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An introduction to

CATEGORY THEORY

in four easy movements

A. Schalk and H. Simmons

Mathematical Foundations Group
The University
Manchester

http://www.cs.man.ac.uk/mfg

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Department of Mathematics
The University of Manchester

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Contents

What is category theory?

In many parts of mathematics we study 'structures' of various kinds. These may be algebraic, topological, geometric, or a mixture of of each. As well as looking at each structure in isolation we also consider how two structure can be compared using morphisms, maps, translations, or whatever. In fact, whenever we have a collections of structures of a like kind we should always try to isolate the appropriate comparison gadgets.

Category theory codifies these general matters. Thus a category consists of objects to take the role of structures, and arrows to take the role of the comparison gadgets. However, if this codification was all it did then category theory would be rather a superficial subject.

Each category is itself a structure, so how should two categories be compared? We match objects against objects and arrows against arrows. The resulting comparison gadget is a functor.

It turns out that many constructions used in mathematics are functorial, and have been around since before category theory was developed. Furthermore, once we see that a construction is functorial we begin to understand it in a better way.

In the first instance category theory was devised (around 1945) to explain why certain manipulations are 'natural' and others are not. This uncovered the notion of a natural transformation which is the appropriate comparison gadget between functors.

Later category theory uncovered the idea of an adjunction which helps to unify many different results in mathematics.

Any comments, typos, or corrections can be sent to Harold Simmons at

hsimmons@manchester.ac.uk

Part I

Development and Exercises

Chapter 1

Categories

! ARROWS COMPARE OBJECTS !

This chapter gives the definition of a category and the definitions of some basic, but subsidiary, notions. These are illustrated by a collection of examples. However, so as not to disturb the flow, most of the subsidiary definitions are gathered together in the final section. Some of these notions are mentioned in the examples in which case they are HIGHLIGHTED in this way. This indicates that the notion is defined in the final section, or for the more complicated notions is merely mentioned in passing. New notions that are defined at or near that point are HIGHLIGHTED in this way.

THUS THE CHAPTER IS NOT ARRANGED IN A LINEAR FASHION, AND YOU MAY NEED TO FLIT ABOUT WHEN YOU READ IT FOR THE FIRST COUPLE OF TIMES.

1.1 Categories defined

This section contains nothing more than the definition of a category and a few bits of terminology.

1.1 DEFINITION. A category $\boldsymbol{C}$ consists of

- a collection Obj of entities called objects
- a collection Arw of entities called arrows
- two assignments $\quad Arw \underset{target}{\overset{source}{\rightrightarrows}} Obj$
- an assignment $\quad Obj \xrightarrow{Id} Arw$
- a partial composition $\quad Arw \times Arw \longrightarrow Arw$

where this data is subjected to certain restrictions as described below. ■

For each arrow f the two assigned objects

$$A = source(f) \qquad target(f) = B$$

are called (naturally enough) the source and target of the arrow. This information is concisely conveyed in a diagram as follows.

$$A \xrightarrow{\ f\ } B$$

(This is a very small diagram, but we will meet some bigger ones later.) Sometimes the source is called the domain and the target called the codomain of the arrow. However, the word 'domain' is used for at least two other notions which can be confused with the idea of a source. An arrow is sometimes called a morphism, but again in some circumstances this can be confusing. This will be explained more fully later.

The arrow $Id(A)$ assigned to the object A is called the identity arrow of A. It is often written

$$id_A \quad \text{or} \quad 1_A$$

whichever is convenient. (Some silly people even write 'A' for this arrow.)

There are four restrictions on this data.

- For each object A the source and target of the arrow $id_A = 1_A$ must be A, that is

  $$A \xrightarrow{id_A \quad 1_A} A$$

  in the concise diagram notation.

- Certain arrows can be combined using compositions to form a third arrow. Two arrows

  $$A \xrightarrow{\quad f \quad} B_1 \qquad\qquad B_2 \xrightarrow{\quad g \quad} C$$

  are said to be compatible (in this order) if $B_1 = B_2$. The composition of two arrows is defined precisely when they are compatible. The resultant arrow is written $g \circ f$ or simply gf and its source and target must be A and B respectively. Thus

  $$A \xrightarrow{g \circ f \quad gf} B$$

  in diagram form.

- This composition must be associative (as far as is possible). Thus for a compatible triple of arrows

  $$A \xrightarrow{\quad f \quad} B \qquad B \xrightarrow{\quad g \quad} C \qquad C \xrightarrow{\quad h \quad} D$$

  the two possible composites

  $$h \circ (g \circ f) \quad \text{and} \quad (h \circ g) \circ f$$

  must be equal. The two components of the parallel pair

  $$A \underset{(h \circ g) \circ f}{\overset{h \circ (g \circ f)}{\rightrightarrows}} D$$

  of arrows are equal.

- The identity arrows behave as neutral arrows in the sense that for each arrow

$$A \xrightarrow{f} B$$

the two equalities

$$f \circ 1_A = f = 1_B \circ f$$

must hold.

Let's have a look at some simple examples of categories. Most of these will be developed in more detail in this chapter or later in the notes.

Category	Objects	Arrows
Set	sets	functions
RelA	sets	binary relations
RelH	sets with a relation	relation respecting functions
Grp	groups	morphism
Sgp	semigroups	morphism
Mon	monoids	morphism
Rng	rings	morphism
$\boldsymbol{Vec}_K$	vectors spaces over a given field K	linear transformations
––	structured sets	structure preserving functions
Pos	posets	monotone maps
$\boldsymbol{Pos}^{pp}$	posets	projection embedding pairs
$\boldsymbol{Pos}^{\dashv}$	posets	poset adjunctions
Top	topological spaces	continuous maps
$\boldsymbol{Top}^{\text{open}}$	topological spaces	continuous open maps
Grf	graphs	
––	developing sets	
––	presheaves	natural transformations

There are a couple of things you should notice about this table. Firstly, it is the common practice to name a category after its objects. However, as we will see, it is the arrows that are more important and should provide the name, but this has never caught on. As with sets or posets or topological spaces, two categories can have the same objects but very different arrows. In most of these examples an arrow is a function of a certain kind, and then arrow composition is just function composition. In more exotic examples arrows are certainly not functions. This is hinted at in the last three examples (where the arrows are not easy to describe in a few words).

We will use $\boldsymbol{C}, \boldsymbol{D}, \ldots$ for arbitrary categories. It is the common practice to use a different type face to distinguish between the objects and the arrows of a category. So far we have used upper case Roman for objects and lower case Roman for arrows, but at other times a different convention will be more convenient (especially when there is more than one category around).

For two objects A and B of a category $\boldsymbol{C}$ we write

$$\boldsymbol{C}[A, B]$$

for the family of all arrows of the form

$$A \longrightarrow B$$

from A to B. This family could be empty or very large. For certain technical work it is sometimes necessary to insist that this family is a set (rather than a class). When the category is understood (for instance, when there is only one category around), we usually write $[A, B]$ for $\boldsymbol{C}[A, B]$.

This family $\boldsymbol{C}[A, B]$ is often called a hom set. This is a very bad piece of terminology. In the original examples of categories the arrows were morphisms which were then called homomorphism, and it wasn't realized that this family could be very large. (Some out and out category theorists still don't realize the significance of this. On the other hand, some off the wall set theorists don't realize the significance of category theory.)

Exercises

1.1 Show that each PRESET is a category, where the objects are the elements and each arrow indicates a comparison.

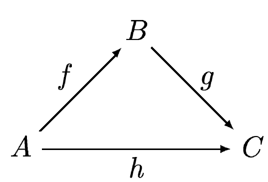
Show that each MONOID is a category with exactly one object and where the arrows are the elements of the monoid.

> Each category is a mixture of a (possibly large) preset and a collection of monoids indexed by the preset. Thus a knowledge of both these kinds of gadgets helps to understand a little bit of category theory. However, it is the mix of these gadgets (rather than the gadgets themselves) that makes a category what it is.

1.2 Diagram chasing

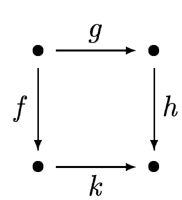
The crucial aspect of a category is that two appropriately compatible arrows can be combined (by composition) to form a third arrow. This leads to an algebra of arrows. Given two compound arrows with the same sources and targets, when are they the same arrow? Such equalities are often verified by diagram chasing to obtain commuting diagrams.

For instance, we say the triangle commutes



$$h = g \circ f$$

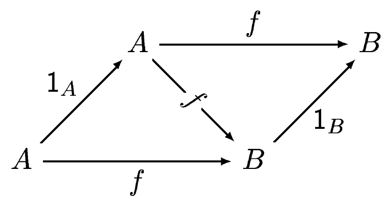
precisely when the equality holds. Similarly the square commutes



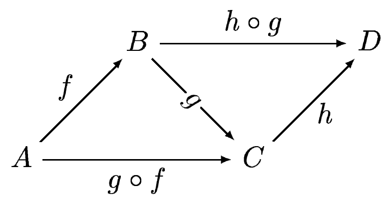
$$k \circ f = h \circ g$$

precisely when the equality holds. In this second example we have indicated each object by • because it is not important that we know what they are. Notice that this convention does *not* mean that all four objects are the same.

The two axioms of arrow algebra (that composition is associative and that identity arrows are neutral) can be expressed by commuting diagrams. The neutral property says that in the diagram

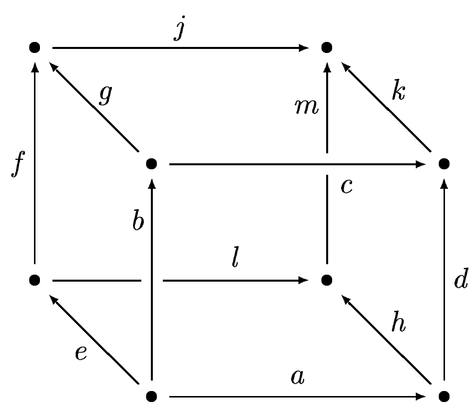


the two triangles commute. The associativity law is a bit more complicated. Consider the diagram



where two of the arrows are composites (to ensure that the two triangles commute). The associativity says that all the composite arrows from A to D produce the same arrow.

It is often more convenient to argue with such diagrams than with the algebraic manipulations. For example, suppose we are given 12 arrows $a, b, \ldots, l, m$



in the form of a cube as shown. Suppose we know that all except the back face commutes, in other words that

$$
\begin{array}{ccc}
 & j \circ g = k \circ c & \\
g \circ b = f \circ e & c \circ b = d \circ a & k \circ d = m \circ h \\
 & l \circ e = h \circ a &
\end{array}
$$

hold. Suppose we also know that the arrow e is EPIC. Under these conditions it follows that the back face also commutes, that is

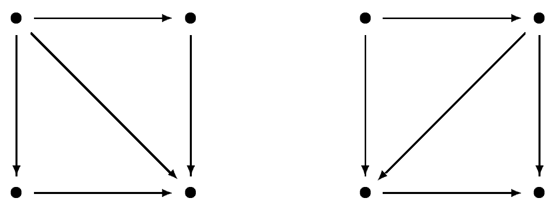
$$j \circ f = m \circ l$$

holds. A proof of this can be worked out using only the given five identities and the epic property. However, the proof is much easier to understand if we chase round the diagram using the faces Left, Top, Front, Right, Bottom in this order.

Exercises

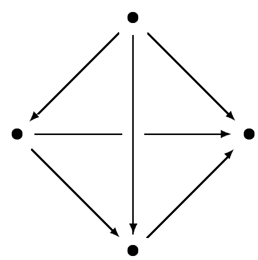
1.2 (a) Consider the diagram related to associativity or arrow composition. Label all the edges and say which four triangles must commute to ensure that the parallel pair agree.

(b) Consider the following two diagrams.



In both cases, show that if the two triangles commute, then so does the outer square.

(c) Consider the triangular pyramid of arrows.



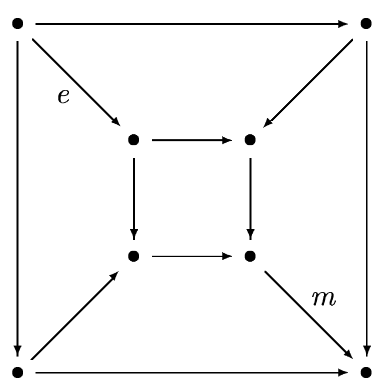
Given that the three other faces commute, show that the back face also commutes.

1.3 Consider the cube of arrows $a, b, \ldots, l, m$.

(e) Show that if e is EPIC and if the other five faces commute, then the back face commutes.

(m) Show that if m is MONIC and if the other five faces commute, then the bottom face commutes.

1.4 Consider the diagram



and suppose the four trapeziums commute.

Show that if the inner square commutes then so does the outer square.

Conversely, show that if e is EPIC, m is MONIC, and the outer square commutes, then so does the inner square.

1.3 Categories of structured sets

In this section we will look at several particular examples of categories, of a similar nature. In most of these categories an object is a set furnished with some 'algebraic' gadgetry, and an arrow is a function which preserves these furnishings. We won't try to set down a general explanation of what this means for the idea will become clear enough after a few examples.

Most of the examples will be like this, but one of the first two isn't.

1.3.1 Two categories of sets

In this subsection we construct two categories ***Set*** and ***RelA*** with the same objects but with different arrows. The objects of both are all possible sets (including the empty set).

A ***Set***-arrow

$$A \xrightarrow{\ f\ } B$$

is a function f from a set A to a set B. In other words, ***Set***$[A, B]$ is just this set of all arrows from A to B (and could be written $(A \longrightarrow B)$ in a different context.)

The composition of arrows is the usual function composition, so there isn't much work to do when verifying that this does give a category.

You should fill in the details of this brief description of ***Set*** for yourself, and observe that something (rather obvious) has been omitted from the description.

There is a point here that it worth expanding on.

Note that since each arrow f of a category must have a uniquely defined source and target (A and B in the above example), we can not treat a ***Set***-arrow solely in terms of its graph (that is as the set of ordered pairs such that ...). For instance, consider the function which sends a real number x to its integer floor $\lfloor x \rfloor$. This can be viewed as a ***Set***-arrow in (at least) two ways, namely as

$$\mathbb{R} \longrightarrow \mathbb{Z} \qquad\qquad \mathbb{R} \longrightarrow \mathbb{R}$$

with different targets. The right hand one is the left hand one composed with the insertion

$$\mathbb{Z} \hookrightarrow \mathbb{R}$$

which embeds $\mathbb{Z}$ into $\mathbb{R}$. In most mathematical situations this embedding isn't visible, but that doesn't mean it should always be ignored.

Actually the common practice of identifying a function with its graph is very silly. If we must do that then the next example shows how it should be done.

The objects of the category ***RelA*** are all possible sets, as with ***Set***.

A ***RelA***-arrow

$$A \xrightarrow{\ F\ } B$$

is a subset $F \subseteq B \times A$ which we can think of as a relation from A to B. In other words, ***RelA***$[A, B]$ is just this set of all such relations from A to B.

Before we can claim this is a category we must first define the composition of these arrows, and then check that the axioms are satisfied. To do this we use a bit of flashy notation.

Consider an arrow F as above, so $F \subseteq B \times A$, For $a \in A$ and $b \in B$ we write bFa for $(b, a) \in F$. For two composible arrows

$$A \xrightarrow{\quad F \quad} B \xrightarrow{\quad G \quad} C$$

we defined the composition $G \circ F$ by

$$c(G \circ F)a \iff (\exists b \in B)[cGbFa]$$

for $a \in A, b \in B$. Thus we show that a is $G \circ F$ related to c by passing through a common element $b \in B$.

It is now straight forward to check that this composition is associative, and that the equality relation on a set gives the identity arrow.

These two categories are related in a certain way (which will be explained in more detail later). There is a canonical way

$$A \xrightarrow{\quad f \quad} B \qquad \longmapsto \qquad A \xrightarrow{\quad \Gamma(f) \quad} B$$

of converting a ***Set***-arrow into a ***RelA***-arrow with the same source and target. We simply take the graph of the function, that is we let

$$b\Gamma(f)a \iff b = fa$$

for $a \in A, b \in B$.

Exercises

1.5 (m) Show that an arrow of ***Set*** is MONIC precisely when it is injective (as a function).

(e) Show that an arrow of ***Set*** is EPIC precisely when it is surjective (as a function).

1.6 Consider the construction $\Gamma(\cdot)$ from ***Set***-arrows to ***RelA***-arrows. Show that

$$\Gamma(g \circ f) = \Gamma(g) \circ \Gamma(f)$$

for each pair of composable ***Set***-arrows.

(This result more or less shows that Γ is a COVARIANT FUNCTOR from ***Set*** to ***RelA***. This notion is discussed in the next chapter.)

1.3.2 Groups, monoids, and semigroups

! SEMIGROUPS AND MONOIDS ARE THE MATHEMATICS OF COMPOSITION !

A semigroup is a structure

$$(A, \star)$$

where A is a set (which may be empty) and $\star$ is an associative binary operation on A. Almost always when working in this structure the operation symbol $\star$ is omitted. Thus we write

$$(ab)(cd) \quad \text{for} \quad (a \star b) \star (c \star d)$$

(where $a, b, c, d \in A$). Furthermore, since the operation is associative it is usually safe to leave out the punctuating brackets, so the term above can be written

$$abcd$$

for it hardly ever matters which way the brackets should be inserted.

A monoid is a structure

$$(A, \star, a)$$

where $(A, \star)$ is a semigroup and $a \in A$ is neutral for the operation $\star$, that is

$$ax = x = xa$$

holds for all $x \in A$. As an exercise you should show that if

$$(A, \star, a) \qquad (A, \star, b)$$

are two monoids (that is, the same semigroup enriched to a monoid in two ways) then, in fact, $a = b$. Thus a semigroup can be enriched to a monoid in at most one way.

1.2 EXAMPLE. (1) Each of

$$\begin{array}{cccc} (\mathbb{N}, +, 0) & (\mathbb{Z}, +, 0) & (\mathbb{Q}, +, 0) & (\mathbb{R}, +, 0) \\ (\mathbb{N}, \times, 1) & (\mathbb{Z} \times, 1) & (\mathbb{Q}, \times, 1) & (\mathbb{R}, \times, 1) \end{array}$$

is a monoid.

(2) Let $\mathbb{N}^+$ and $2\mathbb{N}$ be, respectively, the set of strictly positive natural numbers and the set of even natural numbers. Then $(\mathbb{N}^+, +)$ and $(2\mathbb{N}, \times)$ are two semigroups neither of which contains a neutral element.

(3) For each set X the set $(X \longrightarrow X)$ of all functions from X to X under composition is a monoid. More generally, for each object X of a category $\boldsymbol{C}$ the hom set $\boldsymbol{C}[X, X]$ is a monoid under composition (provided, of course, that it is a set and not a class).

(4) For any set X the power set $\mathcal{P}X$ under either $\cup$ or $\cap$ gives a monoid.

(5) For any set X the set X^* of all words on X is a monoid under concatenation, $\frown$, with the empty word, ϵ, as the neutral element. ■

Semigroups and monoids are the objects of the two categories ***Sgp*** and ***Mon***, respectively. The arrows are the structure preserving functions, the morphisms as described in more generality in subsection 1.3.3.

1.3 EXAMPLE. Each of

$$(X^*, \frown, \epsilon) \xrightarrow{\;\;length\;\;} (\mathbb{N}, +, 0)$$

$$(\mathcal{P}X, \cup, \emptyset) \xrightarrow{\;\;(\cdot)'\;\;} (\mathcal{P}X, \cap, X)$$

$$(\mathbb{N}, +, 0) \xrightarrow{\;\;2^{(\cdot)}\;\;} (\mathbb{N}, \times, 1)$$

$$(\mathbb{N}, \times, 1) \xrightarrow{\text{prime factors of}} (\mathcal{P}\mathbb{N}, \cup, \emptyset)$$

is a ***Mon***-arrow. ■

Each monoid gives a semigroup by forgetting the neutral element. Furthermore, each ***Mon***-arrow

$$A \xrightarrow{f} B$$

is a function f which is a ***Sgp***-arrow. Thus, for monoids A and B we have

$$\boldsymbol{Mon}[A, B] \subseteq \boldsymbol{Sgp}[A, B]$$

that is, we may regard ***Mon*** as a SUBCATEGORY of ***Sgp***. However, this insertion is not FULL.

A group is a monoid $(A, \star, a)$ for which each element x has an inverse, that is

$$x \star y = a = y \star x$$

for some element y. A group morphism between two groups is just a monoid morphism between the groups (viewed as monoids).

The category ***Grp*** of groups and group morphisms can be used to illustrate many results and techniques of category theory. This category has many interesting SUBCATEGORIES, such as the category of abelian groups.

Exercises

1.7 (a) Show that for groups A, B we have $\boldsymbol{Grp}[A, B] = \boldsymbol{Mon}[A, B]$.

(b) Show that for monoids A, B we need not have $\boldsymbol{Mon}[A, B] \neq \boldsymbol{Sgp}[A, B]$, that is there may be a function from A to B which is a semigroup morphism but not a monoid morphism.

1.8 Let A be an arbitrary semigroup and set $B = A \cup \{\omega\}$ where ω is not a member of A. Define an operation $*$ on B by

$$x * y = xy \qquad x * \omega = x = \omega * x \qquad \omega * \omega = \omega$$

for all $x, y \in A$. Show that $(B, *, \omega)$ is a monoid (and hence each semigroup can be extended to a monoid).

1.9 For a set A let $FA = A^*$ be the monoid of words on A under concatenation. Also let

$$A \xrightarrow{\eta} FA$$

be the obvious insertion of A into FA. Show the following.

(i) For each ***Set***-arrow

$$A \xrightarrow{f} B$$

there is a ***Mon***-arrow

$$FA \xrightarrow{F(f)} FB$$

such that

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & FA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F(f)} \\ B & \xrightarrow[\eta_B]{} & FB \end{array}$$

commutes.

