Dynamo Theory and Earth's Magnetic Field

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1 Introduction

The Earth's magnetic field belongs in that class of physical phenomena which are commonplace yet also very complex. It is commonplace in that almost everyone knows that it exists, yet not very many people (probably including a good percentage of physicists) know how it works. The theory behind the generation of a magnetic field, known as dynamo theory, is still an active area of research. The equations describing the physics are well known, and most of the work lies in constructing a good model system or approximation. As people have been working on this for a long time, these have gotten very complex.

Since both the problem and the solutions are so complicated, this paper can only begin to scratch the surface of dynamo theory. I've included two models of disc dynamos, which are not directly related to planetary or other fluid dynamos, but they illustrate the basic principles at work (induction and feedback). There is a section that goes through some of the basics of what is known as the kinematic theory (basically a stability analysis), but only a brief mention of the full, dynamic theory is given here. I'll start with a short description of some of the observed properties of Earth's magnetic field.

2 Earth's Field

The Earth's magnetic field is basically a dipole, which is aligned off the rotation axis by about 11 degrees. Historically, this was believed to be caused by some kind of permanent magnetization of material in the earth, and dynamo theory was orginally put forward to explain the sun's magnetic field. However, there are several pieces of evidence which point away from such a theory. Through the combination of seismic measurements (which

give density), and the solar system's abundance of elements, it is theorized that the center of the Earth (known as the core) is primarily made up of iron. There are two sections, a solid inner core, surrouded by a liquid outer core. If there were a permanent magnetization, it would likely be concentrated here. The temperature of the core, though, is estimated to be on the order of 4000 K, which is far above the Curie point of iron (Merrill [3], however, reminds us to not be too hasty about extrapolating our knowledge of materials properties to such extreme temperature and pressure). A much more dramatic demonstration that a permanent magnet theory is wrong was provided by paleomagnetic samples. As rock cools down past its own Curie point, whatever the present magnetic field is gets frozen in. Studies of these samples show not only significant change of the field over time, but actual polarity flips. The flips occur at pretty randomly distributed intervals, which average about $2*10^5$ years, but have been as long as $5*10^7$ years. The flips themselves are short compared to these time scales, taking several thousand years (Moffatt [1]).

This evidence points out the need for a more dynamic theory of magnetic field generation, and the best candidate is known as dynamo theory, or magnetohydrodynamics (MHD). A conductive fluid moving through a magnetic field will feel induced currents due to the changing flux. These will in turn alter the fluid motion. If all this happens in such a way as to reinforce the original magnetic field, we have a succesful dynamo.

3   Single Disc Dynamo

Before we dive into the mess of equations and approximations that describe how fluid motion and magnetic field interact, it is useful to demonstrate a very simple system which exhibits dynamo action. Consider the system shown in Figure 1.

In this setup, a metal disc of radius a is spinning with angular velocity ω due to a constant torque (N). If a magnetic field is present, there will be a induced emf ($\mathcal{E}$) across the disc, which is proportional to the magnetic flux (Φ) times ω. We can show this by integrating the Lorentz force (working in cylindrical coordinates ρ, ϕ, z):

$$\mathcal{E} = \int_0^a (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r} = \omega \int_0^a B_z \rho d\rho = \frac{\omega}{2\pi} \int \mathbf{B} \cdot d\mathbf{a} = \frac{\omega \Phi}{2\pi}$$

A wire (not rotating) is connected electrically via a sliding contact, such as a brush, to the edge of the disc. It is also connected back to the axle to
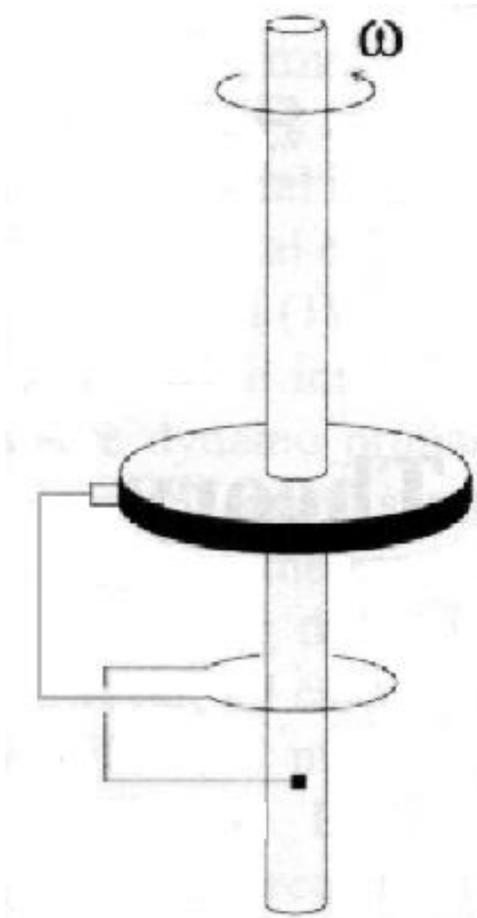
Figure 1: Single-disc dynamo, from Merrill

