of $m - v^2/2$ is zero. Specifically, for a geometric Brownian motion, $X_t = X_0 \exp\{\delta t + \beta W_t\}$, it is well known that if $\delta = 0$, then $\inf_{0 \le t < \infty} X_t = 0$ and $\sup_{0 \le t < \infty} X_t = \infty$.

6. Games with discounting

In this section we consider games where one player wants to maximize the expected discounted reward achieved upon outperforming his opponent, while the other is trying to minimize the same quantity. Symmetry again implies that we need only consider one game, and we will again consider only the maximizing game in terms of Investor A. Specifically, we consider the game where Investor A wants to maximize the expected discounted reward of reaching the upper goal, u, while Investor B wants to minimize the same quantity. Single-player games with related objectives have been studied in Orey *et al.* (1988) and Browne (1995, 1997, 1999).

Let $F^*(z)$ denote the value of this game—should it exist. Specifically, let

$$F^*(z) = \sup_f \inf_g \mathrm{E}_z(\mathrm{e}^{-\lambda \tau_u^{f,g}}) = \inf_g \sup_f \mathrm{E}_z(\mathrm{e}^{-\lambda \tau_u^{f,g}}), \qquad \text{for} \qquad z < u. \tag{6.1}$$

Theorem 3.1 applies here with $c = 0$, $\lambda(z) = \lambda > 0$ in (3.7), and setting $h(u) = 1$. Specifically, by Theorem 3.1, $F^*(z)$ must be the fast-increasing concave solution to

$$\frac{zF_z(z)^2}{2\Gamma F(z)}\theta_2^2[(1-\kappa^2)F_z(z) - (1+\kappa^2 - 2\rho\kappa)(F_z(z) + zF_{zz}(z))] - \lambda F(z) = 0, \qquad \text{for} \qquad z < l, \tag{6.2}$$

with $F^*(u) = 1$. Solutions to the non-linear Dirichlet problem of (6.2) are of the form $(z/u)^\eta$, where η is a root to the quadratic

$$\eta^2[\theta_2^2(1+\kappa^2-2\rho\kappa) + 2\lambda(1-\rho^2)] - \eta\theta_2^2(1-\kappa^2) - 2\lambda = 0. \tag{6.3}$$

The discriminant of this quadratic is

$$D = [\theta_2^2(1-\kappa^2)]^2 + 8\lambda[\theta_2^2(1+\kappa^2-2\rho\kappa) + 2\lambda(1-\rho^2)]$$

which is positive. As such, the quadratic of (6.3) admits the two real roots $\eta^+(\lambda;\kappa,\rho)$ and $\eta^-(\lambda;\kappa,\rho)$, where

$$\eta^{+,-} = \frac{\theta_2^2(1-\kappa^2) \pm \sqrt{D}}{2[\theta_2^2(1+\kappa^2-2\rho\kappa) + 2\lambda(1-\rho^2)]}. \tag{6.4}$$

Moreover, these roots are of different sign (since $\lambda > 0$, and $[\theta_2^2(1+\kappa^2-2\rho\kappa)+2\lambda(1-\rho^2)] > 0$) with $\eta^- < 0 < \eta^+$ for $\lambda > 0$.

Since we require $F_z > 0$, as well as $2F_z + zF_{zz} > 0$, it is the positive root, η^+, that is relevant here. However, concavity of F ($F_{zz} < 0$) requires that $\eta^+ < 1$. This in turn is equivalent to the condition $Q_2(\kappa) > 0$, where

$$Q_2(\kappa) := \kappa^2\theta_2^2 - \kappa\rho\theta_2^2 - \lambda\rho^2. \tag{6.5}$$

(The equivalence follows from the elementary fact that for the quadratic equation ax^2+bx+c, with $a>0$, the requirement that the larger root be less than 1, i.e. $[-b+\surd(b^2-4ac)]/(2a)<1$, is algebraically equivalent to the requirement $a+b+c>0$, which for the quadratic of (6.3) reduces to $Q_2(\kappa)>0$.)