(ii) For each monoid S and ***Set***-arrow

$$A \xrightarrow{f} S$$

there is a unique ***Mon***-arrow

$$FA \xrightarrow{f^\sharp} S$$

such that

$$\begin{array}{ccc} A & \xrightarrow{f} & S \\ & {\scriptstyle \eta}\searrow \quad \nearrow{\scriptstyle f^\sharp} & \\ & FA & \end{array}$$

commutes.

(iii) For each monoid S there is a ***Mon***-arrow

$$FS \xrightarrow{\epsilon} S$$

such that for each ***Mon***-arrow

$$A \xrightarrow{f} S$$

the composite

$$FA \xrightarrow{F(f)} FS \xrightarrow{\epsilon} S$$

is the fill-in required in (ii).

This exercise illustrates many of the attributes of category theory, namely functorality, naturality, and adjointness. We will look at all of these in later chapters.

1.3.3 Categories of algebras

Many categories where the objects are 'algebras' are produced in the same way. In this subsection we describe part of the general idea. (You should be warned that the word 'algebra' also has a technical meaning in category theory which doesn't entirely agree with the more general usage.)

For what we do here a 'structure' is a set furnished with several distinguished operations (usually, but not always, binary) and several distinguished elements. We may also want to impose some conditions on the way these furnishings interact with each other. For example think of the various kinds of rings.

Given two such structures of the same signature (where the meaning of this you will have to guess at) there is an obvious notion of a morphism from one to the other. This is just a function that preserves the furnishings. Let's look at a couple of simple examples of this.

The objects of the category ***Bin*** are the structures

$$(A, \star, a)$$

where A is a set and $\star$ is a binary operation on A and $a \in A$. A ***Bin***-morphism

$$(A, \star_A, a) \longrightarrow (B, \star_B, b)$$

is a function $f : A \longrightarrow B$ which respects the furnishings in the sense that

$$f(x \star_A y) = (fx) \star_B (fy) \qquad fa = b$$

for all $x, y \in A$. (Here we have rather tediously distinguished between the two different carried operations. Some people find this necessary, but I'm sure you won't.)

It is routine to check that the function composite of two morphisms is itself a morphism. In this way we see that these structures and morphisms form the objects and arrows of a category.

The second example ***Pno*** is very similar, but in the end has far more significance. The objects of ***Pno*** are the structures (A, α, a) where A is a set, $\alpha : A \longrightarrow A$, and $a \in A$. Thus we have replaced the binary operation by a singulary operation.

A ***Pno***-morphism

$$(A, \alpha, a) \longrightarrow (B, \beta, b)$$

is a function $f : A \longrightarrow B$ which preserves the structure in the sense that

$$f \circ \alpha = \beta \circ f \qquad fa = b$$

hold.

Exercises

1.10 (a) Show that $(\mathbb{N}, \mathrm{succ}, 0)$ is a ***Pno***-object.

(b) Show that for each ***Pno***-object (A, α, a) there is a unique function $f : \mathbb{N} \longrightarrow A$ which gives a ***Pno***-arrow.

1.4 Categories of posets

! PRESETS AND POSETS ARE THE MATHEMATICS OF COMPARISON !

This section is concerned mainly with categories whose objects are posets or presets. However, before that we look at a category ***RelH*** of a more general nature and analogous to ***Bin*** and ***Pno***.

The objects of ***RelH*** are the structures (that is, relational structures rather than algebraic structures)

$$(A, R)$$

where A is a set and R is a binary relation on A, that is $R \subseteq A \times A$. We write xRy to indicate that $(x, y) \in R$ (where $x, y \in A$).

A ***RelH***-arrow

$$(A, R) \xrightarrow{f} (B, S)$$

is a function $f : A \longrightarrow B$ which respects the carried relations, that is such that

$$xRy \Longrightarrow f(x)Sf(y)$$

holds for all $x, y \in A$. Notice that this is an implication and not an equivalence.

This category ***RelH*** has many interesting SUBCATEGORIES obtained by restricting the nature of the relation. Thus we may consider reflexive, transitive, symmetric, anti-symmetric, ... relations, or any combination of these. In particular if we insist that the relation is just equality, then we retrieve the category ***Set***.

1.4.1 The categories *Pre* and *Pos*

In this and the next subsection we look at various categories built using posets and the less well known presets. Incidentally, although 'poset' is the standard terminology, 'preset' isn't and some people think there should be a better name. The usual fiver won't be given to the best suggestion.

We begin with the definitions.

A preset (pre-ordered set) is a structure $(A, \leq)$ where A is a set and $\leq$ is a pre-order on A. This is a binary relation on A which is reflexive and transitive, that is it satisfies

$$x \leq x \qquad x \leq y \leq z \Longrightarrow x \leq z$$

for all $x, y, z \in A$. Sometimes a pre-order is called a quasi-order (but in Bolton this is regarded as more than a little twee).

A poset (partially ordered set) is a preset $(A, \leq)$ where the comparison $\leq$ is anti-symmetric, that is with

$$x \leq y \wedge y \leq x \Longrightarrow x = y$$

for all $x, y \in A$. Thus the comparison is a partial order.

These structures occur naturally when entities are compared.

1.4 EXAMPLE. (1) For each set S the structure $(\mathcal{P}S, \subseteq)$ is a poset.

(2) Let X be any set (the alphabet) and let $A = X^*$ be the set of words (finite strings) on X. These are partially ordered by extension. Thus for words a, b we have $a \leq b$ precisely when a is an initial segment of b. This comparison is sometimes called the the prefix ordering (usually by people only on the fringes of mathematics).

(3) Let A be the set of all real functions $\mathbb{R} \longrightarrow \mathbb{R}$. For $f, g \in A$ let

$$f \leq g \quad \text{mean} \quad (\forall x \in \mathbb{R})[f(x) \leq g(x)]$$

that is impose the pointwise comparison on A. This partially orders A.

(4) More generally, let S be any partially order set and let $A = (S \longrightarrow S)$, the set of all functions on S. The pointwise comparison on A is given by

$$f \leq g \iff (\forall x \in S)[f(x) \leq g(x)]$$

(for $f, g \in A$) and this partially orders A. There are several variants of this idea.

(5) Let S be any set and consider the set A of all partial functions on S, that is the set of all pairs (f, X) where $X \subseteq S$ and $f : X \longrightarrow S$. These functions are compared by extension. Thus, for two such functions (f, X) and (g, Y) we let

$$(f, X) \leq (g, Y) \quad \text{means} \quad X \subseteq Y \text{ and } f \text{ agrees with } g \text{ on } X$$

to obtain a partial order.

(6) For an arbitrary set S let $A = \mathcal{P}S$. For $X, Y \in A$ let

$$X \leq Y \quad \text{mean} \quad \text{There is a finite } D \in A \text{ with } X \subseteq Y \cup D$$

(where the difference set D may depend on X, Y). This gives a preset which, in general, is not a poset. In particular, if S is finite, then the preset is indiscrete, that is $X \leq Y$ holds for all X, Y.

(7) Let A be the set of all functions $f : \mathbb{N} \longrightarrow \mathbb{N}$ which are both inflationary and monotone, that is with

$$x \leq y \Longrightarrow x \leq fx \leq fy$$

for all $x, y \in \mathbb{N}$. Let

$$f \leq g \quad \text{mean} \quad \begin{cases} g \text{ eventually dominates } f \text{ that is} \\ \quad (\exists n \in \mathbb{N})(\forall x \geq n)[fx \leq gx] \end{cases}$$

for $f, g \in A$. Again this gives a preset which is not a poset. (This is an important idea in the measurement of the complexity of functions.)

(8) Most logical systems $\vdash$ pre-order their formulas $\phi, \psi, \theta, \cdots$ by $\phi \vdash \psi$. ■

Presets and posets form the objects of two categories ***Pre*** and ***Pos*** both of which are SUBCATEGORIES of ***RelH***. In both cases the arrows are the monotone maps. Thus given two presets $(A, \leq)$ and $(B, \leq)$, a ***Pre***-arrow

$$(A, \leq) \longrightarrow (B, \leq)$$

is a function $f : A \longrightarrow B$ such that

$$x \leq y \Longrightarrow fx \leq fy$$

for all $x, y \in A$. The ***Pos***-arrows are define in the same way. (You should check that these do form categories.)

Each ***Pos***-object is also a ***Pre***-object. Furthermore, for posets A and B we have

$$\boldsymbol{Pos}[A, B] = \boldsymbol{Pre}[A, B].$$

This shows that ***Pos*** is a FULL SUBCATEGORY of ***Pre***.

The three categories ***Pos***, ***Pre***, ***Set*** can be used to illustrate various functorial constructions and adjointness properties. We deal with these general notions in later chapters, but we can look at some of the particular details here.

For each preset $(A, \leq)$ we can forget the carried trappings to produce a nude set A. This gives a simple example of a 'forgetful' functor

$$\boldsymbol{Pre} \xrightarrow{\text{¿}} \boldsymbol{Set}$$

which has both a left and a right adjoint. (In the next section we will look at miniature versions of these notions. The general notions are dealt with elsewhere.)

Each set S may be furnished as a preset in two extreme ways

$$(S, =) \qquad (S, ||)$$

using equality and the relation that holds everywhere. These are, respectively, the

discrete indiscrete

presets on S. These two D and I constructions are functorial, and we find that

$$\boldsymbol{Set} \underset{I}{\overset{D}{\rightrightarrows}} \boldsymbol{Pre} \qquad \boldsymbol{Set} \xleftarrow{\text{¿}} \boldsymbol{Pre}$$

are a pair of adjunctions. (Don't worry about what this means just yet.)

Each poset is a preset, so there is another 'forgetful' functor

$$\boldsymbol{Pos} \xrightarrow{\text{¿}} \boldsymbol{Pre}$$

(which simply 'forget' the extra poset property). This functor has a left adjoint.

$$\boldsymbol{Pre} \underset{\text{¿}}{\overset{F}{\rightleftarrows}} \boldsymbol{Pos}$$

The idea is that F converts each preset A into a poset in the 'best possible' way, or the 'free-est possible' way. We don't need to go into the technical meaning of these words, but you will already know the construction.

Given a preset $(A, \leq)$, let $\approx$ be the relation on A defined by

$$x \approx y \iff x \leq y \wedge y \leq x$$

(for $x, y \in A$). You should check that $\approx$ is an equivalence relation on A which is equality precisely when A is a poset. Next, for each $a \in A$, let $[a]$ be the block, the equivalence class, to which a belongs. Let $A/\approx$ be the set of all such blocks, the corresponding partition of A. It is not to hard to convert $A/\approx$ into a poset which is tightly connected with the original preset A. This construction has various properties which are dealt with in the exercises.

Exercises

These exercises illustrate three fundamental notions of category theory, namely FUNCTOR, NATURAL TRANSFORMATION, and ADJOINT. These notions are defined and developed in later chapters. The assignment F is the object part of a functor from ***Pre*** to ***Pos*** whose arrow part is given by the final construction. The assignment η is a natural transformation, and the commuting triangle shows that F is left adjoint to the insertion of ***Pos*** into ***Pre***.

1.11 (a) Show that $A/\approx$ carries a partial ordering $\sqsubseteq$ given by

$$[a] \sqsubseteq [b] \iff a \leq b$$

for $a, b \in A$. (Warning: this is not as trivial as it looks.)

(b) Show that

$$\begin{array}{ccc} A & \xrightarrow{\ \eta\ } & A/\approx \\ a & \longmapsto & [a] \end{array}$$

is a ***Pre***-arrow. (This really is trivial.)

We set $F(A, \leq) = (A/\approx, \sqsubseteq)$.

1.12 (a) To show that $F(A, \leq)$ is the free-est poset generated by A consider any other poset and ***Pre***-arrow

$$A \xrightarrow{\ f\ } B$$

where B is a poset. Show there is a *unique* ***Pos***-arrow

$$FA \xrightarrow{\ f^\sharp\ } B$$

such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\qquad f \qquad} & B \\ & {\scriptstyle \eta}\searrow \qquad \nearrow{\scriptstyle f^\sharp} & \\ & FA & \end{array}$$

commutes.

(b) To conclude let

$$A \xrightarrow{\ f\ } B$$

be any ***Pre***-arrow. For clarity write

$$\approx \qquad \asymp$$

for the induced equivalence relations on A and B, respectively, and

$$[\cdot] \qquad < \cdot >$$

for the corresponding equivalence classes. Show that

$$\begin{array}{ccc} A/\approx & \xrightarrow{\ F(f)\ } & B/\asymp \\ [a] & \longmapsto & < f(a) > \end{array}$$

is a well defined ***Pos***-arrow from $F(A)$ to $F(B)$.

1.4.2 The categeories $\boldsymbol{Pos}^{\dashv}$ and $\boldsymbol{Pos}^{pp}$

So far for almost all the categories we have looked at the arrows have been functions of some kind or other. The only example where this is not the case is ***RelA***, and even there the arrows are relations. In this subsection we begin to see more sophisticated examples of non-function arrrows.

The objects of $\boldsymbol{Pos}^{\dashv}$ and $\boldsymbol{Pos}^{pp}$ are just posets, but for both cases an arrow is a *pair* of functions with certain properties. In particular, they move in oposite directions.

We deal with $\boldsymbol{Pos}^{\dashv}$ first.

Given two posets A and B, an adjunction from A to B (that is a poset adjunction from A to B) is a pair of monotone functions

$$A \underset{g}{\overset{f}{\rightleftarrows}} B$$

such that

$$fa \leq b \Longleftrightarrow a \leq gb$$

for all $a \in A$ and $b \in B$. (This is a miniature instance of a more general notion which we look at later.) By convention we call f the left adjoint and g the right adjoint of the pair, and write

$$A \xrightarrow{f \dashv g} B$$

to indicate this relationship. (There are other bits of notation asssociated with adjunctions which we explain shortly.) In particular, we think of an adjunction as pointing from A to B in the direction of the left adjoint.

The $\boldsymbol{Pos}^{\dashv}$-objects are just posets, and the $\boldsymbol{Pos}^{\dashv}$-arrows are the poset adjunctions. More precisely, for posets A and B we let $\boldsymbol{Pos}^{\dashv}[A, B]$ be the set of adjunctions from A to B.

Before we can claim that $\boldsymbol{Pos}^{\dashv}$ is a category we must at least construct a composition of adjunctions. Thus, given a pair

$$A \xrightarrow{f \dashv g} B \xrightarrow{h \dashv k} C$$

of adjunctions with a common poset B, the composite

$$A \longrightarrow C$$

is defined to be the pair

$$A \underset{g \circ k}{\overset{h \circ f}{\rightleftarrows}} C$$

obtained by composing the left and the right components. Note, however, the orders of these function compositions. It is routine to check that this does produce an adjunction, and the construction gives a category. The relevant details are dealt with in the exercises.

We often think of an adjunction as a single arrow

$$A \xrightharpoonup{f} B$$

with two components

$$A \underset{f_\flat}{\overset{f^\sharp}{\rightleftarrows}} B$$

the left adjoint $f^\sharp$ and the right adjoint $f_\flat$. Sometimes the Stockhausen notation, f^* for $f^\sharp$ and f_* for $f_\flat$, is used for these components. (These conventions and notations are not always adhered to, especially in the older literature. Also, the harpoon arrow '$\rightharpoonup$' is sometimes used to indicate a partial function, and this is quite different.)

A monotone map between posets may or may not have a left adoint, it may or may not have a right adjoint, and it can have one without having the other. This is concerned with the completeness properties of the situtation.

What is perhaps more surprising is that it is possible to have a pair of adjunctions

$$A \;\begin{array}{c} \xrightarrow{f^\sharp} \\ \xleftarrow{f\natural} \\ \xrightarrow[f_\flat]{} \end{array}\; B$$

with a common component where the two extremes are not the same. Thus $f^\sharp \dashv f\natural \dashv f_\flat$ with $f^\sharp \neq f_\flat$. In fact, even more surprising things can happen.

The objects of $\boldsymbol{Pos}^{pp}$ are again just posets. A $\boldsymbol{Pos}^{pp}$-arrow

$$A \xRightarrow{(f,g)} B$$

is a $\boldsymbol{Pos}^{\dashv}$-arrow

$$A \xrightharpoonup{f \dashv g} B$$

for which $g \circ f = 1_A$. You should check that these are closed under composition. The arrows of $\boldsymbol{Pos}^{pp}$ are sometimes called projection pairs.

Exercises

1.13 Consider any pair

$$A \underset{g}{\overset{f}{\rightleftarrows}} B$$

of $\boldsymbol{Pos}$-arrows.

(a) Show that $f \dashv g$ precisely when both $1_A \leq g \circ f$ and $f \circ g \leq 1_B$.

(b) Show that if $f \dashv g$ then

$$f \circ g \circ f = g \qquad g \circ f \circ g = g$$

and hence $g \circ f$ is a closure operation on A and $f \circ g$ is a coclosure operation on B.

1.14 Consider the ordered sets $\mathbb{Z}$ and $\mathbb{R}$ as posets, and let

$$\mathbb{Z} \xrightarrow{\ \iota\ } \mathbb{R}$$

be the insertion.

(a) Show there are (unique) maps

$$\mathbb{R} \underset{\rho}{\overset{\lambda}{\rightrightarrows}} \mathbb{Z}$$

such that

$$\mathbb{Z} \xrightarrow{\ \iota \dashv \rho\ } \mathbb{R} \xrightarrow{\ \lambda \dashv \iota\ } \mathbb{Z}$$

are adjunctions.

(c) Show also that this composite is $1_{\mathbb{Z}}$ in $\boldsymbol{Pos}^{\dashv}$ and the other composite, on $\mathbb{R}$, is idempotent.

(d) Show that $\iota \dashv \rho$ is a $\boldsymbol{Pos}^{pp}$-arrow, but $\lambda \dashv \iota$ is not.

1.15 For a poset S let $\mathcal{L}S$ be the poset of lower sections under inclusion.

(a) For a monotone map

$$T \xrightarrow{\ \phi\ } S$$

between posets, show that setting $f = \phi^{\leftarrow}$ (the inverse image map) produces a monotone map

$$\mathcal{L}T \xleftarrow{\ f = \phi^{\leftarrow}\ } \mathcal{L}S$$

in the opposite direction.

(b) Show that f has both a left adjoint and a right adjoint

$$f^{\sharp} \dashv f \dashv f_{\flat}$$

where, in general, these are different.

1.16 (a) Using the construction of Exercise 1.15 each poset adjunction

$$S \xrightarrow{\ \phi \dashv \psi\ } T$$

induces a pair

$$f^{\sharp} \dashv f \dashv f_{\flat} \qquad g^{\sharp} \dashv g \dashv g_{\flat}$$

of double adjunctions

$$\mathcal{L}S \;\underset{f_{\flat}}{\overset{f^{\sharp}}{\rightleftarrows}}\; \mathcal{L}T \;\underset{g_{\flat}}{\overset{g^{\sharp}}{\leftrightarrows}}\; \mathcal{L}S$$

(with middle arrows $\mathcal{L}S \xleftarrow{\ f\ } \mathcal{L}T \xrightarrow{\ g\ } \mathcal{L}S$; $f^{\sharp}, f_{\flat} : \mathcal{L}S \to \mathcal{L}T$ and $g^{\sharp}, g_{\flat} : \mathcal{L}S \to \mathcal{L}T$)

where the central arrows are

$$f = \phi^{\leftarrow} \quad , \quad g = \psi^{\leftarrow}$$

respectively.

How are these related?

(b) Show how to generate a quadruple poset adjunction

$$\begin{array}{ccc} & \xrightarrow{\quad f^{\sharp} \quad} & \\ & \xleftarrow{\quad f^{\sharp} \dashv f \quad} & \\ A & \xrightarrow{\quad f \dashv g \quad} & B \\ & \xleftarrow{\quad g \dashv h \quad} & \\ & \xrightarrow{\quad h \dashv h_{\flat} \quad} & \\ & h_{\flat} & \end{array}$$

that is

$$f^{\sharp} \dashv f \dashv g \dashv h \dashv h_{\flat}$$

for suitable posets A, B and monotone maps $f^{\sharp}$, f, g, h, $h_{\flat}$ between A and B. (In the diagram each adjoint pair $\cdot \dashv \cdot$ is written between the corresponding pair of arrows, and the extreme arrows are $f^{\sharp}$ and $h_{\flat}$.)

(c) Can you extend this construction to obtain even longer chains of adjunctions?

1.5 Some other categories

In this section we gather together a random collection of examples and observation. The first three subsections are comparatively straight forward. However, subsection 1.5.4 gives a result that is perhaps surprising when first seen, and the final two subsections discuss two families of rather more complicated categories.

1.5.1 Some less obvious categories

In this subsection we look at four examples of categories (or seven depending on how you count them). The three examples are not related, but this is the most convenient place to put them.

We begin with a category whose objects look very like categories.

1.5 EXAMPLE. A directed graph, or simply a graph for short, is a pair

$$G = (V, E)$$

of sets together with a pair of assignments

$$E \underset{t}{\overset{s}{\rightrightarrows}} V$$

(as with a category). Each member of V is a node or a vertex, and each member of E is an edge. Naturally, for each $e \in E$ we call

$$se \qquad te$$

the source node and the target node of e, respectively, and let

$$a \xrightarrow{e} b$$

indicate that $se = a$ and $te = b$. In general there are no other conditions on these edges and nodes. In particular, there is no notion of composing edges. Notice that (modulo the size) each category is a graph.

A morphism of graphs

$$(V, E) \xrightarrow{f} (W, F)$$

is a pair of functions

$$V \xrightarrow{f_0} W \qquad E \xrightarrow{f_1} F$$

such that

$$s \circ f_1 = f_0 \circ s \qquad t \circ f_1 = f_0 \circ t$$

hold. Of course, there are two different source and two different target assignments here.