complete the circuit, through which flows a current (I). The wire is wound in such a way as to create a magnetic flux $\Phi = 2\pi MI$ through the disc, where $2\pi M$ is the mutual inductance between the disc and the wire. We can put these together into a equation for the current:

$$\mathcal{E} = L\frac{dI}{dt} + RI$$

$$L\frac{dI}{dt} + RI = M\omega I \tag{1}$$

In 1, L is the self-inductance of the system, and R is the total resistance.

The motion of the disc is described by $C\frac{d\omega}{dt} = N_{tot}$. C is the moment of inertia of the disc. N_{tot} is the total torque on the disc, and includes both our constant driving torque, N, and the reaction of the current to the magnetic field. We can calculate the second part as follows. Torque, in general, is $\mathbf{N} = \mathbf{r} \times \mathbf{F}$. The force on a infinitesimal section of current ($d\mathbf{l}$) is given by $d\mathbf{F} = Id\mathbf{l} \times \mathbf{B}$. Performing some integrals gives:

$$\mathbf{N} = \int \mathbf{r} \times d\mathbf{F} = -I\hat{\mathbf{z}}\int_0^a B_z\rho d\rho = -\hat{\mathbf{z}}\frac{I\Phi}{2\pi} = -\hat{\mathbf{z}}MI^2$$

Putting this back into N_{tot} gives the second first-order differential equation we need to describe the system:

$$C\frac{d\omega}{dt} = N - MI^2 \tag{2}$$

It is easy to see that this system will spontaeously generate a magnetic field. Imagine starting with $I(0) = 0$. The disc will speed up until eventually $\omega > R/M$. As soon as this condition is met, Eq. 1 is unstable – any small fluctuation in I will be amplified, creating a nonzero $\mathbf{B}$. The system has a equilibrium point at:

$$\omega_{eq} = \frac{R}{M}, \quad I_{eq} = \pm\sqrt{\frac{N}{M}}$$

This is found by setting $\dot{\omega} = \dot{I} = 0$. It is easy to check that this point is stable by linearizing about it. In this steady state situation, the power supplied by the constant torque exactly balances the power lost to ohmic heating. Modifying Eq. 2 by adding a frictional term, $-\kappa\omega$, won't change the situation much, and will probably cause the system to reach equilibrium faster.

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This model obviously isn't meant to be directly analogous to a complicated physical situation like the Earth's core. It is hard to identify the various pieces of our disc dynamo with any realistic fluid motion. However, it demonstrates the basic physics at work in the dynamo effect – motion through an initial magnetic field creates a current, which creates its own magnetic field, reinforcing the original. Moreover, as Moffat [1] points out, even this simple model has characteristics which show up in more complicated theories as well. First, the system exhibits differential rotation. That is, different parts of the conductor are rotating with different angular velocity (the disc with ω, the wire not rotating). This becomes important in a conductive fluid, where it contributes to stretching and distorting the field. Second, the disc dynamo has a certain lack of symmetry. It can support a current (and hence magnetic field) in either direction, as $I \rightarrow -I$ doesn't affect our equations above. However, the rotation and torque must be positive: $\omega \rightarrow -\omega$, $N \rightarrow -N$ changes the equations to a set which don't support a dynamo. If $\omega < 0$ always, then we will also always have $\dot{I} \propto -I$, so no steady-state nonzero current can exist. This shows up later in the so-called "anti-dynamo" theorems, which show how various symmetric situations can never support a dynamo.

