CONFORMAL THEORIES, CURVED PHASE SPACES, RELATIVISTIC WAVELETS AND THE GEOMETRY OF COMPLEX DOMAINS

R. COQUEREAUX[1] and A. JADCZYK[2]

Centre de Physique Théorique, Section 2, Case 907, Luminy, 13288, Marseille, France.

Received 28 December 1989
Revised 24 April 1990

We investigate some aspects of complex geometry in relation with possible applications to quantization, relativistic phase spaces, conformal field theories, general relativity and the music of two and three-dimensional spheres.

1. Introduction

Complex manifolds and in particular classical domains have been studied for many years by mathematicians and theoretical physicists. The very old division of branches of Mathematics between Algebra, Analysis and Geometry is rather arbitrary since all these aspects are inter-related but it remains that it has some deep psychological influence which explains why it is more a classification of mathematicians than a classification of mathematics. For instance, we cannot say that the study of complex domains (and in particular Cartan classical domains) belongs more to the realm of analysis than to the one of algebra or of geometry but, it is clear that most mathematical articles dealing with the subject fall into one of these three families. Often, articles belonging to a given category do not refer to papers dealing with the same subject but written from a different point of view. The same mathematical objects (Cartan classical domains) have been studied—often without noticing it explicitly—by theoretical physicists interested in a variety of different topics: particle physics, quantum field theory, quantum mechanics, statistical mechanics, geometric quantization, accelerated observers, general relativity and even harmony and sound analysis. The present paper is written for those who like cross-disciplinarity both in mathematics and in physics. Most of the results that we will give are already known by some people (sometimes by many) but we hope that the references to be found here will help those who believe that looking at a familiar object from a different point of view can be fruitful. Among those topics that we looked at for some domains and have not found elsewhere, let us mention the following: Poincare-Cartan momentum map and action of the conformal group in the future tube taken as a phase-space (contrasted with the conformal group

[1] Bitnet: Coque at FRCPN11
[2] On leave from the University of Wrocław, Wrocław, Poland.

Reviews in Mathematical Physics Volume 2 No 1 (1990) 1–44

action on space-time), Riemannian geometry of the future tube for its natural Kähler metric, generalization of wavelet analysis to arbitrary dimensions (relativistic music!) in relation with Bergman kernels, Weyl-Berezin calculus on Cartan domains (coherent states). Adopting a particular style for the present paper was not easy because we do not know, a priori, the motivations of the potential reader; also, the main mathematical object to be discussed here has many definitions; our task is precisely to try to present them. Because of a personal prejudice (and because we want to go from the general to the particular), we have decided to start from the geometry of complex manifolds and then specialize to the case of homogeneous spaces for Lie groups; we get classical domains in this way. Their relation to space-time (the Shilov boundary of such a domain) or to phase-space, is discussed at a later stage.

2. Cartan Domains

2.1. *From complex manifolds to classical domains*

2.1.1. *Terminology and a few standard theorems on complex manifolds*

Here, we suppose that the reader is familiar with differential and riemannian geometry but not with complex geometry (cf. [34], [11], [8]). An almost complex manifold is a manifold endowed with an almost complex structure. An almost complex structure on a manifold M is a field of tensors J which, for every x in M is an endomorphism of the tangent space T_xM such that $J^2 = -1$. Alternatively, it can be considered as a reduction of the frame bundle of M from the structure group $GL(2n, R)$ to the group $GL(n, C)$. An almost complex manifold is automatically even dimensional and orientable (actually oriented). A map between two almost complex manifolds is almost complex if its tangent intertwines the almost complex structures of the source and of the target. A manifold is a complex manifold if it is defined by an atlas with holomorphic change of charts. An almost complex manifold is complex if the Nijenhuis bracket N of the almost complex structure with itself vanishes, i.e. if the almost complex structure is integrable. When N is not zero, it is called the Nijenhuis torsion of the almost complex structure. An almost complex map between two complex manifolds is automatically holomorphic. An almost complex structure allows one to define a bi-graduation in the algebra of exterior forms (or more generally bundle-valued p-forms) but, in a complex manifold, one can furthermore decompose the exterior derivative d as $d = \delta + \bar{\delta}$ where $\delta : \Lambda^{p,q} \to \Lambda^{p+1,q}$ and where $\bar{\delta} : \Lambda^{p,q} \to \Lambda^{p,q+1}$. If M is a complex manifold and $z^1, \ldots z^n (z^i = x^i + iy^i)$ a chart on M, one introduces $dz^i = dx^i + idy^i$, $d\bar{z}^i = dx^i - idy^i$ and $\partial/\partial z^i = 1/2(\partial/\partial x^i - i\partial/\partial y^i)$, $\partial/\partial\bar{z}^i = 1/2(\partial/\partial x^i + i\partial/\partial y^i)$ that are bases for the co-tangent and tangent spaces $T_x^*(1,0)$, $T_x^*(0,1)$, $T_x(1,0)$, $T_x(0,1)$. An infinitesimal automorphism of an almost complex structure J is a vector field X such that the Lie derivative of J with respect to X vanishes. The space of such infinitesimal automorphisms is a Lie algebra A (possibly infinite dimensional). If X is an infinitesimal automorphism, JX is not necessarily such, unless the almost complex structure is actually complex; in this last case, the algebra A is a complex Lie algebra. Also, in this case, if X is an infinitesimal automorphism, then $Z = 1/2(X - iJX)$ is a holomorphic vector field, i.e. a vector field of type $(1,0)$ such that Zf is holomorphic for every locally defined holomorphic function f; this also establishes an isomorphism between the algebra of infinitesimal automorphisms and the algebra of holomorphic vector fields. There is a similar result for anti-holomorphic vector fields. Notice that two complex manifolds that are diffeomorphic (in the category of smooth manifolds) are not necessarily holomorphically isomorphic (in the category of complex manifolds) this is for example well known in the case of complex tori (quotient of C^n by a lattice). Complex and almost-complex manifolds can be compact or non-compact. We are particularly interested in those manifolds that carry a coset space structure, i.e. that are homogeneous spaces for Lie groups of holomorphisms (we will always suppose that they are connected). Let us give a few examples of compact manifolds that admit a complex structure: the Grassmann manifold $SU(p+q)/S(U(p) \times U(q))$ of p-planes in C^{p+q}, the complex tori—they are also complex Lie groups—the spheres S^1 (the circle) and S^2 (notice that S^6 is only almost-complex), the products of two odd-dimensional spheres (in particular the Hopf manifolds $S^{2p+1} \times S^1$). Examples of non-compact complex manifolds: C^n itself, complex Lie groups (except complex tori), Cartan classical domains (for instance $SO(n,2)/SO(n) \times SO(2)$). Many other examples can be found in [22], [8], [65]. When an almost complex structure has been chosen, any principal connection in the bundle of the corresponding complex linear frames is called an almost complex connection and the covariant derivative of J with respect to any such connection vanishes automatically (this is the analogue of a corresponding result for Riemannian connections compatible with a given metric structure). Also, in Riemannian geometry, we know that, given a metric structure, i.e. a reduction of the frame bundle from the real linear group to an orthogonal subgroup, there exists a very special connection on the corresponding orthogonal frame bundle which is torsionless, the Levi-Civita connection. Here, we have the same phenomena, in the sense that, given an almost complex structure, i.e. a bundle with structure group $Gl(n, C)$, there is a very special connection whose torsion T—as of a linear connection—is proportional to the Nijenhuis torsion N of the almost complex structure (actually $N = 8T$). In particular, if the manifold is complex, $N = 0$ and one can find a complex connection with $T = 0$ but conversely, if it is possible to find an almost complex connection with $T = 0$, one can prove that $N = 0$. An (almost) complex manifold M is called (almost) Hermitian if it is endowed with a Riemannian metric h invariant under the complex structure J, i.e. if $h(JX, JY) = h(X, Y)$ for any vector fields X and Y. Metrics restrict the structure group $Gl(2n, R)$ to the orthogonal group $O(2n)$. In the same way Hermitian metrics are in one-to-one correspondence with the reductions from the bundle of complex linear frames ($Gl(n, C)$-frames) to the bundle of unitary frames (the structure group being the unitary group $U(n) = O(2n) \cap Gl(n, C)$). From the Hermitian metric h, we can build a Hermitian scalar product H as follows: $H(X, Y) = 1/2(h(X, Y) - ih(JX, Y))$. In any (almost) complex manifold, we can associate to any symmetric tensor field b of type $(1, 1)$ a two-form β of type $(1, 1)$ via the relation $\beta(X, Y) = b(JX, Y)$, and conversely. In particular, on any (almost) Hermitian manifold we can define, from the metric h (a symmetric tensor of type $(1, 1)$) a two-form $\omega(X, Y) = h(JX, Y)$ called the Kähler form of the (almost) Hermitian structure or the "fundamental symplectic two-form". Notice that $[\operatorname{Im} H](X, Y) = -1/2\omega(X, Y)$. The almost complex structure is not, in general, parallel with respect to the Riemannian connection defined by the Hermitian metric h (i.e. the covariant derivative of J does not necessarily vanish); in other words, although the bundle of frames can be reduced, the Riemannian connection does not restrict in general. When it does, the Riemannian connection associated with h (the Levi-Civita connection) is (almost) complex and the manifold itself is called a Kähler manifold (it is then automatically complex since the corresponding torsions vanish). A necessary and sufficient condition for M to be Kähler is $N = 0$ and $d\omega = 0$ (i.e. the Kähler two-form ω has to be closed). However, it can be that $d\omega = 0$ but that N is not zero (so the manifold is almost complex, not complex); such manifolds are called almost Kähler manifolds. To simplify the discussion we shall only discuss complex manifolds from now on. The Kähler condition can be expressed locally in terms of a set of differential equations that show that, locally, the metric h can be written as $h_{\alpha\bar{\beta}} = \partial^2 f/\partial z^\alpha \partial \bar{z}^\beta$ where f is some real-valued function called the Kähler potential. We will return to this later. Also, one can prove that the Ricci tensor of a Kähler manifold is invariant under the complex structure: $R(JX, JY) = R(X, Y)$. We set $G = \det(h_{\alpha\bar{\beta}})$, $K_{\alpha\bar{\beta}} = -\partial^2 \log G/\partial z^\alpha \partial \bar{z}^\beta$ (Ricci tensor). Most reasonable complex manifold carry a Hermitian structure (they only have to be paracompact, the constraint is the same as in Riemannian geometry) but many Hermitian manifolds cannot carry a Kähler structure (from the vanishing of $d\omega$, one can show that a necessary condition is that even dimensional Betti numbers vanish); for instance, products of two odd-dimensional spheres cannot be given a Kähler structure (besides $S^1 \times S^1$). One can find, however, many examples of Kähler manifolds. Examples of compact manifolds admitting a Kähler structure: Riemann surfaces, complex tori or complex grassmannian. In the non-compact case, we shall be particularly interested in the fact that arbitrary bounded domains (not closed) in C^n can be given a Kähler metric by the so-called Bergman construction. We will see many explicit examples later. This important construction is actually valid for an arbitrary complex manifold (not necessarily a bounded domain of C^n) but leads to a symmetric tensor of type $(1, 1)$ which is not necessarily a metric in the sense that it may be degenerated. The construction goes as follows (we only sketch this method here since we will return to it in a forthcoming section). One starts with an n-dimensional complex manifold $\mathscr{D}$ and consider the Hilbert space H of holomorphic n-forms ϕ which are square integrable. Taking any orthonormal basis $\phi_0, \phi_1, \ldots \in H$, one builds $K(z, \bar{\xi}) = i^{n^2} \Sigma \phi(z) \wedge \phi(\bar{\xi})$, then $K(z, \bar{z})$ is a form of degree (n, n) called the Bergman form and is independent of the choice of the basis. Taking a complex local coordinate system $z^1, \ldots, z^n$ in M, we write $K(z, \bar{z}) = i^{n^2} k(z, \bar{z})\, dz^1 \wedge \ldots dz^n \wedge d\bar{z}^1 \wedge \ldots dz^n$ where k is a non-negative "function" called the Bergman kernel (actually it is not a function but a scalar density since its definition depends upon the chosen chart: one gets $k_V(z, z) = |J_V^U|^2 k_U(z, \bar{z})$ where J_V^U is the Jacobian associated to the change of charts). From now on we shall suppose that the complex manifold $\mathscr{D}$ is such that its Bergman form vanishes nowhere; such a manifold is called a "normal" complex manifold. Then we set $t_{\alpha\bar{\beta}} = \partial^2 \log k/\partial z^\alpha \partial \bar{z}^\beta$ and we get a symmetric $(1, 1)$ tensor $t(X, Y)$ called the Bergman tensor which is compatible with J, which is not necessarily positive definite and whose associated $(1, 1)$ real 2-form $\tau(X, Y) = t(JX, Y)$—which has no name—is automatically closed. Let h be a Hermitian metric on $\mathscr{D}$, then since both the Bergman form and the volume element of the metric h are forms of degree (n, n), they differ multiplicatively by a real function λ; of course, Bergman kernel k of $\mathscr{D}$ and square-root of the determinant G of h are related in the same way. In the particular case where $\mathscr{D}$ is a bounded domain of C^n (i.e. it is open and connected), it happens that the tensor t is positive-definite. Since t is positive definite and τ is closed, the Bergman tensor t can be considered as a Hermitian metric for $\mathscr{D}$ which is Kähler and called "the Bergman metric of the domain $\mathscr{D}$". In this last case the Kähler potential f coincides with the Bergman kernel k. Then, previous results imply that Ricci tensor and metric are proportional which means that the Bergman metric of a bounded domain of C^n is automatically an Einstein metric.

2.1.2. *Some results on homogeneous complex manifolds*

We now continue our summary of standard results in the case where the complex manifold $\mathscr{D}$ is homogeneous, i.e. when the group G of holomorphic transformations of $\mathscr{D}$ is transitive on $\mathscr{D}$. Such a manifold can therefore be written as a coset space G/K where K is a Lie subgroup of G. The theory now splits into two: the case where $\mathscr{D}$ is compact and the case where it is not. If $\mathscr{D}$ is compact and because of the transitivity of the group of transformation, a holomorphic n-form either vanishes nowhere or is zero everywhere; from that, one can show that either the complex manifold is not normal but the Bergman form is zero everywhere or it is normal but the Bergman tensor t is zero [40]. We are mainly interested here in applications of the Bergman construction where neither k nor t are zero, so we leave here the theory of compact homogeneous manifolds. Returning to non-compact homogeneous Hermitian manifolds, we then restrict our attention to those which are symmetric (as homogeneous spaces G/K) - so, they are necessarily simply connected— but then a theorem [22] states that such homogeneous spaces equipped with the metric inherited from the Killing form of the algebra Lie(G) are holomorphically diffeomorphic with bounded symmetric domains of C^n equipped with their Bergman metric -- a domain is symmetric iff each point in $\mathscr{D}$ is an isolated fixed point of an involutive holomorphism (holomorphic diffeomorphism) of $\mathscr{D}$. They are therefore automatically Kähler. But simply connected Kähler manifolds admit a De Rham decomposition into a product of irreducible complex manifolds which are Kähler; it is then natural to restrict our attention further to the case of irreducible Hermitian symmetric spaces of the non-compact type. These spaces will only be called "Cartan classical domains" in the sequel or just "Cartan domains", the terminology coming from the holomorphic diffeomorphism already mentioned previously between abstract homogeneous spaces for Lie groups and particular domains of C^n. Such classical domains can be classified for instance by using the classification of Lie groups and this is the starting point that we will choose. We will mention later other constructions (for instance as complex domains!) for those classical domains that are of particular interest for us. The classification of irreducible Hermitian symmetric spaces of non-compact type can be found for instance in [22], [55] but it is unfortunate that there is no standard terminology. Let us only mention that there are four series of classical domains that are quotients of non-compact groups by compact groups that we will call $\mathscr{A}(p,q) = SU(p,q)/S(U(p) \times U(q))$, $\mathscr{B}(n) = SO^*(2n)/U(n)$, $\mathscr{C}(n) = \mathrm{Sp}(2n,R)/U(n)$, $\mathscr{D}(n) = SO(n,2)/SO(n) \times$

$SO(2)$ and two "exceptional" ones, namely $E_6/SO(10) \times U(1)$ and $E_7/E_6 \times U(1)$. We will also write $\mathscr{A}(n) = \mathscr{A}(n, n)$. Members of the $\mathscr{A}(n)$ families are $2n^2$-dimensional and often called "non-compact complex grassmannians", those of the $\mathscr{C}(n)$ family are called "Siegel half-planes". Members of the $\mathscr{D}(n)$ family are $2n$-dimensional and they are also called "Lie balls" for reasons discussed later (we will omit parenthesis in the sequel). Low dimensional isomorphism of Lie groups belonging to different series imply isomorphisms between the lowest members of the above families. For instance, the two-dimensional $\mathscr{D}(1) = SO(1,2)/SO(2) = SL(2,R)/SO(2) = \mathscr{A}(1) = U(1,1)/U(1) = \mathscr{C}(1) = \mathrm{Sp}(2,R)/U(1)$ is the familiar unit disk, a model for the geometry of Lobatchevski (it can also be seen as one of two sheets hyperboloid or as the Poincare upper half-plane); the four-dimensional domain $\mathscr{D}(2) = SO(2,2)/SO(2) \times SO(2)$ is the only one which is not irreducible (it should not appear in the list!) since it is isomorphic with $\mathscr{D}(1) \times \mathscr{D}(1)$—this comes from the local isomorphism between $SO(4)$ and $SU(2) \times SU(2)$; the six-dimensional domain $\mathscr{D}(3) = SO(3,2)/SO(3) \times SO(2)$ coincides with $\mathscr{C}(2)$, the eight-dimensional domain $\mathscr{D}(4) = SO(4,2)/SO(4) \times SO(2)$ coincides with the domain $\mathscr{A}(2) = SU(2,2)/S(U(2) \times U(2))$—this comes from the local isomorphism between $SO(6)$ and $SU(4)$. Finally, we have $\mathscr{D}(6) = \mathscr{B}(4)$ and $\mathscr{A}(3,1) = \mathscr{B}(3)$. There are no more accidental isomorphisms. We will be mainly interested in the study of the $\mathscr{A}(n)$ and $\mathscr{D}(n)$ series and all our explicit examples will involve either $\mathscr{A}(1) = \mathscr{D}(1)$, i.e. the unit disk (because it is easy to visualize and because it is used in wavelet analysis) or $\mathscr{A}(2) = \mathscr{D}(4)$, i.e. the eight-dimensional Lie ball because of its direct relation with space-time geometry. Notice that there are several possible "generalizations" of the unit disk: as Lie balls, as non-compact complex grassmannian or as Siegel half-planes. We will give later several realizations of these Cartan domains. Before ending this section, we would like to mention that there are also non-compact pseudo-Hermitian irreducible symmetric spaces (they are pseudo-Riemannian manifolds) on which little is known but where many of the previous and following results could possibly be generalized (cf. [59]).