(There is a common notation whereby the collections of objects and arrows of a category $\boldsymbol{C}$ are $\boldsymbol{C}_0$ and $\boldsymbol{C}_1$, respectively. Some of this notation as been taken over here. There are extensions of the notion of a category, to those of a 2-category, a 3-category, 4-category,..., an ω-category, in which there are collections $\boldsymbol{C}_0, \boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_3, \ldots$ with interacting properties. Believe me, you don't want to know about these just yet.) ■

So far for most of the categories we have seen the arrows have been functions or pairs of functions or something similar. Only ***RelA*** has been different. Here is another example like that.

1.6 EXAMPLE. The objects of this category are the finite sets. An arrow

$$A \xrightarrow{f} B$$

is a function

$$f : A \times B \longrightarrow \mathbb{R}$$

(with no imposed conditions). For each pair

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of arrows we define

$$g \circ f : A \times C \longrightarrow \mathbb{R}$$

by

$$(g \circ f)(a, c) = \sum \{f(a, y)g(y, c) \mid y \in B\}$$

for $a \in A, b \in B$. With a little work we see that this produces a category. ■

There are several constructions which convert one category into another. We look first at a simple example. As the exercises indicate, this idea can be generalized quite a lot.

1.7 EXAMPLE. The arrows of a category $\boldsymbol{C}$ themselves for the objects of another category variously denoted by

$$\boldsymbol{C}^{\rightarrow} \qquad \boldsymbol{C}^2 \qquad [2, \boldsymbol{C}]$$

and called the arrow category of $\boldsymbol{C}$.

As indicated, the objects of $\boldsymbol{C}^{\rightarrow}$ are the arrows

$$\begin{array}{c} A \\ \downarrow \\ B \end{array}$$

of $\boldsymbol{C}$. For this example it is convenient to write these arrows vertically. A morphism of arrows

$$\begin{array}{ccc} A & & C \\ f\downarrow & \longrightarrow & \downarrow g \\ B & & D \end{array}$$

is a pair of $\boldsymbol{C}$-arrows

$$A \xrightarrow{\alpha} C$$

$$B \xrightarrow[\beta]{} D$$

such that the square

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & C \\ f\downarrow & & \downarrow g \\ B & \xrightarrow[\beta]{} & D \end{array}$$

commutes.

By composing these morphisms in the 'obvious' way we find that arrows and arrow morphisms do form a category. ■

Finally, we look at two constructions which, in a sense, are duals.

1.8 EXAMPLE. Let K be a fixed object of a category $\boldsymbol{C}$. We form two new categories

$$K\backslash\boldsymbol{C} \qquad \boldsymbol{C}/K$$

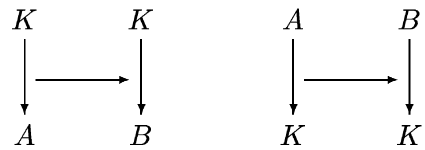
called the slice

under over

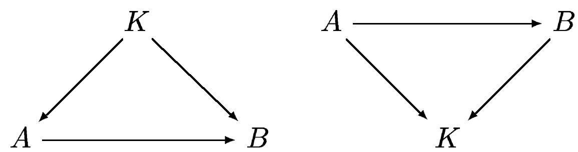
K, respectively. The objects are all arrows

$$\begin{array}{ccc} K & & A \\ \downarrow & & \downarrow \\ A & & K \end{array}$$

for varying objects A. An arrow



of the constructed category is an arrow of the parent category



such that the triangle formed commutes. ■

The slice construction $\boldsymbol{C}/K$ is often used to make sense of a 'local' property. Thus if 'nice' is a property which a category may or may not have, then $\boldsymbol{C}$ is 'locally nice' if the slice category $\boldsymbol{C}/K$ is 'nice' for each object K.

Exercises

1.17 Show that graphs and graph morphisms do form a category.

1.18 Starting with a graph with nodes $a, b, c\ldots$ and with edges $\alpha, \beta, \gamma, \ldots$ we form the **path graph** of the parent graph. This has the same nodes. A path from node a to node b is a list of edges

$$a = a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_1} a_2 \quad \cdots \quad a_{l-1} \xrightarrow{\alpha_l} a_l = b$$

going through the graph. This path has length l, and paths of length 0 are allowed.

Given two paths

$$a \longrightarrow \cdots \longrightarrow b \qquad\qquad b \longrightarrow \cdots \longrightarrow c$$

with a common node, as shown, the composite of the two paths is obtained by following one by the other.

Show that nodes and paths form a category.

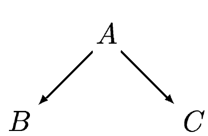
Sort out a decent notation for paths and path composition.

1.19 Show that the construction of Example 1.6 does give a category.

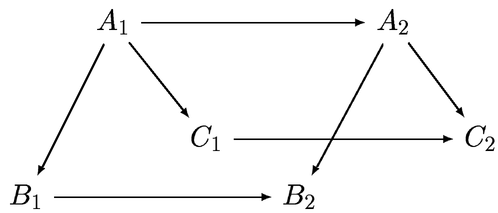
Show that in this category the empty set is both in INITIAL and FINAL.

1.20 Show that the construction of Example 1.7 does convert a category $\boldsymbol{C}$ into a category $\boldsymbol{C}^{\rightarrow}$.

1.21 A wedge in a category $\boldsymbol{C}$ is a pair or arrows



as shown. A wedge morphism



is a triple of arrows, as indicated, which make the two asssociated squares commute.

Show that wedges and wedge morphisms form a category.

Generalize this construction using other diagram templates.

1.22 Show that the constructions of Example 1.8 do produce categories.

Let $1, 2$ be, respectively, a 1-element set and a 2-element set. Show that $1\backslash\boldsymbol{Set}$ is the category of pointed sets, and $\boldsymbol{Set}/2$ is the category of sets with a distinguished subsets.

1.5.2 An arrow need not be a function

For most of the examples of a category produced so far an arrow is a function of some kind. This is not always the case, and we have seen some examples of this already. Later, in Subsection 1.5.6 we will produce a category where each arrow is a whole bunch of functions that fit together in a certain way. Here we give another rather simple example.

The objects of this category are the strictly positive integers $1, 2, 3, \cdots$. For two such integers m, n an arrow

$$n \longrightarrow m$$

is an $m \times n$ matrix A (with real entries). Given two compatible matrices

$$n \xrightarrow{\;B\;} k \qquad\qquad k \xrightarrow{\;A\;} m$$

the composite

$$n \xrightarrow{\;AB\;} m$$

is the matrix product of A and B. You should check that this gives a category.

Exercises

1.23 Make a list of all the categories you have seen where an arrow is not just a single function.

1.24 Show that the example of this subsection is a bit of a cheat. Thus show there is a different description of the category in which the objects are structured sets and the arrows are the structure preserving functions.

1.5.3 Epics need not be surjective; monics need not be injective

When first seen the categorical notion of an EPIC arrow seems to be an attempt to capture the notion of a surjection. Certainly when arrows are functions it is usually the case that a surjective arrow is EPIC. However, the converse need not be true, and there are some easy examples.

In a similar way when first seen the categorical notion of an MONIC arrow seems to be an attempt to capture the notion of an injection. Certainly when arrows are functions it is usually the case that an injective arrow is MONIC. However, the converse need not be true, but examples are harder to find. Such an example is described in the next subsection.

Here is a technique which sometimes can be used to show that a MONIC is injective (assuming this makes sense). An object S of a category $\boldsymbol{C}$ is a separator (or sometimes a generator) of for each parallel pair

$$A \rightrightarrows B$$

of distinct arrows, there is at least one arrow $\mathsf{S} \longrightarrow A$ such that the composite pair

$$S \longrightarrow A \rightrightarrows B$$

are distinct. Sometimes, when the objects are structured sets, the category does have a separator S and the arrows $\mathsf{S} \longrightarrow A$ are in bijective correspondence with the members of A. In such a situation each MONIC is injective.

Exercises

1.25 Show that for the category ***Set*** every MONIC is injective and every EPIC is surjective, and hence ***Set*** is BALANCED.

1.26 Consider the category ***Pos***.

(a) Using the singleton poset $\{\bullet\}$ show that each MONIC ***Pos***-arrow is injective.

(b) To show that each EPIC ***Pos***-arrow is surjective, consider an EPIC arrow

$$A \xrightarrow{e} B$$

and, by way of contradiction, suppose that e is not surjective. Let $b \in B - e[A]$ and set $C = B \cup \{c\}$ where $c \notin B$. Extend the ordering of B to C by $b \mid c$ (i.e. make c incomparable with b) and

$$\begin{aligned} x < b &\Longrightarrow x < c \\ b < x &\Longrightarrow c < x \\ x \mid b &\Longrightarrow x \mid c \end{aligned}$$

for all $x \in B - \{b\}$. This has the effect of replacing b by a double element. Using the obvious two ***Pos***-arrows

$$B \rightrightarrows C$$

derive the required contradiction.

(c) Show that ***Pos*** is BALANCED.

1.27 Consider the category ***Mon***.

(a) Using the additive monoid N to separate elements, show that each MONIC ***Mon***-arrow is injective.

(b) Show that the insertion

$$\mathsf{N} \hookrightarrow \mathsf{Z}$$

(between additive monoids) is EPIC, but clearly is not surjective.

(c) Show that ***Mon*** is not BALANCED.

1.28 Show that for the category ***Abg*** every MONIC is injective and every EPIC is surjective, and hence ***Abg*** is balanced.

1.5.4 A monic that is not injective

We work in a certain subcategory of the category ***Abg*** of abelian groups. Following the usual convention we write such a group additively. Thus we consider groups $(A, +, 0)$ where the carried operation $+$ is commutative.

Given an abelian group A and some $a \in A$ we may iteratively combine a with itself to get an element

$$a + a + \cdots + a$$

using the element a certain number of times, say $m \in \mathbb{N}$. It is convenient to write

$$ma$$

for this compound element, thus

$$0a = 0 \qquad 1a = a \qquad (m+1)a = ma + a$$

for each $a \in A$ and $m \in \mathbb{N}$. (In the equality '$0a = 0$' the left hand '0' is zero and the right hand '0' is the neutral element of A.) We say A is divisible if for each $a \in A$ and non-zero $m \in \mathbb{N}$ there is some $b \in A$ with $mb = a$.

Let ***Dag*** be the subcategory of ***Abg*** of those abelian groups which are divisible. The arrows are just the group morphisms between these divisible abelian groups.

For instance $(\mathbb{Q}, +, 0)$ is an object of ***Dag***. We connect this object $\mathbb{Q}$ with another special object carried by the set $\mathbb{O}$ of rationals q with $0 \leq q < 1$.

We need a bit of notation. Each rational q can be written as

$$q = \lfloor q \rfloor + \langle q \rangle$$

for some unique $\lfloor q \rfloor \in \mathbb{Z}$ (called the integer floor of r) and some unique $\langle q \rangle \in \mathbb{O}$ (which doesn't seem to have a common name, nor a common notation).

Let $\oplus$ be the operation on $\mathbb{O}$ given by

$$p \oplus q = \langle p + q \rangle$$

for $p, q \in \mathbb{O}$. To compute $p \oplus q$ we add p to q as normal and then throw away the integer floor. You should check that $(\mathbb{O}, \oplus, 0)$ is a divisible abelian group, and that there is a morphism

$$\mathbb{Q} \xrightarrow{\ \mu\ } \mathbb{O}$$

such that

$$p \oplus q = \mu(p + q)$$

for all $p, q \in \mathbb{O}$. This $\mathbb{O}$ is a variant of the circle group and the operation $\oplus$ is addition mod 1. (Strictly speaking, the circle group uses all reals in $[0, 1)$.)

The arrow μ is surjective, and hence is EPIC. We show that it is MONIC in ***Dag***, and hence is a surjective BIMORPHISM in ***Dag***. However, it is not injective, and so can not be an ISOMORPHISM in ***Dag***. In particular, ***Dag*** is not BALANCED.

Consider a parallel pair

$$A \underset{g}{\overset{f}{\rightrightarrows}} \mathbb{Q}$$

of ***Dag***-arrows with $\mu \circ f = \mu \circ g$. Thus $fa - ga \in \mathbb{Z}$ for all $a \in A$. If $fa \neq ga$ then, by symmetry, we may suppose that $fa < ga$ and so obtain $0 \neq m = ga - fa \in \mathbb{N}$. By considering any $b \in A$ with $2mb = a$ a simple calculation leads to a contradiction.

Exercises

1.29 Fill in the proofs missing from this subsection.

Does a similar proof show that ***Abg*** is not BALANCED?

1.30 Consider the set $[0, 1)$ of reals. Convert this into an abelian group with $\mathbb{O}$ as a subgroup. Can you find a simpler, more geometric, description of this group.

1.5.5 Monoid actions

For each field K (thought of as the domain of scalars) the vector spaces over K form the objects of a category where the arrows are the linear transformations between the spaces. These examples can be generalized by replacing the field by a ring R, in which case the analogue of a space is called a module. The distinctive feature of these examples is that a scalar and a vector can be combined to form another vector. This operation is called the 'action'. When the ring is non-commutative there are two kinds of modules, depending on whether the action is on the left or the right.

These are the concrete examples of ABELIAN CATEGORIES. We won't even attempt to explain what these are, for we are not going in that direction. However, an enfeebled version of the construction is worth looking at. This gives us categories which have something in common with abelian categories, and are also TOPOSES which are categories intimately related with higher order logic (and other things).

We replace the ring by a monoid

$$R = (R, \star, 1)$$

and construct two categories

$$R\text{-}\boldsymbol{Set} \qquad \boldsymbol{Set}\text{-}R$$

called, respectively, the category of

left right

R-sets.

In both cases an object is a set A together with an action

$$\begin{array}{rcl} R \times A & \longrightarrow & A \\ (r,a) & \longmapsto & ra \end{array} \qquad \begin{array}{rcl} A \times R & \longrightarrow & A \\ (a,r) & \longmapsto & ar \end{array}$$

where these are subject to certain axioms. Thus we require, respectively,

$$\begin{array}{c} (sr)a = s(ra) \\ 1a = a \end{array} \qquad \begin{array}{c} a(rs) = (ar)s \\ a = a1 \end{array}$$

for all $a \in A$ and $r, s \in R$. Notice that there are two different 'multiplications' here; that in R and the action. You should write out these axioms with explicit operation symbols.

In both cases an arrow

$$A \xrightarrow{\ f\ } B$$

is a function (as shown) which is 'linear' in the sense that

$$f(ra) = r(fa) \qquad f(ar) = (fa)r$$

for all $a \in A$ and $r \in R$. You should check that these do give two categories.

Exercises

These exercises give you some practice in the algebraic manipulations used with R-sets. Thus we fix a monoid R and look at the category ***Set***-R of right R-sets (because in the final analysis these are neater then left R-sets). This category is a TOPOS and as such has a SUBOBJECT CLASSIFIER Ω and a process of SUBOBJECT CLASSSIFICATION. These exercises set up some of these facilities without going into a detailed explanation.

1.31 An ideal (or, more precisely, a right ideal) of R is a subset I such that

$$s \in I \Longrightarrow sr \in I$$

holds for all $r, s \in I$.

Given an ideal I and an element $s \in R$ we use

$$r \in I : s \Longleftrightarrow sr \in I$$

(for $r \in R$) to extract a subset $I : s$ of R.

(a) Show that the union and the intersection of a family of ideals are ideals.
(b) Show that $I : t$ is an ideal for each ideal I and $t \in R$.
(c) Show that $I : t = R$ precisely when $t \in I$.

1.32 Let $\Omega(R)$ be the family of all ideals.

Show that

$$\begin{array}{rcl} \Omega(R) \times R & \longrightarrow & \Omega(R) \\ (I, s) & \longmapsto & I : s \end{array}$$

furnishes $\Omega(R)$ with an action to convert it into an R-set.

1.33 For an R-set A a sub-R-set is a subset $B \subseteq A$ such that

$$a \in B \Longrightarrow ar \in B$$

holds for all $a \in A$ and $r \in R$. Such a B is itself an R-set with the restricted action.

For each such sub-R-set B and element $a \in A$ we use

$$r \in B : a \Longleftrightarrow ar \in B$$

to extract a subset $B : a$ of R.

(a) What are the sub-R-sets of R?

(b) Show that if B is a sub-R-set of A then $B : a$ is an ideal and

$$a \in B \Longleftrightarrow B : a = R$$

for each $a \in A$.

(c) Show that if B is a sub-R-set of A then

$$\begin{array}{ccc} A & \longrightarrow & \Omega(R) \\ a & \longmapsto & B : a \end{array}$$

is linear (that is, an arrow in the category). This is called the character of B in A.

1.34 Show that for an R-set A each linear function

$$A \xrightarrow{\beta} \Omega(R)$$

is the character of precisely one sub-R-set of A.

1.5.6 Developing sets

Let $\mathbb{S}$ be a partially ordered set. We construct the category of 'sets developing over $\mathbb{S}$'. At first this looks quite complicated. In fact, it is a simple example of a PRESHEAF CATEGORY and as such is a TOPOS. When seen in a more general context it structure becomes quite simple.

An object of this category is a pair

$$A = (\mathcal{A}, \mathsf{A})$$

where $\mathcal{A}$ is an $\mathbb{S}$-indexed family

$$\mathcal{A} = (A(s) \mid s \in \mathbb{S})$$

of sets, and A is a certain compatible family of functions between these sets.

More precisely, for each pair of indexes $r \leq s$ (from $\mathbb{S}$) there is a function

$$A(s) \xrightarrow{A(r,s)} A(r)$$

such that

$$A(s,s) = id_{A(s)} \qquad A(r,s) \circ A(s,t) = A(r,t)$$

for each $s \in \mathbb{S}$ and each triple $r \leq s \leq t$ from $\mathbb{S}$. Notice that these functions point down the poset $\mathbb{S}$. If you think the composition requirement looks a little odd, then you should write out the diagram.

This gives the objects which, as you can see, have quite a lot of internal structure.

An arrow

$$(\mathcal{A}, \mathsf{A}) \xrightarrow{f} (\mathcal{B}, \mathsf{B})$$

in this category is an indexed family of functions

$$A(s) \xrightarrow{f_s} B(s) \qquad (s \in \mathbb{S})$$

such that the square

$$\begin{array}{ccc} A(s) & \xrightarrow{f_s} & B(s) \\ {\scriptstyle A(r,s)}\downarrow & & \downarrow{\scriptstyle B(r,s)} \\ A(r) & \xrightarrow[f_r]{} & B(r) \end{array}$$

commutes for all $r \leq s$ from $\mathbb{S}$.

Exercises

1.35 The notation for an object $A = (\mathcal{A}, \mathsf{A})$ can be simplified, but does take a bit of getting used to. For indexes $r \leq s$ let us write

$$\begin{array}{ccc} A(s) & \longrightarrow & A(r) \\ a & \longmapsto & a|r \end{array}$$

for the behaviour of the connection function $A(r, s)$. Think of the value $a|r$ as the 'restriction of s to r'.

(a) Write down the properties required of these restriction maps.

(b) Write down the properties required of an arrow

$$(\mathcal{A}, \mathsf{A}) \xrightarrow{f} (\mathcal{B}, \mathsf{B})$$

in terms of these restriction maps.

(c) Can you see a similarity between this category and the category of ***Set***-R for a monoid R?

1.6 Some simple notions

In this final section of this chapter we gather together the definitions of the simple notions and some observations that occur at the beginning of a development of category theory. Some of these notions are mentioned earlier in the previous sections where they are HIGHLIGHTED to indicate that the precise definitions can be found here. Of course, we could have introduced each notion just before it was required, but in the long run it is more convenient to have the information in one place.

1.6.1 Opposites and duality

Each category $\boldsymbol{C}$ is a collection of objects and a collection of arrows with certain properties. In particular, each arrow

$$A \xrightarrow{\ f\ } B$$

has an assigned source and an assigned target. A formal trick converts $\boldsymbol{C}$ into another category $\boldsymbol{C}^{\mathrm{op}}$ called the opposite of $\boldsymbol{C}$. This category $\boldsymbol{C}^{\mathrm{op}}$ has the same objects as $\boldsymbol{C}$. Each arrow f of $\boldsymbol{C}$, as above, is turned into its formal opposite

$$B \xrightarrow{\ f^{\mathrm{op}}\ } A$$

to produce an arrow of $\boldsymbol{C}^{\mathrm{op}}$. The formal composition of these formal arrows is defined by

$$f^{\mathrm{op}} \circ^{\mathrm{op}} g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}$$

for each composable pair

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

of arrows from the parent category $\boldsymbol{C}$.

A routine exercise (which you should go through at least once in your life) shows that $\boldsymbol{C}^{\mathrm{op}}$ is a category.

The process $f \longmapsto f^{\mathrm{op}}$ we doesn't actually do anything to the arrow. We merely decide that the words 'source' and 'target' should mean their exact opposites. Thus the change is merely formal rather than actual.

(There are some natural language words which have gone through, or seem to be going through, a similar process, or even simultaneously mean something and the opposite. The word 'wan' used to mean dark and perhaps leathery, as the skin of a person who contracted the black death would go. After that plague disappeared the word was still used to describe someone who looked ill, as in the expression 'pale and wan', but with the exact opposite of meaning. It should be transparent that we could dust down a few more examples, but let's draw a curtain over that.)

This trick shows there is a lot of duality in category theory. Notions often come in dual pairs

dog god

where a dog of a category $\boldsymbol{C}$ is nothing more than a god of its opposite $\boldsymbol{C}^{\mathrm{op}}$. We will see many example of this.