The quadratic equation $Q_2(\kappa)=0$ admits the two roots

$$\tilde{\kappa}^-(\lambda)=\tfrac{1}{2}\rho\left(1-\sqrt{1+4\lambda/\theta_2^2}\right) \quad \text{and} \quad \tilde{\kappa}^+(\lambda)=\tfrac{1}{2}\rho\left(1+\sqrt{1+4\lambda/\theta_2^2}\right) \tag{6.6}$$

with $Q_2(\kappa)>0$ only for $k<\tilde{\kappa}^-(\lambda)$ and for $k>\tilde{\kappa}^+(\lambda)$. Note that $\tilde{\kappa}^-(0)=0$ and $\tilde{\kappa}^+(0)=\rho$.

As such, we now have the requisite condition for a value to exist, and can therefore now use (3.10) and (3.11) of Theorem 3.1 to find the competitively optimal control functions, which once again turn out to be constant proportion strategies.

Theorem 6.1. *Suppose the measure of advantage parameter, κ, satisfies*

$$\kappa>\tilde{\kappa}^+ \quad \textit{and} \quad \kappa<\tilde{\kappa}^- \tag{6.7}$$

where $\tilde{\kappa}^+$ and $\tilde{\kappa}^-$ are defined in (6.6). Then the value of the discounted game of (6.1) is given by

$$F^*(z)=\left(\frac{z}{u}\right)^{\eta^+} \quad \textit{for } z<u \tag{6.8}$$

where η^+ is defined in (6.4), and the associated saddlepoint is given by

$$f_F^*(z)=\frac{\theta_1}{\sigma_1}\left[\frac{(\rho/\kappa-1)\eta^+-1}{(1-\rho^2)(\eta^+)^2-1}\right] \quad \textit{and} \quad g_F^*(z)=\frac{\theta_2}{\sigma_2}\left[\frac{(1-\rho\kappa)\eta^+-1}{(1-\rho^2)(\eta^+)^2-1}\right]. \tag{6.9}$$

Remark 6.1. Observe that for $\rho<0$ we have $\tilde{\kappa}^+<\rho<0<\tilde{\kappa}^-$, while for $\rho>0$, we have $\tilde{\kappa}^-<0<\rho<\tilde{\kappa}^+$. Thus if $\rho<0$, condition (6.7) becomes $\tilde{\kappa}^+<\kappa<\tilde{\kappa}^-$, while for $\rho>0$, condition (6.7) becomes $\kappa>\tilde{\kappa}^+$. Since in the latter case we must also have $\tilde{k}^+>\rho$, we see that for the discounted game of (6.1) to have a value, we require Investor A to have a *greater* degree of advantage parameter κ, than was required for the probability maximizing game to have a value. (Recall that (4.5) required that $\kappa>\rho$.)

Remark 6.2. Observe further that by letting the discount factor λ go to zero, we obtain $\eta^-(0;\kappa,\rho)=0$ and $\eta^+(0;\kappa,\rho)=\gamma$, where γ is the parameter defined earlier in (4.3). As such we also find that in this case the strategies in (6.9) reduce to the strategies obtained previously in (4.7) and (4.8) for the probability maximizing game of Theorem 4.1. (A similar analysis from the minimizer's point of view will show that the resulting optimal strategies will reduce to the growth-optimal strategies of the previous section.)

Remark 6.3. For the symmetric case, the root η^+ reduces to

$$\eta^+(\lambda;1\rho)=\left[\frac{\lambda}{(1-\rho)[\theta^2+\lambda(1+\rho)]}\right]^{1/2}$$

and the condition for a value to exist becomes $\theta^2(1-\rho)>\lambda\rho^2$.

7. Utility-based games

So far, the objectives considered have related solely to the achievement of relative performance goals and shortfall levels, and the games considered allowed only one winner. In this section, we consider games of a fixed duration T, where both investors receive utility (or disutility) from the ratio of the wealth processes (i.e. from the relative performance of their respective wealths).