2.2. *Boundaries of Cartan domains*

2.2.1. *Classical domains as differentiable manifolds*

We already know that Cartan classical domains are diffeomorphic with simply connected and connected bounded open subspaces of C^n (although bounded, classical domains are not compact since not closed in C^n). They are therefore diffeomorphic (not holomorphically) with R^{2n} and are therefore $2n$-dimensional manifolds without boundary, in the sense that every point posesses a neighborhood diffeomorphic with an open set of R^{2n}. For instance the open unit disk $\mathscr{D}1$ has clearly no boundary.

2.2.2. *Boundaries of Cartan classical domains as subspaces of C^n*

However, if we realize an abstract Cartan domain as a topological subspace of C^n, it has a topological boundary (the complement of its interior in its closure). For instance, the topological boundary of the open disk in C is the circle and the topological boundary of $\mathscr{D}4$ in R^8 is diffeomorphic with a standard seven-sphere ($\mathscr{D}4$ is an open ball for a particular distance that we shall introduce later). However, such a definition is not very satisfactory because it relies on a given realization of the Cartan domain—namely its embedding in C^n.

2.2.3. *Boundaries via the Borel-Harish-Chandra embedding*

In order to give an abstract definition for the "boundary" of a Cartan domain defined as a homogeneous space of Lie groups, one can use the Borel embedding (also called the Harish-Chandra embedding). The observation is that, if $\mathscr{D} = G/K$, one can embed G into the complex group G^c, K into the complex group K^c and finally the non-compact quotient G/K into the compact quotient G^c/K^c which happens to be isomorphic with the quotient $\bar{G}/K$ where $\bar{G}$ is the corresponding real compact Lie group (K was already compact). In this way one embeds each classical domain (an irreducible Hermitian symmetric space of non-compact type) into a corresponding irreducible Hermitian space of compact type, for instance the non-compact space $\mathscr{D}1 = SO(1,2)/SO(2)$ into the compact space $S^2 = SO(3)/SO(2)$, i.e. $\mathscr{D}1$ is identified with the lowest open hemisphere of a two-dimensional sphere. The boundary of $\mathscr{D}1$ in S^2 is then an equatorial circle. In the same way, the eight-dimensional Lie ball $\mathscr{D}(4) = SO(4,2)/SO(4) \times SO(2)$ is identified with a subspace of the compact grassmannian $SO(6)/SO(4) \times SO(2) = SU(4)/S(U(2) \times SU(2))$. The topological boundary of the Cartan domain embedded as previously in its dual compact space is called the "weak boundary" of the domain. From the fact that K is the isotropy group (the little group) of the origin of $\mathscr{D} = G/K$—a point belonging to $\mathscr{D}$, not to its weak boundary—and since $\mathscr{D}$ is embeded in $\bar{G}/K$, it is clear that K acts also on the weak boundary! But, in general, K has no reason to be transitive on it (it is transitive in the case of the unit disk $\mathscr{D}1$), so the boundary will be stratified under the action of K and one can study the orbit structure of this stratification [65]. In the case of the domain $\mathscr{D}n$, of dimension $2n$, one of the strata has one orbit only and it has a dimension n; this orbit is a compact manifold called the Shilov boundary $\check{\mathscr{D}}n$ of the Cartan domain $\mathscr{D}n$. More generally, for all classical domains, it happens that one of the strata consists of one orbit only and that this orbit has a dimension precisely equal to half the dimension of the domain itself. It is the Shilov boundary of the domain. When represented as a bounded domain of C^n, the Shilov boundary can be given an alternative definition. Let us recall the more general (analytic) definition of a Shilov boundary. Let $\mathscr{A}$ be a set of (non-constant) holomorphic functions in a bounded domain $\mathscr{D}$ of C^n and continuous on $\bar{\mathscr{D}}$. Then $\check{\mathscr{D}} \subset \partial\mathscr{D}$ is the smallest closed subset of the boundary $\partial\mathscr{D}$ such that every $f \in \mathscr{A}$ reaches its maximum (in module) on $\check{\mathscr{D}}$. In the case of $\mathscr{D}1$, one gets $\check{\mathscr{D}}1 = S^1$. In the case of $\mathscr{D}4$, one gets $\check{\mathscr{D}}4 = S^3 \times_{Z_2} S^1$. More generally, in the case of $\mathscr{D}n$, one gets $\check{\mathscr{D}}n = S^{n-1} \times_{Z_2} S^1$ (so that $\check{\mathscr{D}}n$ is a higher dimensional analogue of the Klein bottle).

2.2.4. *Geometry on the Shilov boundary of Cartan domains*

We know that Cartan domains are Riemannian manifolds for their Kähler metric. This metric, of course does not coincide with the induced metric that they would acquire from their embedding into C^n. It is natural to try to define a metric structure on the Shilov boundary by taking some limit of the Riemannian metric in the domain itself; however such a limit is singular (for instance the Lobatchevski distance in the upper half-plane blows up when we approach the real line). What is remarkable is that it is nevertheless possible to define unambiguously a related structure on the Shilov boundary: it is a conformal Lorentzian structure. For instance, if we start with a Riemannian metric on $\mathscr{D}4$ invariant under the conformal group $SO(4,2)$ and we go to the Shilov boundary $\check{\mathscr{D}}4$, we do not obtain an invariant metric but a conformal class of Lorentzian metrics. This is almost intuitive since it is well-known that the conformal group does not leave the Minkowski metric invariant but modifies it by an x-dependent factor. We can choose then a unique —up to constant scale— flat Minkowski metric from this class [19].

2.3. *Global and local charts on Cartan domains*

Since Cartan domains are $2n$-dimensional manifolds that can be represented as open subsets of C^n, they admit a global complex chart, namely, the one coming from such an embedding. However, there are other charts that usually do not cover the whole of the domain but are nevertheless interesting. For instance, there is a global chart sending the "abstract" domain $\mathscr{D}1 = \mathrm{Sl}(2, R)/U(1)$ to the unit disk of the complex plane, but the Cayley transformation is a holomorphic transformation of C sending the unit disk to the upper half-plane (and its Shilov boundary—the circle—to the real line). This transformation is however singular on the boundary and this last chart does not cover the whole of $\bar{\mathscr{D}}1$ since one of its points is sent to infinity. The disk is called a bounded realization of $\mathscr{D}1$ and the upper half-plane, an unbounded realization of $\mathscr{D}1$. One also says that the disk is a "compactified upper half-plane" (and the circle is a "compactified real line"). In the same way, there exist higher dimensional analogues of the Cayley transformation for the other Cartan domains [36]. This transformation can be considered as a bi-holomorphic change of charts mapping a bounded realization to an unbounded one (or the opposite); it is singular on the boundary. We will study this transformation in the case of $\mathscr{D}n$ and $\mathscr{A}n$ but we can already give the following result: the Cayley transformation maps the Lie ball $\mathscr{D}4$ to the so-called "Cartan tube" $\mathscr{T}4$ (mathematically isomorphic with the "forward tube" of particle physicists): it is the space of all $z = x + iy$ such that $x \in R^4$ and $y \in R^4$ with the constraint that y lies in the forward light cone ($y_0^2 - y_1^2 - y_2^2 - y_3^2 > 0$). Under this transformation, the "finite part" of the Shilov boundary $S^3 \times_{Z_2} S^1$ of $\mathscr{D}4$ is mapped onto R^4 (which, topologically can be identified with Minkowski space-time M). Independently of any relation with Cartan domains or with Shilov boundaries, the manifold $S^3 \times_{Z_2} S^1$ is often called "compactified space-time" in the physical literature. It can be obtained from usual Minkowski space-time by adding a cone at infinity.

2.4. *Cayley transformation for the A and $\mathscr{D}$ series*

2.4.1. *On $\mathscr{A}1 = \mathscr{D}1$*

The Cayley transformation in C, $z \in \mathscr{D}1 \to w \in \mathscr{T}1$ where

$$w = \frac{-i(z + i)}{(z - i)}$$

associates the upper half-plane $\mathcal{T}1 = \{w \in C, \operatorname{Im}(w) > 0\}$ to the unit disk $z \in \mathcal{D}1$ of C. The inverse transformation is

$$w \in \mathcal{T}1 \to z = \frac{i(w - i)}{(w + i)} \in \mathcal{D}1 .$$

2.4.2. *On $\mathcal{D}n$*

The generalized Cayley transform τ goes from the bounded realization of the domain $\mathcal{D}n$ to the Cartan tube $\mathcal{T}n$(the forward tube). Setting $z = (z_0, z_1, \ldots, z_{n-1}) \in \mathcal{D}n$ and $w = (w_0, w_1, \ldots, w_{n-1}) \in \mathcal{T}n$, we get $w_k = -2iz_k/Z$, $w_0 = -i + 2(z_0 - i)/Z$ where $Z = (z_0 - i)^2 + z_1^2 + z_2^2 + \cdots + z_{n-1}^2$. The inverse transformation is $z_k = -2iw_k/W$, $z_0 = i - 2(w_0 + i)/W$ where $W = -(w_0 + i)^2 + w_1^2 + w_2^2 + \cdots + w_{n-1}^2$. Notice that $W = -4Z^{-1}$ and that the Jacobian of the transformation is

$$\frac{D(z_1, z_2, \ldots, z_n)}{D(w_1, w_2, \ldots, w_n)} = -2^n(-i)^{n+1}Z^{-n} .$$

2.4.3. *On $\mathcal{A}n$*

We suppose here $\mathcal{A}n$ given by a bounded realization in terms of matrices Z parametrizing the quotient $\mathcal{A}(n) = SU(n,n)/S(U(n) \times U(n))$. The Cayley transformation, mapping this bounded realization to an unbounded one (called $\mathcal{T}n$), is given as follows. $Z \to W$ where $W = W(Z) = i(1 - Z)(1 + Z)^{-1}$. The inverse mapping is $Z = Z(W) = (1 - iW)^{-1}(1 + iW)$. In the case $n = 2$, the Lebesgue measures on the bounded realization $|dZ| = \Pi d\operatorname{Re}(z)\, d\operatorname{Im}(z)$ and on the unbounded realization $|dW| = \Pi d\operatorname{Re}(w)\, d\operatorname{Im}(w)$ are connected by a Jacobian $|dZ| = J_1|dW|$ with $J_1 = 2^{-4}|\det(1 + Z)|^8 = 2^{12}|\det(1 - iW)|^{-8}$. Similarly, the $K = SO(4) \times SO(2)$-invariant measure $|d\mu|$ on the bounded realization of the Shilov boundary (compactified space-time) is related to the Poincare-invariant measure on unbounded space-time $|d^4u| = |d\operatorname{Re} W|$ by $|d\mu| = J_2|d^4u|$ with $J_2 = (2/\pi)^3|\det(1 - iU)|^{-4}$. In the case $n = 1$, besides unimportant constant factors, the Jacobians are $J_1 = |1 - iw|^{-2} = (1 + iw^+)^{-1}(1 - iw)^{-1}$ and $J_2 = |1 - iu|^{-2} = (1 + u^2)^{(-2)}$.

2.5. *Matrix realizations of the Cartan domains*

2.5.1. *The $\mathcal{A}(n)$ series*

We define the group $G = SU(n,n)$ as follows. We consider complex $2n \times 2n$ matrices $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A, B, C, D are $n \times n$ submatrices. M is assumed to satisfy the constraint $M^+H = HM^{-1}$ with $H = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. The maximal compact subgroup $K = S(U(n) \times U(n))$ consists of the matrices $\begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$ with the constraint $\det(K_1K_2) = 1$ and K_1, $K_2 \in U(n)$. Elements of the quotient G/K can be considered as $2n \times 2n$

complex matrices Z such that $1 - Z^+Z > 0$. The group G acts on Z by $Z \to Z' = (AZ + B)(CZ + D)^{-1}$ where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$. The Shilov boundary of $\mathscr{A}(n)$ is $\check{\mathscr{A}}(n) = U(n) = SU(n) \times_{Z_2} U(1)$. In particular, we recover the fact that $\check{\mathscr{A}}(2) = U(2) = SU(2) \times_{Z_2} U(1) = S^3 \times_{Z_2} S^1$.

2.5.2. *The $D(n)$ series*

The reader may, for instance, refer to [10].

3. Geometrical Aspects of the Bergman and Szegö Kernels

3.1. *General vector bundles above Cartan domains*

Let $\mathscr{D}$ be a Cartan domain, for instance $\mathscr{D}4$. Then, as a homogeneous space G/K, it is automatically the base of a principal bundle whose total space is G and the fiber is K. Complex-valued functions over the domain are a particular example of sections of vector bundles that one can construct (those associated with the trivial representation of the structure group K). More general vector or tensor valued functions (or even p-forms) can be constructed by choosing particular representations of the structure group K, which, in the case of $\mathscr{D}4$ is $SO(4) \times SO(2)$ which, locally, is the same that $SU(2) \times SU(2) \times U(1)$. Representations of this "little" group are characterized by an integral label n and two spin labels j_1, j_2. These vector bundles are equivariant under the action of the conformal group G itself and the spaces of sections of these bundles provide representation spaces for G. It is then possible to define a scalar product in such a space of sections and one can then look for subspaces of holomorphic (or antiholomorphic), square-integrable sections $\mathscr{H}_{p,j_1,j_2}$ carrying an irreducible unitary representation of the conformal group. Such spaces can, of course, be zero for some values of (p, j_1, j_2). For instance, it is shown, in [56], that, if $j_1 = j_2 = 0$, the spaces $\mathscr{H}_p(\mathscr{D}4)$ reduce to zero if $p < 2$. It is possible to define a Bergman kernel for each of these spaces and to investigate the (distributional) boundary values of those sections on the Shilov boundary. It should even be possible to generalize the theory further and consider spinor fields on such domains. Let us mention here that the theory of harmonicity cells recalled at the end of this article in relation with functions (and the Laplace operator) can also be developed for more general classes of pseudo-differential operators [39]. We will restrict our study here to a particular class of vector bundles: the "densities of conformal weight l".