4   Kinematic Dynamo Theory

The starting point on the journey into the fluid dynamo problem is what is known as kinematic dynamo theory. This approach considers a conducting fluid which is undergoing motion described by a velocity field $\mathbf{u}$. The approximation is to take $\mathbf{u}$ as given, and calculate what effect it has on the magnetic field. We don't consider the field feeding back (via Lorentz forces) to affect $\mathbf{u}$. Referring back to the simple one-disc dynamo, this approach would be analagous to taking ω as fixed, and solving Eq. 1 for I.

This may sound like a huge oversimplification, and in some sense it is. The theory can't tell us anything about long-term behavior of a dynamo system – it is only valid while $\mathbf{B}$ is weak enough to not affect the fluid motion significantly. However, it does have its uses as well. One huge advantage of the approach is that when we consider $\mathbf{u}$ to be independent of $\mathbf{B}$, the equation describing the system (as we will see) is a *linear* PDE for $\mathbf{B}$. This means, of course, that we can apply all our powerful linear algebra techniques (e.g. spherical harmonic decompositions, eigenvalue analysis, etc. . .) and get analytic results for specific problems. Indeed, up until the recent advances in numerical computing power, most dynamo research concentrated

on kinematic theory, since the full, non-linear theory ("dynamic", or MHD theory) is so hard to work with analytically.

Perhaps the most useful aspect of kinematic theory is that it can tell us what systems *aren't* dynamos. Childress [2] likens the situation to that of stability analysis of critical points in dynamical systems. We apply our given flow $\mathbf{u}$ to a small initial field $\mathbf{B}$, and see whether the system causes $\mathbf{B}$ to grow, or to be damped out. If $\mathbf{B}$ (or some measure of it, such as the total magnetic energy) grows, we say the system is capable of dynamo action, or simply that it is a dynamo. If $\mathbf{B}$ tends to damp out, the system is sometimes called a non-dynamo.

With these remarks in mind, let's go through the basics of this theory.

4.1   The Induction Equation

The first set of equations we need to consider is simply Maxwell's equations:

$$
\begin{aligned}
\nabla \cdot \mathbf{E} &= \rho_{ch}/\epsilon \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\partial \mathbf{B}/\partial t \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J}
\end{aligned}
$$

The only modification that has been made to the standard set of equations is to drop the displacement current ($\partial \mathbf{E}/\partial t$) term in Ampere's Law. Hence we are working in some sort of magnetostatic approximation, where radiation isn't important. This is usually justified by saying that the time scale of the processes we are considering is much longer than the time light takes to travel across the system.

The next equation we need is referred to as the generalized Ohm's Law. It is basically Ohm's Law for a conductor moving with velocity $\mathbf{u}$ through fields $\mathbf{E}$ and $\mathbf{B}$ (see for example, Jackson [4], §5.15). In this case, the expression for the current takes the form:

$$
\mathbf{J} = \sigma \left(\mathbf{E} + \mathbf{u} \times \mathbf{B} \right)
$$

Eliminate $\mathbf{J}$ to get:

$$
\nabla \times \mathbf{B} = \mu_0 \sigma (\mathbf{E} + \mathbf{u} \times \mathbf{B})
$$

Take the curl of the above equation, and use the identity $\nabla \times \nabla \times \mathbf{B} = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -\nabla^2 \mathbf{B}$. This gives:

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$$\frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B} = \nabla \times \mathbf{E} + \nabla \times (\mathbf{u} \times \mathbf{B})$$

Therefore, we have what is known as the induction equation, which is the fundamental equation of kinematic dynamo theory:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \tag{3}$$

In this equation, $\eta = 1/(\mu_0 \sigma)$ is called the magnetic diffusivity. Thus, as promised, we have a linear PDE for $\mathbf{B}$. In general, $\mathbf{u}(\mathbf{x}, t)$ is an arbitrary function. The only requirement it needs to satisfy is continuity:

$$\partial \rho / \partial t + \nabla \cdot (\rho \mathbf{u}) = 0$$

In addition, a useful approximation is often that of a incompressible fluid, for which $\nabla \cdot \mathbf{u} = 0$.

To get a feeling for what each of the terms in Eq. 3 do, we will consider two limits. The first, with $\mathbf{u} = 0$ is purely diffusive. The second, with $\eta = 0$ is purely advective.

