Generating Functions

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1 The Moment Generating Function

Definition 1 *The Moment Generating Gunction (mgf) of the random variable X with cdf F_X is the Laplace Transform of F_X,*

$$M(t) := E\left[e^{Xt}\right] = \int_{-\infty}^{\infty} e^{xt} dF,$$

defined for those values of t for which the integral exists. If the expectation does not exist even in a neighborhood of 0, we say that the moment generating function does not exist.

Note that $\frac{d}{dt}M(t) = \frac{d}{dt}\int_{-\infty}^{\infty} e^{xt}dF = \int_{-\infty}^{\infty} \frac{de^{xt}}{dt}dF = \int_{-\infty}^{\infty} xe^{xt}dF = E\left[Xe^{tX}\right]$. Thus, $\frac{d}{dt}M(t)\big|_{t=0} = E\left[Xe^{tX}\right]\big|_{t=0} = E[X]$. Analogous, we can stablish that $\frac{d^n}{dt^n}M(t)\big|_{t=0} = E\left[X^n e^{tX}\right]\big|_{t=0} = E[X^n]$[1].

Theorem 2 *If X has mgf $M_X(t)$, then*

$$E[X^n] = M_X^{(n)}(0) = \left.\frac{d^n}{dt^n}M_X(t)\right|_{t=0}.$$

Remember that two random variables with different pdfs can have the same moments (all of them), unless they have bounded support. That is, the moment generating function does not uniquely determines the distribution[2].

Theorem 3 *Let F_X and F_Y be two cdfs all of whose moments exist: (i) if X and Y have bounded support, then $F_X = F_Y$ if and only if $EX^r = EY^r, r = 0, 1, 2, ...$; (ii) if the moment generating function exist and $M_X(t) = M_Y(t)$ for all t in some neighborhood of 0, then $F_X = F_Y$.*

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[1] See Casella (2002) for more on deriving under the integration sign (Leibniz´s Rule).

[2] A sufficient condition for the moment sequence to be unique is Carleman´s Condition (Chung, 1974): $\sum_{r=1}^{\infty} \frac{1}{\left(\mu'_{2r}\right)^{\frac{1}{2r}}} = +\infty$.

Theorem 4 $M_{aX+b}(t) = e^{bt} M_X(at), \forall a, b \in \mathbb{R}$.

Theorem 5 ***(Convergence of mgfs)*** *Let $\{X_i\}$ be a sequence of random variables, each with mgf $M_{X_i}(t)$ and suppose that $\lim_{i\to\infty} M_{X_i}(t) = M_X(t)$, for all t in a neighborhood of 0, and that $M_X(t)$ is a mgf. Then there is a unique cdf F_X whose moments are determined by $M_X(t)$ and, for all x where $F_X(x)$ is continuous, we have $\lim_{i\to\infty} F_{X_i}(x) = F_X(x)$. That is, convergence, for $|t| < h > 0$, of mgfs implies convergence of cdfs.*

2 Cumulant Generating Function

For a random variable X, the cumulant generating function $S(t)$ is defined as $S(t) := \ln[M_X(t)]$. This function can be used to generate the cumulants of X. The cumulants are (rather circuitously) the coefficients in the Taylor series of the cumulant generating function. The first three cumulants are equal to the central moments[3]. That is,

$$k_1 = \left.\frac{d}{dt}S(t)\right|_{t=0} = EX = \mu_1,$$

$$k_2 = \left.\frac{d^2}{dt^2}S(t)\right|_{t=0} = VarX = \mu_2,$$

$$k_3 = \left.\frac{d^3}{dt^3}S(t)\right|_{t=0} = E\left[(X-EX)^3\right] = \mu_3,$$

and, more generally,

$$k_r := \left.\frac{d^r}{dt^r}S(t)\right|_{t=0}.$$

3 The Characteristic Function

Definition 6 *The Characteristic Function (cf) of the random variable X with cdf F_X is the Fourier Transform of F_X,*

$$\phi(t) := E\left[e^{iXt}\right] = \int_{-\infty}^{\infty} e^{ixt} dF,$$

where $i^2 = -1$, that is, the complex i.