3.2. *The spaces $H_l^2(\mathscr{D})$ and their associated kernels*

As already mentioned, the frame bundle of a complex manifold is naturally reduced from $GL(2n, R)$ to $GL(n, C)$-bundle of frames of type $(1,0)$. A holomorphic density ϕ of weight l (l integer) is a holomorphic section of the associated bundle corresponding to the 1-dimensional representation of $GL(n, C): GL(n, C) \in A \mapsto \det(A)^l$. Alternatively, ϕ can be (in general, locally) defined as a holomorphic function in each coordinate system with the rule

$$z^{\alpha} \mapsto z^{\alpha'} \Rightarrow \Phi'(z') = \left[\det\left(\frac{\partial z}{\partial z'}\right)\right]^{l} \Phi(z).$$

In particular holomorphic densities of weight 0 are just holomorphic functions, while densities of weight 1 are holomorphic n-forms. Indeed an n- form ϕ of type $(1,0)$ can be uniquely be written as

$$\phi = \Phi(z)\, d^n z = \Phi'(z')\, d^n z'$$

which implies the transformation rule

$$\Phi'(z') = \left[\det\left(\frac{\partial z}{\partial z'}\right)\right]\Phi(z).$$

Anti-holomorphic densities of weight l could be defined similarly. Using quite a different language, the properties of these densities, for the case of the domain $D1$, have been investigated in an article by [5], and [62].

Assume now that the complex manifold $\mathscr{D}$ admits a coordinate system that maps it onto a bounded domain in C^n. The natural Hilbert space $\mathscr{H}^2(\mathscr{D})$ associated to $\mathscr{D}$ consists of holomorphic densities of weight 1 which are square integrable w.r.t. the measure $d^n z\, d^n z$ of C^n. Notice that the scalar product in $\mathscr{H}^2(\mathscr{D})$

$$(\phi, \psi) = \int \Phi(z)\bar{\Psi}(z) dvol(z) \left(= \int \Phi(z)\, d^n z \Psi(z)\, d^n \bar{z} \right)$$

is independent of the coordinate system. The measure $dvol(z)$ is the euclidean measure on C^n and does not coincide with the intrinsic measure of the domain $d\mu(z) = k(z, \bar{z}) dvol(z)$ associated with its Bergman metric. The Bergman kernel function of $\mathscr{D}$ is then defined as

$$k(z, w) = \Sigma \psi_n(z) \bar{\psi}_n(w)$$

where ψ_n is any orthonormal basis in $\mathscr{H}^2(\mathscr{D})$. The kernel k is a holomorphic (resp. anti-holomorphic) density of weight 1 w.r.t. z (resp. w). It has the reproducing property

$$f(z) = \int_{\mathscr{D}} f(\xi) k(z, \xi) dvol(\xi).$$

Notice that if we set $z = w$, then $K(z, \bar{z})$ becomes an (n, n)-form. Given the Bergman kernels for classical domains we can introduce now a family of (possibly trivial, i.e. $= \{0\}$) Hilbert space $\mathscr{H}_l^2(\mathscr{D})$ consisting of holomorphic densities of weight l, square integrable w.r.t. the measure $d\mu_l(z) = k(z, \bar{z})^{1-l} dvol(z) = k(z, \bar{z})^{-l} d\mu(z)$. The scalar product

$$(\phi, \psi)_l = \int \Phi(z)\Psi(\bar{z})\, d\mu_l(z) = \int \frac{\Phi(z)\Psi(z)}{k(z,z)^l}\, d\mu(z)$$

is again independent of the coordinate system. (One can define also $\mathscr{H}_l^2(\mathscr{D})$ for non-integer l, but then, for topologically non-trivial $\mathscr{D}$, it can depend on coordinates, because of the problems with definition of a holomorphic density of non-integer weight). Notice that the space $\mathscr{H}_0^2(\mathscr{D})$ of square-integrable functions on $\mathscr{D}$ for the Bergman measure itself is usually trivial. Each $\mathscr{H}_l^2$ carries a natural representation U_l of the group G of holomorphism of $\mathscr{D}$:

$$(U_l(g)\Phi)(z) = \left[\det\left(\frac{\partial g^{-1}z}{\partial z}\right)\right]^l_{g^{-1}z} \Phi(g^{-1}z).$$

For $\mathscr{D} = G/K = SO(n,2)/SO(n) \times SO(2)$ in the bounded domain realization the action of the $SO(2)$ subgroup of K is just $z \mapsto e^{i\phi}z$. The Jacobian for this action is $(e^{i\phi})^n$, n being the complex dimension of $\mathscr{D}$. It follows that the spaces $\mathscr{H}_l^2(\mathscr{D})$ can be identified with the spaces of holomorphic sections of the bundle associated to $G \to G/K$ via the representation $e^{i\phi} \mapsto e^{ip\phi}$ of K, with $p = nl$. This shows in particular that the structure group of the corresponding principal bundle (only the $U(1)$ part of K matters here) is either $U(1)$ itself (in which case p has to be an integer), or a quotient of $U(1)$ by a Z_m (in which case p is a multiple of m) or even a cover of $U(1)$ by a subgroup (in which case p can be non-integer). There are no problems related to extensions of the structure group here, since the manifolds we study are topologically trivial. The values of l for which $\mathscr{H}_l^2$ is not zero may have some physical interpretation and this set is not necessarily an interval (this is discussed in [6]). Also, the value of l is related to what is called the canonical dimension of the fields in Lagrangian field theories. Each space $\mathscr{H}_l$ has its reproducing Bergman kernel $k^l(z,\bar{z}) = k(z,\bar{z})^l$.

3.3. *Coherent states on complex manifolds and the Bergman kernels*

All spaces $\mathscr{H}_l^2(\mathscr{D})$, $\mathscr{D}$ a bounded domain, have associated natural families of coherent states (the following may be considered as a definition). Indeed, for each $\xi \in \mathscr{D}$ the evaluation function $\Phi \mapsto \Phi(\xi)$ vanishes on a hyperplane and is continuous. It follows that there exists a vector e_ξ in $\mathscr{H}_l^2(\mathscr{D})$, unique up to a factor, which is $\perp$ to all Ψ such that $\Psi(\xi) = 0$. In each coordinate system $\{z\}$ there is a natural choice of e_ξ:

$$e_\xi(z) \doteq k_l(z, \bar{\xi}).$$

The coherent states $|\xi\rangle$ (of unit norm) differ from e_ξ just by a normalization factor. We can think of these coherent states on bounded domains as generalizations of Bergman coherent states on C (or C^n).

3.4. *Classical domains and the Szegö kernel*

The Shilov boundary $\check{\mathscr{D}}$ for a bounded domain $\mathscr{D}$ is defined as a minimal subset of $\partial\mathscr{D}$ which has the property that every bounded holomorphic function on $\mathscr{D}$ reaches its maximum at some point of $\check{\mathscr{D}}$. This property characterizes $\check{\mathscr{D}}$ uniquely. In case of bounded symmetric domain G/K, its Shilov boundary is a homogeneous space not only of G but also of K. By a general theorem of Gleason, holomorphic bounded functions in $\mathscr{D}$ admit a representation of the form

$$\psi(z) = \int_{\check{\mathscr{D}}} \psi(\zeta) s_\mu(z,\zeta)\, d\mu(\zeta),$$

where μ is a measure on $\check{\mathscr{D}}$, and $s_\mu(z,\zeta)$ is holomorphic in z and integrable with respect to μ in ζ. The kernel function $s_\mu(z,\zeta)$ is called the Szegö kernel of the domain. It can be computed as

$$s_\mu(z,u) = \sum \psi_n(z)\bar{\psi}_n(u),$$

for z, $u \in \mathscr{D}$, where $\psi_n(z)$ is a basis of holomorphic functions on $\mathscr{D}$ which is μ-orthonormal:

$$\int_{\check{\mathscr{D}}} \psi_n(\zeta)\bar{\psi}_m(\zeta)\, d\mu(\zeta) = \delta_{mn}.$$

When μ is the measure induced by the Euclidean measure of a simply connected domain there is usually a simple relation between the Szegö and the Bergman kernel of the domain:

$$\left(\frac{s(z,\bar{u})}{s(u,\bar{u})}\right)^2 = \frac{k(z,\bar{u})}{k(u,\bar{u})}.$$

One can generalize the concept of the Szegö kernel so as to describe more general spaces of not necessarily bounded holomorphic functions, for instance the spaces $\mathscr{H}_l{}^2$. It is important to remember that the Szegö kernel depends on the measure μ that one is using at the Shilov boundary. For instance in the case of $\mathscr{D}4$ in the Cartan tube realization there are two natural measures on the finite part of the Shilov boundary: the measure

$$d\mu = cte \times d^4x/((1+(r-t)^2)^2(1+((r+t)^2)^2),$$

which is the unique invariant measure for the group $SO(4) \times SO(2)$—the stability group of $\mathscr{D}4$; the other is the Lebesgue measure d^4x, invariant up to scale under the stability group of a distinguished point called ∞. To relate the two Szegö kernels one has to use the corresponding Radon-Nikodym derivative.

3.5. *The case of $\mathscr{D}1$ and $\mathscr{D}4$*

3.5.1. *Bergman kernels*

In the case of $\mathscr{D}1$ the Bergman kernel is given by

$$k(z_1, z_2) = (1/\pi)\frac{1}{(1 - z_1\bar{z}_2)^2}$$

in the bounded representation (the disk) and

$$k(w_1, \bar{w}_2) = (1/16)(1/2i)\frac{1}{(w_1 - \bar{w}_2)^2}$$

in the unbounded one (upper half-plane).

In the case of $\mathscr{D}4$ or more generally on the Lie ball $\mathscr{D}n$ we have correspondingly $k(z_1, \bar{z}_2) = d_n/(1 + z_1^2\bar{z}_2^2 - 2z_1\bar{z}_2)^n$ in the bounded domain realization and $k(w_1, \bar{w}_2) = cte/[(w_1 - \bar{w}_2)^2]^n$ in the tube realization (as usual, in the tube realization, the "square" is computed with the Lorentzian metric). The Bergman constant d_n is equal to the inverse of the volume of the corresponding domain. Namely $d_n = 2^{n-1}n!/\pi^n$.

In the case of Cartan domains $\mathscr{A}n$, we have the expression $k(z_1, \bar{z}_2) = a_n[\det(1 - Z_1\bar{Z}_2)]^{2n}$ for the bounded realization and $k(w_1, \bar{w}_2) = (1/16)\det[(-i/2)(w_1 - \bar{w}_2)]^{-2n}$. As already mentioned several times, the $\mathscr{A}2$ and $\mathscr{D}4$ cases coincide (in which case $a_4 = 12/\pi^4$) [56].

3.5.2. *Szegö kernels*

In the case of $\mathscr{D}1$ the Szegö kernel is given by $s(Z, x) = c_1/(1 - Z\bar{x})$ in the bounded representation ($x \in S^1, Z \in \mathscr{D}1$) and $s(W, u) = c_1/(u - W)$ in the unbounded one ($u \in R$, $W \in$ {upper half plane}). The relation between both expressions is the following. $s(Z, x) = (1 - iW)(1 + iu)s(W, u)$. Indeed, the relation between the measures, in the domain itself and on its Shilov boundary, in both realizations are $d\mu = (1 + iu)^{-1} \cdot (1 - iu)^{-1}\, du$ and $dZ = (1 + iW^+)^{-1}(1 - iW)^{-1}\, dW$.

In the case of $\mathscr{D}4$ or more generally $\mathscr{D}n$, we have correspondingly $s(xe^{i\phi}, Z) = c_n/[(x - e^{i\phi}Z)(x + e^{i\phi}Z)]^{n/2}$ in the bounded realization–here $xe^{i\phi} \in S^{n-1} \times_{Z_2} S^1$, i.e. $x \in S^{n-1} \subset R^n$, [24], and $s(u, W) = c_n/[(u - W)^2]^{n/2}$ in the tube realization [56]. The Szegö coefficient c_n is equal to the inverse of the volume of the Shilov boundary. It is given by $c_n = \Gamma(n/2)/(2\pi^{n/2+1})$.

3.6. *Square-integrable functions, distributions and hyperfunctions on the Shilov boundary*

The correspondence between "functions" on the domain $\mathscr{D}$ and "functions" on its Shilov boundary $\check{\mathscr{D}}$ can be studied in both ways. Elements of $\mathscr{H}_l^2$ defined in the domain usually approach a distribution when their argument tends to the Shilov boundary.

For a given l, only the elements of a (small) subspace of H_s^2 approach a square integrable function defined on the Shilov boundary for the measure $|d\mu|$ (invariant under K when we write $\check{\mathscr{D}}$ as K/H). Conversely, elements of this subspace can be gotten from (analytic) elements of $L^2(\check{\mathscr{D}})$ by means of the Szegö kernel. More generally, distributions on $\check{\mathscr{D}}$ can be extended — via the Szegö kernel — to elements of holomorphic (or anti-holomorphic) elements of $\mathscr{H}_l^2$, the value of l depending actually upon the kind of singularity of the distribution. If one chooses an unbounded realization rather than a bounded one, one usually takes tempered distributions on the Shilov boundary. The holomorphic extension of (tempered) distributions on space-time into the future tube and generalization of such results to the n-point functions of quantum field theory has been studied by a whole generation of particle physicists and field theorists. Let us only mention the book [61]. In the case of Lie balls $\mathscr{D}n$, the correspondence between spaces of distributions-dual spaces of $C^\infty(\check{\mathscr{D}})$, hyperfunctions-dual space of the space of real analytic functions in $\check{\mathscr{D}}$ and their extensions to the domain $\mathscr{D}^4$ is studied in [56] [46].

3.7. *One-dimensional wavelets and relativistic wavelets*

The following discussion of the wavelet transform is based on an example taken from [55]. The affine group $L = R^+ \times R$,

$$(a, b)(x, y) = (ax, ay + b)$$

acts on $L^2(R, dp)$,

$$(U(a, b)\psi)(p) = a^{1/4} e^{-ibp^2/2} \psi(a^{1/2} p)$$

by unitary transformations. This representation has two irreducible components corresponding to odd and even functions $\psi(p)$. Let us consider the odd case. The analyzing wavelet is

$$\phi_0(p) = pe^{-p^2/2}.$$

By definition, it is such that

$$\int |U(g)\phi_0, \phi_0|^2 \, d\mu(g) < \infty,$$

$d\mu(g)$ being the left-invariant measure on the — non-unimodular — affine group L. The wavelet transform of $\psi(p)$:

$$L_0(\psi) = \frac{1}{\sqrt{c_\phi}} (U(a, b)\phi_0, \psi)$$

is then of the form $a^{3/4} \times f(z)$, where $f(z) \sim \int pe^{izp^2} \psi(p) \, dp$ is a function holomorphic in the upper half-plane $\mathrm{Im}(z) > 0$, and square integrable with respect to the measure $\mathrm{Im}(z)^{1/2}\,dz\,d\bar{z}$. The Bergman kernel $k(z,u)$ of the upper half-plane being $\sim 1/(z-\bar{u})^2$, the invariant measure is $\mathrm{Im}(z)^{-2}\,dz\,d\bar{z}$. Thus the holomorphic factors of the wavelet transform form up the Hilbert space $\mathscr{H}^2_{3/4}$. Notice that the holomorphic wavelet transform of the analyzing wavelet itself is

$$\int p^2 e^{i(z+i)p^2/2}\,dp \sim \frac{1}{(z+i)^{3/2}}\,.$$

This coincides with the coherent state $e_O(z) \sim 1/(z+i)^{3/2}$ of $\mathscr{H}^2_{3/4}$ at the origin $O = 0 + i$. A similar discussion of the odd case gives weight $k = 5/4$. Notice that the coherent state at z is defined here by the property of being orthogonal to all holomorphic functions in the Hilbert space that vanish at z. The natural action of the affine group on the upper half-plane extends naturally to the group $SL(2,R) \approx SU(1,1) \approx SO(1,2)$ (up to discrete factors), which is the conformal group of the time axis R. After Cayley transform the affine subgroup becomes the stability subgroup of ∞ (cf. 5.2) in the Shilov boundary. The holomorphic factor of the wavelet transform can be obtained via the Szegö kernel from its boundary limit. The only role of the (non-holomorphic) factor c_ϕ in the wavelet transform is to ensure that the transformation is an isometry from $\mathscr{L}^2(R,dp)$ to $\mathscr{L}^2(L,d\mu)$. It follows that the essential part of the wavelet transform that contains information about scaling behaviour of the analyzed signal is contained in the Szegö transform. This observation opens the way for a natural generalization of the wavelet transform to higher dimensions. These can be either of time character, or space-time character. In particular the relativistic wavelet transform analyzes fields on space-time in terms of coherent states from Hilbert spaces of holomorphic functions on the domain $D_4 = SO(4,2)/SO(4) \times SO(2)$. The affine group is replaced here by the semi-direct product of the Poincaré group $E(3,1)$ and dilations R^+ (it contains translations, dilations, and Lorentz rotations). A possible extra inversion would replace translations with special conformal transformations, while the extra Fourier transform will replace positions with momenta. The relevant Hilbert spaces are naturally labeled by the representations of the stability group $SO(4) \times SO(2)$. When applied to scalar fields only the $SO(2)$ is relevant. It should be however kept in mind that in higher dimensions the wavelet transform can apply to vector and tensor-valued signals as well.

