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Metrology Triangle Using a Watt Balance, a Calculable Capacitor, and a Single-Electron Tunneling Device ‡

Mark W. Keller[1], **François Piquemal**[2], **Nicolas Feltin**[2], **Barthélémy Steck**[2], **and Laurent Devoille**[2]

[1]National Institute of Standards and Technology (NIST), 325 Broadway, Boulder, CO 80305-3328

[2]Laboratoire National de Metrologie et d'Essais LNE, Trappes F78197, France

E-mail: mark.keller@boulder.nist.gov, francois.piquemal@lne.fr

Abstract. The combination of a Watt balance, a calculable capacitor, and a single-electron tunneling device forms a triangle that yields a value for the single-electron charge quantum Q_{S} in terms of the SI coulomb. Importantly, this result is independent of the Josephson and quantum Hall effects, and thus avoids the possible confounding corrections from these two effects that arise in the traditional quantum metrology triangle. This new triangle can be used to test for corrections to the expected relation $Q_{\mathrm{S}} = e$, where e is the elementary charge. Combining existing results for Watt balances, calculable capacitors, and an electron counting capacitance standard yields $(Q_{\mathrm{S}}/e) - 1 = (-0.09 \pm 0.92) \times 10^{-6}$.

1. Introduction

Quantum electrical standards for voltage and resistance based on the Josephson effect and the quantum Hall effect, respectively, revolutionized electrical metrology almost 20 years ago [1]. They have allowed dramatic improvements in consistency among national measurement institutes [2]. The question of the exactness of these quantum effects has been the subject of many studies, both theoretical and experimental, and these have been summarized recently in [3]. A well known test for exactness is the quantum metrology triangle (QMT), which is made possible by adding a quantum electical standard of current or charge based on single-electron tunneling effects. As described in detail in the next section, the QMT tests for corrections in all three quantum electrical standards at once. While this idea is elegant, it has the weakness that corrections to two (or even three) of the effects could cancel each other. Thus it is important to have methods to test each "leg" of the QMT *individually.* Such methods exist for both the Josephson and quantum Hall legs [3], and in this paper we describe a test for the single-electron leg alone. This test has been mentioned previously as part of broader discussions [4, 5]. Here we present more details to highlight the essential aspects of this test, and we compute a value for Q_{S} from the best experimental results to date.

We begin with a review of the traditional QMT to provide context. We then explain how a Watt balance, calculable capacitor, and electron counting capacitance standard can be combined to form a new triangle. We do this in terms of an intuitive (but impractical) thought experiment, in order to make the idea clear. Finally, we take the best existing results for the three legs of this triangle and show how they can be combined to yield a value for Q_{S}.

2. The Quantum Metrology Triangle

The quantum metrology triangle (QMT) comprises standards for voltage U, resistance R, and current I. Each of the standards relates one of these quantities to the Planck constant h and/or the elementary charge e:

(i) A Josephson voltage standard (JVS) driven at a frequency f_{J} and operating on the nth step produces a voltage

$$U_{\mathrm{JVS}} = nf_{\mathrm{J}}/K_{\mathrm{J}} \ \text{ with } K_{\mathrm{J}} = \frac{2e}{h}(1+\varepsilon_{\mathrm{J}}). \tag{1}$$

(ii) A quantum Hall resistance (QHR) standard quantized on the ith plateau has a resistance

$$R_{\mathrm{QHR}} = R_{\mathrm{K}}/i \ \text{ with } R_{\mathrm{K}} = \frac{h}{e^2}(1+\varepsilon_{\mathrm{K}}). \tag{2}$$

(iii) A single-electron tunneling (SET) current standard driven at a frequency f_{S} produces a current

$$I_{\mathrm{SET}} = Q_{\mathrm{S}}f_{\mathrm{S}} \ \text{ with } Q_{\mathrm{S}} = e(1+\varepsilon_{\mathrm{S}}). \tag{3}$$

In each case, a possible deviation from the expected quantum relation is parametrized by ϵ.