4.2 Limiting Case I – Free Decay

In this limit, we will set $\mathbf{u} = 0$ everywhere. Then Eq. 3 reduces to:

$$\partial \mathbf{B} / \partial t = \eta \nabla^2 \mathbf{B} \tag{4}$$

Everyone should recognize this as the standard (vector) diffusion equation. We can look at the time depedence of solutions by breaking the solution up into a sum over decay modes, $\mathbf{B}^\alpha$, each of which has a time dependence $e^{p_\alpha t}$. Say that our conductor is bounded in some region $\mathcal{V}$. Then, inside $\mathcal{V}$, we will have the Helmholtz equation for each mode:

$$(\nabla^2 - p_\alpha / \eta) \mathbf{B}^\alpha = 0 \tag{5}$$

Outside the conductor, we have $\nabla \times \mathbf{B} = 0$, and at $r \to \infty$, $\mathbf{B} \propto 1/r^3$ (ie, a dipole field). By dotting $\mathbf{B}^\alpha$ onto both sides of Eq. 5, and integrating over all space (making clever use of some vector identities – see Jackson [4], inside front cover), we can prove the relation:

$$\frac{p_\alpha}{\eta} = \frac{-\int (\nabla \times \mathbf{B}^\alpha)^2 d^3 x}{\int (\mathbf{B}^\alpha)^2 d^3 x} < 0$$

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So all the modes α decay to zero as $t \to \infty$. A system without fluid motion cannot support a magnetic field indefinitely.

The system where $\mathcal{V}$ is the sphere of radius R can be solved exactly in terms of Bessel functions. I won't go through the math (see Moffatt [1], §2.7), but simply state the result $p_\alpha = -\eta(x_{nq}/R)^2$, where x_{nq} are zeros of $J_{n+1/2}(x)$. We can use this result to estimate the decay time $(\tau = 1/p_{min})$ of a field in the Earth's core. Using values given by Moffatt for the core, $R = 3500 km$, $\eta = 2.6 m^2/s$, we get $\tau \approx 1.5 * 10^4$ years. Since the age of the Earth, and the geological record of the field span a much longer time than this, we can take this calculation as evidence of a fluid dynamo in the core acting to perpetuate the field.

As a last note on free decay, we observe that while the solution to the diffusion equation is a sum of negative exponentials, this doesn't mean that the field decays monotonically to zero everywhere. There can be a initial period where the field grows as a function of time. For example, the function $f(t) = e^{-t} - e^{-2t}$ starts with a positive slope at $t = 0$ before maxing out and decaying as $t \to \infty$. While it might appear at first that diffusion only works to counteract dynamo action, it is actually necessary (as we will see in the next section) in order for the field to escape a bounded conductor by diffusing into the free space surrounding it.

4.3 Limiting Case II – Frozen Flux

The second limiting case of Eq. 3 is that of $\eta = 0$. This is known as the perfectly conducting limit, since we can achieve it by setting $\sigma \to \infty$. As one gets deeper into the theory, there are some subtleties associated with whether you take "$\eta \to 0$" or "$\eta = 0$" (see Childress [2]) – the limit of infinitesimal diffusion can be qualitatively different from that of zero diffusion. However, for our purposes here, it is ok to take the zero diffusion limit, as it will show more clearly the action of the advective term. In this limit, Eq. 3 becomes:

$$\partial \mathbf{B}/\partial t = \nabla \times (\mathbf{u} \times \mathbf{B}) \tag{6}$$

If we expand out the right hand side, and use the constraint mentioned earlier, that $\mathbf{u}$ satisfy the continuity equation, we can get Eq. 6 into the form:

$$\frac{d(\mathbf{B}/\rho)}{dt} = (\mathbf{B}/\rho) \cdot \nabla \mathbf{u} \tag{7}$$

In this equation, $(d/dt) = (\partial/\partial t) + \mathbf{u} \cdot \nabla$. This is known as the material (or convective, or co-moving, ...) derivative, and represents the derivative

of a quantity as seen by an observer moving with the flow of fluid. If the flow is incompressible, the ρ's disappear from Eq. 7.