Notice that the upper half-plane $\mathscr{D}_1$ can be identified with the group L itself. However, in higher dimensions, one cannot identify the domain $\mathscr{D}_n$ with the semi-direct product of the Poincare group $E(n-1,1)$ and dilations (for instance D_4 has dimension 8, and the corresponding group has dimension $11 = (6+4)+1$. For this reason, the holomorphic wavelet transform can be defined (as in [16]) in terms of unitary representations of the non-unimodular group L, or, as here, as an analytic extension from $\mathscr{L}^2(\check{\mathscr{D}})$ to a space $\mathscr{H}_l^2(\mathscr{D})$ where $\check{\mathscr{D}}$ is the Shilov boundary of $\mathscr{D}$. Both definitions are different but directly related (they agree in the case of $\mathscr{D}1$). Notice that one can "correct" the holomorphic transform by a non-holomorphic factor in such a way that the transformation becomes an isometry.

It is traditional in physics and in particular in space-time physics to analyse signals (for instance elementary particles) in Fourier space. However, like in music, there are no "perfect", infinitely lasting sounds — particles are created and destroyed. Relativistic wavelet transforms could provide a way to analyse signals localized in space and time.

4. Lie Balls and the Action of the Conformal Group

Here, we analyse in more detail the group theoretical aspects of the conformal group $SO(4,2)$ acting on $\mathscr{D}4$ or on its Shilov boundary. Everything can be generalized to other members of the $\mathscr{D}n$ series in a straightforward way and, with proper care, to the other Cartan domains. We also review what happens in the $n = 1$ case (the upper half-plane) since the situation there is rather degenerated.

4.1. *The conformal group and its subgroups*

In an n-dimensional real vector space endowed with a non-degenerated pseudo-euclidean scalar product of signature (p, q), $n = p + q$, conformal transformations are defined as transformations that preserve the angles, not the scalar product itself. One can show that this group of transformations is finite dimensional as soon as $n > 2$ and is isomorphic with $O(p + 1, q + 1)$. In the case of a Lorentzian metric of signature $(3, 1)$, we get $O(4, 2)$. It acts via a composition of rotations (also Lorentz rotations, i.e. boosts), translations, dilations and inversion with respect to the origin. The fifteen dimensional group $SO(4, 2)$ itself does not contain the inversion and is generated by 6 rotations (3 pure rotations and 3 boosts), 4 translations, 4 special-conformal translations (obtained by composing an inversion, a translation and again an inversion) and the dilation. However, its action is singular (see explicit formulae below) on Minkowski space-time; we know that it acts however in a non-singular way—but non-linearly—on compactified Minkowski, i.e. on the Shilov boundary of $\mathscr{D}4$. We know that $\mathscr{D}n = G/K$ where $G = SO(n, 2)$ and $K = SO(n) \times SO(2)$ is the little group of the origin o of the domain; in the same way, we can write the compactified Minkowski $\check{\mathscr{D}}n = G/L$ where L is the little group of a chosen point on $\check{\mathscr{D}}n$ called ∞. It is clear that this little group L is made of rotations, translations and dilations. More precisely L is the semi-direct product of the Poincare subgroup times dilations (the Poincare subgroup being itself the semi-direct product of the Lorentz group times translations). L is the direct analogue of the "$ax + b$" group that appears in the geometry of the upper half-plane or its boundary, the real line. However we know that K also acts transitively on the Shilov boundary $\check{\mathscr{D}}n$; this comes from the general theory (Harish-Chandra realization) but also from the obvious fact that $K = SO(n) \times SO(2)$ acts in an obvious way on $S^{n-1} \times S^1$. So we can write $\check{\mathscr{D}}n = K/H$ where $H = SO(n - 1)$. The case $n = 4$ is itself rather special since S^3 possesses a group structure ($S^3 = SU(2)$) and that $SO(4)$ is locally isomorphic to $SU(2) \times SU(2)$ to that $\check{\mathscr{D}}4 = SO(4) \times SO(2)/SO(3)$; the Lie algebra of this $SO(3)$ subgroup is here the diagonal Lie subalgebra of $\mathrm{Lie}(SU(2) \times SU(2))$. In all cases, we can write the domain itself $\mathscr{D} = G/K$ and its Shilov boundary $\check{D} = G/L = K/H$ where K and H are compact Lie groups but G and L are not compact and L itself not even semi-simple.

4.2. *Group theoretical aspects of the unit disk*

In this case the situation is slightly degenerated. We know that $\mathscr{D} = G/K$ with $G = SO(1,2) = \mathrm{Sl}(2,R)$ and $K = SO(2) = U(1)$. The Shilov boundary is $S^0 \times_{Z_2} S^1 = S^1$ (since $S^0 = Z_2$) but $S^1 = G/L$ where L is the "$ax + b$" group (non-unimodular) and also $S^1 = K/H$ where $H = 1$. Notice that "space-time" here is just the time axis and that the unbounded realization of this domain is indeed defined by $y > 0$ so that we get the upper half-plane as the inside of the light cone (the forward tube). A particular feature of this one-dimensional case is that the domain itself (think of it as the upper half-plane) can be identified with the subgroup of the affine group $L =$ "$ax + b$" with positive dilation parameter $a > 0$. This plays an important role in the theory of wavelets.

4.3. *The Lie algebra of $SO(4,2)$*

Let $J_{M,N}$ denote the generators of the Lie algebra of $SO(4,2)$. They satisfy the following commutation relations

$$[J_{M,N}, J_{K,P}] = g_{MK}J_{NP} + g_{NP}J_{MK} - g_{NK}J_{MP} - g_{MP}J_{NK}$$

where the indices run from 0 to 6, i.e. $M \in \{0,1,2,3,4,5,6\}$ and g_{MN} denotes a (6×6) bilinear symmetric form with signature $(+---+-)$. In order to make the link with the conventional generators for the Poincare group, it is convenient to set $J_{\mu 5} + J_{\mu 6} = -2K_\mu$ (conformal translations) $J_{\mu 5} - J_{\mu 6} = -P_\mu$ (translations) $J_{56} = D$ (dilation). Then, commutation relations read

$$[J_{\mu\nu}, J_{\rho\sigma}] = -g_{\mu\rho}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\rho} + g_{\mu\sigma}J_{\nu\rho} + g_{\nu\rho}J_{\mu\sigma}$$

$$[J_{\mu\nu}, P_\rho] = -g_{\mu\rho}P_\nu + g_{\nu\rho}P_\mu$$

$$[P_\mu, P_\nu] = 0$$

$$[P_\mu, K_\nu] = -2J_{\mu\nu} + 2Dg_{\mu\nu}$$

$$[K_\rho, J_{\mu\nu}] = g_{\mu\rho}K_\nu - g_{\nu\rho}K_\mu$$

$$[P_\mu, D] = P_\mu$$

$$[K_\mu, D] = -K_\mu$$

$$[K_\mu, K_\nu] = 0$$

$$[J_{\mu\nu}, D] = 0$$

where $\mu \in \{0, 1, 2, 3\}$. It is also convenient to introduce generators for the 3-dimensional rotations $J_i = 1/2\varepsilon_{ijk}J_{jk}$ and for the Lorentz boosts $N_i = J_{0i}$, $i \in \{1, 2, 3\}$. $G = SO(4, 2)$ is then generated by the fifteen generators, $\mathbf{J}$, $\mathbf{N}$, P_μ, K_μ, D. The domain $\mathscr{D}$ is G/K where the Lie algebra of the stabilizer of the origin is $LieK = Lie(SO(4) \times SO(2)) = Lie(SO(3)) + Lie(SO(3)) + Lie(U(1))$ is generated by $\mathbf{R} = \mathbf{J} + (\mathbf{P} - \mathbf{K})$, $\mathbf{L} = \mathbf{J} - (\mathbf{P} - \mathbf{K})$ and $P_0 + K_0$. Writing $Lie(G) = Lie(K) + T(\mathscr{D})$, we see that the tangent space $T(\mathscr{D})$ to $\mathscr{D}$ at the origin is generated by the 8 generators $\mathbf{N}$, $\mathbf{P} + \mathbf{K}$, $P_0 - K_0$ and $\mathscr{D}$. A maximal set of generators in $\mathscr{D}$ is $\{D, N_3\}$ (the Lie balls are rank two spaces); the only arbitrariness is the choice of the direction in the 3-dimensional $\mathbf{N}$ space. A Cartan basis for $LieG$ (which is of rank 3) can be gotten by completing the previous set, we get $\{D, N_3, J_3\}$. If we write the Shilov boundary of $\mathscr{D}4$ as $\check{\mathscr{D}}4 = K/H$, then H is the diagonal subgroup of $SO(4) \in K$ so that $Lie(H)$ is generated by $\mathbf{J} = 1/2(\mathbf{R} + \mathbf{L})$. Writing $Lie(K) = Lie(H) + T(\check{D})$, we find that the tangent space to $\check{D}$ is generated by $P_i - K_i$ and $P_0 + K_0$. Considering now the action of $G = SO(4, 2)$ on the Shilov boundary, we know that $\check{D} = G/L$ where L is the little group of ∞ and is the semi-direct product of the Poincare group times dilation. L is therefore generated by $\mathbf{J}$, $\mathbf{N}$, P_μ and D so that the tangent space at the infinity point of $\check{\mathscr{D}}$ is generated by the conformal translations $\mathbf{K}$. It can be shown that choosing an origin in the domain $\mathscr{D}$ and a point at infinity in the Shilov boundary $\check{\mathscr{D}}$ implies the choice of an "origin" of $\check{\mathscr{D}}$ (cf. 5.2). The little groups of the infinity and of the origin $\check{\mathscr{D}}$ are exchanged by replacing translations by conformal translations, and conversely.

4.4. *Conformal transformations on space-time*

In the case of Minkowski space-time considered as the Shilov boundary of $\mathscr{D}4$ in its unbounded realization, we already know the action of rotations $exp(\theta . J)$, Lorentz boosts $exp(\alpha . L)$ and translations $exp(a . P)$ on space-time. One can then show that

$$exp(dD)(x_\mu) = d \,.\, x_\mu ,$$

$$exp(s \,.\, K)(x_\mu) = \frac{[x_\mu + s_\mu x^2]}{[1 + 2s \,.\, x + s^2 x^2]} .$$

Notice that the action of conformal translations is singular (and not linear). Notice also that if we set $x' = exp(s \,.\, K)x$, we get

$$x'^2 = \frac{x^2}{[1 + 2s \,.\, x + s^2 x^2]}$$

which shows that conformal translations (and dilations) do not preserve the line element but they do preserve light-like intervals; the conformal group therefore preserves light-cones and the causal structure of space-time. The fundamental vector fields associated with this action are the following:

$$P_\mu = \partial_\mu$$

$$J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$$

$$D = x^\mu \partial_\mu$$

$$K_\mu = 2x_\mu D - x^2 P_\mu .$$

They are not Killing vector fields since they do not leave invariant the Minkowski metric $\eta_{\mu\nu}$ of space-time but they are conformal Killing vector fields. Let $C(\tau)$ be a line in Minkowski space (Lorentz metric $\eta_{\mu\nu}$) with tangent vector $X = \dot{x}^\mu \partial_\mu$ and let us compute the angles between this line and the conformal Killing vector fields (this information will be used later). One gets

$$\eta(P_\rho, X) = \dot{x}_\rho$$

$$\eta(L_{\rho\sigma}, X) = \dot{x}_\sigma x_\rho - \dot{x}_\rho x_\sigma$$

$$\eta(D, X) = x^\mu \dot{x}_\mu$$

$$\eta(K_\rho, X) = 2x_\rho x^\mu \dot{x}_\mu - x^2 \dot{x}_\rho .$$

4.5. *Riemannian geometry of $\mathscr{D}n$*

4.5.1. *General remarks*

The holonomy group of a generic metric on $\mathscr{D}4$ is $SO(8)$. The holonomy group of a generic Hermitian metric on $\mathscr{D}4$ is $U(4)$. The holonomy group of the $SO(4,2)$-invariant Kähler metric on $\mathscr{D}4$ is $SO(4) \times SO(2)$.

4.5.2. *The Lobatchevski metric on $\mathscr{D}1$, the product metric on $\mathscr{D}2 = \mathscr{D}1 \times \mathscr{D}1$ and the Bergman metric on $\mathscr{D}n$*

It is well known that the Lobatchevski metric on $\mathscr{D}1$, considered as the upper half plane ($z = u + iv, v > 0$) is

$$ds^2 = \frac{(du^2 + dv^2)}{v^2} .$$

The direct product metric on $\mathscr{D}2 = \mathscr{D}1 \times \mathscr{D}1$ is

$$ds^2 = \frac{(du_1^2 + dv_1^2)}{v_1^2} + \frac{(du_2^2 + dv_2^2)}{v_2^2} = \frac{v_2^2(dz_1\, d\bar{z}_1) + v_1^2(dz_2\, d\bar{z}_2)}{(v_1^2)(v_2^2)} .$$

Let us set $w_1 = (z_1 + z_2)/2$, $w_2 = (z_1 - z_2)/2$, $w_1 = x_1 + iy_1$ and $w_2 = x_2 + iy_2$. We get

$ds^2 = 2/(y^2) \times \{2y_i y_j dw^i d\bar{w}^j - y^2 dw_j d\bar{w}^j\}$. In the previous formula, the indexes are contracted with a Lorentz metric, i.e. $y^2 = y_i y^i = y_1^2 - y_2^2$, $y_i dw^i = y_1 dw^1 - y_2 dw^2$, etc. but ds^2 is an Euclidean metric! This expression can be generalized to all the $\mathscr{D}n$. Introducing the Lorentz metric $\eta_{\mu\nu} = (-1\,1)$ in the case of $\mathscr{D}2$ or $\eta_{\mu\nu} = (-1\,-1\,-1\,1)$ in the case of $\mathscr{D}4$— and removing the irrelevant numerical factor two—we can write the Kähler Euclidean metric $ds^2 = g_{\mu\nu} dw^\mu d\bar{w}^\nu$ of the Cartan domains $\mathscr{D}n$ as

$$ds^2 = [\eta_{\mu\nu}/y^2 - 2y_\mu y_\nu/(y^2)^2]\, dw^\mu d\bar{w}^\nu.$$

Also,

$$ds^2 = [\eta_{\mu\nu}/y^2 - 2y_\mu y_\nu/(y^2)^2][dx^\mu dx^\nu + dy^\mu dy^\nu].$$

We will see later that it is useful to set

$$y_\mu = p_\mu/p^2$$

then, $dy^\mu = dp^\mu/p^2 - 2p^\mu(p\,.\,dp)/(p^2)^2$ and the metric reads

$$ds^2 = [\eta_{\mu\nu} p^2 - 2p_\mu p_\nu][dx^\mu dx^\nu + dp^\mu dp^\nu/(p^2)^2].$$

Notice also that the Kähler metric $g_{\mu\nu}$ is equal to the commutator of operators x_μ and $y_\mu = p_\mu/p^2$ where x_μ and p_μ are the position and momentum operators of quantum mechanics (use $[x_\mu, p_\nu] = -i\eta_{\mu\nu}$).

4.5.3. *The connection coefficients*

The non-zero connection coefficients are $\Gamma^{\bar{\mu}}_{\bar{\nu}\bar{\rho}} = -\Gamma^\mu_{\nu\rho}$ with

$$i\Gamma^\mu_{\nu\rho} = 1/y^2(y^\mu \eta_{\nu\rho} - \eta^\mu_\nu y_\rho - \eta^\mu_\rho y_\nu) + 2/(y^2)^2(y^\mu y_\nu y_\rho).$$

4.5.4. *The Riemann tensor on $\mathscr{D}n$*

The non-vanishing components are

$$R^\mu_{\nu\rho\bar{\sigma}} = R^{\bar{\mu}}_{\bar{\nu}\bar{\rho}\sigma} = R^\mu_{\nu\bar{\rho}\sigma} = R^{\bar{\mu}}_{\bar{\nu}\rho\bar{\sigma}}$$

with

$$\begin{aligned}R^\mu_{\nu\rho\bar{\sigma}} = 1/2\{&1/y^2(\eta^\mu_\sigma \eta_{\nu\rho} - \eta^\mu_\nu \eta_{\rho\sigma} - \eta^\mu_\rho \eta_{\nu\sigma})\\ &+ 2/(y^2)^2(\eta^\mu_\sigma y_\rho y_\sigma + \eta^\mu_\rho y_\nu y_\sigma + \eta^\mu_\sigma y_\nu y_\rho + \eta_{\rho\sigma} y^\mu y_\nu)\\ &- 8/(y^2)^3 y^\mu y_\nu y_\rho y_\sigma\}.\end{aligned}$$

4.5.5. *Geodesics*

The geodesic equation is $dw^\mu/d\tau^2 + \Gamma^\mu_{\nu\rho}(dz^\nu/d\tau)(dz^\rho/d\tau) = 0$ with $w = x + iy$. We will not discuss this equation here because we will use another technique to study geodesics.