In its original form [6], the three quantities are linked by Ohm's law, $U = IR$. In another form [7, 8], in which charge Q is used instead of current and R is linked to a capacitance C *via* a quadrature bridge, the quantities are linked by $Q = CU$. Both forms of the QMT lead to the same result:§ a value for the dimensionless product $K_\mathrm{J}R_\mathrm{K}Q_\mathrm{S}$, which equals exactly 2 if the three quantum relations are exact. Carrying through the possible corrections gives

$$\begin{aligned} K_\mathrm{J}R_\mathrm{K}Q_\mathrm{S} &= 2\,(1+\varepsilon_\mathrm{J})\,(1+\varepsilon_\mathrm{S})\,(1+\varepsilon_\mathrm{K}) \\ &\approx 2(1+\varepsilon_\mathrm{J}+\varepsilon_\mathrm{S}+\varepsilon_\mathrm{K}), \end{aligned} \tag{4}$$

where the second line relies on the fact that each ε term is much less than 1.

Because equation 4 involves possible corrections to all three quantum electrical effects, a specific result for a QMT experiment is subject to interpretation. How one treats the possibility of multiple corrections of opposite sign depends on how much confidence one has in each of the individual legs. Since the current status of each leg is quite different [3], this confidence is a function of the uncertainty assigned to the experimental result that one wants to interpret. Thus while the QMT is a compact and elegant way to express the effect of possible corrections, in practice it must be supplemented by tests that isolate one leg at a time.

3. A Triangle to Measure Q_S

Figure 1 illustrates the triangle that is the focus of this paper. The principle that links it together, analogous to Ohm's law for the original form of the QMT, is the equivalence of electrical and mechanical power. The most accurate realization of this equivalence to date is the moving-coil Watt balance. This experiment was first proposed in 1976 [9] and is reviewed in detail in [10, 11]. The equivalence realized by a Watt balance can be written as

$$mgv = \frac{U^2}{R}, \tag{5}$$

where mgv is the mechanical power of a mass m moving at velocity v in a gravitational field with acceleration g, and U^2/R is the electrical power in a coil of wire moving at velocity v through a magnetic field gradient. The equivalence of these powers is established indirectly by performing the measurement in two phases: (1) a static phase in which the current U/R in the coil balances the force mg, and (2) a moving phase in which the motion of the coil at velocity v induces a voltage U in the coil. The key to the accuracy of the Watt balance is the fact that the geometrical factor related to the magnetic field gradient is the same for both phases, and thus cancels out of the final result [10, 11].

§ We ignore the factors n, f_J, i, and f_S here because in practice they are known with negligible uncertainty.

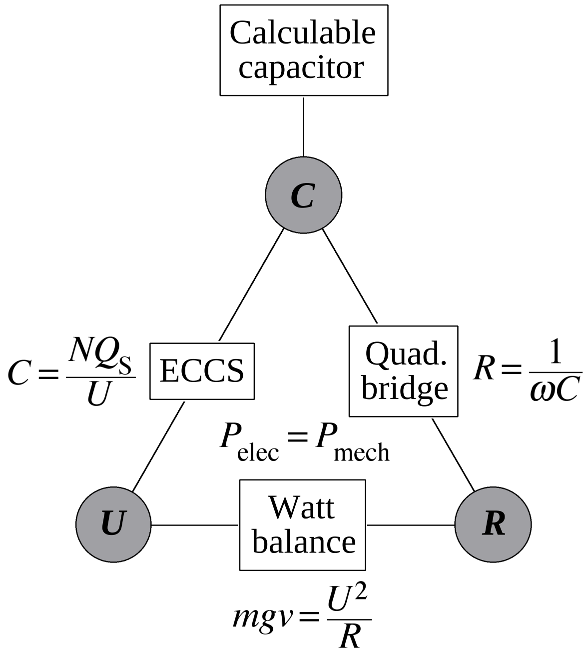

Figure 1. Metrology triangle involving a Watt balance, a calculable capacitor, and an electron counting capacitance standard. This triangle is built on the equivalence of electrical and mechanical power.