Let's examine the action of the fluid flow on two neigbouring fluid particles, at positions $\mathbf{x}$ and $\mathbf{x} + d\mathbf{x}$. After a time dt, the two particles will be at $\mathbf{x} + \mathbf{u}(\mathbf{x})dt$ and $\mathbf{x} + d\mathbf{x} + \mathbf{u}(\mathbf{x} + d\mathbf{x})dt$, respectively. We can see that $d\mathbf{x}$ has been translated by the flow, and has undergone a stretching given by $d\mathbf{x} \rightarrow d\mathbf{x} + d\mathbf{x} \cdot \nabla\mathbf{u}dt$. Rewriting this as a derivative gives:

$$\frac{d}{dt}(d\mathbf{x}) = d\mathbf{x} \cdot \nabla\mathbf{u}$$

However, this is exactly the same equation as Eq. 7 for the magnetic field. This tells us that if we pick our small line of fluid, $d\mathbf{x}$, to be parallel to $\mathbf{B}$ at time t_0, they will remain parallel for all t. The magnetic field is simply carried along with the flow of fluid. This field is usually referred to as being frozen or tied into the flow.

As an extension of this idea, imagine a small tube around a certain magnetic field line such that through any cross-section of the tube passes a certain flux, Φ. Now imagine evolving the tube under the flow. As the field lines are simply carried along with the flow, so is the flux tube. However, since the tube (and any section of it) is a perfect conductor, $d\Phi/dt = 0$ (this can be seen easily by using Faraday's Law combined with Ohm's Law and taking the $\sigma \rightarrow \infty$ limit). If the flow tends to stretch out the tube, so that its cross-sectional area decreases, the field lines running through it will have to strengthen in order to keep Φ constant. This is one basic mechanism for how a dynamo acts to strengthen magnetic fields, and is discussed in detail in Childress and Gilbert [2]. For example, image a flux tube that is a torus, which gets stretched, twisted once, and folded back onto itself. We end up with a torus of the same volume but a increased flux. This is the stretch-twist-fold mechanism. Notice that it has somewhat the same type of asymmetry as the single disc dynamo – the twist and fold must be in the correct direction for the magnetic field to reinforce itself, rather than cancel itself out.

Let's now consider the case of a perfect condcutor bounded in a (simply-connected) region $\mathcal{V}$, with surface $\partial\mathcal{V}$. While the system may indeed be amplifying magnetic fields through processes such as those mentioned above, it can't amplify fields outside $\mathcal{V}$. Since field lines are tied to the flow, any closed field line inside $\mathcal{V}$ will remain inside. Also, since the component of $\mathbf{B}$ normal to the surface (B_n) must be continuous, the flux through any piece of the surface is conserved (as it moves with the flow). We can therefore

sum up the regions where B_n is positive and negative separately, and get a conserved quantity (called the pole strength):

$$F_n = \int_{\partial \mathcal{V}} |B_n| dS$$

All the system can do is rearrange the flux coming out of it, not create new flux. Therefore, the utility of the perfect conductor in modelling actual physical systems is limited.

4.4 Anti-Dynamo Theorems

As mentioned earlier, many of the results of kinematic theory are restrictions on what combinations of motion and magnetic field will produce a dynamo. This kind of thinking was initiated by T. Cowling back in 1934, who proved that a dynamo can't exist if both $\mathbf{u}$ and $\mathbf{B}$ are axisymmetric about the same axis (say $\hat{\mathbf{z}}$). Some more of these "anti-dynamo" theorems are given in Moffatt [1] – purely toroidal motion (ie, $\mathbf{u} = u(\mathbf{x}, t)\hat{\phi}$ in spherical coordinates) can't produce a dynamo, and neither can purely planar motion ($\mathbf{u} = u_x \hat{\mathbf{x}} + u_y \hat{\mathbf{y}}$). These theorems are useful, but also frustrating, as they eliminate a lot of the easy choices one would like to study.

4.5 Modelling

There are many complicated models that try to represent an actual planetary dynamo system. Rather than go into all the gory details, we'll try to use some qualitative understanding of the induction equation to describe what's going on (at least, what's *possibly* going on).