Sometimes the opposite category $\boldsymbol{C}^{\mathrm{op}}$ has properties rather different to the parent $\boldsymbol{C}$. For instance $\boldsymbol{Set}^{\mathrm{op}}$ is the category of complete, atomic boolean algebra and complete morphisms. As a simpler version of this the opposite of the category of finite sets is the category of finite boolean algebras. (Both of these observations are instances of Stone duality.)

Exercises

1.36 Each poset S is a category. What is the opposite S^{op}?

Each monoid M is a category. What is the opposite M^{op}?

1.6.2 Initial and final objects

In some categories some objects play special roles. An object

$$I \qquad F$$

of a category $\boldsymbol{C}$ is, respectively

initial   final

if for each object A there is a *unique* arrow

$$I \longrightarrow A \qquad A \longrightarrow F$$

as indicated. Here the uniqueness is important.

Sometimes a final object is said to be terminal.

For instance, in the category ***Set*** of sets the empty set $\emptyset$ is initial and any singleton set $\{\star\}$ is final.

In any category of structured sets, if the furnishings do not have a distinguished element (such as posets, semigroups, topological spaces, and so on) then almost certainly the object on the empty carrier is initial. A final object in such a category can be more complicated, and need not exist.

(There is a branch of mathematics, called Universal Algebra, in which structured sets are investigated in some generality. Almost to a man the people who do this kind of thing do not understand that a structure can have an empty carrier. This can lead to a lot of silly messing about. These people are simply wrong.)

When a category has a final object it is essentially unique (as an exercise asks you to show). It common to let 1 be this object. Because of certain special cases where they arise quite naturally, an arrow

$$1 \xrightarrow{\ a\ } A$$

to an object A is a global element of A. For instance, in ***Set*** these pick out the members of a set in the usual sense. In more structured categories these can pick out a special kind of member of an object.

Exercises

1.37 Show that in a category any two initial objects are uniquely isomorphic. That is, if I, J are two initial objects, then there is a unique arrow $I \longrightarrow J$, and this is an isomorphism.

State and prove the dual result concerning terminal objects.

1.38 Suppose that I is initial in $\boldsymbol{C}$. Show that each $\boldsymbol{C}$-arrow of the form

$$A \longrightarrow I$$

is a RETRACTION, and prove the corresponding result for terminal objects. Hence show that if $\boldsymbol{C}$ has both an initial object I and a final object F and there is an arrow

$$F \longrightarrow I$$

then I and T are isomorphic. In such a case we have a zero object.

1.39 Show that the category ***Pno*** has an interesting initial object but a boring final object. What are these objects?

1.40 Show that the category ***Grp*** of groups has both an initial and a final object, and these are the same.

Show that the category ***Rng*** of unital rings has both an initial and a final object, and these are not the same.

What about the categories ***Idm*** and ***Fld*** of integral domains and fields?

1.41 Show that for each set A there is a bijection between the elements of A and the ***Set***-arrows $1 \longrightarrow A$.

Show further that for each pair of ***Set***-arrows

$$A \xrightarrow{f} B \qquad\qquad 1 \longrightarrow A$$

where the second represents the element $a \in A$, the composite

$$1 \longrightarrow A \xrightarrow{f} B$$

represents the element $fa \in B$.

1.42 Let R be a monoid and consider the category of right R-sets.

(a) Show that this category has a final object 1.

(b) Show that for an R-set A the global elements of A (the linear maps $1 \longrightarrow A$) are in bijective correspondence with certain (set theoretic) elements of A.

(c) Can you find a separator for the category? That is a fixed R-set S such that for each R-set A the linear maps $S \longrightarrow A$ are in bijective correspondence with the (set theoretic) elements of A.

1.43 Let $\mathbb{S}$ be a poset and consider the category of presheaves over $\mathbb{S}$ (as described in Subsection 1.5.6).

(a) Show that this category has a final object 1.

(b) Show that for a presheaf $A = (\mathcal{A}, \mathsf{A})$ over $\mathbb{S}$ a global element $1 \longrightarrow A$ is a kind of choice function for the family $\mathcal{A}$. It 'threads' its way through the component sets $A(s)$. You should make precise this notion of 'thread'.

1.6.3 Monic, epic, and the like

In a category some arrows have special properties. An arrow

$$B \xrightarrow{m} A \qquad\qquad A \xrightarrow{e} B$$

is, respectively,

monic epic

if for each parallel pair

$$X \underset{y}{\overset{x}{\rightrightarrows}} B \qquad\qquad B \underset{y}{\overset{x}{\rightrightarrows}} X$$

the implication

$$m \circ x = m \circ y \Longrightarrow x = y \qquad x \circ e = y \circ e \Longrightarrow x = y$$

holds, as appropriate.

Consider a category of structured sets in which each arrow is a function of a certain kind. Almost certainly you will find that each arrow that is an injective function will be monic. Similarly, and arrow that is a surjective function will be epic. However, the converses of these observations do *not* hold in general. Subsections 1.5.3 and 1.5.4 above give some counter-examples. There are some quite general conditions which ensure that one or other of these converses does hold.

A pair of arrows

$$B \xrightarrow{\ s\ } A \qquad A \xrightarrow{\ r\ } B$$

such that

$$r \circ s = 1_B$$

are a

section retraction

respectively (as indicated by the initial letter). Almost trivially, each section is monic and each retraction is epic. As such each such arrow is often referred to has a

split monic split epic

respectively.

Sometimes, in specially circumstances, the couple r, s is called a projection, embedding pair (with r as the projection and s as the embedding). This is not very good terminology to use in general, because of the connotation that a projection is surjective and an embedding is injective. (You should be warned that in a certain area of category theory there is a notion of an 'embedding' and of a 'surjection'. If and when you find out what these are you will realize that this terminology is utterly ludicrous.)

A couple of arrows

$$B \xrightarrow{\ g\ } A \qquad A \xrightarrow{\ f\ } B$$

such that

$$g \circ f = 1_A \qquad f \circ g = 1_B$$

form an inverse pair of isomorphisms and each component is an isomorphism. Trivially, each isomorphism is both a retraction and a section.

In other words, being monic or epic is concerned with having a one-sided cancellation property, being a section or a retraction is concerned with having a one-sided inverse, and being an isomorphism is concerned with having a two-sided inverse.

A bimorphism is an arrow that is both monic and epic. Each isomorphism is a bimorphism, but there can be bimorphisms which are not isomorphisms. A category is balanced if each bimorphism is an isomorphism.

Exercises

1.44 (a) Show that both of the implications

$$\text{section} \Longrightarrow \text{monic} \qquad \text{retraction} \Longrightarrow \text{epic}$$
$$\text{section+epic} \Longrightarrow \text{iso} \qquad \text{retraction+monic} \Longrightarrow \text{iso}$$

hold, that is if an arrow satisfies the hypothesis then it satisfies the conclusion.

(b) Show that if arrows

$$B \xrightarrow{\ g\ } A \xrightarrow{\ f\ } B \xrightarrow{\ h\ } A$$

satisfy

$$h \circ f = 1_A \qquad f \circ g = 1_B$$

then $g = h$, and each arrow is an isomorphism.

1.45 Can you find a reasonable large category which is balanced but where each isomorphism is an identity arrow.

1.46 Consider a monoid viewed as a category.

Which of the elements (when viewed as arrow) are monic, epic, a retraction, a section, iso?

What is a balanced monoid?

1.47 Consider a composable pair of arrows

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

in some category.

Show that if both f, g are monic then so is $g \circ f$.

Show that if $g \circ f$ is monic then so is f.

Obtain similar results (where possible) for the other classes of arrows discussed in this subsection.

Chapter 2
Functors and natural transformations

! ARROWS COMPARE OBJECTS !
! FUNCTORS COMPARE CATEGORIES !

This chapter gives the definitions of the crucial notions of functor and natural transformation, together with a collection of examples of these gadgets. The idea of a *natural* transformation (rather than a mere transformation) is one of the original motivations for inventing categories. Whatever these natural transformations are, they have to pass between two things. It turns out that these things are functors, are these must pass between two other things. These other things are categories. In other words, categories are there to carry functors, which are there to carry natural transformations.

2.1 Functors defined

Given two categories $\boldsymbol{S}$ and $\boldsymbol{T}$, a functor from $\boldsymbol{S}$ to $\boldsymbol{T}$

$$\boldsymbol{S} \xrightarrow{F} \boldsymbol{T}$$

is a pair of assignments. It assigns a $\boldsymbol{T}$-object FA to each $\boldsymbol{S}$-object A, and a $\boldsymbol{T}$-arrow $F(f)$ to each $\boldsymbol{S}$-arrow f. These assignments must satisfy certain conditions, but before we look at those let's sort out some notation.

Although there are two assignments involved (from objects to objects and from arrows to arrows) the same symbol is used for both. Rarely does this cause any confusion. Here we will indicate which is which by the use of brackets. Thus we write FA for the object assigned to A and $F(f)$ for the arrow assigned to f. This is not a standard convention, but it can be useful.

Because of the direction, we think of $\boldsymbol{S}$ as the source category and $\boldsymbol{T}$ as the target category of the functor.

In fact, there are two kinds of functors distinguished by the way they treat arrows. A functor F (as above) is

covariant contravariant

if for each source arrow

$$A \xrightarrow{f} B$$

the target arrow has the form

$$FA \xrightarrow{F(f)} FB \qquad\qquad FA \xleftarrow{F(f)} FB$$

respectively. In other words, a covariant functor preserves the direction of arrows, whereas a contravariant functor reverses the direction of arrows.

It doesn't make sense to have a functor that is partly covariant and partly contravariant. This is because of the other conditions that a functor must satisfy. It must respect composition of arrows.

Let's look at these composition restrictions in parallel, with the covariant conditions on the left and the contravariant conditions on the right.

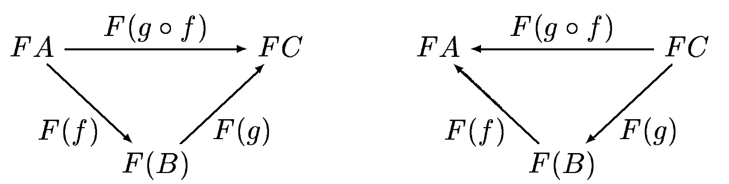
- For each composite

  $$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

  in the source category $\boldsymbol{S}$, the equality

  $$F(g \circ f) = F(g) \circ F(f) \qquad\qquad F(g \circ f) = F(f) \circ F(g)$$

  must hold in the target category $\boldsymbol{T}$. That is, the target triangle

  must commute.

- For each $\boldsymbol{S}$-object A the equality

  $$F(1_A) = 1_{FA}$$

  must hold. That is, each identity arrow must be sent to the identity arrow of the translated parent object.

It is only with arrow composition that the variance of a functor appears. However, the behaviour on arrow composition is by far the most important attribute of a functor.

This definition has an immediate consequence concerning commuting diagrams. Suppose we have a commuting diagram in the source category and we hit each component (object and arrow) with the same functor. Then we get a diagram in the target category, and furthermore this diagram commutes.

In the next few sections will look at various examples of functors. You might want to move directly to those sections. Before that, in the remainder of this section, the examples we look at are rather straight forward,

Let's look at a collection of examples of functors all of which are 'forgetful' since they forget either some structure or some property or a bit of both . (There is a technical definition of a forgetful functor, and not all of the following examples meet that condition. We won't look at that definition, not because of the technicalities but because the precise notion isn't much use whereas the informal notion is.)

There are covariant functors

$$\boldsymbol{Pos} \longrightarrow \boldsymbol{Pre} \longrightarrow \boldsymbol{Set} \qquad\qquad \boldsymbol{Grp} \longrightarrow \boldsymbol{Mon} \longrightarrow \boldsymbol{Sgp} \longrightarrow \boldsymbol{Set}$$

all of which forget something or other.

Each poset A is a preset, so can be thought of as such. Let us write $¿A$ to indicate we are thinking of it in this way. Furthermore, each ***Pos***-arrow

$$A \xrightarrow{f} B$$

is a function of a certain kind, and is still a ***Pre***-arrow

$$¿A \xrightarrow{¿f} ¿B$$

when we change our point of view. This gives a covariant functor

$$\boldsymbol{Pos} \xrightarrow{¿} \boldsymbol{Pre}$$

and there is almost nothing to check. (The symbol '¿' we use here isn't standard. In fact, a forgetful functor isn't usually given a symbol. However, sometimes 'U' is used for a forgetful functor from a category to the category ***Set*** of sets. This 'U' stands for 'Underlying', or strictly speaking 'Unterliegende'.)

In the same way there is a functor

$$\boldsymbol{Grp} \xrightarrow{¿} \boldsymbol{Sgp}$$

which forgets some of the properties of groups (namely, that inverses exist).

The other three functors

$$\boldsymbol{Pre} \longrightarrow \boldsymbol{Set} \qquad \boldsymbol{Mon} \longrightarrow \boldsymbol{Sgp} \longrightarrow \boldsymbol{Set}$$

all forget structure rather than property.

Pedantically, each preset is a pair $(A, \leq)$ carried by a set A, or a set furnished with a relation. If we forget this relation we obtain a set. Let us write $¿(A, \leq)$ to indicate this process (so $¿(A, \leq) = A$). Now, each ***Pre***-arrow

$$(A, \leq) \xrightarrow{f} (B, \leq)$$

is a function of a certain kind

$$A \longrightarrow B$$

between the carriers. Let us write $¿(f)$ for the ***Pre***-arrow viewed in this way as a function. Thus each ***Pre***-arrow, as above, gives a ***Set***-arrow

$$¿(A, \leq) \xrightarrow{¿(f)} ¿(B, \leq)$$

by forgetting the carried structure. This gives a covariant functor

$$\boldsymbol{Pre} \xrightarrow{¿} \boldsymbol{Set}$$

and again there is very little to check.

In the same way there are functors

$$\boldsymbol{Mon} \xrightarrow{¿} \boldsymbol{Sgp} \xrightarrow{¿} \boldsymbol{Set}$$

which forget the unit in the left hand case and the operation in the right hand case.

As you can see, these functors are almost invisible in everyday mathematics, and certainly would not be given names, that is denoted by a symbol. However, when we start to analyse how they interact with other functors, then we have to be a bit more careful. So here we have given them a generic name (which might be a bit non-pc but it suits our purpose).

For other examples of this kind think of the categories ***Fld***, ***Idm***, ***Rng***, ***Grp***, and ***Mon*** of fields, integral domains, unital rings, groups, and monoids (where you should work out what the first three are). There are functors

$$\begin{array}{ccccccc} & & & & & \boldsymbol{Grp} & \\ & & & & \nearrow & & \searrow \\ \boldsymbol{Fld} & \longrightarrow & \boldsymbol{Ind} & \longrightarrow & \boldsymbol{Rng} & \longrightarrow & \boldsymbol{Mon} \end{array}$$

forgetting either property or structure. The functors

$$\boldsymbol{Rng} \longrightarrow \boldsymbol{Grp} \qquad \boldsymbol{Rng} \longrightarrow \boldsymbol{Mon}$$

forget the multiplicative structure and the additive structure of a ring, respectively. In particular, the triangle of functors does not commute. (We will make sense of this later.)

For the final two examples consider the category $\boldsymbol{Pos}^{\dashv}$ of posets and adjunctions. Each arrow in this category is a pair of ***Pos***-arrows, one going in the other direction. Thus we obtain two functors

$$\boldsymbol{Pos}^{\dashv} \xrightarrow{\ \mathit{left}\ } \boldsymbol{Pos} \qquad \boldsymbol{Pos}^{\dashv} \xrightarrow{\ \mathit{right}\ } \boldsymbol{Pos}$$

which select the left component and the right component of an adjunction pair, respectively. One of these is covarant and the other is contravariant.

Exercises

2.1 (a) Let S, T be posets viewed as categories. What is a covariant functor $S \longrightarrow T$? What is a contravariant functor between these two?

(b) Let R, S, be monoids viewed as categories. What is a covariant functor $R \longrightarrow S$? What is a contravariant functor between these two?

2.2 (a) Show that the opposite construction can be viewed as a contravariant endofunctor on the parent category.

(b) Show that each contravariant functor can be viewed as a covariant functor using the opposite of one of the categories.

2.3 Consider the two selection functors $\boldsymbol{Pos}^{\dashv} \longrightarrow \boldsymbol{Pos}$ describe above. Which of these is covariant and which is contravariant?

2.4 For a group A let δA be the derived subgroup (generated by the commutators). In particular, $A/\delta G$ is an abelian group. Show that each of the two object assignments

$$A \longmapsto \delta A \qquad A \longmapsto A/\delta A$$

is part of a functor.

2.5 Show that categories and functors themselves form category (and don't worry about the size of the thing).

2.2 Some power set functors

For each set A let $\mathcal{P}A$ be the power set of A, that is the set of all subsets of A. This has quite a lot of algebraic structure which we will use to convert $\mathcal{P}$ into the object assignment of several functors on ***Set*** (as source and target category).

For a category $\boldsymbol{C}$, an endofunctor on $\boldsymbol{C}$ (or sometimes of $\boldsymbol{C}$) is a functor where $\boldsymbol{C}$ is both the source and target. Such an endofunctor can have either variance. Thus we are going to look at some endofunctors of ***Set***.

First we need to fix some notation.

For a parent set A and $X, Y \subseteq A$ we write

$$X \cup Y \qquad X \cap Y$$

for the union and intersection of the two sets. We write

$$Y - X$$

for the difference (the set of elements of Y which are not in X). We write

$$A - X \qquad X'$$

for the complement of X in A. The left hand notation is used only when omitting 'A' may cause confusion.

Consider a function

$$A \xrightarrow{\ f\ } B$$

from a set A to a set B. For each $X \subseteq A$

$$f[X] = \{fx \mid x \in X\}$$

is the direct image of X across f. This converts a subset of A into a subset of B. We can also go the other way. For each $Y \subseteq B$

$$f^{\leftarrow}(Y) = \{x \in A \mid fx \in Y\}$$

gives the inverse image of Y across f. Note that

$$x \in f^{\leftarrow}(Y) \iff fx \in Y$$

for $x \in A$. This is often easier to use than the previous description.

(You may know already a quite ridiculous notation for inverse image. If so, then you should forget it at once.)

Consider a function f, as above, and think of this as a ***Set***-arrow. We define three ***Set***-arrows

$$\mathcal{P}A \xrightarrow{\ \exists(f)\ } \mathcal{P}B \qquad\qquad \mathcal{P}A \xleftarrow{\ \mathcal{P}(f)\ } \mathcal{P}B \qquad\qquad \mathcal{P}A \xrightarrow{\ \forall(f)\ } \mathcal{P}B$$

by

$$\exists(f)(X) = f[X] \qquad \mathcal{P}(f) = f^{\leftarrow} \qquad \forall(f)(X) = f[X']'$$

for each $X \in \mathcal{P}A$. (Notice the 2-step process. Both $\exists$ and $\forall$ first convert a function f into another function which then converts subsets into subsets.)

This produces three endofunctors on ***Set*** where central one is contravariant but the other two are covariant. You should check the details of this.

It turns out that the central one, the contravariant one, is usually more important than the other two, and so is often called the power set functor. However, sometimes this description is applied to one of the other two.

You might be wondering about the use of '$\exists, \forall$' here. It is not just a fad. When these constructions are generalized in a suitable way they really do have something to do with quantification.

There is an obvious way to compose two functors, even those of different variance. This produces a functor where the variance is determined by the parity of the two component variances. In particular, we may iterate the contravariant power set functor to obtain a covariant endofunctor

$$\boldsymbol{Set} \xrightarrow{\Pi = \mathcal{P} \circ \mathcal{P}} \boldsymbol{Set}$$

on ***Set***.

Thus for each set A we let

$$\Pi A = \mathcal{P}(\mathcal{P}A)$$

be the set of all collections $\mathcal{X}$ whose elements are subsets X of A. Each ***Set***-arrow

$$A \xrightarrow{f} B$$

gives first a ***Set***-arrow

$$\mathcal{P}A \xleftarrow{\mathcal{P}(f) = f^{\leftarrow}} \mathcal{P}B$$

in the opposite direction, and then a ***Set***-arrow

$$\mathcal{P}(\mathcal{P}A) \xrightarrow{\mathcal{P}(\mathcal{P}(f)) = \mathcal{P}(f)^{\leftarrow}} \mathcal{P}(\mathcal{P}B)$$

in the same direction. This is the arrow assignment

$$\Pi A \xrightarrow{\Pi(f)} \Pi B$$

of the composite functor.

This looks quite complicated. It is, but becomes clearer when you write out the constructions in more detail. There are similar situations in mathematics which are far more complicated. Category theory helps to organize such situations and to tame some of these complications.

Exercises

2.6 Let

$$A \xrightarrow{f} B$$

be an arbitrary function.

(a) Write down a quantificational definition of $\exists(f)$ and $\forall(f)$, and hence explain the notation.

(b) Show that in general $\exists(f)$ and $\forall(f)$ are different.

(c) Viewing the power sets $\mathcal{P}A$ and $\mathcal{P}B$ as posets (with inclusion as the comparison), show that the three functions

$$\mathcal{P}A \; \underset{\forall(f)}{\overset{\exists(f)}{\rightleftarrows}} \; \mathcal{P}B \quad (\text{with } f^{\leftarrow} : \mathcal{P}B \to \mathcal{P}A)$$

are monotone with $\exists(f) \dashv f^{\leftarrow} \dashv \forall(f)$.

> The results of this exercise are not just superficial observations with no more than a curiosity value. When generalized to an appropriate setting they have some deep consequences about the nature of higher order logic.

2.7 Show that the three power set constructions are functorial. That is, show that for each pair of functions

$$A \xrightarrow{\;f\;} B \xrightarrow{\;g\;} C$$

the three equalities

$$\exists(g \circ f) = \exists(g) \circ \exists(f) \qquad (g \circ f)^{\leftarrow} = f^{\leftarrow} \circ g^{\leftarrow} \qquad \forall(g \circ f) = \forall(g) \circ \forall(f)$$

hold together with the appropriate identity conditions.