Geodesics on T^*M

When M is a Riemannian or pseudo-Riemannian manifold, there are three kinds of "natural metrics" on the cotangent bundle T^*M, but when M is flat (for instance in the case of Minkowski), these metrics coincide. In this last case, let us write $g = g_{\mu\nu}\,dx^\mu\,dx^\nu + g_{\mu\nu}\,dp^\mu\,dp^\nu$ and we have an obvious identification between TM and T^*M via the relation $p_\mu = g_{\mu\nu}p^\mu$. In the very simple case where $M = R^n$, the equation for geodesics in T^*M is $\ddot{x} = 0$, $\ddot{p} = 0$ and we get two kinds of geodesics:

1. The first class of geodesics ($\ddot{x} = 0$, $p = \dot{x}$ so that $\dot{x} = p = cte$) are lift of geodesics of M ($\ddot{x} = 0$) by the Levi Civita of the metric of T^*M.
2. The second class of geodesics ($\ddot{p} = 0$, $x = \dot{p}$, so that $\dot{p} = x = cte$) are orthogonal to those of the previous class and, in relativity describe deviation of geodesics of space-time (cf. Jacobi fields) in M.

These very elementary remarks show that there are two very different kinds of geodesics even in a standard phase-space such as T^*M. We will see later why the Cartan domain $\mathscr{D}n$ is also a "phase-space"; we should therefore keep in mind the fact that, as in the classical case, there are several types of geodesics in a phase-space and that their physical signification can be very different from one class to another one.

Geodesics in $\mathscr{D}n$

Geodesics can be studied by solving the equation for geodesics—which is of second degree but in the present case, where we have a group acting by isometries, it is much more handy to write that the angle between geodesics and Killing vector fields are conserved quantities because, in this way, we have already introduced the appropriated constants of motion. We shall see later what the Killing fields are. When interpreted as a phase space, some of these geodesics can be interpreted in terms of trajectories of space-time but others, exactly like in the case of $T * M$, describe relative accelerations. This can also be described in group theoretical terms [27].

4.6. *Conformal transformations on the Lie ball $\mathscr{D}4$*

4.6.1. *Action on $\mathscr{D}4$ (direct calculation)*

We know what the fundamental fields for the conformal group action on space-time are. We have also given previously the action itself. The action on the domain $\mathscr{D}4$ itself, in its unbounded realization (forward tube) is gotten by replacing x_μ by $z_\mu = x_\mu + iy_\mu$. For instance, in a conformal translation of vector s, the point z goes to z' with $z'_\mu = (z_\mu + s_\mu z^2)/(1 + 2s.z + s^2z^2)$. We have then to compute the real and imaginary part of z' in terms of those of z. It is convenient to set

$$y_\mu = hp_\mu/p^2$$

where h is a constant with the dimension of the Planck constant and where p will be interpreted as a momentum (cf. next section). Expressions are lengthy, so we will only give the transformation associated with a conformal translation of vector s (which is the more complicated but the more interesting). One gets

$$x' = (1/\delta)\{[2h^2(s^2x\,.\,p + s\,.\,p)]p + [h^4s^2 + h^2((4x\,.\,p)(s^2x\,.\,p + s\,.\,p) - (1 + 2s\,.\,x + s^2x^2)p^2) + x^2(p^2)^2(1 + 2s\,.\,x + s^2x^2)]s + [h^2(-p^2s^2) + (p^2)^2(1 + 2s\,.\,x + s^2x^2)]x\}$$

where

$$\delta = h^4(s^2)^2 + h^2[4(s\,.\,p + s^2(x\,.\,p))^2 - 2s^2p^2(1 + 2s\,.\,x + s^2x^2)] + (p^2)^2[1 + 2s\,.\,x + s^2x^2]^2$$

and

$$p' = (1/p^2)[-h^2s^2 + p^2x^2s^2 + 2p^2s\,.\,x + p^2]p + (1/p^2)[2h^2s\,.\,p - 2p^2x^2s\,.\,p + 4p^2(x\,.\,p)(s\,.\,x) + 2p^2x\,.\,p]s + [-2s^2x\,.\,p - 2s\,.\,p]x.$$

As expected, the transformation of x and p depend on both x and p (one should of course not set $p = m\dot{x}$ here!). We should now investigate what happens at the limit h goes to zero. One gets

$$x'_{cl} = (x^2s + x)/[1 + 2s\,.\,x + s^2x^2].$$

This expression coincides with the well-known action of the conformal group on space-time (already given previously). Taking $h \to 0$ in the case of the momentum leads to

$$p'_{cl} = [1 + 2s\,.\,x + s^2x^2]p + [-2x^2s\,.\,p + 4(x\,.\,p)(x\,.\,s) + 2x\,.\,p]s + [-2s^2x\,.\,p - 2s\,.\,p]x.$$

The important observation is to remark that this last expression is exactly what one gets classically. Indeed, if we set $u = \dot{x} = dx/d\tau$, $u' = \dot{x}'_{cl} = dx'_{cl}/d\tau'$ and compute $d\tau/d\tau'$ from the transformation law for x, we find

$$u' = \{[1 + 2s\,.\,x + s^2x^2]u + [-2x^2s\,.\,u + 4(x\,.\,u)(x\,.\,s) + 2x\,.\,u]s + [-2s^2x\,.\,u - 2s\,.\,u]x\}/[1 + 2s\,.\,x + s^2x^2]^{(1/2)}.$$

So that if we set classically $p = mu$, we obtain $p'_{cl} = m'u'$ where $m'^2 = [1 + 2s.x + s^2x^2]m^2$ as it should.

This striking result is another indication that the imaginary part of $z = x + iy$ should indeed be considered as the inverse of a momentum. Another argument using the Poincaré-Cartan map will be given in the next section. We therefore have a map from $\mathscr{D}4$ to the co-tangent bundle of its Shilov boundary (space-time), namely $x + iy \to (x, p = hy/y^2)$ and the action of the conformal group on the domain commutes, at the $h \to 0$ limit with the action of the conformal group on space-time itself. Notice that, in the domain itself, x and p are independent variables. Notice also that, in the domain, the Jacobian of the transformation $z \to z'$ is a 4×4 matrix whose components can be related, in the $h \to 0$ limit, to the four-acceleration of a classical curve in space-time. We will examine further some relations between accelerated motion and the conformal group in a coming section.

4.6.2. *Killing vector fields*

In the unbounded realization of the Cartan domain $\mathscr{D}4$, i.e. the forward tube ($z = x + iy, y^2 > 0$, (or < 0, depending upon Lorentz signature), the fundamental vector fields associated with the action of the conformal group are Killing vector fields for the Bergman metric. Their complex extension (as elements of the complexified tangent bundle) are given by the same expressions as in Minkowski space-time, but we have to replace $\partial/\partial x$ by $\partial/\partial z$ where $\partial/\partial z = 1/2(\partial/\partial x + 1/i\partial/\partial y)$. The (real) vector fields themselves are then obtained by taking (half of) the real part of their complex extension. For instance the complexified conformal translations read

$$K^c_\mu = 2z_\mu z^\rho \partial/\partial z^\rho - z^2 \partial/\partial z^\mu .$$

The corresponding real vector fields, obtained by taking their real part, are given as follows:

$$P_\mu = \partial/\partial x^\mu$$

$$J_{\mu\nu} = (x^\mu \partial/\partial x^\nu - x^\nu \partial/\partial x^\mu) + (y^\mu \partial/\partial y^\nu - y^\nu \partial/\partial y^\mu)$$

$$D = x^\mu \partial/\partial x^\mu + y^\mu \partial/\partial y^\mu$$

$$K_\mu = 2x_\mu D - 2y_\mu(y^\rho \partial/\partial x^\rho - x^\rho \partial/\partial y^\rho -) - (x^2 - y^2)P_\mu - 2x^\rho y_\rho \partial/\partial y^\mu .$$

Notice that these vector fields coincide with those of space-time when we set $y = 0$. Warning: $x = 0$, $y = 0$ is not a point of the domain $\mathscr{D}4$, it is the origin of its Shilov boundary; the origin of the domain itself is $x = 0$, $y = (1000)$. Notice also that the vector fields that vanish at the origin of the domain are $\mathbf{J}$, $\mathbf{K} - \mathbf{P}$, $K_0 + P_0$, i.e. the tangent vectors that generate the isotropy group $K = SO(4) \times SO(2)$ of the origin. This also shows that the tangent space at the origin of the domain (considered as a homogeneous space) is generated by a set of vector fields orthogonal to the previous ones. For the same reason, we can identify those generators of *LieG* that span the little group of the origin of space-time when we write it as G/L. They are those vector fields that vanish at $x = y = 0$.

4.6.3. *Conserved quantities*

Angles between geodesics of the domain and Killing vector fields are conserved quantities in $\mathscr{D}4$. Let us call $X = \dot{x}^\mu \partial/\partial x^\mu + \dot{y}^\mu \partial/\partial y^\mu$ the tangent vector to a geodesic at the point $x(\tau)$, $y(\tau)$. The metric is the Kähler metric given previously. We express these conserved quantities in terms of x_μ and $p_\mu = y_\mu/y^2$:

$$\mathbf{P}_\rho = g(P_\rho, X) = p^2\dot{x}_\rho - 2(\dot{x}\,.\,p)p_\rho$$

$$\mathbf{D} = g(X, D) = p^2(\dot{x}\,.\,x) + (\dot{p}\,.\,p)/p^2 - 2(\dot{x}\,.\,p)(x\,.\,p)$$

$$\mathbf{K}_\rho = g(X, K\rho) = 1/p^2\{2[(p^2)^2(\dot{x}\,.\,x) + (\dot{p}\,.\,p) - 2(\dot{x}\,.\,p)(x\,.\,p)p^2]x_\rho$$

$$+ 2[(x\,.\,\dot{p}) + p^2(x\,.\,x)(\dot{x}\,.\,p)]p_\rho - [(p^2)^2x^2 - p^2]\dot{x}_\rho - 2(x\,.\,p)\dot{p}_\rho\}$$

$$\mathbf{J}_{\rho\sigma} = p^2[(\dot{x}_\sigma x_\rho - \dot{x}_\rho x_\sigma)] + [(\dot{p}_\sigma p_\rho - \dot{p}_\rho p_\sigma)]/p^2 - 2(\dot{x}\,.\,p)[(p_\sigma x_\rho - p_\rho x_\sigma)].$$

Rather than discussing geodesics of the domain $\mathscr{D}4$ by using the geodesic equation itself, it is much more convenient to use the fact that the previous quantities are conserved (for instance, in Minkowski space-time, we would write that the angle $\mathbf{P}_\mu = \eta(\dot{x}^\rho \partial/\partial x^\rho, P_\mu = \partial/\partial x^\mu) = \dot{x}_\mu$ is a conserved quantity and this precisely describes geodesics). We will not carry this analysis here but want to mention that while discussing these equations, it is also convenient to impose that particular conserved quantities (corresponding to particular directions in the domain) vanish and to restore the constant h, i.e. to set $z = x + ihp/p^2$.

4.7. *Cones, spheres and hyperboloids*

4.7.1. *Spaces of two-spheres in S^3 and in B_4*

Consider the space of spheres S^n in R^q. A sphere is characterized by its center (q coordinates), its radius (1 coordinate) and the position of the $(n + 1)$-plane of R^q inside which it lies. This position depends therefore on $(n + 1)(q - (n + 1))$ parameters—the dimension of the Grassmannian $SO(q)/SO(n + 1)SO(q - (n + 1))$. So the dimensionality of this space of spheres is $d(n, q) = q + 1 + (n + 1)(q - (n + 1)) = 2q - n^2 + n(q - 2)$. For instance $d(2, 3) = 4$, $d(2, 4) = 8$, $d(3, 4) = 5$, $d(3, 5) = 10$.

The fact that $d(2, 3) = 4$ is not surprising since we know that if we consider a fixed S^3 hypersurface $t = t_0$ in compactified Minkowski $M = S^3 \times_{Z_2} S^1$ and the two-spheres inside this S^3, each two-sphere determines a causality diamond (past and future light-cones) with apexes $z1$, $z2$ in Minkowski $M_{3,1}$. Therefore, up to a Z_2 factor, the space of S^2's is just $M_{3,1} = SO(4,2)/(P \times R^+)$ where the little group is the semi-direct product of the Poincare group P and dilations R^+. The conformal group acts on these spheres as follows: call $S_{t_0}(x)$ the two-sphere $t = t_0 \cap Cone(x)$. Then $SO(4,2)$ acts on points of M, so that we know what $x' = \Lambda x$ is and also what the $Cone(\Lambda x)$ is. The transformed sphere is $S_{t_0}(x' = \Lambda x)$.

The 3-sphere S^3 is the boundary of the euclidean ball $B4$ in R^4; it is therefore not too surprising to discover that the space of two-spheres in $B4$ coincides with the eight-dimensional Cartan domain $\mathcal{D}4$; we will see that this is indeed the case (cf. section devoted to harmonicity cells and to the Lelong map).

4.7.2. *Spaces of light-cones*

A light-cone in (compactified) Minkowski M is fully characterized by its origin so that the space of these light-cones is M itself so it can be identified with $M = SO(4,2)/(P \times R^+)$.

4.7.3. *Spaces of hyperboloids*

Here, hyperboloids are thought of as three-dimensional spheres of compactified Minkowski $M_{3,1}$. In the same way that $SO(4,2)$ acts on points of $M_{3,1}$ and that its action on two-spheres of S^3 by cutting light-cones of $M_{3,1}$ by $t = t_0$, we find that $SO(5,2)$ acts on points of $M_{4,1}$, i.e. on its light cones (of equation $t^2 - x^2 - y^2 - z^2 - x_5^2 = 0$). If we cut such a light cone by an hypersurface $\{x_5 = cte\}$, we get a two-sheeted hyperboloid of $M_{3,1}$. The space of two-sheeted hyperboloids can therefore be identified with $M_{4,1}$ itself so that it is equal to $SO(5,2)/(P(5) \times R^+)$ where $P(5)$ denotes the Poincare group in five dimensions. Notice that the space of 1-sheeted hyperboloids is not a quotient of $SO(5,2)$ but of $SO(4,3)$; indeed we start from $M_{3,2}$ and analyse the action of its conformal group $(SO(4,3))$ on its light cones ($t^2 - x^2 - y^2 - z^2 + x_5^2 = 0$) that we cut by $\{x_5 = cte\}$, so that we get $t^2 - x^2 - y^2 - z^2 = -x_5^2 < 0$.

To summarise: The space of two-spheres in the four-dimensional open ball B_4 can be identified with the Cartan domain $\mathcal{D}4$ (up to a discrete Z_2 factor). The space of two-spheres in the three-sphere S^3 can be identified with its Shilov boundary, the compactified Minkowski space-time $M_{3,1}$ (up to a discrete Z_2 factor). The space of light-cones in $M_{3,1}$ is a quotient of the conformal group $SO(4,2)$, it has dimension 4. The space of 2-sheeted hyperboloids of $M_{3,1}$ is a quotient of $SO(5,2)$, it has dimension 5. The space of one-sheeted hyperboloids of $M_{3,1}$ is a quotient of $SO(4,3)$, it has also dimension 5.

Notice that $SO(4,2)$ is a subgroup of both $SO(5,2)$ and $SO(4,3)$, therefore it acts also on the spaces of hyperboloids but not transitively.

The typical light cone itself in $M_{3,1}$ is a quotient of $SO(3,1)$ by the Euclidean group $E(2)$. The typical two-sheeted hyperboloid in $M_{3,1}$ is a quotient of $SO(3,1)$ by $SO(3)$. The typical one-sheeted hyperboloid in $M_{3,1}$ is a quotient of $SO(3,1)$ by $SO(2,1)$.