The left side of the triangle in figure 1 involves an electron counting capacitance standard (ECCS) [12, 13, 14] in which a known number of electrons N is placed onto a cryogenic, vacuum-gap capacitor [15], generating a voltage U across the capacitor. From the definition of capacitance, the defining relation for this side is $C = NQ_S/U$. The right side of the triangle uses a quadrature bridge at angular frequency ω to balance the imaginary impedance $1/\omega C$ of a capacitor with the real impedance R of a resistor [16]. Since the quadrature bridge necessarily operates at finite frequency (typically $\omega = 10^4$ rad/s), while C in the ECCS and R in the Watt balance are effectively dc values, the frequency dependence of both C and R must be known. For C, an upper bound for the frequency dependence has been determined using a combination of direct measurements and a model for the dielectric films presumed responsible for the frequency dependence [17]. For R, a coaxial resistor having a calculable ac/dc difference can be used [18, 16].

The following thought experiment illustrates how the triangle in figure 1 yields a value for Q_S. A Watt balance is operated in the static phase with a known mg and with R chosen so that the current in the coil at the balance condition produces a voltage U across R. In the moving phase, the velocity v is chosen so that the voltage induced in the coil is again U. From equation 5, we have

$$U = (mgvR)^{1/2}. \tag{6}$$

A quadrature bridge is then used to link R with a capacitance C, giving

$$U = \left(\frac{mgv}{\omega C}\right)^{1/2}. \tag{7}$$

Finally, the capacitance C is charged by an SET device, with the number of charge quanta N chosen so that the voltage generated is again precisely U. This gives

$$U = \frac{NQ_\mathrm{S}}{C}, \tag{8}$$

and setting this equal to equation 7 yields

$$\frac{NQ_\mathrm{S}}{C} = \left(\frac{mgv}{\omega C}\right)^{1/2} \tag{9}$$

$$Q_\mathrm{S} = \frac{1}{N}\left(\frac{mgvC}{\omega}\right)^{1/2}. \tag{10}$$

It is clear that equation 10 will give a value for Q_S in SI units if all quantities on the right side are measured in SI units. The mechanical quantities coming from the Watt balance are measured in terms of the SI kilogram, meter, and second [10, 11]. A Thompson-Lampard calculable capacitor [19] gives a capacitance of $(\epsilon_0 \ln 2/\pi)L$, where ϵ_0 is the electric constant and L is the displacement of its guard electrode, and thus gives C in terms of the SI farad if L is measured in meters. For the purposes of this illustration we can choose L so that the calculable capacitor gives the same value of C used in the other parts of the triangle. It is worth noting that the calculable capacitor is not essential here. Other means of bringing an SI electrical unit (volt, ampere, *etc.*) into the triangle would also yield an SI value for Q_S. In other words, the triangle in figure 1 needs to be "anchored" to an SI electrical unit, and this could be done at any of the three vertices. However, all other realization experiments for SI electrical units currently have much larger uncertainty than calculable capacitors.

The thought experiment leading to equation 10 intentionally avoided any use of the Josephson and quantum Hall effects in order to make it clear that the final result is completely independent of any assumptions about the exactness of these effects. In an actual experiment, the limitations of the Watt balance, ECCS, and calculable capacitor do not allow the same values of voltage and capacitance to be used for all parts of the experiment. Thus in practice one would use a programmable JVS [20] for some of the voltage measurements, and a QHR system might be involved in the link between the Watt balance and the calculable capacitor (especially if different parts are far apart in space and time). However, in this case it is only the *stability and universality* of the JVS and QHR that are exploited, not their connections to h and e.

Single-electron tunneling enters the triangle in figure 1 as a *charge* source that links the Watt balance directly to the calculable capacitor. It is also possible to obtain a value for Q_S using an SET *current* source. This approach, which is less direct than figure 1 but leads to a result that is equally independent of the Josephson and quantum Hall relations, is described in the Appendix.