Since we are going to have to consider both diffusive and advective action, we need some measure of the relative importance of the two terms. This is given by what's known as the magnetic Reynolds number, R_m:

$$\frac{\nabla \times (\mathbf{u} \times \mathbf{B})}{\eta \nabla^2 \mathbf{B}} \sim \frac{uL}{\eta} = R_m$$

In this (somewhat shady) definition, we say that the system has some characteristic speed u and length L. We then combine these with η to get the dimensionless R_m. Large R_m implies that advection dominates, while small R_m implies diffusion dominates. It seems to be the commonly held belief that most astrophysical systems (planets, stars) have large R_m, due to the large sizes and conductivities, so this is the view we'll take. When doing a more careful analysis, it would be good to be more specific about
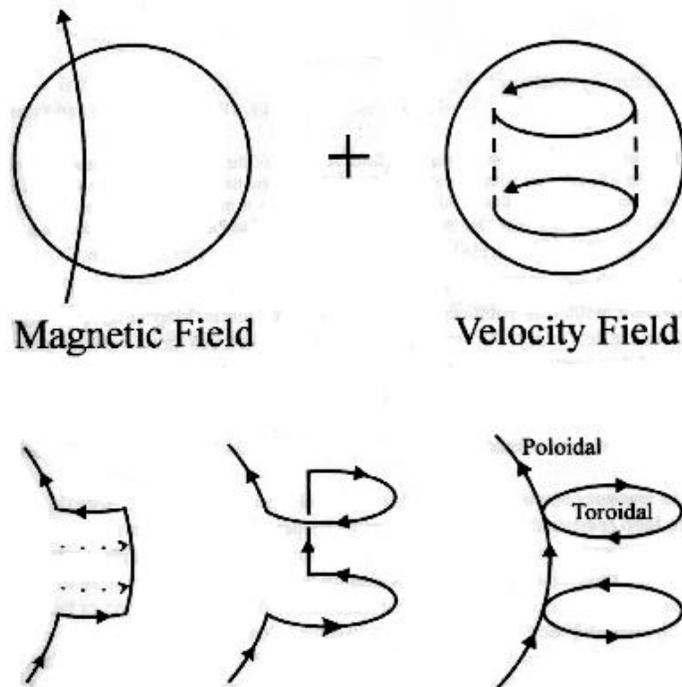
Figure 2: The ω effect, from Merrill

things like "length scale", since a problem could easily have several length scales of interest.

The effect we'll describe is mainly advective – the diffusion won't be mentioned explicitly, but it serves to transport flux out of the conductor and to smooth out small-scale knottedness of the field lines by breaking and reconnecting them (a process which can't happen without diffusion). The field won't become arbitrarily convoluted.

Refer to Figure 2. This shows a rotating sphere of conducting fluid. Imagine it has some differential rotation – for example, it could be rotating faster at the center. The inital magnetic field (known as a "poloidal" field) gets dragged along by the fluid motion, and sheared due to the differential rotation. This eventually winds around, creating a toroidal field. This is known as the ω-effect. The next part of the process (not shown) involves a upwelling of fluid, such as in thermal convection. This upwelling gains some helicity from the Coriolis force, and hence it pushes out and twists a section

of the toroidal field, which ends up reinforcing the original poloidal field. This is one version of what's called the α-effect. Together, this is called the $\alpha\omega$-effect (clever, huh?).

One question which comes up is that of where the energy to drive this process comes from. The answer (of course) is that this is still a debated question. One possibility which would work well with our $\alpha\omega$ model is that iron is solidifying from the (liquid) outer core onto the (solid) inner core. This releases some latent heat of crystallization, as well as previously trapped lighter elements, both of which would contribute to a upwelling. Of course, there are many other possible energy sources such as tidal heating, and radioactive decays. Merrill [3] lists most of them.

5   Dynamic Theory

If any true progress is to be made in understanding the dynamo process, one must look at the full dynamic theory. In this theory, the induction equation, Eq. 3 still holds, but we now need to add in the Navier-Stokes equation to describe the motion of the fluid. It will look something like:

$$\rho\frac{d\mathbf{u}}{dt} = -2\rho(\Omega \times \mathbf{u}) - \nabla P + \nu\nabla^2\mathbf{u} - \rho\nabla\phi_g + \mathbf{J} \times \mathbf{B} \qquad (8)$$

The terms I've put in above are (from left to right, after the equals): Coriolis force, pressure, viscosity, gravity, and the Lorentz force. One could easily modify this to include more things, such as buoancy and heat flows. Obviously this is a big mess which is best left to computers to solve. Some other people who were thinking along those lines were Glatzmaier and Roberts [5], who performed a huge simulation in 1995 which came out looking a lot like the actual field of the Earth, and even included a reversal. Some neat movies from their simulation can be found at http://www.psc.edu/research/graphics/gallery/geodynamo.html.