5. Cartan Domains as Spaces of Symmetries, $\mathcal{D}4$ as a Phase-space and Berezin-Weyl Calculus

5.1. *Algebra and geometry of the $SU(n,n)/S(U(n) \times U(n))$ family*

In this section we will study geometry of the symmetric domains $A_n \doteq SU(n,n)/S(U(n) \times U(n))$ by algebraic methods. To this end we will identify A_n with a submanifold $S(V)$ of the vector space of all linear operators acting in a $2n$ dimensional vector space **V**.

Let V be a complex vector space of complex dimension $2n$, equipped with a non-degenerate sesquilinear form $\langle v, w\rangle$ of signature (n,n). We denote by $A \to A^*$ the conjugation $\langle Av, w\rangle = \langle v, A^*w\rangle$ in $L(V)$. We call an operator $S \in L(V)$ a symmetry iff $S = S^*$, $S^2 = 1$, and $\langle v, Sv\rangle \geq 0$ for all $v \in V$. The set of all symmetries is denoted by $S(V)$. For each $S \in S(V)$, the positive definite scalar product $(v,w)_S$ is defined by

$$(v,w)_S = \langle v, Sv\rangle .$$

Given a symmetry S one defines the orthogonal projection $E_S \doteq \frac{1}{2}(2S + I)$. In this way one gets a one-to-one correspondence between the set of all symmetries and a subset of the Grassmannian of positive n-planes. The unitary group $U(V) \approx U(n,n)$ acts transitively on $S(V)$ by the natural action $S \to USU^*$ with the isotropy group $\approx U(n) \times U(n)$. Since the central circle group of $U(n)$ acts trivially on $S(V)$, we get the isomorphism $S(V) \approx A_n \doteq SU(n,n)/S(U(n) \times U(n))$. Notice that the Lie algebra $Lie(U(V))$ can be identified with anti-Hermitian operators on V. When considered as tangent vectors to $S(V)$, they can also be identified with fundamental vector fields on $S(V)$.

The fact that A_n is now embedded into an algebra allows us to use the algebraic machinery for studying differential geometry of these domains. First of all, the relations $S = S^2, S = S^*$ allow us to identify complex tangent vectors at $S \in S(V)$ with operators W such that $WS + SW = 0$. Real tangent vectors are characterized by the extra condition $W = W^*$. A natural almost complex structure J on $S(V)$ is given by

$$W \overset{J_S}{\mapsto} J_S W \doteq iSW .$$

The Kähler metric h on $S(V)$ is simply given by

$$h_S(W_1, W_2) = -Tr(W_1 W_2) .$$

The symplectic form ω is

$$\omega_S(W_1, W_2) = iTr(W_1 S W_2),$$

for W_1, W_2 tangent at S to $S(V)$. Both h and ω are evidently invariant under the action of $U(n)$. The symplectic form is closed $d\omega = 0$. To see that the almost complex structure J is covariantly constant under the Levi-Civita connection of h, it is again convenient to use the algebraic machine that provides an easy tool for describing the geodesic parallel transport on $S(V)$.

Given two points $S_1, S_2 \in S(V)$, the operator $S_1 S_2$ is positive with respect to the p.d. scalar products $(v, w)_{S_i}$, $i = 1, 2$. The operator $t_{1,2} \doteq (S_1 S_2)^{1/2}$ is then unambigously defined, positive for both scalar products, and an isometry of V; we have $t_{1,2}{}^* = t_{1,2}{}^{-1} = t_{2,1}$. Moreover,

$$t_{1,2} S_2 t_{1,2}{}^* = S_2 ,$$

and $t_{1,2}$ maps the n-plane V_{S_2} onto V_{S_1}. The most interesting property of $t_{1,2}$ is that when applied to tangent vectors to $S(V)$ at S_2 it maps them into the tangent vectors at S_1 obtained by parallel transport along the unique geodesic connecting the two points. To see this one uses the fact that geodesics on $S(V)$ are trajectories of one-parameter subgroups of $U(n)$. The transport operators t preserve now the almost complex structure J on $S(V)$.

5.2. *Boundary map and Cayley transform*

The Shilov boundary $\check{S}(V)$ of $S(V)$ consists of isotropic n-planes. Let us fix one such plane denoted ∞. Each S, being in particular a symmetry in $S(V)$, reflects ∞ onto another isotropic n-plane

$$\Pi_\infty(S) \doteq S_\infty .$$

The map $\Pi_\infty : S(V) \to \check{S}(V)$ is equivariant with respect to the stability group at ∞. If now an origin O is fixed in $S(V)$, its image under Π_∞ is called the antipode of ∞—or the origin o of $\check{S}(V)$.

There exists a parametrization of $S(V)$ and of $\check{S}(V)$ which makes the complex structure of $S(V)$ transparent, and the boundary map Π_∞ simple. To this end fix an origin O in $S(V)$ and ∞ in $\check{S}(V)$. One has $\infty \perp O\infty$, therefore $V = \infty \oplus O\infty$, and so one can identify $\infty \oplus \infty$ with V by the map

$$(v_1, v_2) \mapsto v_1 + iOv_2 .$$

With this identification O has the form

$$O = antidiag(i, -i),$$

and the indefinite scalar product of V reads

$$\left\langle \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle = i\{(v_1, w_2)_O - (v_2, w_1)_O\},\ v_i, w_i \in \infty,\ i = 1, 2.$$

The p.d. scalar product $(\ ,\ )_O$ is given by

$$(v, w)_O = v_1{}^\dagger w_1{}^\dagger + v_2{}^\dagger w_2{}^\dagger,$$

for $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$. We use the "†" to denote the adjoint with respect to the p.d. scalar product induced on ∞. This scalar product will be simply written as $v^\dagger w$. The Lie algebra $Lie(SU(V))$ can be parametrized as

$$\begin{pmatrix} d + L & T \\ A & -d - L^* \end{pmatrix}$$

where $d \in R$, $T = T^\dagger$, $A = A^\dagger$. The $n(n + 1)/2$-parameter Abelian subgroup generated by T—s leaves ∞ (but not o) fixed—we call it "translation subgroup". To each linear operator Z in V one associates now a subspace V_Z of V by

$$V_Z = \{(Zv, v) : v \in \infty\} = (Z + iO)\infty .$$

It follows that V_Z is in $S(V)$ iff

$$i(Z^* - Z) > 0,$$

and V_Z is in the Shilov boundary iff

$$(Z^* - Z) = 0.$$

Every subspace $E_S V$, $S \in S(V)$ is of the form V_Z for a unique Z. S is expressed in terms of Z via

$$S = \begin{pmatrix} Z & 0 \\ 0 & 1 \end{pmatrix} \frac{2}{Z - Z^*} \begin{pmatrix} 1 & -Z^* \\ 1 & -Z^* \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

A necessary and sufficient condition for a subspace of V to be of the form V_Z is that it is transversal to ∞. The correspondence $Z \leftrightarrow V_Z$ is called Cayley transform. It maps selfadjoint operators Z onto the Shilov boundary of $S(V)$ minus "cone at infinity". The boundary map Π_∞ is now given by

$$Z \mapsto \frac{Z + Z^*}{2} .$$

Twistors. The above study can be applied, in a particular case, to the study of twistors (flag manifolds in C^4) [25].

5.3. *The momentum map*

For a symplectic manifold $(\mathscr{S}, \omega)$ with a symplectic action of a Lie group G one defines a momentum mapping (Poincare-Cartan form) as a function $J : \mathscr{S} \to Lie(G)^*$

satisfying the condition

$$d(\hat{J}(X)) = i_{\bar{X}}\omega,$$

for all $X \in Lie(G)$, where for all $s \in \mathscr{D}$, the function $\hat{J}(X)$ is defined by $\hat{J}(X)(s) \doteq \langle J(s), X \rangle$, and $\bar{X}$ is the fundamental vector field associated to X. An explicit knowledge of the momentum mapping is useful for a physical interpretation of the geometrical quantities. We shall compute the momentum map by using the algebraic technique introduced in the previous section. In our case the momentum mapping is given by a simple formula

$$\hat{J}(X)(S) = 2iTr(SX),$$

where S is in $S(V)$ and $X = -X^*$ is in $Lie(G)$. Taking $V = C^{2n}$,

$$\langle v, w \rangle \doteq v^\dagger G w,$$

where G is the Hermitian matrix $G \in Mat_{2n \times 2n}(C)$

$$G = i\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

we have for the scalar product

$$\langle v, w \rangle = i({v_1}^\dagger w_2 - {v_2}^\dagger w_1),$$

for $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$. For $n = 2$ we can write the generators T of the translation group in terms of the Pauli matrices $T = p^\mu \sigma_\mu$. The momentum map reads then

$$p_\mu = \hat{J}(T = \sigma_\mu)(Z) = Tr\left(\frac{1}{Z - Z^\dagger}\sigma_\mu\right).$$

Writing $Z = (x^\mu + iy^\mu)\sigma_\mu$ we get $p^\mu = y^\mu / y^2$. This explains why, in the forward tube realization of the domain $\mathscr{D}4 = \mathscr{A}2$, the imaginary part of $z = x + iy$ can be interpreted as the inverse of the momentum.

5.4. *Berezin-Weyl calculus*

The classical Weyl calculus associates Hilbert space operators to their phase-space function symbols (see [58]). The fact that each Kähler manifold $\mathscr{D}$ inherits its metric from the canonical immersion in the associated (projective) Hilbert space $P\mathscr{H}^2$ of 1-densities [40] endowed with the Fubini metric, makes the Weyl calculus on such a manifold almost automatic. Denoting by e_z the evaluation functional and by $|z\rangle$ the coherent state at $z \in \mathscr{D}$, so that

$$|z\rangle = \frac{e_z}{\|e_z\|},$$

$$k(z, \bar{u}) = \langle e_z, e_u \rangle$$

one associates to each function $f(z)$ on $\mathscr{D}$ the operator

$$F \doteq \int f(z)|z\rangle\langle z|\, d\mu,$$

where $d\mu = k(z, z)\, dz\, d\bar{z}$ is the natural invariant measure on $\mathscr{D}$. Equivalently, one ean use the Bergman projection P_B given by the Bergman kernel to produce the Toeplitz operator $F = P_B \hat{f} P_B$ out of the multiplication operator $\hat{f}$. Notice that if the function f is itself holomorphic, then no projection is necessary since $(\hat{f}\Psi)(z) \doteq f(z)\Psi(z)$ is holomorphic. Many of the relevant physical observables will be however represented by non-holomorphic functions. In such a case $(\hat{f}\Psi)(z)$, even if square integrable, is not holomorphic and the projection back onto the holomorphic functions is necessary. It is by this projection that the non-Abelian algebra is created out of the Abelian algebra of classical functions on $\mathscr{D}$. This approach to the Weyl calculus is natural in the case where $\mathscr{D}$ is considered as the phase space. For $\mathscr{D}$ the classical domains it has been first investigated by Berezin [6] without understanding however the role of the Shilov boundary for the physical interpretation (see also [62] where, for the domains $\mathscr{D}_n$ and instead of the Shilov boundary, the light cone plays the role of the configuration space). The domain $\mathscr{D}_4$ has been proposed explicitly as a conformal relativistic phase space by [50]. The domain $\mathscr{D}3$ has also been studied in the same spirit (in relation with De Sitter space-time) in [2] and [4]. It is important to understand that although the Weyl calculus for these domains looks just like a standard game with coherent states and Bergman kernels, it is the physical identification of space-time with the Shilov boundary of the domain that makes it into a new and open subject. Anyhow, every two separable Hilbert spaces are isomorphic, but it is the interpretation of the Hilbert space states and operators that makes the difference between the study of a hydrogen atom and a cup of coffee.

6. Conformal Invariance in Physics and Mathematics

6.1. *The breaking of conformal invariance in physics*

The conformal group $SO(4, 2)$ is the biggest invariance group of the Maxwell equations, of the massless Klein-Gordon equation, of the massless Dirac equation and more generally of all massless equations for particles of spin j. This means that if we have a solution of one of these equations, we can obtain others by action of the conformal group. At the infinetisimal level, this can be proven, for example in the case of the Klein-Gordon equation, by showing that the commutator of the Dalembertian with a generator of the conformal group is either zero or proportional to the Dalembertian itself. Massless free field equations are not the only ones that are invariant under the action of the conformal group. For instance, this is also the case in four dimensions for the nonlinear equation $\Box\phi + \phi^3 = 0$. One possible technique to manufacture four dimensional lagrangians that are invariant under the conformal group is to use a projective formalism in six dimensions [44]. Introduction of mass terms in the classical lagrangian usually breaks the invariance under the conformal groups and this is why it is often said that conformal invariance is broken in physics. It remains that the conformal group acts on space-time itself and that one can write its action also on fields defined on space-time. As usual, these fields will transform under a representation of the little group of a chosen origin L = Lorentz transformations × Conformal translations × Dilations. Therefore, each field will be characterized not only by their spin but by other "quantum numbers" corresponding to the other generators (for instance by their conformal weight). The action of these generators on fields is given as follows:

$$P_\mu \varphi(x) = i\partial_\mu \varphi(x)$$

$$J_{\mu\nu}\varphi(x) = \big(i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \Sigma_{\mu\nu}\big)\varphi(x)$$

$$D\varphi(x) = (ix_\mu \partial^\mu + \Delta)\varphi(x)$$

$$K_\mu \varphi(x) = \big(i(2x_\mu x_\nu \partial^\nu - x^2\partial_\mu - 2ix^\nu(\eta_{\mu\nu}\Delta + \Sigma_{\mu\nu})) + k_\mu\big)\varphi(x).$$

Here, the matrices $\Sigma_{\mu\nu}$, Δ and k_μ represent the generators of the little group (Lorentz, dilations and conformal translations).

For any choice of Lagrangian, one can then define the corresponding Noether currents and the associated charges [44]. Invariance (if any) of the classical Lagrangian under the conformal group is then expressed by the vanishing of the divergence of those currents. When quantized, it happens usually that the classical equations for the (non) conservation of currents are modified. One then defines "anomalous dimensions" for the quantum fields and describe physics by renormalization group equations (cf. [9] and references therein). Without going to the quantum level, it is already clear that conformal symmetry is "broken" from the fact that, for instance, the Klein-Gordon equation with mass is not invariant, unless mass itself is not treated as a constant but as an x-dependent field. However, this last possibility is incompatible with what we mean physically by "mass", as measured in a locally Lorentzian frame.

6.2. *Quantum mechanics in arbitrary frames*

On the other hand, and according to the general philosophy of general relativity, the laws of physics should be "the same" in any coordinate system. More precisely, if somebody knows how to describe physics in a particular class of coordinate system, we can choose any other—not necessarily gotten from the previous one by a Lorentz transformation—and describe the same physics. For instance, such a change of coordinate system $O \to O'$ can be a conformal transformation (in general it will not be a Lorentz transformation), in which case, a (constant) mass m in the first coordinate system will become a (non-constant) mass $m'(x)$ in the new. This is not surprising and has nothing to do with the previous discussion related to what is called the breaking of conformal invariance. One thing is to notice that mass is not a constant quantity when we decide to perform a conformal change of coordinates, another one is to decide to actually perform such an arbitrary change of coordinates (for instance conformal) and to describe physics in this new system. General relativity, in a sense, contains two parts. The first describes how gravitation itself is created, the other tells us how test particles react in a given geometry and tells us that laws of (non-quantum) physics are invariant under diffeomorphisms of space-time. Quantization of general relativity should also contain two parts. The first would describe quantization of the gravity field itself (but maybe one should quantize only part of it or not do it at all), the next should describe how to do quantum mechanics in a given gravity background, or more generally, in an arbitrary coordinate system (not necessarily Lorentzian). Both questions are related but it may be easier to answer the last first. There were several attempts to describe quantum mechanics in some particular accelerated systems (for instance, the Hawking effect, i.e., the fact that a vacuum state appears as a temperature state, when analysed by an observer in uniformly accelerated motion, has been studied in [20]. Assuming that the global geometry of space-time is given (for instance that it is a flat Minkowski space-time, or better, a compactified Minkowski space-time $S^3 \times_{Z_2} S^1$) it seems reasonable to describe how quantum mechanics looks when analysed from a coordinate system gotten from a Lorentz one by an arbitrary conformal transformation. Why to stop there and not to consider an arbitrary diffeomorphism? The answer is that we certainly should, but the conformal group is the biggest finite dimensional Lie subgroup of the group of all diffeomorphisms (which is infinite dimensional) and it is transitive on space-time, so this should be considered as a reasonable (and essential) first step. We believe that some of the information gathered in the present paper should be useful in this respect.