4. Best Value for Q_S to Date

The measurements needed to realize the triangle in figure 1 have in fact been completed, although not for this purpose and not at the same time and place. They are the following:

(i) Several results for R_{K} in SI units have been obtained by using a quadrature bridge and ac/dc resistor to compare the impedances of a calculable capacitor and a QHR device. The weighted mean of five such results is (see equation 200 of [21])

$$R_{\mathrm{K}} = 25\ 812.808\ 18(47)\ \Omega \quad [18 \times 10^{-9}]. \tag{11}$$

(Here the number in parentheses is the standard uncertainty referred to the last digits of the quoted value, and the number in square brackets is the relative standard uncertainty. All uncertainties given in this paper are standard uncertainties.)

(ii) Three Watt balance results are included in the most recent CODATA adustment of fundamental constants [22]. The Josephson and quantum Hall effects are always used in a Watt balance experiment (this is what allows it to link the kilogram to the Planck constant [10, 11]), and it is common to express the result of the experiment as a value for the quantity $K_{\mathrm{J}}^2 R_{\mathrm{K}}$. It is then clear that combining the value of R_{K} in equation 11 with the weighted mean of the three Watt balance results will yield a weighted mean value for K_{J}. This has been done, and the result is given in equation 290 of [22],

$$K_{\mathrm{J}} = 483\ 597.8865(94)\ \mathrm{GHz/V} \quad [19 \times 10^{-9}]. \tag{12}$$

It is important to note that this value of K_{J} is independent of the relations $K_{\mathrm{J}} = 2e/h$ and $R_{\mathrm{K}} = h/e^2$, whereas the recommended value of K_{J} resulting from a least-squares adjustment is not. ‖

(iii) An ECCS experiment has recently been completed [14]. The result can be expressed as a ratio of the values for a cryogenic capacitor measured in two ways. C_0 is determined by comparison to a calculable capacitor with an ac bridge, while C_{ECCS} is determined by charging the capacitor with a known number of charge quanta and measuring the resulting voltage. Furthermore, C_{ECCS} is defined for $Q_{\mathrm{S}} = e$ and $K_{\mathrm{J}} = 2e/h$ exactly. The ratio is then [14]

$$\frac{C_0}{C_{\mathrm{ECCS}}} = \frac{Q_{\mathrm{S}} K_{\mathrm{J}}}{2e^2/h} = (1 + \varepsilon_{\mathrm{S}})\,(1 + \varepsilon_{\mathrm{J}})\,, \tag{13}$$

and the experimental result for this ratio is [14]

$$\frac{C_0}{C_{\mathrm{ECCS}}} - 1 = (-0.10 \pm 0.92) \times 10^{-6}. \tag{14}$$

‖ We caution the reader that the values of input data in the tables of [21] and [22] have been truncated to show only the significant digits, while calculations of weighted means (and the least-squares adjustment itself) rely on values having more digits. The calculations also take account of covariances when appropriate. Thus calculating an accurate weighted mean for several input data requires more information than is contained in the tables. The complete values of most input data, as well as their covariances, are available at *http://physics.nist.gov/constants*.

Combining equations 12, 13, and 14, and using the 2006 recommended value for $2e^2/h$ [22], ¶ we obtain the following value for Q_S,

$$Q_\mathrm{S} = 1.602\ 1763(15) \times 10^{-19}\ \mathrm{C}\ \ [0.92 \times 10^{-6}]. \tag{15}$$

Although JVS and QHR standards were involved for practical reasons in the three types of experiments used to obtain this value of Q_S, it is nevertheless independent of the relations $K_\mathrm{J} = 2e/h$ and $R_\mathrm{K} = h/e^2$. Finally, using equation 3 and the 2006 recommended value for the elementary charge [22], we find the following value for a possible correction to the SET charge quantum,

$$\epsilon_\mathrm{S} = (-0.09 \pm 0.92) \times 10^{-6}. \tag{16}$$

Unlike the value of Q_S in equation 15, this value of ϵ_S is not strictly independent of the Josephson and quantum Hall relations. The recommended value of e depends strongly on the value of h, and the Watt balance results that determine h are treated, for the purposes of the least-squares adjustment, under the assumption that the Josephson and quantum Hall relations are exact. However, there is little reason to question this assumption at the current uncertainty of Q_S [3, 22], so we may still draw the conclusion that $Q_\mathrm{S} = e$ within 0.9 parts in 10^6.