Specifically, for given concave-increasing utility functions $\beta(z)$ and $U(z)$, and for a given fixed terminal time T, let $J^{f,g}(t,z)$ be the expected payoff function under the policy pair f, g, defined by

$$J^{f,g}(t,z) = \mathrm{E}_{t,z}\left(\int_t^T \beta(Z_s^{f,g}) \exp\left\{-\int_t^s \lambda(Z_v^{f,g})\,\mathrm{d}v\right\} \mathrm{d}s + U(Z_T^{f,g}) \exp\left\{-\int_t^T \lambda(Z_s^{f,g})\,\mathrm{d}s\right\}\right). \tag{7.1}$$

(Here we use the notation $\mathrm{E}_{t,z}(\cdot)$ as shorthand for $\mathrm{E}(\cdot \mid Z_t = z)$.) Once again we assume that A is trying to maximize this quantity while B is trying to minimize it.

Let $J(t,z)$ denote the value of this game, should it exist, i.e.

$$J(t,z) = \inf_g \sup_f J^{f,g}(t,z) = \sup_f \inf_g J^{f,g}(t,z), \tag{7.2}$$

and let $f_J(t,z)$ and $g_J(t,z)$ denote the associated optimal strategies. Note that in this case we have time-dependence, which will lead to a non-linear Cauchy problem, as opposed to the Dirichlet problem of Theorem 3.1. An analysis similar to that of Theorem 3.1 and its proof (see next section) will show that, if $\Upsilon(t,z) : [0,T] \times (0,\infty) \mapsto \Re$ is a $\mathcal{C}^{1,2}$ concave and sufficiently fast-increasing solution (in z) to the non-linear Cauchy problem:

$$\Upsilon_t + \frac{z\Upsilon_z^2}{2\Gamma\Upsilon}\theta_2^2[(1-\kappa^2)\Upsilon_z - (1+\kappa^2-2\rho\kappa)(\Upsilon_z + z\Upsilon_{zz})] + \beta - \lambda\Upsilon = 0 \tag{7.3}$$

with $\Upsilon(T,z) = U(z)$, then subject to the appropriate regularity conditions (e.g. that $\Upsilon(z)$ satisfies conditions (i), (ii) and (iii) of Theorem 3.1), $\Upsilon(t,z)$ is the competitively optimal value function of the game in (7.2), i.e. $\Upsilon(t,z) = J(t,z)$, and in this case the competitively optimal control functions are given by

$$f_J^*(t,z) = \frac{\theta_1}{\sigma_1}\left(\frac{\Upsilon_z(t,z)}{\Gamma\Upsilon(t,z)}\right)\left[\left(\frac{\rho}{\kappa} - 1\right)(\Upsilon_z(t,z) + z\Upsilon_{zz}(t,z)) - \Upsilon_z(t,z)\right] \tag{7.4}$$

$$g_J^*(t,z) = \frac{\theta_2}{\sigma_2}\left(\frac{\Upsilon_z(t,z)}{\Gamma\Upsilon(t,z)}\right)[(1-\rho\kappa)(\Upsilon_z(t,z) + z\Upsilon_{zz}(t,z)) - \Upsilon_z(t,z)]. \tag{7.5}$$

(The proof of this result is in fact easier than that of its Dirichlet counterpart, Theorem 3.1, and so we leave it for the reader to fill in the missing details.)

As an example of a utility-based game, consider the case where $\beta(z) = \lambda(z) = 0$, and where $U(z) = z^\alpha$, for $0 < \alpha < 1$. (The logarithmic case treated earlier would correspond to the limiting case of $\alpha = 0$, since $\lim_{\alpha\to 0}(z^\alpha - 1)/\alpha = \ln(z)$.)