2.8 (a) Show that for each function

$$A \xrightarrow{\;f\;} B$$

the image arrow

$$\begin{array}{ccc} \Pi A & \xrightarrow{\;\Pi(f)\;} & \Pi B \\ \mathcal{X} & \longmapsto & \Pi(f)(\mathcal{X}) \end{array}$$

of the double power set functor is given by

$$Y \in \Pi(f)(\mathcal{X}) \iff f^{\leftarrow}(Y) \in \mathcal{X}$$

for each $\mathcal{X} \in \Pi A$ and each $Y \in \mathcal{P}B$.

(b) Check directly that this gives a functor, that is it Π commutes with function composition and preserves identities.

2.3 Some section functors

The material of the previous section can be generalized so that the carrier is a poset not just a set. (In fact, we could use a preset, but we won't bother with that.) In a way this makes the situation more complicated, but to compensate for that we begin to see more clearly what is going on. We find there are several different generalizations, and these distinctions are not obvious in the power set case.

We need a bit of terminolgy and notation.

Given a poset S, a subset X of S is, respectively,

a lower section   an upper section

if for all $x, y \in S$ with $x \leq y$ the implication

$$y \in X \Longrightarrow x \in X \qquad x \in X \Longrightarrow y \in X$$

holds. In other words, once we are in the section and we move in the appropriate direction, then we remain in the section. Sometimes a lower section is called an initial section and an upper section is called a final section.

A subset X of S is a convex section if for all $x, y, z \in S$ with $x \leq z \leq y$ the implication

$$x, y \in X \Longrightarrow z \in X$$

holds. Convex sections are also called intervals, especially when the poset is linear.

Every lower section and every upper section is convex. The two extremes $\emptyset$ and S are both upper and lower sections. For each $a \in S$, the singleton $\{a\}$ is convex but, in general, is neither a lower nor an upper section (unless a has a special position in S).

We let

$$\mathcal{L}S \qquad \Upsilon S$$

be the sets of

lower   upper

sections of S, respectively. (Once you know that Υ is upsilon you will see where this notation comes from.)

Both $\mathcal{L}S$ and ΥS are posets using inclusion as the comparison. Furthermore, when S is discrete both are just the power set $\mathcal{P}S$.

We need to generate some sections.

For each $a \in S$

$$x \in {\downarrow}a \Longleftrightarrow x \leq a \qquad a \leq x \Longleftrightarrow x \in {\uparrow}a$$

defines, respectively, the

principal lower section   principal upper section

generated by a. For each subset H of S

$$x \in {\downarrow}H \Longleftrightarrow (\exists a \in H)[x \leq a] \qquad (\exists a \in H)[a \leq x] \Longleftrightarrow x \in {\uparrow}H$$

gives the

lower section generated by   upper section generated by

H, respectively. There is a slight descrepancy in the notation here since

$${\downarrow}a = {\downarrow}\{a\} \qquad {\uparrow}a = {\uparrow}\{a\}$$

but that hardly matters. Once we accept this we see that

$${\downarrow}H = \bigcup\{{\downarrow}a \mid a \in H\} \qquad {\uparrow}H = \bigcup\{{\uparrow}a \mid a \in H\}$$

hold.

The two object assignments

$$S \longmapsto \mathcal{L}S \qquad S \longmapsto \Upsilon S$$

on posets S can be filled out to give *six* different endofunctors; three for $\mathcal{L}$ and three for Υ. There are two contravariant functors and four covariant functors, split evenly. Each of the two triples gives an appropriate generalization of the three power set functors.

Let

$$T \xrightarrow{\ \phi\ } S$$

be a ***Pos***-arrow. It is easy to check that the inverse image function $\phi^{\leftarrow}$ transfers lower section of S into lower sections of T, and similarly transfers upper sections of S into upper sections of T. Thus we may set

$$\mathcal{L}(\phi)(X) = \phi^{\leftarrow}(X) \qquad \Upsilon(\phi)(U) = \phi^{\leftarrow}(U)$$

for each $X \in \mathcal{L}S$ and $U \in \Upsilon S$ to obtain functions

$$\mathcal{L}(\phi) : \mathcal{L}S \longrightarrow \mathcal{L}T \qquad \Upsilon(\phi) : \Upsilon S \longrightarrow \Upsilon T$$

(in a contravariant direction). It is easy to check that these constructions are functorial to produce contravariant endofunctors on ***Pos***.

For each ***Pos***-arrow

$$S \xrightarrow{\ \phi\ } T$$

let $\exists(\phi)$ and $\forall(\phi)$ be the functions

$$\mathcal{L}S \longrightarrow \mathcal{L}T$$

given by

$$\exists(\phi)(X) = {\downarrow}\phi[X] \qquad \forall(\phi)(X) = ({\uparrow}\phi[X'])'$$

for each $X \in \mathcal{L}S$. It is useful to observe that

$${\downarrow}(\phi[{\downarrow}H]) = {\downarrow}\phi[H] \qquad {\uparrow}(\phi[{\uparrow}H]) = {\uparrow}\phi[H]$$

holds for all subsets H of S. Using this it can be checked that the assignments

$$S \longmapsto \mathcal{L}S \quad \phi \longmapsto \exists(\phi) \qquad\qquad S \longmapsto \mathcal{L}S \quad \phi \longmapsto \forall(\phi)$$

give two covariant endofunctors on ***Pos***. You should check all the relevant details and observe that these are almost the same as for the power set functors.

We call these functors

$$\exists \qquad \forall$$

respectively. You should not confuse these with the similarly named endofunctors on ***Set*** (although they do a similar job).

Exercises

2.9 Let S be an arbitrary poset.

(a) Show that for each $H \subseteq S$

$$\downarrow H \qquad \uparrow H$$

are, respectively, the least

lower section upper section

which include H.

Show that

$$\between H = \downarrow H \cap \uparrow H$$

is the least convex section which includes H.

(b) Show that for each $H \subseteq A$

$$(\uparrow H')' \qquad (\downarrow H')'$$

are, respectively, the largest

lower section upper section

included in H.

2.10 For an arbitrary subset H of a poset what are

$$\bigcup\{\downarrow a \mid a \in H\} \qquad \bigcup\{\uparrow a \mid a \in H\}$$
$$\bigcap\{\downarrow a \mid a \in H\} \qquad \bigcap\{\uparrow a \mid a \in H\}$$

respectively?

2.11 For an arbitrary poset S let

$$\leq^\flat \qquad \leq^\natural \qquad \leq^\sharp$$

be the relations on $\mathcal{P}A$ given by

$$X \leq^\sharp Y \iff (\forall y \in Y)(\exists x \in X)[x \leq y]$$

$$X \leq^\natural Y \iff \begin{cases} (\forall y \in Y)(\exists x \in X)[x \leq y] \\ \text{and} \\ (\forall x \in X)(\exists y \in Y)[x \leq y] \end{cases}$$

$$X \leq^\flat Y \iff (\forall x \in X)(\exists y \in Y)[x \leq y]$$

for $X, Y \in \mathcal{P}A$.

Show that each of these is a preordering on $\mathcal{P}A$.

Can you find a neat description of the associated partial orderings?

[These are called the upper, convex, and lower ordering on $\mathcal{P}A$. They are also have other, less informative, names. Refined versions of these orderings are used to construct power domains.]

2.12 (a) Re-do Exercise 1.15.

(a) Show that for each monotone map

$$T \xrightarrow{\ \phi\ } S$$

between posets, the inverse imape function

$$\Upsilon S \xrightarrow{\ f = \phi^{\leftarrow}\ } \Upsilon T$$

(which is monotone) has both a left and a right adjoint

$$f^{\sharp} \dashv f \dashv f_{\flat}$$

and describe what these are.

2.4 Some other functors

In this section we gather together some examples of functors you should know about, and some other that are interesting (but perhaps not important).

2.4.1 Hom functors

Let $\boldsymbol{C}$ be any category and let K be any object of $\boldsymbol{C}$. We use this to produce a pair of functors

$$\boldsymbol{C} \underset{[\cdot, K]}{\overset{[K, \cdot]}{\rightrightarrows}} \boldsymbol{Set}$$

where the upper one is covariant and the lower one is contravariant. Here we have used one of the standard names for the object assignement of either functor. You will see why shortly.

For each $\boldsymbol{C}$-object A let

$$FA = \boldsymbol{C}[K, A]$$

that is the set of $\boldsymbol{C}$-arrows from K to A. (The word 'set' here may not be strictly correct. There are some exotic examples of categories where this collection may be very large. However, for most examples we are likely to come across the collection is a set.)

This is the object assignment of the functor. For each $\boldsymbol{C}$-arrow

$$A \xrightarrow{\ f\ } B$$

let $F(f)$ be the function

$$\begin{array}{ccc} \boldsymbol{C}[K, A] & \xrightarrow{\ F(f)\ } & \boldsymbol{C}[K, B] \\ p & \longmapsto & f \circ p \end{array}$$

obtained by arrow composition. It is routine to check this gives a functor $\boldsymbol{C} \longrightarrow \boldsymbol{Set}$. Because of the terminolgy 'hom-set' this functor is often referered to as the covariant hom-functor induced by K. This functor is usually indicated by $\boldsymbol{C}[K,\cdot]$ or by $[K,\cdot]$ when $\boldsymbol{C}$ is understood.

A similar construction gives the contravariant hom-functor. For each $\boldsymbol{C}$-object A let

$$FA = \boldsymbol{C}[A,K]$$

that is the set of $\boldsymbol{C}$-arrows from A to K. For each $\boldsymbol{C}$-arrow

$$A \xrightarrow{\ f\ } B$$

let $F(f)$ be the function

$$\begin{array}{ccc} \boldsymbol{C}[B,K] & \xrightarrow{F(f)} & \boldsymbol{C}[A,K] \\ p & \longmapsto & p \circ f \end{array}$$

obtained by arrow composition. It is routine to check this gives a functor $\boldsymbol{C} \longrightarrow \boldsymbol{Set}$, but this time it is contravariant.

These examples may look a bit artificial. However, they are important, especially the second one. These constructions often underlie a representation result (where an abstract object is represented in some concrete form). In many situations it is possible to enrich the set $[A,K]$ so that it becomes an object, perhaps in the same category or perhaps in another category. In such circumstances the construction produces a functor from the parent category to the enriching category. A simple example of this is when K is a field and $\boldsymbol{C}$ is the category of vector spaces over K. Then $\boldsymbol{C}[A,K]$ is the dual space of the space A.

Exercises

2.13 Show that these two constructions of this subsection do give functors.

2.14 Show that for all posets S,T the arrow set $\boldsymbol{Pos}[S,T]$ becomes a poset under the pointwise comparison.

Show that both the hom-functors of $\boldsymbol{Pos}$ become endofunctors on $\boldsymbol{Pos}$.

2.15 Let S be a poset viewed as a category, and let $k \in S$. What are the two hom functors induced by k? Remember to described the behaviour on arrows.

2.4.2 Functors and arrow categories

Let $\boldsymbol{C}$ be any category and let $\boldsymbol{C}^{\to}$ be its arrow category as described in Example 1.7. There are three associated functors

$$\boldsymbol{C} \xrightarrow{\ \Delta\ } \boldsymbol{C}^{\to} \qquad\qquad \boldsymbol{C}^{\to} \xrightarrow{\ S,T\ } \boldsymbol{C}$$

with fairly obvious behaviour.

The two right hand ones S,T select the source and target of an object in $\boldsymbol{C}^{\to}$, that is an arrow of $\boldsymbol{C}$. This is the object assignment, and the arrow assignment can be only one thing.

The diagonal functor Δ sends the object A to the arrow 1_A, and sends the $\boldsymbol{C}$-arrow

$$A \xrightarrow{\ f\ } B$$

to the pair of arrows

$$1_A \underset{f}{\overset{f}{\rightrightarrows}} 1_B$$

which then form a $\boldsymbol{C}^{\rightarrow}$-arrow.

Exercises

2.16 (a) Describe the arrow assignment of S, T and show that the two constructions do give functors.

(b) Check that the construction Δ does give a functor.

2.17 For an arbitrary category $\boldsymbol{C}$ let $\boldsymbol{C}^{\wedge}$ be the category of wedges in $\boldsymbol{C}$ (as described in Exercise 1.21.

(a) Show there are three functors $\boldsymbol{C}^{\wedge} \longrightarrow \boldsymbol{C}$ each of which selects a particular node of the wedge.

(b) Set up a functor $\boldsymbol{C} \longrightarrow \boldsymbol{C}^{\wedge}$ which converts each object into a trivial wedge.

2.4.3 Comma categories

In Example 1.8 we used an object K of a category $\boldsymbol{C}$ to produce two slice categories $K \backslash \boldsymbol{C}$ and $\boldsymbol{C}/K$ under and over K. These two constructions have a common generalization.

Let

$$\boldsymbol{A} \xrightarrow{\ L\ } \boldsymbol{C} \qquad\qquad \boldsymbol{C} \xleftarrow{\ R\ } \boldsymbol{B}$$

be a given pair of covariant functors. We produce a category

$$\big(L, \boldsymbol{C}, R\big)$$

where the objects are the triples

$$\big(A, L(A) \xrightarrow{f} R(B), B\big)$$

where A is a $\boldsymbol{A}$-object, B is a $\boldsymbol{B}$-object, and f is a $\boldsymbol{C}$-arrow. An arrow of this category

$$\big(A, L(A) \xrightarrow{f} R(B), B\big) \longrightarrow \big(X, L(X) \xrightarrow{g} R(Y), B\big)$$

is a pair of arrows

$$A \xrightarrow{\ l\ } X \qquad\qquad B \xrightarrow{\ r\ } Y$$

from $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively,

$$\begin{array}{ccc} L(A) & \xrightarrow{L(l)} & L(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ R(B) & \xrightarrow[R(r)]{} & R(Y) \end{array}$$

commutes. Composition of these arrows is defined in the obvious way.

This $\big(L, \boldsymbol{C}, R\big)$ is a comma category, although the devil knows why.

Exercises

2.18 Show that this construction does produce a category.

2.19 (a) Let 1 be the category with just one object and just one arrow. Let

$$1 \xrightarrow{L} \boldsymbol{C} \qquad\qquad \boldsymbol{C} \xrightarrow{R} \boldsymbol{C}$$

be, respectively, any functor and the identity functor.

What is $(L, \boldsymbol{C}, R)$?

(b) Describe $\boldsymbol{C}/K$ as a comma category.

(c) What is $(L, \boldsymbol{C}, R)$ where both L and R are the identity functor on $\boldsymbol{C}$?

2.20 For convenience let $\boldsymbol{Com}$ be the comma category $(L, \boldsymbol{C}, R)$, as above. Construct three forgetful functors

$$\boldsymbol{Com} \longrightarrow \boldsymbol{A} \qquad \boldsymbol{Com} \longrightarrow \boldsymbol{C}^{\rightarrow} \qquad \boldsymbol{Com} \longrightarrow \boldsymbol{B}$$

using the arrow category in the central one.

2.5 Natural transformations defined

! Arrows compare objects !
! Functors compare categories !
! Natural transformations compare functors !

Natural transformations are the reason that category theory was invented. In a category the objects are compared by the arrows. Categories themselves are compared by functors. But how do we compare functors? That might seem a silly question (since, on the face of it, there is no good reason why we want to compare functors). However, there are some comparisons in mathematics which seems to be between objects but have a certain uniformity and 'naturality' about them. It was an attempt to explain this that give rise to category theory.

A natural transformation compares a parallel pair

$$\boldsymbol{S} \underset{G}{\overset{F}{\rightrightarrows}} \boldsymbol{T}$$

of functors of the same variance. Thus, by definition, a natural transformation

$$F \xrightarrow{\eta} G$$

from F to G assigns to each $\boldsymbol{S}$-object A a $\boldsymbol{T}$-arrow

$$FA \xrightarrow{\eta_A} GA$$

with a certain property. We think of η as an indexed family

$$\big(\eta_A \mid A \in Obj(\boldsymbol{S})\big)$$

of $\boldsymbol{T}$-arrows. It could be better, but this is the standard notation.

The extra property required of η depends on whether F, G are both covariant or both contravariant. Thus, for each $\boldsymbol{S}$-arrow

$$A \xrightarrow{\;f\;} B$$

the appropriate $\boldsymbol{T}$-square

$$\begin{array}{ccc} FA & \xrightarrow{\eta_A} & GA \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ FB & \xrightarrow[\eta_B]{} & GB \end{array} \qquad\qquad \begin{array}{ccc} FA & \xrightarrow{\eta_A} & GA \\ {\scriptstyle F(f)}\uparrow & & \uparrow{\scriptstyle G(f)} \\ FB & \xrightarrow[\eta_B]{} & GB \end{array}$$

commutes.

This is a quite short definition, but has some subtleties. Several examples will help, and these will take up most of the rest of this chapter. Before that let's give a possible informal explanation of the idea.

Consider two $\boldsymbol{T}$-objects which somehow 'arise from the same parent by two different constructions'. It's not clear what this phrase should mean, but here we take it to mean that the two $\boldsymbol{T}$-objects are FA and GA for some $\boldsymbol{S}$-object A. Thus we assume that the two functors F, G have been set up and these are the 'two different constructions'. The $\boldsymbol{S}$-object A is 'the same parent'.

We now ask whether the one $\boldsymbol{T}$-object FA can be compared with the other $\boldsymbol{T}$-object GA in a way that is uniform and without depending too much on the particular parent. In other words we ask for a $\boldsymbol{T}$-arrow

$$FA \xrightarrow{\;\eta_A\;} GA$$

which 'varies smoothly and naturally' as A varies through $\boldsymbol{S}$.

We decide to make this precise by the notion of a natural transformation.

We conclude with a definition which will make more sense once you have seen a few examples.

2.1 DEFINITION. Consider a parallel pair

$$\boldsymbol{S} \underset{G}{\overset{F}{\rightrightarrows}} \boldsymbol{T}$$

of functors of the same variance. An inverse pair of natural isomorphisms between F, G is a pair

$$F \underset{\lambda}{\overset{\eta}{\rightleftarrows}} G$$

such that for each $\boldsymbol{S}$-object A the $\boldsymbol{T}$-arrows

$$FA \underset{\lambda_A}{\overset{\eta_A}{\rightleftarrows}} GA$$

are an inverse pair of isomorphisms in $\boldsymbol{T}$.

Two functors are naturally isomorphic or sometimes naturally equivalent if there is an inverse pair of natural isomorphisms between them. ■

Informally, two functors (with concrete constructions) are naturally isomorphic if they are essentially the same except for the irrelevant inner workings of the two constructions.

Exercises

2.21 Show that a natural transformation

$$F \xrightarrow{\ \eta\ } G$$

is a natural isomorphism if and only if each component η_A is a isomorphism.

2.6 Examples of natural transformations

In this section we look at some examples of natural transformations using some of the functors set up in the earlier sections of this chapter.

2.6.1 Using the power set and section functors

It will be convenient later if each of the examples is given a number.

2.2 EXAMPLE. Consider the two endofunctors

$$\boldsymbol{Set} \underset{\mathcal{P}^{\exists}}{\overset{I}{\rightrightarrows}} \boldsymbol{Set}$$

where the top one is the identity functor and the bottom one is the existential power set functor (which uses the direct image behaviour on functions). For each set A consider the singleton function

$$\begin{array}{rcl} A = IA & \xrightarrow{\ \eta^{\exists}_A\ } & \mathcal{P}A \\ a & \longmapsto & \{a\} \end{array}$$

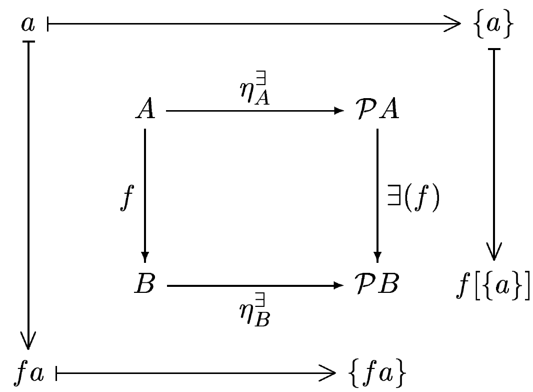
that is

$$x \in \eta^{\exists}_A(a) \iff x = a$$

for each $x, a \in A$. To show this is a natural transformation we need to show that each function

$$A \xrightarrow{\ f\ } B$$

induces a certain commuting square. Consider the following.



We must show that the inner square commutes. To do that we start from an arbitrary element $a \in A$ at the top left corner and track it round the two paths to the bottom right corner. We must show that the two subsets

$$\{fa\} \qquad f[\{a\}]$$

of B are the same. However, this is a trivial observation. ■

You might wonder if anything similar can be done with the power set functor with the universal behaviour on functions. Certainly something similar can be done with the double power set functor.

2.3 EXAMPLE. Consider the two endofunctors

$$\textbf{\textit{Set}} \underset{\Pi}{\overset{I}{\rightrightarrows}} \textbf{\textit{Set}}$$

where the top one is the identity functor and the bottom one is the composite $\mathcal{P} \circ \mathcal{P}$ of the contravariant power set functor. For each set A consider the 'dual transform' function

$$A = IA \xrightarrow{\eta_A} \Pi A$$

given by

$$X \in \eta_A(a) \Longleftrightarrow a \in X$$

for $a \in A$ and $X \in \mathcal{P}A$. Thus η_A attaches to each $a \in A$ all those subsets of A of which a is a member. In particular, $\eta_A(a)$ is a collection of subsets of A, and hence $\eta_A(a) \in \mathcal{P}(\mathcal{P}A) = \Pi A$.