5. Conclusion

The triangle described here provides the first measurement of Q_S in SI units that is independent of any assumptions about possible corrections to the Josephson and quantum Hall effects. Using the best existing results yields a value that agrees with the recommended value of e within a relative standard uncertainty of slightly less than 1 part in 10^6. A direct test now exists for each of the legs of the original quantum metrology triangle, which should facilitate the interpretation of forthcoming experimental results.

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Appendix A. Determining Q_S by using an SET current source

In the ECCS, an SET pump is programmed to operate for a specified number of cycles, but this same device can also be run continuously to produce a specified current. Other types of SET devices that operate only in the continous mode may eventually be able

¶ Since $2e^2/h = 4\alpha/\mu_0 c$, with the magnetic constant μ_0 and the speed of light c defined constants in the SI, it is proportional to the fine structure constant α. The results that determine the recommended value of α come from experiments involving the electron magnetic moment anomaly and the photon recoil of atoms [22], and are completely independent of the Josephson and quantum Hall relations.

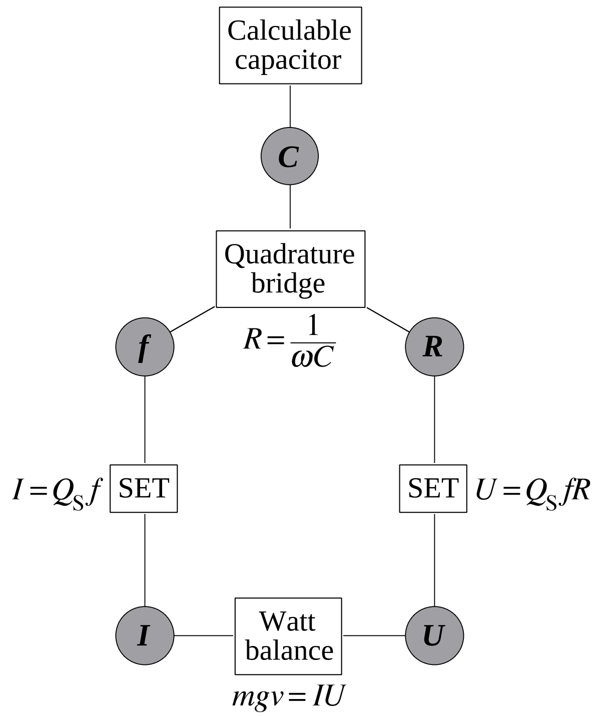

Figure A1. Links involving a Watt balance, a calculable capacitor, and an SET current source that can be used to determine Q_S. We assume a known relation between f and ω.

to deliver much larger currents than an SET pump [8, 5]. Thus a measurement of Q_S based on an SET current source is also of interest, and figure A1 shows the links needed to realize such a measurement. The Watt balance relation is expressed in terms of a current and a voltage, and an SET device links the current to a frequency on the left side and the voltage to a resistance on the right side. The frequency and resistance are then linked to a calculable capacitor (which again anchors the experiment to an SI electrical unit) via a quadrature bridge. The equations for these links are

$$\begin{aligned} mgv &= (Q_S f)(Q_S f)R \\ &= (Q_S f)^2(1/\omega C), \end{aligned} \tag{A.1}$$

$$Q_S = \left(\frac{mgv\omega C}{f^2}\right)^{1/2}. \tag{A.2}$$

In an idealized thought experiment, the SET source could drive the current through the coil of the Watt balance in the static phase with f chosen to balance the force mg. The velocity in the moving phase and the other parameters could be chosen to close the loop, analogous to the discussion in section 3. In a real experiment, JVS and QHR devices would again be used for their stability and universality, but the result for Q_S is completely independent of the exactness of the quantum relations for these effects.

The measurements with SET current sources needed to determine a value for Q_S from figure A1 have not yet been done. Ongoing efforts at LNE and NIST, as well as other national measurement institutes, are aimed at performing such measurements in the future. As always, having multiple paths to the same result will allow checks of systematic errors, which will be substantially different for the experiments shown in figures 1 and A1.

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