For this case we find the value function

$$J(t,z) = \mathrm{e}^{q(\alpha)(T-t)} z^\alpha \tag{7.6}$$

where $q(\alpha)$ is defined by

$$q(\alpha) := \alpha\theta_2^2 \frac{(1-\kappa^2) - \alpha(1+\kappa^2 - 2\rho\kappa)}{2[(1-\rho^2)\alpha^2 - 1]}.$$

The associated optimal strategies are again constant proportions, with

$$f_J^*(t,z) = \frac{\theta_1}{\sigma_1}\left[\frac{(\rho/\kappa - 1)\alpha - 1}{(1-\rho^2)\alpha^2 - 1}\right] \quad \text{and} \quad g_J^*(z) = \frac{\theta_2}{\sigma_2}\left[\frac{(1-\rho\kappa)\alpha - 1}{(1-\rho^2)\alpha^2 - 1}\right]. \tag{7.7}$$

Comparison of the policies of (7.7) with those obtained previously for goal-based games will provide obvious further analogues between objective criteria and utility function (see Browne (1995, 1997, 1999) for other equivalences in single-player games).

8. Proofs

To prove Theorem 3.1, we first exploit the the Hamilton–Jacobi–Bellman (HJB) equations of dynamic programming for *single*-player games (cf. Krylov (1980)) to obtain candidate value functions and equilibrium portfolio control strategies for the two-player games considered here. These controls and value functions will then be verified to be in fact competitively optimal via an extension of a fairly standard martingale argument.

To proceed, observe that for any given policy function $g(z)$ used by Investor B, the HJB optimality equation for Investor A for maximizing $\nu^{f,g}(z)$ of (3.2) over control policies $\{f_t\} \in \mathcal{G}$, to be solved for a function $\nu^{*,g}$ is (see e.g. Krylov (1980), Theorem 1.4.5):

$$\sup_f\{\mathcal{A}^{f,g}\nu^{*,g} + c - \lambda\nu^{*,g}\} = 0, \qquad \nu^{*,g}(l) = h(l),\ \ \nu^{*,g}(u) = h(u), \tag{8.1}$$

where $\mathcal{A}^{f,g}$ is the generator given by (2.10). The infimum of this, $\overline{\nu(z)} = \inf_g \nu^{*,g}(z)$, is the upper-value function (see e.g. Fleming and Souganides (1989)).

Similarly, for any given policy function $f(z)$ used by Investor A, the HJB optimality equation for Investor B for minimizing $\nu^{f,g}(z)$ of (3.2) over control policies $\{g_t\} \in \mathcal{G}$, to be solved for a function $\nu^{f,*}$ is

$$\inf_g\{\mathcal{A}^{f,g}\nu^{f,*} + c - \lambda\nu^{f,*}\} = 0, \qquad \nu^{f,*}(l) = h(l),\ \ \nu^{f,*}(u) = h(u). \tag{8.2}$$

The supremum of this, $\underline{\nu}(z) = \sup_f \nu^{f,*}(z)$, is the lower-value function.

Assuming now that (8.1) admits a classical solution with $\nu_{zz}^{*,g} < 0$, we may use calculus to optimize with respect to f in (8.1) to obtain the maximizer (as a function of g)

$$\tilde{f}(z:g) = -\frac{\theta_1}{\sigma_1}\left(\frac{\nu_z^{*,g}}{z\nu_{zz}^{*,g}}\right) + g(z)\rho\frac{\sigma_2}{\sigma_1}\left(1 + \frac{\nu_z^{*,g}}{z\nu_{zz}^{*,g}}\right). \tag{8.3}$$

Similarly, if we assume that (8.2) admits a classical solution with $2z\nu_z^{f,*} + z^2\nu_{zz}^{f,*} > 0$, its minimizer will be given by

$$\tilde{g}(z:f) = \frac{\theta_2}{\sigma_2}\left(\frac{z\nu_z^{f,*}}{2z\nu_z^{f,*} + z^2\nu_{zz}^{f,*}}\right) + f(z)\rho\frac{\sigma_1}{\sigma_2}\left(\frac{z\nu_z^{f,*} + z^2\nu_{zz}^{f,*}}{2z\nu_z^{f,*} + z^2\nu_{zz}^{f,*}}\right). \tag{8.4}$$

(Observe that B's second-order condition is the basis of the fast-increasing condition of (3.5).) The optimizers $\tilde{f}(z : g)$ and $\tilde{g}(z : f)$ of (8.3) and (8.4) are also referred to as the optimal *reaction* functions.