To show that η is natural we must show that for each function

$$A \xrightarrow{f} B$$

the square

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \Pi A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \Pi(f)} \\ B & \xrightarrow[\eta_B]{} & \Pi B \end{array}$$

commutes. We do this by tracking an arbitrary element $a \in A$ round the two paths. Each produces a family of subsets Y of B, and these must be the same. It is easy to check this, provided you get the notation under control. ■

We will return to this example later on.

Some of this material can be generalized using of section functors on ***Pos***.

2.4 EXAMPLE. Consider the two endofunctors

$$\boldsymbol{Pos} \underset{\mathcal{L}^{\exists}}{\overset{I}{\rightrightarrows}} \boldsymbol{Pos}$$

where the top one is the identity functor and the bottom one is the lower section set functor with the existential behaviour on maps. For each poset S consider the function

$$\begin{array}{rcl} S & \xrightarrow{\eta^{\exists}_S} & \mathcal{L}S \\ a & \longmapsto & {\downarrow}a \end{array}$$

which converts an element into the principal lower section generated by that element. It is routine to check that this is monotone.

To show that $\eta^{\exists}$ is natural consider ***Pos***-arrow

$$S \xrightarrow{\ \phi\ } T$$

that is a monotone map. This induces a square

$$\begin{array}{ccccc} a & & \longmapsto & & {\downarrow}a \\ \Big\downarrow & S & \xrightarrow{\eta^{\exists}_S} & \mathcal{L}S & \Big\downarrow \\ & {\scriptstyle\phi}\Big\downarrow & & \Big\downarrow{\scriptstyle\exists(\phi)} & \\ & T & \xrightarrow[\eta^{\exists}_T]{} & \mathcal{L}T & {\downarrow}\phi[{\downarrow}a] \\ \phi a & & \longmapsto & & {\downarrow}\{\phi a\} \end{array}$$

which we must show commutes. In this case there is just a little more to the proof. ■

So far we have considered mainly existential behaviour, but universal behaviour can be handled as well.

2.5 EXAMPLE. Consider the two endofunctors

$$\boldsymbol{Pos} \underset{\mathcal{L}^{\forall}}{\overset{I}{\rightrightarrows}} \boldsymbol{Pos}$$

where the top one is the identity functor and the bottom one is the lower section set functor with the universal behaviour on maps. For each poset S consider the function

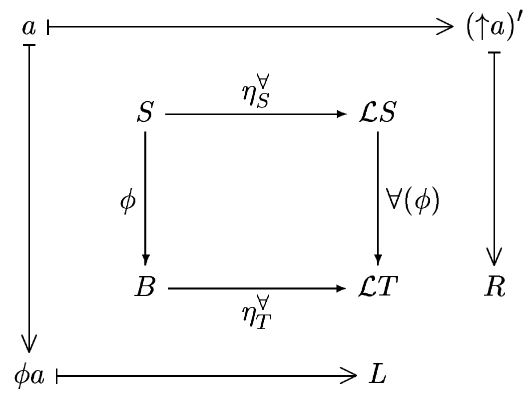
$$\begin{array}{rcl} S & \xrightarrow{\eta^{\forall}_S} & \mathcal{L}S \\ a & \longmapsto & ({\uparrow}a)' \end{array}$$

which converts an element into the complement of the principal upper section generated by that element. This is a ***Pos***-arrow.

Consider ***Pos***-arrow

$$S \xrightarrow{\ \phi\ } T$$

that is a monotone map. This induces a square



where

$$L = \big(\uparrow(\phi a)\big)' \qquad R = \forall(\phi)\big((\uparrow a)'\big) = \big(\uparrow\phi[\uparrow a]\big)' = \big(\uparrow(\phi a)\big)'$$

are the two section resulting from the element $a \in S$. This simplification of R shows that the square commutes, and hence $\eta^{\forall}$ is a natural transformation. ■

As you can probably imagine, there are many variations on these ideas, much more than we could look at here.

Exercises

2.22 Complete the proofs for Example 2.2. That is, show that the required square does commute.

2.23 Show there is a natural transformation $I \longrightarrow \mathcal{P}^{\forall}$ where I is the identity functor on ***Set*** and $\mathcal{P}^{\forall}$ is the power set functor with the universal behaviour on functions.

2.24 Complete the proofs for Example 2.3.

2.25 Complete the proofs for Example 2.4.

2.26 (a) Show there are natural transformations $I \longrightarrow \Upsilon^{\exists}$ and $I \longrightarrow \Upsilon^{\forall}$ (where $I, \Upsilon^{\exists}, \Upsilon^{\forall}$ are endofunctors on ***Pos***).

(b) Make sure you construction for part (a) is correct.

2.27 (a) Show that taking the opposite of a poset produced an endofunctor $\mathcal{O}$ on ***Pos***.

(b) Show there is a natural isomorphism

$$\mathcal{L}^{\exists} \circ \mathcal{O} \cong \mathcal{O} \circ \mathcal{L}^{\forall}$$

and find a concrete description of the left hand endofunctor.

2.6.2 Using hom functors

Remember that each object K of a category $\boldsymbol{C}$ gives us two hom functors

$$\boldsymbol{C}[K,-] \qquad \boldsymbol{C}[-,K]$$

from $\boldsymbol{C}$ to $\boldsymbol{Set}$, The left hand one is covariant and the right hand one is contravariant. These are important tools when 'representations' are involved.

2.6 EXAMPLE. Let $\boldsymbol{C}$ be an arbitrary category, let

$$\boldsymbol{C} \xrightarrow{F} \boldsymbol{Set}$$

be an arbitrary functor, and let K be an arbitrary $\boldsymbol{C}$-object. What can a natural transformation

$$[K,-] \longrightarrow F$$

look like?

For an arbitrary element $k \in FK$ consider the family of functions

$$\begin{array}{rcl} [K,A] & \xrightarrow{\epsilon_A} & A \\ p & \longmapsto & F(p)k \end{array}$$

indexed by the $\boldsymbol{C}$-objects A. Notice how each such ϵ_X is obtained by 'evaluation at k'.

This family ϵ is a natural transformation

$$[K,-] \longrightarrow F$$

(where, of course, $[K,-]$ is $\boldsymbol{C}[K,-]$). To see this consider a $\boldsymbol{C}$-arrow

$$A \xrightarrow{f} B$$

and the square

$$\begin{array}{ccc} [K,A] & \xrightarrow{\epsilon_A} & FA \\ {\scriptstyle f\circ -}\downarrow & & \downarrow{\scriptstyle F(f)} \\ [K,B] & \xrightarrow[\epsilon_B]{} & FB \end{array}$$

induced by this arrow. By tracking an arbitrary $p \in [K,A]$ along the two paths we see that the square does commute. The calculation depends on the functorial properties of F.

This gives us many examples of natural transformations from $[K,-]$ to F. Are there any more? To answer this consider any natural transformation

$$[K,-] \xrightarrow{\epsilon} F$$

of this kind. From the component

$$[K,K] \xrightarrow{\epsilon_K} FK$$

we may set

$$k = \epsilon_K(1_K)$$

to obtain a member of FK. Now consider any arbitrary $\boldsymbol{C}$-object A and arbitrary $p \in [K, A]$. Since

$$K \xrightarrow{\ p\ } A$$

we have a commuting square

$$\begin{array}{ccc} [K,K] & \xrightarrow{\epsilon_K} & FK \\ {\scriptstyle p\circ -}\downarrow & & \downarrow{\scriptstyle F(p)} \\ [K,A] & \xrightarrow[\epsilon_A]{} & FA \end{array}$$

induced by the naturality of ϵ. By tracking round this square we see that ϵ is nothing more than 'evaluation at k'.

These two constructions set up a bijection between the natural transformations $[K, -] \longrightarrow F$ and the elements of FK. ■

When the natural transformation

$$[K, -] \xrightarrow{\ \epsilon\ } F$$

induced by $k \in FK$ is a natural isomorphism, we say the pair (K, k) is a pointwise representation of F. We say F is representable when it has at least one pointwise representation.

In subsection 1.5.6 we showed how each poset can be used to produce what looks like a rather intricate category; the category of sets developing over a poset. In fact, that construction can be generalized to replace the poset by any category. In this generalization some things become clearer.

2.7 EXAMPLE. Each category $\boldsymbol{C}$ has an associated category $\boldsymbol{C}^\frown$ of presheaves. A ($\boldsymbol{Set}$-valued) presheaf on $\boldsymbol{C}$ is a contravariant functor

$$\boldsymbol{C} \xrightarrow{\ F\ } \boldsymbol{Set}$$

and these form the objects of $\boldsymbol{C}^\frown$. An arrow

$$F \xrightarrow{\ \eta\ } G$$

of $\boldsymbol{C}^\frown$ is a natural transformation between two presheaves.

You should check that these do form a category.

For each $\boldsymbol{C}$-object A the hom-functor

$$A^\frown = \boldsymbol{C}[-, A]$$

is an object of $\boldsymbol{C}^\frown$. You should check that this is the object assignment of a functor

$$\boldsymbol{C} \xrightarrow{\ F\ } \boldsymbol{C}^\frown$$

(where the arrow behaviour needs to be described). ■

This construction $(\cdot)^\frown$ is the Yoneda embedding of the category $\boldsymbol{C}$. It is a kind of completing process and has applications in many parts of category theory. Presheaves, and the more refined subclass of sheaves are important in algebraic geometry and representation theory. They lead to the notion of a topos which is a kind of category with many nice properties.

When used in a certain way with a special kind of object the contravariant hom-functor can explain many things. To complete this section we look at a simple case where the base category is $\boldsymbol{Pos}$. This will also gives us a hint of the idea of enrichment where the hom-sets themselves carry certain structure.

2.8 DEFINITION. Let 2 be the 2-element poset $\{0,1\}$ ordered by $0 < 1$. For an arbitrary poset A a monotone map

$$A \xrightarrow{\ p\ } 2$$

is a character of A. ■

In other words $\boldsymbol{Pos}[A,2]$ is the set of characters of A. What is going on here? You already know this idea. Think of the similar construction where the poset A is replaced by an arbitrary set, and then look at the following observation.

2.9 LEMMA. *For each poset A there are bijective correspondences*

$$\begin{array}{ccccc} \mathcal{L}A & & \boldsymbol{Pos}[A,2] & & \Upsilon A \\ X & \longleftrightarrow & p & \longleftrightarrow & U \end{array}$$

given by

$$a \in X \iff pa = 0 \qquad pa = 1 \iff a \in U$$

for each $a \in A$.

We have set up both $\mathcal{L}$ and Υ as contravariant endofunctors of $\boldsymbol{Pos}$. By composing with the forgetful functor we can temporarily view them as functors $\boldsymbol{Pos} \longrightarrow \boldsymbol{Set}$.

2.10 LEMMA. *The three functors*

$$\boldsymbol{Pos} \underset{\mathcal{L}}{\overset{\Upsilon}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xrightarrow{[-,2]}}}}} \boldsymbol{Set}$$

are naturally isomorphic via the assignments of Lemma 2.9.

This shows that as $\boldsymbol{Set}$-valued functors these three gadgets are essentially the same. However, we know that $\mathcal{L}A$ and ΥA do have more structure. In particular, they are both posets, and both $\mathcal{L}$ and Υ are endofunctors of $\boldsymbol{Pos}$. We find that much of this structure is controlled by the hom-functor, provided it is suitably enriched.

For each poset A the set $\boldsymbol{Pos}[A,2]$ carries the pointwise comparison given by

$$q \leq p \iff (\forall x : A)[qx \leq px]$$

for characters p, q. It is routine to check that this furnishes $\boldsymbol{Pos}[A,2]$ as a poset. In fact, there is much more.

2.11 THEOREM. *When enriched* $\boldsymbol{Pos}[-,2]$ *is an endofunctor on* $\boldsymbol{Pos}$, *and is naturally isomorphic to* Υ.

The proof of this is quite a long series of simple calculations. Some of these have been alluded to already. What else must be done?

We know that $\boldsymbol{Pos}[-,2]$ is functor to $\boldsymbol{Set}$ and we have seen how to enrich each $\boldsymbol{Pos}[A,2]$ as a poset. What we must show is that for each $\boldsymbol{Pos}$-arrow

$$B \xrightarrow{\ \phi\ } A$$

the induced function

$$\boldsymbol{Pos}[A,2] \xrightarrow{-\circ\phi} \boldsymbol{Pos}[B,2]$$

is monotone relative to the enrichments. This shows that $\boldsymbol{Pos}[-,2]$ is an endofunctor on $\boldsymbol{Pos}$.

Next we must show that Υ and $\boldsymbol{Pos}[-,2]$ are naturally isomorphic. We have set up the bijections and we have checked the naturality. It still remains to show that each bijection

$$\begin{array}{ccc} \boldsymbol{Pos}[A,2] & & \Upsilon A \\ p & \longleftrightarrow & U \end{array}$$

is a poset isomorphism, that is that both assignments are monotone.

There is nothing very difficult here. The only problem is remembering to do everything that has to be done.

To conclude this section let's look at a simple example of a contravariant adjunction. We present it using the functor Υ, but again it can be explained using 2.

2.12 EXAMPLE. For each pair of posets A, S there is a bijective correspondence

$$\begin{array}{ccc} \boldsymbol{Pos}[A,\Upsilon S] & & \boldsymbol{Pos}[S,\Upsilon A] \\ f & \longleftrightarrow & \phi \end{array}$$

given by

$$s \in fa \Longleftrightarrow a \in \phi s$$

for $a \in A, s \in S$.

Furthermore, when both the hom-sets are enriched this bijection is a poset isomorphism.

Finally, the bijection is natural for variations of both A and S. ■

Again there are several things to be checked here, and none is very difficult.

We won't explain exactly what is going on here, for that is better done in a more general setting, and we aren't in a position to do that just yet. However, this idea gives the categorical support for several major results in different parts of mathematics.

Exercises

2.28 Complete the details of Example 2.6

2.29 Show that if (K, k) and (L, l) are both pointwise representations of a functor

$$\boldsymbol{C} \xrightarrow{F} \boldsymbol{Set}$$

then there is an arrow

$$K \xrightarrow{p} L$$

of $\boldsymbol{C}$ with $F(p)k = l$.

2.30 (a) Show that the identity functor on $\boldsymbol{Set}$ is representable.
(b) Show that the forgetful functor $\boldsymbol{Mon} \longrightarrow \boldsymbol{Set}$ is representable.

2.31 Fill in all the details of Example 2.7.

2.32 Let $\boldsymbol{S}$ be a poset viewed as a category. Show that a presheaf on $\boldsymbol{S}$ is nothing more than a developing set as in subsection 1.5.6

2.33 Fill in the missing details for Lemmas 2.9, 2.10 an Theorem 2.11.

2.34 (a) Sort out the details for Example 2.12.
(b) For each pair of posets A, S there are bijections

$$[A, [S, 2]] \longleftrightarrow [A, \Upsilon S] \longleftrightarrow [S, \Upsilon A] \longleftrightarrow [S, [A, 2]]$$

(where each $[-,-]$ is $\boldsymbol{Pos}[-,-]$). Write down this composite bijection, and hence 'explain' the construction of Example 2.12.
(c) What has this got to do with curry and chips?

2.35 Why was the material around Lemma 2.10 – Example 2.12 done using the functor Υ rather than $\mathcal{L}$?

2.36 (a) Explain the endofunctor Π of $\boldsymbol{Set}$ and the associated natural transformation $I \longrightarrow \Pi$ using characters (for sets).
(b) For each vector space V over a given field K there is a natural embedding $V \longrightarrow V^{**}$ into its second dual. How is this similar to $I \longrightarrow \Pi$?

2.37 When suitably furnished the set 2 can be an object in several different categories. In particular, it can be an object in the category $\boldsymbol{Top}$ of topological spaces. The appropriate topology on 2 is $\{\emptyset, \{1\}, 2\}$, that is the collection of upper section of the poset 2. When furnished in this way 2 is sierpinski space.
For a space S what do the characters $\boldsymbol{Top}[S, 2]$ characterize?

2.7 Adjunctions

In subsection 1.4.2 we looked at the notion of an adjunction

$$S \underset{g}{\overset{f}{\rightleftarrows}} T$$

between posets S, T. In fact, a poset is a miniature kind of category, and a poset adjunction is a miniature example of a larger categorical notion. In this section we first look at two important examples of the general notion. After that we describe the general notion itself without going into too many details.

An adjunction between categories is a pair of functors

$$\boldsymbol{S} \underset{G}{\overset{F}{\rightleftarrows}} \boldsymbol{T}$$

together with certain extra structure where all this fits together in a certain way. Often one or other of these functors is rather trivial, and it is the other one that is important. We are going to look first at the situation where either F or G is 'forgetful'.

Suppose we have a situation

$$\boldsymbol{Set} \xleftarrow[\text{¿}]{} \boldsymbol{C}$$

for some category $\boldsymbol{C}$ where ¿ is a functor which we can call forgetful. As a typical example you may think of $\boldsymbol{C}$ as a category of structured sets $(A, \cdots)$ and ¿ merely extracts the carrier A (by forgetting the structure carried by A).

Is there any way we can convert an arbitrary set A into a $\boldsymbol{C}$-object in 'the best possible way'?

To illustrate this suppose $\boldsymbol{C}$ is the category of groups. How can we convert a set A into a group in 'the free-est possible way'? The answer to this is not merely to impose some group structure on A (for there are many incompatible such structures). What we do is think of A as the set of generators of a group, and consider what such a group can look like if it can not have any special features beyond being a group. What we do is generate the free group over A. This is formed by first extending A to a set of 'words' and then hitting this collection with an equivalence relation. The details of this can be a bit messy, but the general idea becomes much clearer when set in a categorical context.

Let's work with an arbitrary functor ¿, as above. It does no harm to think of this as 'forgetful' but, in fact, what we do works for any functor from $\boldsymbol{C}$ to $\boldsymbol{Set}$. Also, we could replace the target by any category. We will first make precise the notion of a 'free' object relative to the functor ¿ (which we think of as forgetful). Later we will dualize this notion to get what we call a 'co-free' object (for want of a better terminology). It is useful to see the appropriate notions set down in parallel, as in Table 2.1. Later we will extend this material to the more general notion of an adjunction.

2.7.1 Free constructions

In this subsection we consider the left hand column of Table 2.1.

Given a ‘forgetful’ functor

$$\boldsymbol{Set} \xleftarrow[\text{¿}]{} \boldsymbol{C}$$

and $\boldsymbol{Set}$-object A, a $\boldsymbol{C}$-object FA and a $\boldsymbol{Set}$-arrow

$$A \xrightarrow{\eta_A} \text{¿}(FA)$$

provide a free $\boldsymbol{C}$-object over A if for each $\boldsymbol{Set}$-arrow

$$A \xrightarrow{f} \text{¿}S$$

where the target arises from a $\boldsymbol{C}$-object, there is a unique $\boldsymbol{C}$-arrow

$$FA \xrightarrow{f^\sharp} S$$

such that the $\boldsymbol{Set}$-triangle

$$\begin{array}{ccc} A & \xrightarrow{f} & \text{¿}S \\ & {\scriptstyle \eta_A}\searrow \quad \nearrow{\scriptstyle \text{¿}(f^\sharp)} & \\ & \text{¿}(FA) & \end{array}$$

commutes.

Given a ‘co-forgetful’ functor

$$\boldsymbol{Set} \xrightarrow{\text{¿}} \boldsymbol{C}$$

and $\boldsymbol{C}$-object S, a $\boldsymbol{Set}$-object GS and a $\boldsymbol{C}$-arrow

$$\text{¿}(GS) \xrightarrow{\epsilon_S} S$$

provide a co-free $\boldsymbol{Set}$-object over S if for each $\boldsymbol{C}$-arrow

$$\text{¿}A \xrightarrow{g} S$$

where the source arises from a $\boldsymbol{Set}$-object, there is a unique $\boldsymbol{Set}$-arrow

$$A \xrightarrow{g_\flat} GS$$

such that the $\boldsymbol{C}$-triangle

$$\begin{array}{ccc} \text{¿}A & \xrightarrow{g} & S \\ & {\scriptstyle \text{¿}(g_\flat)}\searrow \quad \nearrow{\scriptstyle \epsilon_S} & \\ & \text{¿}(GS) & \end{array}$$

commutes.

Table 2.1: Free and co-free objects

2.13 DEFINITION. Consider a functor

$$\boldsymbol{Set} \xleftarrow[\text{¿}]{} \boldsymbol{C}$$

from some category $\boldsymbol{C}$ to $\boldsymbol{Set}$. The associated notion of a $\boldsymbol{C}$-free object over a $\boldsymbol{Set}$-object A is set out in Table 2.1. The arrow

$$A \xrightarrow{\eta_A} \text{¿}(FA)$$

is called the unit of the construction. ■

The definition gives ‘*a* $\boldsymbol{C}$-free object’ over A. However, it is an easy exercise to show that any two such objects over A are uniquely isomorphic, and consequently we usually say ‘*the* $\boldsymbol{C}$-free object’ over A.

In many situations where this idea is used the forgetful functor ¿ is unnamed. We then find a diagram such as

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & S \\ & {\scriptstyle \eta_A}\searrow \quad \nearrow{\scriptstyle ¿(f^\sharp)} & \\ & FA & \end{array}$$

which appears to compare objects living in different categories. If at first you find this is confusing, then simply give the invisible functor a name. However, the common practice is to work with an unnamed functor.

Notice that this idea applies to each ***Set***-object separately. It can happen for some functors ¿ that some ***Set***-object does have a ***C***-free objects but others don't. Often we find that every ***Set***-object has a ***C***-free object, and then several other things happen.