6.3. *Conformal structures and diffeomorphisms*

Let M be a Riemannian (or pseudo-Riemannian) manifold. Call g_0 a metric, *Diff* the group of diffeomorphisms of M. This group acts on points of M and by pull-back on its space of metrics. Let R be the space of Riemannian structures, i.e., the quotient of the space of metrics by the action of *Diff*. Call $[g_0]$ the orbit of g_0 under this action. Metrics which are conformally related to g_0, i.e. metrics of the kind $\lambda(x)g_0$ are usually called conformal to g_0 or better "punctually-conformal" to g_0. There are however of two possible kinds. Some of them could be gotten from g_0 by the pull-back action of diffeomorphisms -- they are in the same class $[g_0]$. Others cannot be gotten in this way and they define inequivalent Riemannian structures. Finally, we have the case of metrics which can be obtained from g_0 by both a conformal rescaling and a diffeomorphism; they are called "globally conformal" to g_0. Notice that the so-called "confeomorphism"— product of diffeomorphisms times the Weyl group of conformal rescalings -- acts also on the space of metrics and we could also consider the corresponding orbits. Invariance of laws of physics under general change of coordinates amounts to say that physics is independent of the choice of a metric in a given orbit under *Diff*. This should not be confused with a more general principle that would require physics to be invariant under confeomorphisms. Notice that in the particular case where M is the compactified Minkowski space-time, the group $SO(4,2)$ acts by diffeomorphisms, i.e. $SO(4,2) \subset \mathit{Diff}$. Moreover, those metrics that are in the same class $[\eta]$ as the Minkowski metric η and are punctually conformal to it are precisely obtained from η by a transformation of $SO(4,2)$. However, there are metrics that are punctually (or globally) conformal to the Minkowski metric that cannot be obtained from it by a transformation of the conformal group. Incidentally, notice that the solutions of Friedman equations for a spatially closed universe with cosmological constant (Friedman-Lemaitre cosmologies) can be given the topology of $S^3 \times_{Z_2} S^1$. In cosmic coordinates, the metric reads $-dt^2 + R(t)^2 \, d\sigma^2$. The fact that time is also compactified is harmless since the closure of the circle can even take an infinite cosmic time. Notice that these solutions are conformally related with the "flat" Minkowski metric—this is obvious if we use the conformal time $d\tau^2 = dt^2/R(t)^2$.

6.4. *The bundle of conformal frames and the compactified tangent bundle*

A conformal structure on a given manifold M of dimension n is defined by the choice of a conformal class of metrics of signature (p, q). We will choose here $(p, q) = (n-1, 1)$. We will show that Cartan domains of type $\mathscr{D}n$ and their Shilov boundaries are intimately related to the geometry of Lorentzian manifolds. Let such a conformal structure be given. The space of adapted frames of the tangent bundle is therefore a principal bundle with a structure group which is the product of $SO(n-1,1)$—rotations times R^+ (dilations). The structure group can then be extended to $SO(n,2)$—it is always possible to extend a structure group! The principal bundle that we obtain in this way can be called the "bundle of conformal frames". Then, one can build an associated bundle above M whose typical fiber is $SO(n,2)/(P \times R^+)$ where P is the corresponding Poincare subgroup. This typical fiber can therefore be identified as the compactified Minkowski space $S^{n-1} \times_{Z_2} S^1$. This bundle is the "compactified tangent bundle". It could be obtained also from the tangent bundle itself by adding at each point a cone at infinity but this identification is not canonical and depends on the choice of a Riemannian metric from the conformal class. For this compactified tangent bundle one can develop here a theory of parallel transport (Cartan connexions). In the same way, we can build a bundle above M with Cartan domain Dn as a typical fiber. It is constructed by acting with the structure group $SO(n,2)$ of the previously constructed principal bundle (also called bundle of "second order conformal frames" [33] on the homogeneous space $SO(n,2)/SO(n) \times SO(2)$. The case $n = 2$ is exceptional in the sense that any two (pseudo)-Riemannian structures are conformal (up to a diffeomorphism).

Notice that the conformal group $SO(4,2)$ acts globally on "conformal frames" (elements of the corresponding principal bundle) but it does not know how to act, in the non-flat case, on the manifold M itself. Of course, in the particular case where M itself is diffeomorphic with $S^{n-1} \times_{Z_2} S^1$, the conformal group knows to act on M (provided we choose a point called origin).

6.5. *Conformal invariance and accelerated observers*

Action of a conformal transformation on a non-accelerated object.

Let us set $u = \dot{x}$ and let us also call acceleration the quantity $a = \dot{u}$. From the classical law of transformation of x under a conformal translation of vector s in space-time, one can compute how velocity transforms (this was already done previously) and how acceleration itself transforms. This last expression is rather long and is of no interest here but we want to mention that it is of the kind $a' = Aa + Bu + Cx + Ds$ and that, even if we choose $a = 0$ we find $a' \neq 0$, and it will be a function of u, s and x. Actually, if we also impose $x = 0$ and $s \,.\, u = 0$ we find that $a' = 2(u \,.\, u)s$. Therefore, if we observe an unaccelerated object in a Lorentz frame and perform a change of coordinates associated with a conformal translation, the same object looks accelerated in the new frame. However, this acceleration will depend usually upon its position.

Uniformly accelerated objects (hyperbolic motion)

Notice first that a Fermi-Walker transported tetrad ($e_0 = u$, e_i) can be defined for any kind of motion (see for instance [45]). Notice also that "uniform acceleration" can be defined as a motion for which the scalar product $e_i \,.\, a$ is constant. This, in turn, is equivalent to assume that the following equation is satisfied:

$$da/d\tau - a^2 u = 0 \,.$$

The hyperbolic motion has been analysed by several authors and we do not intend to give an account of what has been done here. However, we want to mention that this kind of motion is *not* what one gets when a conformal transformation is applied. This is already clear from the fact that the hyperbolae of the hyperbolic motion do not coincide with those obtained by conformal transformations: the first are asymptotic to the same light cone, whereas those obtained by conformal transformations are parallel! Moreover, according to our general philosophy explained before, we prefer to consider the conformal transformations as "passive" transformations.

7. Harmonicity Cells, Lie Spheres and Lie Balls

7.1. *Definition of a harmonicity cell*

Consider R^n as the subspace of $C^n = \{z = (z_1, z_2, \ldots, z_n), z = \alpha + i\beta\}$ defined by $\beta = 0$. Then, any real analytic function (i.e. existence of a convergent Taylor series) on an open set $\Omega \subset R^n$ can be considered as the trace on Ω of a function holomorphic on an open set Ω_f of C^n with $\Omega_f \cap R^n = \Omega$. We could be tempted into considering the space $\bigcap_L \Omega_f$ where L is the set of all real analytic functions on Ω. Unfortunately, this space is essentially Ω itself (and, more precisely, its interior is the empty set). However, if we do not consider *all* the real analytic functions on Ω but only those that are real harmonic (i.e. solutions of $\Delta f = 0$) in Ω, the set $\mathscr{H}(\Omega) \doteq \overset{0}{\bigcap}_L \Omega_f$ is no longer empty and it is called "the harmonicity cell of the open set" $\Omega \subset R^n$ [3][39]. It can be proven that a harmonicity cell is always a domain of holomorphy (we remind the reader that the domain of holomorphy D of a function g of several complex variables is a domain of

C^n such that g cannot be holomorphically extended beyond D and that a domain is called a domain of holomorphy if it is the domain of holomorphy of some function g).

7.2. *Complex cones and spaces of spheres*

Here we indicate a geometrical construction for the harmonicity cell of a given convex domain in R^n. Only sketchy proofs will be given. Detailed proofs can be found in [3].

Notations:

$$\zeta = \alpha + i\beta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in C^n,$$

$$\bar{\zeta} = \alpha - i\beta = (\bar{\zeta}_1, \bar{\zeta}_2, \ldots, \bar{\zeta}_n) \in C^n,$$

$$Q(\zeta) = \zeta_1^2 + \zeta_2^2 + \cdots + \zeta_n^2 \in C,$$

$$\langle \zeta, a \rangle = \bar{\zeta} a = \bar{\zeta}_1 a_1 + \bar{\zeta}_2 a_2 + \cdots + \bar{\zeta}_n a_n,$$

$$\|\zeta\| = \bar{\zeta}\zeta = \langle \zeta, \zeta \rangle^{1/2}.$$

We also denote by $\Gamma(u)$ the isotropic complex cone of center $u \in R^n$, i.e.

$$\Gamma(u) = \{\zeta \in C^n / Q(\zeta - u) = 0\}$$

and by S^{n-1} the unit sphere in R^n. Notice that $Q(z) \neq \bar{z}z$.

The Lelong transformation

This transformation T maps points of C^n to spheres of dimension $n - 2$ included into R^n. If $\zeta \in C^n$, then $T(\zeta)$ is the S^{n-2}-sphere centered in $\alpha = Re(\zeta)$, of radius $\|\beta\|$ contained in the hyperplane of equation $\langle u - \alpha, \beta \rangle = 0$. Notice that the above hyperplane cuts the S^{n-1}-sphere of center α and radius $\|\beta\|$ along an equatorial S^{n-2} sphere.

According to the definition of $Q(\zeta)$, it means that $T(\zeta) = \{t \in R^n / Q(z - t) = 0\}$, which means that, if $\zeta \in C^n$ and $t \in R^n$ then

$$\zeta \in \Gamma(t) \Leftrightarrow t \in T(\zeta).$$

This establishes a relation between complex cones and spheres.

It is easy to show that the map T is two-to-one, more precisely

$$T(\zeta) = T(\zeta') \Leftrightarrow \zeta' = \bar{\zeta}.$$

Notice that if $\zeta = \alpha \in R^n$ then $T(\alpha) = \alpha$; it is an $(n - 2)$-sphere of dimension zero!

One can show that, if $\Lambda : C^n \to C^n$ is a linear map defined by a (real) orthogonal matrix in the canonical basis of C^n then $T(\Lambda[\zeta]) = \Lambda[T(\zeta)]$.

We already mentioned the fact that the space of S^{n-2}-spheres of R^n is of dimension $2n$ since a given element is determined by the center (n parameters), the radius

(1 parameter) and the choice of the $(n-1)$-hyperplane in which it is embedded ($n-1$-parameters). In particular, if $n=4$ then the space of S^2 spheres of R^4 is 8-dimensional.

Lelong paths and harmonicity cells

Let $\Omega \subset R^n, \Omega \neq \emptyset$, then a Lelong path associated to Ω of origin $\zeta \in C^n$ and extremity $a \in C^n$ is a continuous map $\gamma : [0,1] \to C^n$ such that for any $\tau \in [0,1]$, the sphere $T(\gamma(\tau))$ is contained in Ω.

The following theorem [3] provides a "constructive" definition of the harmonicity cell $\mathscr{H}(\Omega)$.

$\mathscr{H}(\Omega) = \{\zeta \in C^n/$There exist $a \in \Omega$ and a Lelong-path associated with Ω linking a to $\zeta\}$.

Harmonicity cells of convex domains in R^n

Let us suppose now that the open set Ω is a convex domain of R^n (we have in mind the case where Ω is the 4-dimensional euclidean ball), then, the fact that $T(\zeta) \in \Omega$ insures the existence of a Lelong path linking ζ to the points of Ω. Therefore

$$\mathscr{H}(\Omega) = \{\zeta \in C^n/T(\zeta) \subset \Omega\}.$$

More explicitly,

$$\mathscr{H}(\Omega) = \left\{\zeta = \alpha + i\beta \Big/ \underset{\xi \in T(i\gamma)}{Max} \left[\underset{\|a\|=1}{Max} (\langle \alpha + \xi, a\rangle - sup(\langle \alpha, a\rangle)), \{\alpha \in \Omega\} \right] < 0 \right\}.$$

We shall apply this theorem below to the case where Ω is the euclidean ball B_4.

7.3. *Lie norm, Lie distance and Lie balls*

We first introduce another norm in C^n called the Lie norm and show that the unit ball for this norm (called the Lie ball) is the harmonicity cell of the "usual" unit ball of C^n.

The Lie norm on C^n

The Lie norm is defined as the map $L : C^n \to R^+$:

$$\zeta \in C^n \to L(\zeta) = \underset{\xi \in T(\zeta)}{Max} \|\xi\|$$

or, equivalently,

$$L(\zeta) = [\|\zeta\|^2 + \sqrt{\|\zeta\|^4 - |Q(\zeta)|^2}]^{1/2}.$$

Let us just mention the following useful properties of the Lie norm.

L is a norm on C^n:

$$L^2(\alpha + i\beta) = \|\alpha\|^2 + \|\beta\|^2 + 2\sqrt{\|\alpha\|^2\|\beta\|^2 - \langle \alpha, \beta\rangle^2}$$

$\|\zeta\| \leq L(\zeta) \leq \|\alpha\| + \|\beta\|$;

moreover

$$L(\zeta) = \|\zeta\| \Leftrightarrow \|\zeta\|^2 = \|\zeta_1^2 + \cdots + \zeta_n^2\|, \qquad L(\zeta) = \|\alpha\| + \|\beta\| \Leftrightarrow \langle \alpha, \beta \rangle = 0.$$

The previous formulae show that $L(\zeta)$ can be interpreted as the usual euclidean distance between the origin O and the point of the Lelong-sphere $T(\zeta)$ which is at the maximum distance of O.

The Lie distance:

The Lie distance $Ld(\zeta, \xi)$ is defined as $L(\zeta - \xi)$.

The Lie ball:

The Lie ball $D_n(R)$ of center O and of radius R in C^n is of course defined as the ball for the Lie norm L, i.e.

$$D_n(R) = \{\zeta \in C^n / L(\zeta) < R\}.$$

The following equivalence will be useful in the sequel:

$$L^2(\zeta) = \|\zeta\|^2 + \sqrt{\|\zeta\|^4 - |Q(\zeta)|^2} < R^2 \Leftrightarrow [|Q(\zeta)|^2 - 2R^2\|z\zeta\|^2 + R^4 > 0, \text{ and}$$

$$|Q(\zeta) < R^2].$$

Notice that the Lie distance is equivalent to the usual euclidean distance; in other words, the Lie ball as well as the usual eight-dimensional euclidean open ball B_8 are homeomorphic as topological manifolds. Both are also homeomorphic to the space R^8 or C^4.

7.4. *D_4 is the harmonicity cell of the euclidean ball B_4*

The previous geometrical interpretation of the Lie norm and the characteristic property of harmonicity cells of convex domains imply in particular that

$$D_n(O, R) = \{\zeta \in C^n / T(\zeta) \subset B_n(O, R).$$

In other words, the Lie ball D_n is the harmonicity cell of the euclidean ball $B_n(O, R)$. In the following, we will mainly consider the unit balls and denote them by B_n (for the usual euclidean ball) and D_n (in the case of the Lie ball).

Let us recall the analytic definition of the Shilov boundary.

Let $\mathscr{A}$ be a set of (non-constant) holomorphic functions in a bounded domain D of C^n and continuous on $\bar{D}$. Then $\check{D} \subset \partial D$ is the smallest closed subset of the boundary ∂D such that every $f \in \mathscr{A}$ reaches its maximum (in module) on $\check{D}$ and on $\check{D}$ only.

7.5. *The Shilov boundary of the Lie ball D_n (Lie spheres)*

One can prove the following results [3].

The topological boundary of D_n.

Since D_n is, in the case $n = 4$ topologically homeomorphic to the eight-dimensional open ball B_8, it is clear that its boundary is homeomorphic to the seven-sphere S^7. The topological boundary of D_n (of radius 1) is therefore the set

$$\partial D_n = \{\xi \in C^n / L(\xi) = 1\}$$

where L is the Lie norm. More explicitly,

$$\partial D_n = \{\xi \in C^n / \|\xi\|^2 + \sqrt{\|\xi|^4 - |Q(\xi)\|^2} = 1\}.$$

Notice that this "boundary" is here defined as the "topological boundary of D_n in C^n", i.e. as the complement of the interior of D_n in its closure. It is clear that holomorphic functions will reach their maximum on a subset (but as we shall see, not all) of this boundary.

The Shilov boundary $\check{S}_4$.

One can prove that $\check{S}_4$ is the subset of ∂D_4 defined by

$$\check{S}_4 = \{\xi \in C^n / \|\xi\|^2 = |Q(\xi)| = 1\}.$$

It is equivalent to say that this space is the subset of the boundary made of the points where the euclidean distance $\|\xi\|^2$ is equal to the Lie distance $\|L(\xi)\|^2$. Using coordinates, one can also write

$$\check{S}_n = \{\xi = \alpha e^{i\theta} \in C^n / \alpha \in R^n, \|\alpha\| = 1, \theta \in R\}.$$

This last property shows that $\check{S}_n = S^{n-1} \times_{Z_2} S^1$. As already discussed, this generalized Klein bottle (the $n = 2$ case) can be identified, for $n = 4$ with compactified space-time.