Let us assume now that a saddlepoint exists, and that hence the game must have an achievable value with $\nu^{*,\tilde{g}} = \nu^{\tilde{f},*} \equiv \nu$ (see e.g. Elliott (1976), Maitra and Sudderth (1996), Fleming and Souganides (1989)). If this is the case, then we can find the saddlepoint by substituting $\tilde{g}$ into (8.3) and $\tilde{f}$ into (8.4) and solving the resulting linear equations. When we do this we obtain the optimal control functions

$$f^*(z) = \frac{\theta_1}{\sigma_1}\left(\frac{\nu_z(z)}{\Gamma\nu(z)}\right)\left[\left(\frac{\rho}{\kappa} - 1\right)(\nu_z(z) + z\nu_{zz}(z)) - \nu_z(z)\right] \tag{8.5}$$

$$g^*(z) = \frac{\theta_2}{\sigma_2}\left(\frac{\nu_z(z)}{\Gamma\nu(z)}\right)[(1 - \rho\kappa)(\nu_z(z) + z\nu_{zz}(z)) - \nu_z(z)] \tag{8.6}$$

where κ is the measure of advantage parameter defined in (3.6), and where Γ is the differential operator of (3.4).

When the control functions $f^*(z)$, $g^*(z)$ of (8.5) and (8.6) are then in turn substituted back into either (8.1) or (8.2), with $\nu = \nu^{*,g^*} = \nu^{f^*,*}$, we obtain, after some manipulations, the non-linear Dirichlet problem of (3.7), with $\Psi = \nu$.

To complete the argument (i.e. to verify that ν is indeed the value of the game, and is achieved by the policies f^*, g^* of (8.5) and (8.6)), we can now rely on the results of Fleming and Souganides (1989), who provide a quite general verification argument for stochastic differential games, of which the model treated here is a special case. Alternatively, we can construct a verification argument directly, similar to the standard martingale arguments in, for example, Fleming and Soner (1993). To carry out the latter program, define for any admissible policy pair $f, g = \{f_t, g_t, t \geq 0\}$, the process

$$M(t : f, g) := \mathrm{e}^{-\Lambda_t^{f,g}}\Psi(Z_t^{f,g}) + \int_0^t \mathrm{e}^{-\Lambda_s^{f,g}} c(Z_s^{f,g})\,\mathrm{d}s, \qquad \text{for } t \geq 0, \tag{8.7}$$

where Ψ is the concave fast-increasing solution of (3.7), (3.8), and $\Lambda_t^{f,g} := \int_0^t \lambda(Z_s^{f,g})\,\mathrm{d}s$. $M(t : f, g)$ may be interpreted as a conditional (on $\mathcal{F}_t$) expectation of the gain if controls f, g are used up to time t, and the optimal controls thereafter. It can be shown that, under the conditions given in Theorem 3.1, M is a (uniformly integrable) martingale under the pair $\{f_t^*, g_t^*; t \geq 0\}$, but a *supermartingale* under the pair $\{f_t, g_t^*; t \geq 0\}$, for any admissible $\{f_t\}$, and a *submartingale* under the pair $\{f_t^*, g_t; t \geq 0\}$ for any admissible $\{g_t\}$, where f^* and g^* are the policies given in (8.5) and (8.6).

The representation of the function $\Psi(z)$ of Theorem 3.1 as the value of the game, and the competitive optimality of the saddlepoint policies (f^*, g^*) of (8.5) and (8.6), will now follow as a consequence of the following lemma.

Lemma 8.1. *For any admissible policies $f = \{f_t, t \geq 0\}$ and $g = \{g_t, t \geq 0\}$, with $M(t : f, g)$ as defined in (8.7), and f^*, g^* as defined in (8.5) and (8.6), we have*

$$\mathrm{E}_z[M(t \wedge \tau^{f,g^*} : f, g^*)] \leq M(0, f, g^*) \equiv \Psi(z) \qquad \textit{for } t \geq 0 \tag{8.8}$$

$$\mathrm{E}_z[M(t \wedge \tau^{f^*,g} : f^*, g)] \geq M(0 : f^*, g) \equiv \Psi(z), \qquad \textit{for } t \geq 0, \tag{8.9}$$

with equalities holding if and only if $f = f^*$ and $g = g^*$.