2.14 THEOREM. *Consider a functor ȯ, as above, and suppose each **Set**-object has a selected*

$$A \xrightarrow{\ \eta_A\ } ȯ(FA)$$

***C**-free object. Then the selecting object assignment F fills out to a functor*

$$\boldsymbol{Set} \xrightarrow{\ F\ } \boldsymbol{C}$$

*and the selected family η of arrows is a natural transformation $I \longrightarrow (ȯ \circ F)$ (where I is the identity functor on **Set**).*

Sketch proof. We must attach to each ***Set***-arrow

$$A \xrightarrow{\ f\ } B$$

a ***C***-arrow $F(f)$ with appropriate properties. The compound

$$A \xrightarrow{\ f\ } B \xrightarrow{\ \eta_B\ } ¿(FB)$$

compares a ***Set***-object A with a ***C***-object FB. The universal property produces a unique ***C***-arrow

$$FA \xrightarrow{(\eta_B \circ f)^\sharp} FB$$

which we take as $F(f)$. Thus, by construction, the ***Set***-square

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ ¿(FA) & \xrightarrow[¿(F(f))]{} & ¿(FB) \end{array}$$

commutes. The proof is completed by a series of simple arguments. ■

What we have here is a fairly common example of an adjunction

$$\textbf{\textit{Set}} \underset{¿}{\overset{F}{\rightleftarrows}} \textbf{\textit{C}}$$

where the free-functor F is the left adjoint to the forgetful functor. Notice how the functorality, naturality, and universality are intimately entwined. If we have some of one kind of thing then we get some of another kind of thing. In the general analysis of adjunctions given in Subsection 2.7.3 we will see much more of this kind of thing.

2.7.2 Co-free constructions

Freeness in the sense of Subsection 2.7.1 occurs quite often in mathematics, and is easy to recognize. There is also a co-version, which is not so obvious. We can quickly set down the relevant details.

2.15 DEFINITION. Consider a functor

$$\textbf{\textit{Set}} \xrightarrow{¿} \textbf{\textit{C}}$$

from ***Set*** to some category ***C***. The associated notion of a ***Set***-co-free object over ***C***-object S is set out in Table 2.1. The arrow

$$¿(GS) \xrightarrow{\epsilon_S} S$$

is called the co-unit of the construction. ■

You should compare the two Definitions 2.13 and 2.15, and note how one can be obtained from the other by 'reversing arrows'. This is made clearer by perusing the two columns of Table 2.1 in parallel. To test your understanding of this you should work out a proof of the following.

2.16 THEOREM. *Consider a functor*

$$\textbf{\textit{Set}} \xrightarrow{ὸ} \textbf{\textit{C}}$$

and suppose each ***C****-object has a selected*

$$ὸ(GS) \xrightarrow{\epsilon_S} S$$

Set*-co-free object. Then the selecting object assignment G fills out to a functor*

$$\textbf{\textit{Set}} \xrightarrow[G]{} \textbf{\textit{C}}$$

and the selected indexed family ϵ of arrows is a natural transformation $(ὸ \circ G) \longrightarrow I$ (where I is the identity functor on ***C****).*

Once you are reasonably happy with this material you should be able to handle at least a first reading on the following.

2.7.3 Adjunctions

In both the free- and co-free-constructions we said that one of the categories is ***Set*** but never made any use of the internal properties of ***Set***. Similarly, in the first construction we thought of the given functor $\wr$ as 'forgetful' but again never needed to know what this might mean. In fact, there is a more general situation of which the two examples are instances. To conclude this section we set down some of this information but make no serious attempt to analyse it. [`Perhaps we should`]

In an adjoint situation there is a pair of functors

$$\boldsymbol{S} \underset{G}{\overset{F}{\rightleftarrows}} \boldsymbol{T}$$

between two categories. We call F the left adjoint and G the right adjoint of the pair. The notation

$$F \dashv G$$

is used to indicate that F is the left adjoint and G is the right adjoint of an adjunction. By convention we think of $\boldsymbol{S}$ as the source and $\boldsymbol{T}$ as the target of the adjunction and sometimes write

$$\boldsymbol{S} \xrightarrow{F \dashv G} \boldsymbol{T}$$

to indicate this.

As well as the two functors there is also a pair

$$I_{\boldsymbol{S}} \xrightarrow{\eta} G \circ F \qquad\qquad F \circ G \xrightarrow{\epsilon} I_{\boldsymbol{T}}$$

of natural transformations using the two composites of F and G and the appropriate identity functors. Finally, for each $\boldsymbol{S}$-object A and $\boldsymbol{T}$-object S there is an inverse pair

$$\begin{array}{ccc} f & \longmapsto & f^\sharp \\ \boldsymbol{S}[A, GS] & & \boldsymbol{T}[FA, S] \\ g_\flat & \longleftarrow\!\shortmid & g \end{array}$$

between the indicated hom-sets, where these are natural for variation of A and S.

This data is subject to various restrictions. The first two should be read in parallel.

For each $\boldsymbol{S}$-arrow

$$A \xrightarrow{f} GS$$

there is a unique $\boldsymbol{T}$-arrow

$$FA \xrightarrow{f^\sharp} S$$

such that the triangle

$$\begin{array}{ccc} A & \xrightarrow{f} & GS \\ & {\scriptstyle\eta_A}\searrow \quad \nearrow{\scriptstyle G(f^\sharp)} & \\ & (G \circ F)A & \end{array}$$

commutes.

For each $\boldsymbol{T}$-arrow

$$FA \xrightarrow{g} S$$

there is a unique $\boldsymbol{S}$-arrow

$$A \xrightarrow{g_\flat} GS$$

such that the triangle

$$\begin{array}{ccc} FA & \xrightarrow{g} & S \\ & {\scriptstyle F(g_\flat)}\searrow \quad \nearrow{\scriptstyle \epsilon_S} & \\ & (F \circ G)S & \end{array}$$

commutes.

These conditions assert the existence arrows $f^\sharp$ and $g_\flat$. In fact, these turn out to be the arrows mentioned in the pair of inverse bijections.

There are several identities as part of the adjunction. Thus

$$\begin{aligned} \eta_A &= (1_{FA})_\flat & \qquad \epsilon_S &= (1_{GS})^\sharp \\ f^\sharp &= \epsilon_S \circ F(f) & g_\flat &= G(g) \circ \eta_A \\ \epsilon_{FA} \circ F(\eta_A) &= 1_{FA} & G(\epsilon_S) \circ \eta_{GS} &= 1_{GS} \\ F(f) &= (\eta_B \circ f)^\sharp & G(g) &= (g \circ \epsilon_A)_\flat \end{aligned}$$

for the gadgets as above.

There is a lot of data and information here. However, it turns out that once we have some of it the rest follows. Different combinations are useful in different situations.

[`Do we want to do more on this?`]

Exercises

2.38 Show that the forgetful functor ***Abg*** $\longrightarrow$ ***Grp*** (from abelian groups to groups) has a left adjoint F (in the sense of Theorem 2.14).

2.39 Consider the categories ***Pre*** and ***Set***. Show there are two adjunctions

$$\boldsymbol{Set} \; \begin{array}{c} \xrightarrow{\;\;D\;\;} \\ \xleftarrow{\;\;\text{¿}\;\;} \\ \xrightarrow[\;\;I\;\;]{} \end{array} \; \boldsymbol{Pre}$$

where the central functor is forgetful. You should work out what the other functors are.

Chapter 3

Limits and colimits; a universal solution

In this chapter we look at a dual pair of notions which, in a way, test the completeness properties of a category. Of course, we have to say what we mean by completeness, and there are many different possibilites. Each variant is defined in the same way. A category is complete in a certain sense if it has all limits of a certain kind, or all colimits of a certain kind. (Strictly speaking, having colimits is a cocompleteness property.)

In section 3.4 we set down the general notion of a limit and colimit. In fact, we could start with that and then specialize to particular cases. However, it is probably easier if we start with some of these special cases.

Before we look at these particular examples it is worth going through some of the generalities to get used to the terminology. Some of the phrases used look as though they are just informal description whereas they are used in a quite precise way (but without any all purpose definition).

We work entirely within one category $\mathbf{C}$, the parent category. We start with something described as a problem and we are looking for a solution or more precisely the universal solution. Such a solution is always a single object in $\mathbf{C}$ together with a collection of arrows which satisfy certain restrictions. There may be several such solutions, but we look for the 'best' one. This universal solution is characterized by the way it interacts with all other solutions. It generates each other solution via a unique mediating arrow.

We are going to look at several particular examples where each of these four highlighted words has a precise meaning. To get an idea of what will happen let's go through the paragraph again and this time put a bit more meat on the bones.

A problem is always posed by a diagram. This, as usual, is a collection of objects and some arrows between these objects. The diagram need not commute. A particular case of this is when the diagram is a functor from some other category thought of as an indexing gadget. In fact, every problem posed by a diagram can be rephrased in this way (but this doesn't help in particular cases).

Each diagram poses two problems

the left problem   the right problem

and in general these have different solutions. When we deal with this kind of material, by convention we think of each arrow as moving from left to right, with the flight or blunt end at the left and the head or sharp end at the right. The terminology above comes from this way of picturing the situation. However, in many concrete situations we often draw the arrows pointing all over the place, so perhaps

the blunt problem   the sharp problem

would be better terminology.

Given a problem (diagram) we look for

a left solution   a right solution

depending on which one we are interested in. Each such solution is an object X together with a collection of arrows

going from X going to X

and where the other end is an object in the diagram. There is one arrow for each diagram object. These arrows must interact with the diagram arrows in a way you can probably guess.

The universal solution is a special solution in an optimal position, and is called the

limit   colimit

of the diagram. At various stages in the development of the subject these gadgets have been given different names with, perhaps,

left limit   right limit

being the most obvious. However, other names have been, and still are being, used. Qualifiers such as 'inverse', 'direct', 'projective' are applied to 'limit' to indicate one or other of the universal solutions. These terminologies have been around from before the invention of category theory, and are often used in special situations. You will also come across more recent qualifiers arising directly from category theory. More often than not the notions being described have very little use except to a certain strain of category theorist who likes to ponce about showing off his vivid imagination.

In this chapter we give a small catalogue of the more common limits and colimits. Most of the diagrams we look at are finite, but we will consider an infinite diagram towards the end. On the whole we will look at the finite diagrams in a systematic order, via number of objects and number of arrows. To start with we can point out that we have already seen one example.

3.1 THEOREM. *For the empty diagram in the category* $\boldsymbol{C}$, *the*

limit   *colimit*

is the

final   *initial*

object of $\boldsymbol{C}$.

This is a trivial result which can not be proved just yet, because we don't have the formal definition of limit/colimit.

A word of warning before you start reading. The first section is quite long and deals with that example in great detail. This sets down a general format for each of the examples. The later sections follow this format, and you might find the repetition a bit tedious. If you do then it means you have cottoned on to the general idea, and can perhaps go directly to section 3.4 pausing only to read the main definition of the section you are in.

3.1 Products and sums

In this section we look at the universal solution to one of the smallest diagrams, the one with just two objects and no arrows. As we will see, this example (or the left-hand case to be precise) analyses one of the most common concrete situations in mathematics, to extract the essence and disregard the inessential. It is the example which showed the originators of category theory that they were on to something.

Strictly speaking the title of the section is wrong. It should be 'Products and coproducts' but the word 'sum' is often used for 'coproduct' in concrete situations where that terminology is traditional (and more or less correct).

As we have seen in the introduction, universal solutions come in two kinds, left and right. This is not a matter of the good and the evil (such as City and United) but simply a consequence of the internal duality of category theory. In its more general form this helps to simplify and organize many apparently distinct situations.

To illustrate this we will do the two developments in parallel, and at a first reading you can concentrate on one side. (If you do that you may find the left-hand version easier. This deals with products.) At a couple of places we will make some observations about a concrete version of one side that don't readily transfer to the other.

We will go through the material rather slowly and formally. Furthermore, the development will seem somewhat pedantic. This is done to help you get used to the ideas, and so the more complicated examples considered later will be easier to understand.

We work throughout in some arbitrary category $\boldsymbol{C}$, and develop the material in three phases: the abstract generalities, the extracted properties, and some concrete examples.

The general set-up

In this subsection we will look at products and coproduct as instances of a more general notion.

3.2 DEFINITION. Let $\mathbb{D}$ be the diagram

$$A$$

$$B$$

consisting of two objects and no arrows. ■

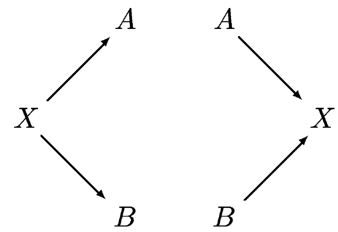
We have drawn the diagram in the 'standard' position. However, in practice we often draw it differently. We see more common versions shortly.

The definition of $\mathbb{D}$ says there are two objects. In fact, A and B could be the same object playing two different roles, like the actor who plays Captain Hook always plays Mr Darling. (There used to be futile philosophical debate about whether the evening star and the morning star are the same thing. They are both Venus.)

3.3 DEFINITION. For the diagram $\mathbb{D}$ (of Definition 3.2) a

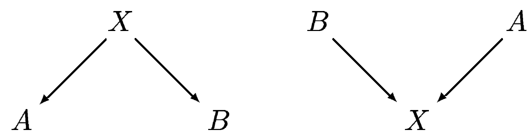
left right

solution is an object X together with a selected pair of arrows



as indicated. ■

These two diagrams are often drawn



and called a

cone   cocone

or sometimes an

under-cone   over-cone

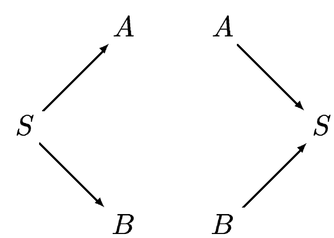
respectively. Here we will call both a wedge to avoid a lot of clumsy language.

A

limit   colimit

of the diagram $\mathbb{D}$ is a universal solution, that is a solution through which every other solution factors uniquely. This is made precise as follows.

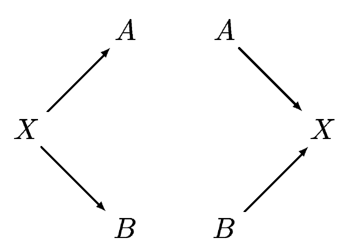
3.4 DEFINITION. A solution



on the

left   right

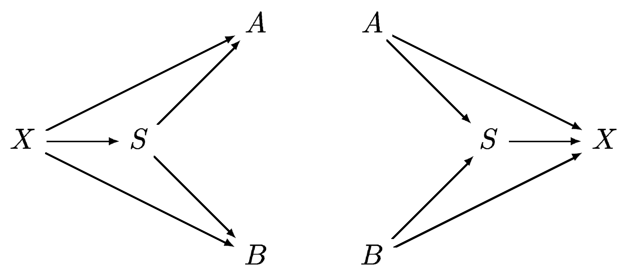
for the diagram $\mathbb{D}$ (of Definition 3.2) is universal (on that side) if for each solution



there is a *unique* arrow

$$X \longrightarrow S \qquad\qquad S \longrightarrow X$$

such that the diagram



commutes. This is the mediating arrow for that solution. ■

Read the definition again. If you can't follow which arrows are supposed to be doing what then try labelling them so you can track each one through the layers of quantifiers. Once you get used to the idea you will find that you don't need the labels.

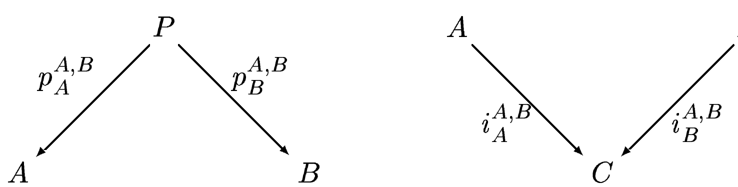
As you can see, this account is a little pedantic, but this will help when we start to look at more general situations. Let's now drop the pedantry and look at the details of this particular pair of constructions.

Here is the succinct version of the previous definition.

3.5 DEFINITION. For a pair A, B of objects in $\boldsymbol{C}$ a

product	coproduct

is a wedge



which is universal for all such wedges. ■

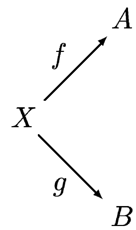
You should note that a product of a pair A, B is *not* just an object P, but an object *together with* a selected pair of arrows. These arrows are called the projections of the product. In spite of this we often write

$$A \times B$$

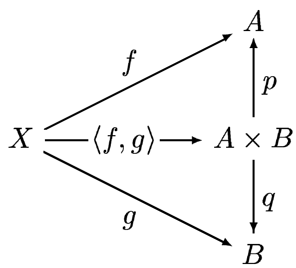
for the product object with a tacit understanding that we know what the projections are. You can probably guess where the notation comes from, and we will go into this later. In the definition these projections have rather arcane decorations to indicate that they depend both on the pair and the particular component. In practice, we drop most, and sometimes all of these sub- and superscripts. We sometimes speak of the left projection or the right projections in the product case. Here 'left' and 'right' refer to the position of the two components in $A \times B$, not to the handedness of the gadget involved. There are also other notations and terminologies which are useful at times. For instance, some form of 'left' and 'right' is useful, or perhaps indexed by 0,1 or 1,2. Sometimes π is used for a projection with some kind of decoration to indicate which is which.

You will find that it is impossible to find a usable notation that works in all situations. For instance, the notation in Definition 3.5 looks reasonable if a little elaborate, but think of what it becomes when A and B are the same object.

There is some notation commonly used with products. Each wedge



induces a unique mediating arrow through the product wedge, and this is written



in keeping with a common set theoretical notation. Sometimes '$\langle f, g\rangle$' is replaced by '(f, g)', but that can be confused with the ordered pair formed from f and g.

Similar remarks apply to the coproduct of a pair A, B. This is often written

$$A + B \quad \text{or} \quad A \amalg B$$

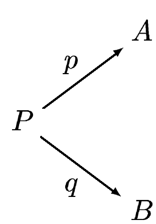
and called a sum when the left hand notation is applicable. The two selected arrows are the insertions of coprojections. There are some quite common notations used with coproducts, but these are far from standard.

Some consequences of universality

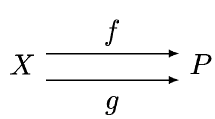
In Definition 3.4 we saw that for a solution to be universal there has to be a *unique* connection to every solution. This uniqueness has several consequences. Let's look at these for products.

We start with the simplest consequence, which has something of the flavour of a monic/epic property.

3.6 LEMMA. *Let*



be a product wedge (with indicated projections). For each parallel pair



of arrows if

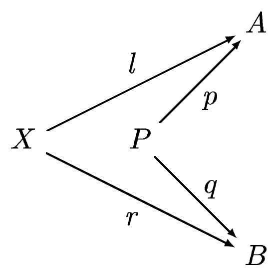
$$p \circ f = p \circ g \qquad q \circ f = q \circ g$$

then, in fact, $f = g$.

Proof. Using the arrows

$$l = p \circ f = p \circ g \qquad r = q \circ f = q \circ g$$

the solution from X

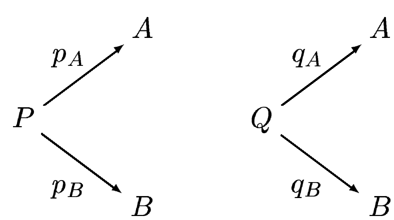


must have a unique mediating arrow. But both f and g have the required mediating property. ■

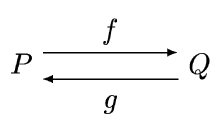
We will see there is a similar property for each limit or colimit, and this leads to the essential uniqueness of the object.

In the statement of the next result we quite deliberately vary the notation used.

3.7 LEMMA. *Let*



be a pair of product wedges for the same objects A, B. *Then* P *and* Q *are uniquely isomorphic over the wedges. That is, there are unique morphisms*



such that

$$p_A = q_A \circ f \quad p_B = q_A \circ f \qquad q_A = p_A \circ g \quad q_B = p_B \circ g$$

and

$$g \circ f = 1_A \qquad f \circ g = 1_B$$

hold.

Proof. Since P is a solution and Q is a universal solution there is a mediating arrow f with the two top left hand properties. Similar, by interchanging the roles of P and Q, we get an arrow g with the top right hand properties.

Next, with X as either A or B, we have

$$p_X \circ g \circ f = q_X \circ f = p_X = p_X \circ 1_P$$

and hence $g \circ f = 1_P$ by an application of Lemma 3.6. In the same way $f \circ g = 1_Q$.
Similar arguments show the uniqueness of this f and this g. ■

In any category any pair of objects may have many product wedges. However, by this result anything we can do with one of them we can do with any other. Thus it is rare that we need to distinguish between these. Accordingly, we often speak of *the* product on the understanding that one particular wedge has been selected (and it doesn't matter which).

Of course, in any particular category a particular pair of objects may not have a product. We will see some concrete examples of this later.

3.8 DEFINITION. A category is cartesian if it has a final object and each pair of objects has a product wedge. ■

There are a couple of points about this notion, one minor and one major.

The minor point is that there are slight variants of the notion. Each variant has products for all pairs, but sometimes other limits are required as well. In the variant given above the limit of the empty diagram (the final object) is required. The differences between these variants are not important, but you should be aware of them.

The major point is concerned with what the definition actually means. Here is what it should mean. Each pair A, B of objects has a product wedge and, furthermore, one such wedge has been selected. Thus whenever the product of A and B is mentioned, it is a reference to this selected wedge. This might seem a bit finicky, and most of the time it doesn't matter too much. However, if at some stage we have to compare products (perhaps in different categories), or we have to transfer products from one place to another, then it begins to make a difference. Quite a lot of the literature on products is not entirely clear on this point.

Here we will take the strict view. Thus, although our phraseology might get a bit sloppy, for us a product of two object is a selected object and a selected pair of projections.

You might be wondering how these selections are suppose to be made, and whether the particular selections make any difference. On the whole they don't. Here is a result which, when first seen, can be a bit surprising. It shows that no matter how certain selections are made there will always be some uniformity in the outcome.