The subject of dynamics is of great importance to the problem, and it is where most of the current research is at. Unfortunately, I wasn't able to learn enough of it in time to say many useful things here. The next section, however, presents a much simpler dynamical system which exhibits some interesting behaviour, including spontaneous polarity reversals.
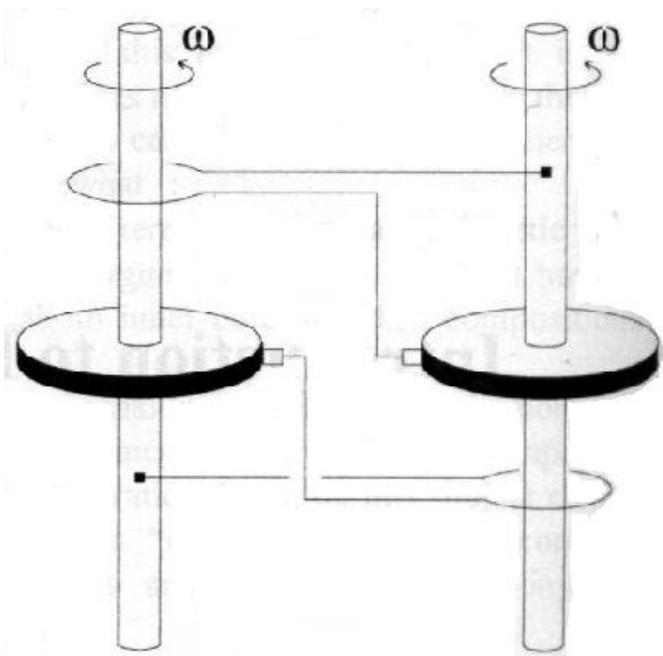
Figure 3: Two-disc dynamo, from Merrill

6 Double-Disc Dynamo

In the face of all this complexity, it's time once again to retreat back to a much simpler, finite-dimensional model. Again, the system is a disc dynamo, only now we've coupled two discs together (refer to Fig. 3 to see the system). This model, like the single disc dynamo described at the beginning of the paper, is hard to relate exactly to the real physical system, but does show us some qualitatively correct effects. It also provides a lot of interesting dynamics. The single disc model, while non-linear, was essentially a one degree of freedom system, so it couldn't act chaotically. Adding another disc keeps the non-linearity, while adding another degree of freedom, so we start seeing chaotic effects. In fact, the behaviour of this system over time looks a lot like the measured behaviour of the Earth's field – it includes both oscillations about average values, and polarity flips at seemingly random intervals.

We'll be considering a symmetric two-disc system – Both disc-loop systems have the same mutual (M)and self (L) inductance, resistance (R), inertia (C) and are acted on by the same constant torques (N). $I_{1,2}$ and $\omega_{1,2}$ refer to the current through and rotation of each disc (1 or 2). The set of equations that we get, following the same kind of steps as in the single disc case is:

$$
\begin{aligned}
L\frac{dI_1}{dt} + RI_1 &= M\omega_1 I_2 \\
L\frac{dI_2}{dt} + RI_2 &= M\omega_2 I_1 \\
C\frac{d\omega_1}{dt} &= N - MI_1 I_2 \\
C\frac{d\omega_2}{dt} &= N - MI_1 I_2
\end{aligned}
$$

The system could, of course, be made less symmetric, but since the symmetric system already shows a lot of interesting behaviour, we'll stick with it for the sake of simplicity.

To help the analysis, we'll go to a set of dimensionless variables, defined by:

$$\tau = (NM/CL)^{1/2}t, \quad X, Y = (M/N)^{1/2}I_{1,2}, \quad Z, V = (CM/NL)^{1/2}\omega_{1,2}$$

This results in the prettier system of equations:

$$\begin{aligned}
\dot{X} + \mu X &= ZY \\
\dot{Y} + \mu Y &= VX \\
\dot{Z} = \dot{V} &= 1 - XY
\end{aligned} \qquad (9)$$

In these, the constant μ is given by $\mu = (C/NLM)^{1/2}R$, and "dot" now refers to $d/d\tau$.