Proof. An application of Itô's formula to the process $M(t : f, g)$ of (8.7) using (2.6) gives

$$M(t \wedge \tau^{f,g} : f, g) = M(0 : f, g) + \int_0^{t\wedge\tau^{f,g}} \mathrm{e}^{-\Lambda_s^{f,g}} Q(f_s, g_s : Z_s^{f,g})\,\mathrm{d}s + \int_0^{t\wedge\tau^{f,g}} \mathrm{e}^{-\Lambda_s^{f,g}} Z_s^{f,g}\Psi_z(Z_s^{f,g})[f_s\sigma_1 dW_s^{(1)} - g_s\sigma_2\,\mathrm{d}W_s^{(2)}], \quad (8.10)$$

where the quadratic form $Q(f, g : z)$ is defined by

$$\begin{aligned} Q(f, g : z) := {} & f^2\sigma_1^2(\tfrac{1}{2}z^2\Psi_{zz}(z)) + g^2\sigma_2^2(\tfrac{1}{2}z^2\Psi_{zz}(z) + z\Psi_z(z)) \\ & + (f\sigma_1\theta_1 - g\sigma_2\theta_2)z\Psi_z(z) - fg\rho\sigma_1\sigma_2(z^2\Psi_{zz}(z) + z\Psi_z(z)) \\ & + c(z) - \lambda(z)\Psi(z). \end{aligned}$$

Observe first that if assumption (i) holds, then the stochastic integral term in (8.10) is a finite-variance martingale, and hence uniformly integrable. Some direct computations will now show that for any given g, the $\max_{f\in\mathcal{G}} Q(f, g : z)$ is achieved at the control function $\tilde{f}(z : g)$ of (8.3), and similarly, for any given f, the $\min_{g\in\mathcal{G}} Q(f, g : z)$ is achieved at the control function $\tilde{g}(z : f)$ of (8.4). Therefore, the minimax value of Q is reached at the policies f^* and g^* of (8.5) and (8.6), and is equal to

$$Q(f^*, g^* : z) = \frac{z\Psi_z^2}{2\Gamma\Psi}\theta_2^2[(1-\kappa^2)\Psi_z - (1+\kappa^2 - 2\rho\kappa)(\Psi_z + z\Psi_{zz})] + c(z) - \lambda(z)\Psi = 0 \quad (8.11)$$

where the last equality follows from (3.7). As such, for f^* of (8.5), g^* of (8.6), and for all other possible f, g, we have

$$Q(f, g^* : z) \le \sup_f Q(f, g^* : z) = Q(f^*, g^* : z) \equiv 0 = \inf_g Q(f^*, g : z) \le Q(f^*, g : z). \quad (8.12)$$

Therefore, from (8.10) we see that for all admissible control functions f, and g^* of (8.6) we have

$$\begin{aligned} & \int_0^{t\wedge\tau^{f,g^*}} \exp\{-\Lambda_s^{f,g^*}\}Z_s^{f,g^*}\Psi_z(Z_s^{f,g^*})[f_s\sigma_1\,\mathrm{d}W_s^{(1)} - g^*(Z_s^{f,g^*})\sigma_2\,\mathrm{d}W_s^{(2)}] \\ & = M(t\wedge\tau^{f,g^*} : f, g^*) - M(0, f, g^*) - \int_0^{t\wedge\tau^{f,g^*}} \exp\{-\Lambda_s^{f,g^*}\}\, Q(f_s, g^* : Z_s^{f,g^*})\,\mathrm{d}s \\ & \ge M(t\wedge\tau^{f,g^*} : f, g^*) - M(0, f, g^*) - \int_0^{t\wedge\tau^{f,g^*}} \exp\{-\Lambda_s^{f,g^*}\}\,\{\sup_f Q(f_s, g^* : Z_s^{f,g^*})\}\,\mathrm{d}s \\ & = M(t\wedge\tau^{f,g^*} : f, g^*) - M(0, f, g^*). \end{aligned} \quad (8.13)$$