Note that

$$\phi^{(r)}(0) = \left.\frac{d^r}{dt^r}\phi(t)\right|_{t=0} = E\left[(iX)^r\right] = i^r \mu_r',$$

that is,

$$\mu_r' = \frac{\phi^{(r)}(0)}{i^r},$$

[3] For further information, look for "cumulant" in http://mathworld.wolfram.com

where $\mu_r' = E[X^r]$ is the rth moment of X[4].

The power series expansion of e^{iXt} is

$$e^{iXt} = \sum_{r=0}^{\infty} \frac{(iXt)^r}{r!},$$

then the characteristic function can also be written as

$$\phi(t) = \sum_{r=0}^{\infty} \frac{E[(iXt)^r]}{r!} = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \mu_r'.$$

Thus, if the rth exists, it is generated as the coeficient of $\frac{(it)^r}{r!}$ in the infinite series expansions of $\phi(t)$.

Because $\left\| e^{ixt} \right\| = \|\cos(xt) + i \sin(xt)\| = 1$ and a probability density integrates 1, the characteristic function always exists even though the moment generating function may not exist. The characteristic function completely determines the distribution, that is, every cdf has a unique characteristic function.

Theorem 7 ***(Uniqueness Theorem)*** *Two distribution functions are identical if and only if their characteristic functions are also identical.*

Theorem 8 *A characteristic function is uniformly continuous on the real line.*[5]

Theorem 9 ***(Inversion Theorem)*** *Let $F(x)$ be the distributon function, assumed continuous, and $\phi(t)$ be the corresponding characteristic function. Then, $\forall x \in \mathbb{R}, \forall h \in \mathbb{R}^+$,*

$$F(x+h) - F(x-h) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{1-e^{-ith}}{it} e^{-itx} \phi(t)\, dt,$$

provided that $x-h$ and $x+h$ are continuity points of $F(x)$.

Theorem 10 ***(Inversion Formula)*** *If a characteristic function $\phi(t)$ is absolutely integrable over $(-\infty, +\infty)$, then the corresponding distribution function $F(x)$ is absolutely continuous and the corresponding density function (which is continuous) is*

$$f(x) = F'(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t)\, dt,$$

which is the Fourier Inverse Transform of $\phi(t)$.

Thus, note that to each characteristic function $\phi(t)$ there exists a unique distribution function $F(x)$.

Theorem 11 ***(Convergence of Characteristic Functions)*** *Let $\{X_i\}$ be a sequence of random variables, each with characteristic function $\phi_{X_i}(t)$ and suppose that $\lim_{i\to\infty} \phi_{X_i}(t) = \phi_X(t)$, for all t in a neighborhood of 0, and that $\phi_X(t)$ is a characteristic function. Then for all x where $F_X(x)$ is continuous, we have $\lim_{i\to\infty} F_{X_i}(x) = F_X(x)$. That is, convergence, for $|t| < h > 0$, of characteristic functions implies convergence of cdfs.*

[4] Several exercises can be found in Ramanathan (1993).

[5] The proof is in Ramanathan (1993).

4 Factorial Moment Generating Function

Definition 12 *The factorial moment generating function of the random variable X is defined as $E\left[t^X\right]$, if the expectation exists. Note that*

$$\left.\frac{d^r}{dt^r}E\left[t^X\right]\right|_{t=1} = E\left[X\left(X-1\right)...\left(X-r+1\right)\right],$$

where the right-hand side is the rth factorial moment of X.

5 Probability Generating Function

Definition 13 *If X is a discrete random variable, then its factorial moment generating function is called the probability generating function and we can write $E\left[t^X\right] = \sum_x t^x P\left(X=x\right)$. Note that the coefficients of the power series give the probabilities,*

$$\left.\frac{1}{k!}\frac{d^k}{dt^k}E\left[t^X\right]\right|_{t=0} = P\left(X=k\right).$$

References

[1] Casella, G. and Roger L. Berger, 2002. Statistical Inference, second edition. Duxbury Advanced Series.

[2] Feller, William, 1957. An Introduction to Probability Theoru and Its Applications. Volume I and II. Wiley International Edition.

[3] Gnedenko, Boris V., 1978. The Theory of Probability. Mir Publishers Moscow.

[4] Ramanathan, Ramu, 1993. Statistical Methods in Econometrics. Academic Press, Inc.