If we take $\xi \in \check{S}_n$—and choose the radius to be R we show easily, using the previous relations that the corresponding Lelong-sphere $T(\xi)$ is an $(n-2)$-sphere *of radius* R; it is therefore included in the Euclidean sphere S^{n-1} of R^n (remember that the other $(n-2)$-spheres corresponding to the points of the domain D_n are included in the euclidean ball B_n, i.e. "inside" the sphere S^{n-1}. The Shilov boundary of the Lie ball (i.e. the compact manifold $S^{n-1} \times_{Z_2} S^1$) is also called a Lie sphere.

7.6. *Non-intrinsicness of the construction, physical meaning of the ball B_4*

We started from a given four-dimensional ball B_4 inclosed into R^4 then built D_4 inclosed in C^4 and its Shilov boundary (space-time) $\check{S}_4$. But there are many ways of putting R^4 into C^4 and we could have started with another one. The choice of the subset R^4 amounts to the following: start with C^4 and choose a conjugacy (a complex antilinear map of square 1), this defines a supspace R^4 (on which this conjugacy acts trivially), now we can choose the euclidean ball B_4 and proceed to the construction of the harmonicity cell D_4 and of its Shilov boundary as before. What we get is independent (c.f. the explicit formulae for the Lie norm) of the choice of R^4 in C^4.

Physically, we have the following situation: start with space-time $\check{S}_4$, we know that, up to a Z_2-factor it has the topology of $S^3 \times S^1$. Let us choose a fixed "event"

$\zeta_0 = (x_0, t_0) \in S^3 \times S^1$ and consider the corresponding 3-sphere $S^3 = S^3 \times \{t_0\}$; this set is a "spacelike" hypersurface and can be considered as the spatial universe at time t_0 (we can even speak of a global time because of the homogeneity of space-time in this description). Now, consider the spatial-universe at time t_0, i.e. this 3-sphere S^3, we can then fill it in (this operation does not take place in "space" but in a "higher" dimension); in this way we get a four-dimensional ball whose boundary is "space" at time t_0. Now we can proceed to the construction of the harmonicity cell D_4. If we now consider a massless scalar field on this space S^3 at time t_0, we can propagate it "in time" to a later time t_1 by using the wave equation (Dalembertian), but we can also extend it to the "inside" of S^3, $t = t_0$ by a solution of the Laplacian in the ball, then extend it to the harmonicity cell D_4 by analytic continuation using a holomorphic map and finally consider its radial limit to the Shilov boundary at a later "time" t_1.

7.7 *The physical meaning of the Lelong map*

Points of $\check{S}_4$

We already know that the image $T(\zeta)$ of a point $\zeta \in \check{S}_4$ is a 2-sphere included in the 3-sphere S^3 but this construction was marking reference to a "fixed" B_4. The previous remark concerning the non-unicity of this choice shows that one can define a Lelong map for each choice of the euclidean ball B_4. We also know that the boundary of this ball is a 3-sphere belonging to the Shilov boundary and that it can be considered as a spacelike hypersurface at a fixed time t_0. The physical interpretation of the two sphere $T(\zeta)$ is clear and was studied in Sec. 4.7.1 (when an observer at time t_0 contemplates the sky and more particularly a sphere centered on him, he should remember that this sphere (a 2-sphere) defines a causality diamond or more precisely, a double-cone with apexes ζ and $\bar{\zeta}$).

Points of D_4

Here again, the Lelong map is defined relatively to the same reference euclidean ball B_4 chosen at time t_0. The difference now is that if we take $\zeta \in D_4$ and not on the Shilov boundary (i.e. not on space-time), the corresponding Lelong 2-sphere lies "in" the euclidean ball B_4 but not "in" its boundary S^3.

Low-dimensional examples

In order to acquire some familiarity with the previous constructions, it may be useful to consider several low dimensional examples. In the case $n = 1$, it is not possible to define D_1 as the harmonicity cell of the interval $B_1 =]-1, +1[$ because this would lead to the entire complex plane. However, the Lie norm coincides in this case with the usual norm, therefore, the Lie ball D_1 coincides with the unit disk.

8. Other Aspects of Classical Domains

Cartan domains—and particularly Lie balls—have many relations with several branches of mathematics and physics, and many independent works have been done in the past in relation with this subject. It was impossible for us to cover everything here but we would like to conclude this paper by giving a brief description of what can be found along with a few other references.

Field theory of extended objects A recurrent idea in physics is to write field theories for extended objects. In some cases, the argument of fields are three-spheres of Minkowski space (i.e. hyperboloids). Along the same lines, we have to mention the famous article [66] where a value for the fine structure constant α was proposed $\left(\text{namely } \alpha = \frac{9}{8\pi^4}\left(\frac{\pi^5}{2^4 5!}\right)^{1/4}\right)$ and interpreted in terms of the Bergman kernels of the domains $\mathscr{D}4$ and $\mathscr{D}5$. This interpretation was actually rather unclear and has never been accepted. The precision of the prediction was actually so impressive (it still is) that it has triggered many papers who try to prove or disprove the result. We will not quote these papers because we think that they are actually even less convincing than the first. It remains that the value of α is an experimental data in particle physics and it could even be that it has something to do with the geometry of Cartan domains!

Super-Hamiltonian formalism and Schwinger proper-time formalism Super-Hamiltonian formalism is described for instance in [45] and Fock-Schwinger proper-time formalism for instance in [26]. In this last case, for example, one introduces an evolution operator $U(x_1, x_2, \tau) = \langle x_1 | U(\tau) | x_2 \rangle$ for a hamiltonian that describes the proper time evolution of some system. Here x_1 and x_2 are space-time events. One has then to use the fact that the commutator of operators $[x^\mu, p^\nu] = -i\eta^{\mu\nu}$. The commutation relation between x^μ and $y^\nu = p^\nu/p^2$ is then exactly the Kähler metric on $\mathscr{D}4$.

Field theory at finite temperature As already mentionned, if we perform a change of coordinates described by a conformal transformation, a non-accelerating object in the first coordinate system will be accelerating in the new coordinate system. Moreover, it is known that quantum mechanics in accelerated systems can be interpreted in terms of thermal effects. This certainly would deserve a more detailed study.

Jordan pairs The theory of Jordan algebras and, more generally, Jordan triple systems and Jordan pairs, may be applied to study the geometry of bounded domains. The reason is that one can define a Jordan triple product by using "structure constants" C_{ijkl} expressed in terms of the fourth derivative of the logarithm of the Bergman kernel. A detailed account of this theory can be found in [42]. The Szegö kernel of the domain G/K admits an explicit (non-abelian) Fourier development with respect to the group K. This development can be explicitly computed for all the Cartan domains in terms of the "function gamma" associated to the corresponding Jordan algebra [38].

Acknowledgement

We would like to thank the referee for a careful reading of the manuscript.

References

[1] A. Aizenberg and A. P. Yuzhakov, "Integral representations and residues in multidimensional complex analysis", *Transl. of Math. Monogr.*, Vol 58, AMS, Providence, 1983.

[2] S. T. Ali, J. P. Antoine, and J. P. Gazeau, "De Sitter to Poincaré contractions and relativistic coherent states", *Annales de l'Institut Henri Poincaré*, 1989.

[3] V. Avanissian. "Cellule d'harmonicite et prolongement analytique complexe". *Coll. Travaux en Cours.*, Hermann, Paris, 1985.

[4] R. Balbinot, A. Elgradechi, J. P. Gazeau, and B. Giorgini, "Phase spaces for De Sitterian and Einsteinian quantum elementary systems", Preprint LPTM, Paris VII, 1989.

[5] E. Baalslev, A. Grossmann, and T. Paul, "A characterisation of dilation-analytic operators", *Annales de l'Institut Henri Poincaré*, Vol. 45 (1986) 277–292.

[6] F. A. Berezin, "Quantization", *Math. USSR Izvestja*, Vol. 8 (1974), No. 5; "Quantization in complex symmetric spaces", *Math. USSR Izvestja*, Vol. 9 (1975), No. 2.

[7] S. Bergman, "The kernel function and conformal mapping", *Math. Surv.* No. 5 (1970) AMS, Providence.

[8] A. Besse, *Einstein Manifolds*, Springer Verlag, 1986.

[9] S. Coleman, *Erice lectures*: *"Dilatations" in "Aspects of Symmetries"*, Cambridge University Press, 1988.

[10] R. Gilmore, *Lie groups and Lie Algebras*, Wiley Intersciences, 1974.

[11] S. Goldberg, *Curvature and Homology*, Academic Press, New York, 1962.

[12] R. L. Ingraham, "Conformal geometry and elementary particles", *Il Nuovo Cimento*, Vol XII, N6 (1954).

[13] S. G. Krantz, *Function Theory of Several Variables*, John Wiley & Sons, New York, 1982.

[14] R. Michael Range, *Holomorphic Functions and Integral Representations in Several Complex Variables*, Springer-Verlag, New York, 1986.

[15] E. M. Stein, "Boundary behavior of holomorphic functions of several complex variables", *Math. Notes*, Princeton University Press, Princeton, 1972.

[16] A. Grossmann, J. Morlet, and T. Paul, "Transforms associated to square integrable representations I", *J. Math. Phys.* **27** (1985) 2473–79.

[17] A. Grossmann, J. Morlet, and T. Paul, "Transforms associated to square integrable representations II", *Ann. Inst. Henri Poincaré* **45** (1986) 293–309.

[18] K. Diederich, "Some recent developments in the theory of the Bergman kernel function: a survey", in "Several Complex Variables", *Proc. Symp. Math.*, Vol. 30 (1977) 127–137.

[19] C. L. Fefferman, "Monge-Ampere equations, the Bergman kernel, and geometry of pseudoconvex domains", *Ann. of Math.* **103** (1976) 395–416.

[20] R. Haag, H. Narnhofer, and Ustein, "On quantum field theory in gravitational background", *CMP* **93** (1984) 219–238.

[21] L. A. Harris, "Bounded symmetric homogeneous domains in infinite dimensional spaces".

[22] S. Helgason, *Differential Geometry of Lie Groups and Symmetric Spaces*, Academic Press, 1978.

[23] U. Hirzebruch, "Halbraeume und ihre holomorphen Automorphismen", *Math. Ann.* **153** (1964) 395–417.

[24] L. K. Hua, "Harmonic analysis of functions of several variables in the classical domains", translation of *Mathematical Monographs* **6**, Amer. Math. Soc., Providence, R. I., 1963.

[25] L. P. Hughston and R. S. Ward. "Advances in twistor theory", *Research Notes in Math* **37**, Pitman Adv. Publ., San Francisco, 1979.

[26] C. Itzykson and J. B. Zuber *Quantum Field Theory*, McGraw Hill, 1980.

[27] F. I. Karpelevič, "Geometry of geodesics and eigenfunctions of the Laplace-Beltrami operator in symmetric spaces", *Trudy Moskov. Mat. Obsv.* **14** (1965) 48–185.

[28] W. Kaup, "Bounded symmetric domains in finite and infinite dimensions – a review", in "Several Complex Variables", *Proc. Int. Conf. in Cortona 1976–1977*, Scuola Normale Superiore, Pisa, 1978.

[29] N. Kerzman and E. M. Stein, "The Cauchy Kernel, the Szegö Kernel, and the Riemann Mapping Function", *Math. Ann.* **236**, (1978) 85–93.

[30] M. Koecher, "Jordan algebras and differential geometry", *Actes Congrès Intern. Math.*, Tome I (1970) 279–283.

[31] A. W. Knapp and B. Speh, "Irreducible unitary representations of SU(2, 2)", *J. Func. Anal.* **45** (1982) 41–73.

[32] S. Kobayashi, *Hyperbolic Manifolds and Holomorphic Mappings*, Marcel Dekker, Inc., New York, 1970.

[33] S. Kobayashi, *Transformation Groups in Differential Geometry*, Springer-Verlag, 1972.

[34] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. I and II, Intersciences, New York, 1963.

[35] A. Koranyi, "The poisson integral for generalized half-planes and bounded symmetric domains", *Ann. Math.* **82** (1965) 332–350.

[36] A. Korányi and J. A. Wolf, "Realization of Hermitian symmetric spaces as generalized half-planes", *Ann. Math.* **81** (1965) 265–288.

[37] A. Korányi, "Harmonic functions on symmetric spaces", in *Symmetric Spaces*, ed. W. Boothby and G. Weiss, 379–412, Marcel Dekker, 1972.

[38] M. Lassalle, "Noyau de Szegö, K-types et algèbres de Jordan", *C. R. Acad. Sc. Paris*, t.303, Serie I, n1 (1986).

[39] P. Lelong, "Fonctions plurisousharmoniques et fonctions analytiques de variables reelles", *Ann. Inst. Fourier* **11** (1961) 263–303.

[40] A. Lichnerowicz, "Variétés complexes et tenseur de Bergmann", *Ann. Inst. Fourier*, Grenoble, Vol. 15, 2 (1965) 345–408.

[41] D. B. Lowdenslager, "Potential theory in bounded symmetric homogeneous complex domains", *Ann. Math.* **67** (1958) 467–484.

[42] O. Loos, "Bounded symmetric domains and Jordan pairs", *Math. Lectures*, Publ. of the Dept. of Maths, University of California at Irvine, 1977.

[43] G. Mack, "All unitary ray representations of the conformal group SU(2, 2) with positive energy", *Commun. Math. Phys.* **55**, (1977) 1–28.

[44] G. Mack and A. Salam, "Finite-component field representations of the conformal group", *Ann. Phys.* **53** (1969) 174–202.

[45] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, Freeman, San Francisco, 1970.

[46] M. Morimoto, "Harmonic analysis on the Lie spheres", *Tokyo J. of Math.*, Vol. 3, N1 (1980).

[47] J. Muggli, "Analytic representations of the conformal group in four and five dimensions", *Helvetica Physica Acta*, Vol. 46 (1973) 253–273.

[48] S. Nakajima, "On invariant differential operators on bounded symmetric domains of type IV", *Proc. Japan Acad.* **58**, Ser. A (1982).

[49] R. Narasimhan, *Several Complex Variables*, Chicago Lectures in Mathematics, University of Chicago Press, 1971.

[50] A. Odzijewicz, "A model of conformal kinematics", *Inter. Jour. of Theor. Phys.*, Vol. 15, No. 8 (1976) 575–593.

[51] A. Odzijewicz, "A conformal holomorphic field theory", *Commun. Math. Phys.* **107** (1986) 561–575.

[52] A. Odzijewicz, "On reproducing kernels and quantization of states", *Commun. Math. Phys.* **114** (1988) 577–597.

[53] B. Oersted, "Composition series for analytic continuations of holomorphic representations of SU(n, n)", *Trans. Amer. Math. Soc.* **260** (1980) 563–573.

[54] E. A. Poletskii and B. V. Shabat, "III. Invariant metrics", in *Several Complex Variables III. Geometric Function Theory*, ed. G. M. Khenkin, Springer Verlag, Berlin, 1987.

[55] I. I. Pyatetskii-Shapiro, *Automorphic Functions and the Geometry of Classical Domains*, Gordon and Breach, New York, 1969.

[56] W. Rühl, "Distributions on Minkowski space and analytic representations of the conformal group", *Commun. Math. Phys.* **27** (1972) 53–86.

[57] W. Schmidt, "Die Randwerte holomorpher Funktionen auf hermitesh symmetrischen Raumen", *Inv. Math.*, 9 (1969) 61–80.

[58] I. E. Segal, "Transforms for operators and symplectic automorphisms over a locally compact abelian group", *Math. Scand.* **13** (1963) 31–43.

[59] R. A. Shapiro, "Pseudo-Hermitian symmetric spaces", *Ann. Maths* (1971) 529–548.

[60] J. M. Souriau, *Structure des Systèmes dynamiques*, Dunod, Paris, 1970.

[61] R. F. Streater and A. S. Wightman, *PCT, Spin, Statistic and All That*, W. A. Benjamin, Inc., New York, 1964.

[62] A. Unterberger, "Analyse Harmonique et Analyse Pseudo-différentielle du Cône de Lumière", *Astérisque* **156** (1987).

[63] H. Upmeier, "Toeplitz C*-algebras on bounded symmetric domains", *Ann. Math.* **119** (1984) 549–576.

[64] R. O. Wells, Jr., *Differential analysis on complex manifolds*, Prentice-Hall, Inc.

[65] J. A. Wolf, "Fine structure of Hermitian symmetric spaces", in *Symmetric Spaces, Pure and Applied Mathematics*, Marcel Dekker, Inc., New York, 1972.

[66] A. Wyler, "L'espace symmetrique du groupe des équations de Maxwell", *C. R. Acad. Sc. Paris*, t.269 Ser.A-743 (1969).