(The inequality in (8.13) following from (8.12).) Observe now that the stochastic integral term on the left-hand side of (8.13) is a continuous local martingale that is in fact a martingale by assumption (i) of Theorem 3.1. Hence, taking expectations on (8.13) directly gives the desired inequality of (8.8). Equality holds in both (8.13) and (8.8) for $f = f^*$.

Similarly, for all admissible control policies g, and f^* of (8.5) we have

$$\int_0^{t\wedge\tau^{f^*,g}} \exp\{-\Lambda_s^{f^*,g}\}\, Z_s^{f^*,g}\Psi_z(Z_s^{f^*,g})\,[f^*(Z_s^{f^*,g})\sigma_1\,\mathrm{d}W_s^{(1)} - g_s\sigma_2\mathrm{d}W_s^{(2)}]$$

$$= M(t\wedge\tau^{f^*,g} : f^*, g) - M(0, f^*, g) - \int_0^{t\wedge\tau^{f^*,g}} \exp\{-\Lambda_s^{f^*,g}\}\, Q(f^*, g_s : Z_s^{f^*,g})\,\mathrm{d}s$$

$$\leq M(t\wedge\tau^{f^*,g} : f^*, g) - M(0, f^*, g) - \int_0^{t\wedge\tau^{f^*,g}} \exp\{-\Lambda_s^{f^*,g}\}\, \{\inf_g Q(f, g : Z_s^{f^*,g})\}\,\mathrm{d}s$$

$$= M(t\wedge\tau^{f^*,g} : f^*, g) - M(0, f^*, g), \tag{8.14}$$

where again, the inequality follows from (8.12). Now once again assumption (i) of Theorem 3.1 shows that the stochastic integral term in (8.14) is a continuous local martingale that is in fact a martingale.

As such, inequality (8.9) is established by taking expectations in (8.14), with equality holding only if $g = g^*$.

Finally, observe that if condition (iii) of Theorem 3.1 holds, then both $zf^*(z)$ and $zg^*(z)$ as well as the function $zm(f^*(z), g^*(z))$, where $m(f, g)$ is defined in (2.7), are all Lipschitz continuous, implying therefore that the drift and diffusion coefficients of the resulting competitively optimal ratio process, $Z^{*,*} := Z^{f^*,g^*}$, are locally Lipschitz continuous; therefore the equation (2.6) with $f = f^*$, $g = g^*$ admits a strong solution. Moreover, since $Q(f^*, g^*; z) \equiv 0$, the process $\{M(t\wedge\tau^{f^*,g^*} : f^*, g^*), t \geq 0\}$ is a (uniformly integrable) martingale under the conditions of Theorem 3.1. Thus we have shown that

$$\mathrm{E}_z[M(t\wedge\tau^{f,g^*} : f, g^*)] \leq \mathrm{E}_z[M(t\wedge\tau^{f^*,g^*} : f^*, g^*)] \leq \mathrm{E}_z[M(t\wedge\tau^{f^*,g} : f^*, g)]. \tag{8.15}$$

Since $\mathrm{E}_z[\lim_{t\to\infty}\inf M(t\wedge\tau^{f,g} : f, g)] = \nu^{f,g}(z)$, we may now complete the proof by sending $t\to\infty$ in (8.15) to obtain $\nu^{f,g^*}(z) \leq \nu^{f^*,g^*}(z) \leq \nu^{f^*,g}(z)$. The passage to the limit is justified for the left-hand side by Fatou's lemma (since everything is bounded from below), and for the other terms by using uniform integrability and the martingale stopping theorem (which is valid by the assumption that $\mathrm{E}_z(\tau^{f,g}) < \infty$ for all admissible f and g).

This completes the proof of Theorem 3.1.

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