We can immediately see one conserved quantity of the system: Since $\dot{Z} - \dot{V} = 0$, we have a conserved quantity $A = Z - V$. This reduces the phase space from four to three dimensions, leading to easier visualization. We can view the system as living in the 3-dimensional phase space (X, Y, Z), with parameters A and μ. The system evolves according to a "velocity field" $\mathbf{u} = (\dot{X}, \dot{Y}, \dot{Z})$. It is easy to work out from the differential equations that $\nabla \cdot \mathbf{u} = -2\mu$. This means that any volume will eventually shrink to zero. As it turns out, there is a flat (seemingly 2-D) attractor for the system, which looks a lot like the famous Lorenz attractor (see Fig. 5).

The steady state of this system can by found by setting all the time derivatives to zero, and solving for the state of the system. We get the two equations:

$$XY = 1, \quad ZV = \mu^2$$

We can write the solution of this as:

$$X = \pm K, \quad Y = \pm K^{-1}, \quad Z = \mu K^2, \quad V = \mu K^{-2}$$

In these equations, K is related to our earlier parameter A by $A = \mu(K^2 - K^{-2})$.

This system has some of the same qualitative features of the single disc dynamo. A (non-zero) steady state can occur at both a positive and negative value of the current, but only for positive values of the rotation. However, the difference between these two systems lies in the dynamics. This system is pretty easy to solve numerically on a computer, and I've produced a couple plots, using the values $\mu = 1$, and $K = 2$. Figure 4 shows the behaviour of the current, X, as a function of time. It is pretty obviously chaotic – it switches from oscillating about the positive equilibrium to the negative one at random looking intervals. Figure 5 shows a projection of the trajectory onto the YZ plane.
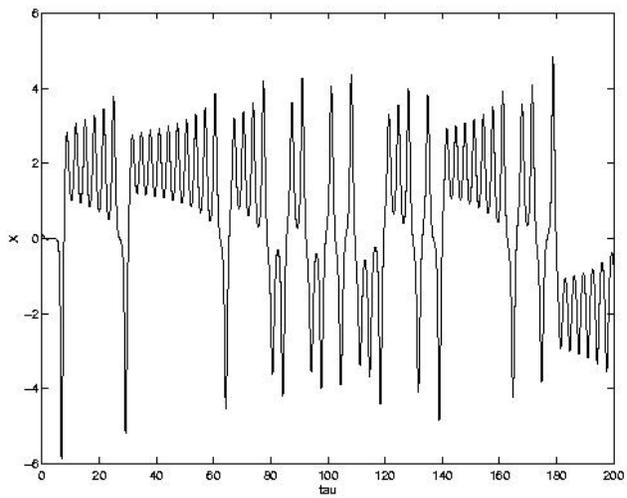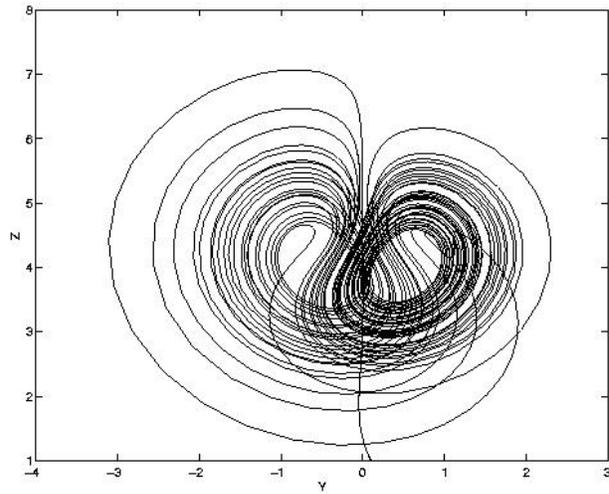
Figure 4: Plot of $X(\tau)$.

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Figure 5: Projection of the trajectory into the YZ plane.

7 Conclusions

Hopefully this paper shed a little light on how planetary magnetic fields are generated. I chose this topic because I realized that I really had no idea how it worked. I think I've at least started to learn the basics of this complicated subject.

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