3.9 THEOREM. *Let $\mathbf{C}$ be a cartesian category and let K be some fixed object. For each object A let*

$$FA = A \times K$$

be the selected product object for the pair A, K. Similarly, let

$$FA \xrightarrow{p_A} A \qquad FA \xrightarrow{q_A} K$$

be the selected projections for that product wedge.

Then there is an arrow assignment $f \longmapsto F(f)$ *such that* F *becomes an endofunctor on* $\boldsymbol{C}$. *Furthermore*

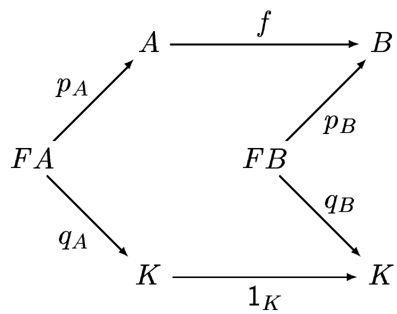
$$F \xrightarrow{\;p\;} I$$

is a natural transformation (where I *is the identity functor on* $\boldsymbol{C}$*).*

Proof. Consider any arrow

$$A \xrightarrow{\;f\;} B$$

for arbitrary object A, B. Using this we produce a diagram



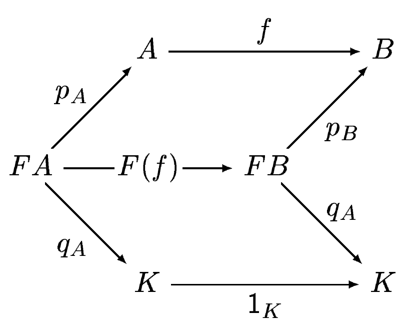
where the various projections have been labelled with the parent object. The composites

$$f \circ p_A \qquad 1_K \circ q_A = q_A$$

provide a solution to the problem posed by (B, K), of which p_B, q_B give the universal solution. Thus there is a unique mediating arrow

$$FA \xrightarrow{\;F(f)\;} FB$$

which we name as indicated, such that the two cells

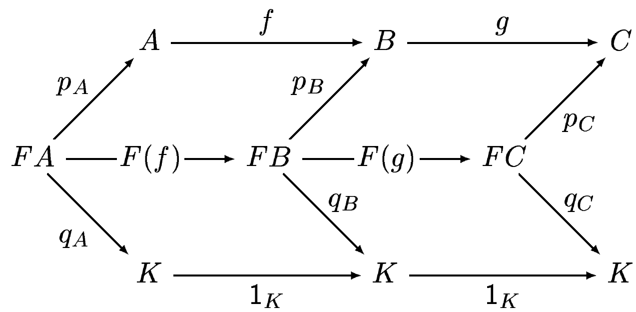


commute. We need to show that this fills out the object assignment F to a functor. Once we have done that the top cell shows that p is natural.

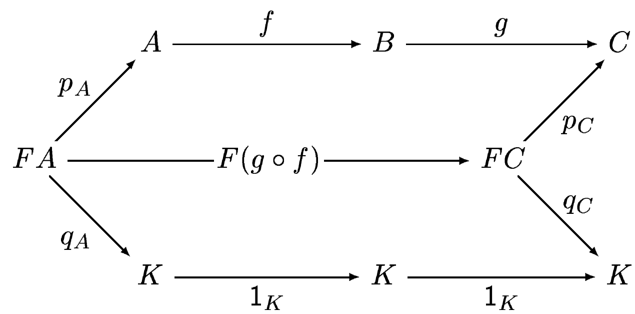
To this end consider an arrow

$$B \xrightarrow{\;g\;} C$$

which can be composed with f. This gives a bigger diagram



which should be compared with the diagram



used to define $F(g \circ f)$. This is the unique arrow which makes the two cells of the lower diagram commute. But, from the upper diagram, the composite $F(g) \circ F(f)$ does this job, and hence

$$F(g \circ f) = F(g) \circ F(f)$$

holds.

This, with a simple argument to show that

$$F(1_A) = 1_{FA}$$

completes the proof. ∎

You should go through this proof again and note that it is the required *uniqueness* of the mediating arrow that makes everything work.

The arrow $F(f)$ induced by an arrow

$$A \xrightarrow{\ f\ } B$$

is often written

$$A \times K \xrightarrow{\ f \times K\ } B \times K$$

but this notation has never won any prizes for style.

There is a generalization of this result which shows that each pair of arrows

$$A \xrightarrow{\ f\ } B \qquad C \xrightarrow{\ g\ } D$$

induce an arrow

$$A \times C \xrightarrow{f \times g} C \times D$$

in a functorial fashion. The details of this are dealt with in the exercises. Once we have this we find that

$$f \times K = f \times 1_K$$

as a particular case.

Concrete products and sums

In this subsection we see how products and sums can be obtained is several concrete categories. In general we see that, when they exists, products are easy to construct, but sums are more complicated.

Consider first the category ***Set*** of sets. We know that for each pair A, B of sets there is a set

$$A \dot{\times} B$$

called the cartesian product, and which is just the set of ordered pairs

$$(a, b)$$

of elements $a \in A, b \in B$. Notice that we have written '$\dot{\times}$' for this concrete construction. This is so we can distinguish it from the categorical product $\times$ in ***Set***. (At this stage we have no reason to believe that the two constructs are related.)

Remember how ordered pairs (a, b) are obtained within the universe of sets. We must construct (a, b) as a set of some kind, so a trick is needed. The usual trick is to let

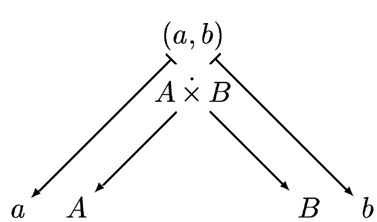
$$(a, b) = \{\{a\}, \{a, b\}\}$$

but there are others. We could equally well let

$$(a, b) = \{\{b\}, \{a, b\}\} \quad \text{or} \quad (a, b) = \{\{a, 0\}, \{b, 1\}\}$$

where $0, 1$ are two distinct tags. What has such a trick got to do with the required properties of pairs and products? Nothing at all, and the categorical description gets rid of these inessential features to expose the heart of the idea.

No matter how we set up ordered pairs we do have two functions



and these with $A \dot{\times} B$ provide a solution to the product problem for A, B in ***Set***. A simple exercise shows that this is a universal solution, and hence the cartesian product provides an implementation of the categorical product in ***Set***.

When we use products of sets all we need to know are the properties given by the categorical description and not the internal details of how ordered pairs are conceived.

It was this example that gave the originators of category theory an indication that perhaps the notion of a universal solution was worth investigating (although at the time they did not use this terminology to describe the situation).

This shows that ***Set*** has all binary products. Recall also that in ***Set*** the singleton $1 = \{\bullet\}$ is the final object, so the category is cartesian.

What about coproducts in ***Set***?

It should come as no surprise that the coproduct in ***Set*** of two sets A, B is implemented by the disjoint union

$$A \dot{+} B$$

which is also called the sum (but not the boolean sum). Whenever we use this in a concrete situation there is always a bit of messing about. We often 'assume the two sets are disjoint' and then take

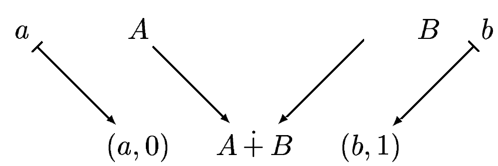
$$A \dot{+} B = A \cup B$$

as the construction. But what if A, B are *not* disjoint? We let

$$A \dot{+} B = (A \dot{\times} \{0\}) \cup (B \dot{\times} \{1\}) = \{(a,0) \mid a \in A\} \cup \{(b,1) \mid b \in B\}$$

where $0, 1$ are distinct tags. Clearly there is a lot of fiddly stuff here which has nothing whatsoever to do with the eventual aim of the construction.

We have two functions



and these with $A \dot{+} B$ provide a solution to the coproduct problem for (A, B) in ***Set***. A simple exercise shows that this is a universal solution, and hence the disjoint sum provides an implementation of the categorical coproduct in ***Set***.

There are two points worth emphasizing. Firstly, the categorical description gets to heart of the two notions, and hides the irrelevant details. Secondly, we see that the two notions are duals, which is something not entirely clear when we look at the concrete constructions.

Similar constructions sometimes work in more structured situations.

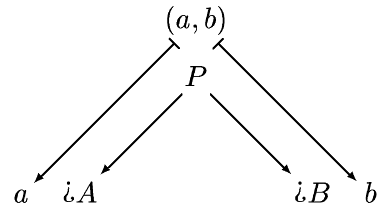
Let ***C*** be a category of structured sets, for example groups, abelian groups, unital rings, posets, topological spaces, and the like. Each object of ***C*** is a set furnished with some attributes, and the arrows are functions which preserve these attributes (in some sense). This is one of the few occasions when referring to a structure by its carrier is confusing. So, for the time being, let us write A for an object of ***C*** with the set $¿A$ as its carrier. In other words

$$\mathbf{Set} \xleftarrow[\;¿\;]{} \mathbf{C}$$

is the forgetful functor. (For most of the categories where the construction we describe does work this forgetful functor will have a left adjoint in the sense of Section 2.7. We don't make much use of that just yet, but we will when we look at coproducts.)

How might we try to produce the product of two objects A, B of ***C***?

We can certainly pass to the sets ¿A, ¿B and construct ¿$A \times$ ¿B as a set of ordered pairs. In many cases this set carries an obvious 'pointwise' structure to become an object of $\boldsymbol{C}$. We write $A \times B$ for this object. Furthermore, we can usually check that the two projections from $P =$ ¿$A \times$ ¿B



are arrows of $\boldsymbol{C}$. Thus we have a solution to the product problem for A, B in $\boldsymbol{C}$. In fact, if we can get this far then it is usually routine to show that we have a universal solution. All that is needed is to check that a mediating function is, in fact, an arrow of $\boldsymbol{C}$.

This observation shows how many concrete product object constructions are instances of one general idea. We are not saying that the pointwise construction always works, but we are saying that when it can be done it should be the first thing to look at.

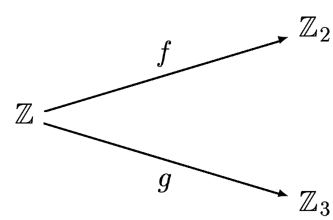
You may have wondered why the product topology for two topological spaces is defined in the way it is. It is to ensure that the space constructed is the categorical product in the category of topological spaces. The topology imposed on the cartesian product of the two spaces is the smallest topology which makes the two projection functions continuous.

This cartesian construction doesn't always work.

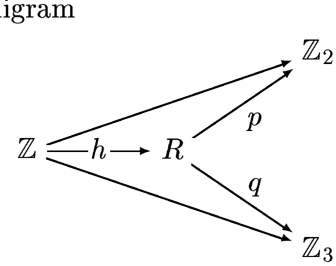
3.10 EXAMPLE. Let ***Rng*** be the category of unital rings and let ***Idm*** be the subcategory of integral domains. Given ***Idm***-objects A, B they are also ***Rng***-objects, and so have a product $A \times B$ in ***Rng*** (which can be obtained as a structured cartesian product). However, in general $A \times B$ is *not* an integral domain, so this can not be the product in ***Idm***. This observation does not show that A, B do not have a product in ***Idm***, only that the obvious construction doesn't work.

We can show that ***Idm*** is not cartesian.

Consider the wedge



in ***Idm*** where f, g are the obvious morphisms. We know that $\mathbb{Z}_2 \times \mathbb{Z}_3$ is not the product of $\mathbb{Z}_2, \mathbb{Z}_3$ in ***Idm***, but there might be some other object R which is a product. If there is then we get a commuting digram



for some projection morphisms p, q and mediating morphism h. Remembering that $\mathbb{Z}$ is a principal ideal domain and by fiddling about with divisibility we can show that any such R must contain zero divisors, and so in not an integral domain.

Thus, although in ***Rng*** the pair $\mathbb{Z}_2, \mathbb{Z}_3$ do have a product, this is not the product in ***Idm*** and, in fact, there is no product in ***Idm***. ■

Let's now look at concrete coproducts. These are not so easy to get at. To illustrate this we use the two categories ***Grp***, ***Abg*** of groups and abelian groups. We do these two cases in unison, so let $\boldsymbol{C}$ be one or other of these two categories. Each $\boldsymbol{C}$-object has a carrier which is a set. We know how to produce coproducts for sets, and we use this to produce the required gadgets in $\boldsymbol{C}$. In other words, we are going to flit about between the two categories $\boldsymbol{C}$ and ***Set***. Usually this is done without any distinguishing notation, but here we will do it formally. Of course, in practice this is not the way to set out the construction without a very good reason for doing so.

We know there is a pair of functors

$$\boldsymbol{Set} \underset{¿}{\overset{F}{\rightleftarrows}} \boldsymbol{C}$$

which, in technical terms form an adjoint pair. We haven't quite formally defined this notion, but we can say what it means here. The forgetful functor ¿ merely selects the carrier of a $\boldsymbol{C}$-object. For each set X the functor F produces the free $\boldsymbol{C}$-object FX generated by X, as explained in Section 2.7. This involves adding to X many new elements and then hitting the larger set with an equivalence relation. We will use the universal property of this free generation.

Consider a pair A, B of $\boldsymbol{C}$-objects. We can transfer to ***Set*** and form the sum $¿A + ¿B$. In general there is no easy way to furnish this as a group, but we do have the free group

$$K = F(¿A + ¿B)$$

generated by the set. This is still not the required coproduct, but we are getting there.

Let

$$¿A \xrightarrow{\ i\ } ¿A + ¿B \qquad ¿B \xrightarrow{\ j\ } ¿A + ¿B \qquad ¿A + ¿B \xrightarrow{\ k\ } ¿K = ¿F(¿A + ¿B)$$

be the functions obtained from the sum construction in the case of i, j and the associated unit of the freeness in the case of k. (The notation is already getting a bit hairy, which is why we usually omit the '¿'.)

Since K is a $\boldsymbol{C}$-object we can consider quotient morphisms (that is surjective morphisms)

$$K \xrightarrow{\ l\ } L$$

to arbitrary groups. (If we are working in ***Abg*** then this L will be automatically abelian.) There is at least one such morphism l such that both the composite functions

$$¿A \xrightarrow{\ i\ } ¿A + ¿B \xrightarrow{\ k\ } ¿K \xrightarrow{\ l\ } ¿L \qquad ¿B \xrightarrow{\ j\ } ¿A + ¿B \xrightarrow{\ k\ } ¿K \xrightarrow{\ l\ } ¿L$$

are morphisms (but where the individual components need not be). For instance we can take the extreme case where L is the trivial group. For the purposes of this construction let us say a quotient morphisms l of K is special if both the composite functions

$$A \xrightarrow{l \circ k \circ i} L \qquad B \xrightarrow{l \circ k \circ j} L$$

are morphisms.

The trick is to find the universal solution of a slightly different kind of problem. If you prefer you can accept the following result on trust.

3.11 LEMMA. *For the situation A, B, K, as above, there is a universal special morphism*

$$K \xrightarrow{\quad m \quad} M$$

that is a special morphism m, as indicated, such that for each special morphism

$$K \xrightarrow{\quad l \quad} L$$

there is a unique factorization of l through m, that is there is a unique morphism

$$M \xrightarrow{\quad n \quad} L$$

such that

$$\begin{array}{ccc} K & \xrightarrow{\quad l \quad} & L \\ & {\scriptstyle m}\searrow \quad \nearrow{\scriptstyle n} & \\ & M & \end{array}$$

commutes.

Proof. Each special morphism l has a kernel in K and this is a certain normal subgroup of K. Let us say a normal subgroup of K is special if it arises in this way. There is an easy characterization of these special subgroups, and we find that the intersection of any family of special subgroups is itself special. Using this we see that there is a unique smallest special normal subgroup, namely the intersection of all special normal subgroups. Thus we can find an extra special quotient of K. This gives the required morphism m. ■

We now define $A \amalg B$ to be the target M of the extra special morphism m. Thus we have a morphism

$$K \xrightarrow{\; m \;} A \amalg B$$

with the universal property given by the lemma. In particular, the two function composites

$$\begin{array}{lllllll} ¿A & & & & & & \\ & {\scriptstyle i}\searrow & & & & & \\ & & ¿A + ¿B & \xrightarrow{\quad k \quad} & ¿K & \xrightarrow{\quad m \quad} & ¿(A \amalg B) \\ & {\scriptstyle j}\nearrow & & & & & \\ ¿B & & & & & & \end{array}$$

are morphisms.

Let's now drop any pretence that the notation '$\imath$' is useful, and continue in the more customary style.

3.12 THEOREM. *For each pair A, B of $\boldsymbol{C}$-objects, the two morphisms*

$$A \xrightarrow{u = m \circ k \circ i} A \amalg B \qquad\qquad B \xrightarrow{v = m \circ k \circ j} A \amalg B$$

form the coproduct in $\boldsymbol{C}$.

Proof. Consider any pair

$$A \xrightarrow{f} G \qquad\qquad B \xrightarrow{g} G$$

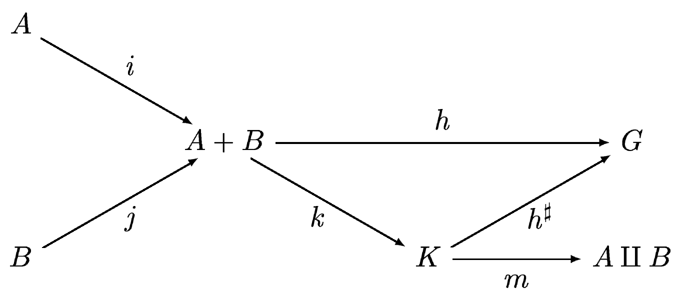
of morphisms. Working in $\boldsymbol{Set}$, that is by first applying the forgetful functor, we obtain a unique *function*

$$A + B \xrightarrow{h} G$$

such that both

$$f = h \circ i \qquad g = h \circ j$$

hold. Using the fact the K is the free object on $A + B$ we obtain a diagram



where $h^\sharp$ is a morphism and the central cell commutes. From this diagram we see that

$$h^\sharp \circ k \circ i = h \circ i = f \qquad h^\sharp \circ k \circ j = h \circ j = g$$

and hence (in the terminology used above) the morphism $h^\sharp$ is special. This gives a factorization

$$h^\sharp = n \circ m$$

for a unique morphism n. Also

$$n \circ u = n \circ m \circ k \circ i = h^\sharp \circ k \circ i = f \qquad n \circ v = n \circ m \circ k \circ j = h^\sharp \circ k \circ j = g$$

so we do have a factorization (in $\boldsymbol{C}$) of the given morphisms f, g.

This doesn't quite complete the proof, for we still have to show that there is only one morphism n which does this job. However, the proof of that is routine. ■

The precise details of this proof are not important here. However, there are three points you should take note of.

Firstly, it is clear from this construction that although products and coproducts are categorically dual notions, they are not equally simple in the real world. Products of groups are easy to produce, but coproducts are not. This doesn't have much to do with groups, for many algebraic structures display the same kind of disparity.

Secondly, much of this concrete construction can be put in a general categorical setting. There are appropriate categorical notions that can be used in place of normal subgroups and the like.

Thirdly, something you have probably missed. The construction above works for both ***Grp*** and ***Abg***. Now suppose A, B are abelian groups. By the construction we can produce the coproduct $A \amalg B$ in ***Abg***. This is a certain abelian group. However, it is *not* the coproduct of A, B in ***Grp***. There is a coproduct in ***Grp*** but, in general, it is not an abelian group. You should worry about this until you find the precise place in the construction where it matters what whether the parent category is ***Grp*** or ***Abg***.

Exercises

3.1 Each poset is a category.
What is the product of two elements?
What is the sum of two elements?

3.2 Suppose the category $\boldsymbol{C}$ is cartesian. Thus for objects A, B, C there are objects

$$A \times B \quad B \times C \quad (A \times B) \times C \quad A \times (B \times C)$$

with selected projections.

(a) Show that the two triple product objects are isomorphic.

(b) Show that the three objects $1 \times A, A, A \times 1$ are isomorphic (where 1 is the terminal object).

3.3 [*This should be split between the two previous chapters*]

Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be categories (which may be the same). We form a new category $\boldsymbol{C} \times \boldsymbol{D}$. The objects are pairs (C, D) where C is a $\boldsymbol{C}$-object and D is a $\boldsymbol{D}$-object. An arrow

$$(A, B) \longrightarrow (C, D)$$

is a pair of arrows

$$A \xrightarrow{\ f\ } C \qquad\qquad B \xrightarrow{\ g\ } D$$

from the component categories. The composition and extra structure is imposed in the obvious way.

(a) Show that this does produce a category.

(b) Show that the two object assignments

$$(C, D) \longmapsto C \qquad\qquad (C, D) \longmapsto D$$

with the obvious arrow assignments form functors.

(c) Show that for each $\boldsymbol{D}$-object K the object assignment

$$C \longmapsto (C, K)$$

fills out to a functor.

3.4 Suppose the category $\boldsymbol{C}$ is cartesian, and consider the 'square' category $\boldsymbol{C}^2 = \boldsymbol{C} \times \boldsymbol{C}$ of pairs from $\boldsymbol{C}$ (as constructed in Exercise 3.3).

Show that the object assignment

$$\begin{array}{ccc} \boldsymbol{C}^2 & \longrightarrow & \boldsymbol{C} \\ (A, B) & \longmapsto & A \times B \end{array}$$

fills out to a functor, and suggest a notation for the arrow assignment.

3.5 Let $\boldsymbol{C}$ be a category with all products and sums. For objects A, B, C let

$$L(A, B, C) = A \times C + B \times C \qquad R(A, B, C) = (A + B) \times C$$

to form two more objects.

(a) Show there is an arrow

$$L \longrightarrow R$$

which is natural for variations of A, B, C.

(b) Explain exactly why this arrow is natural in the technical sense.

(c) Find an example to show that there need not be an arrow $R \longrightarrow L$.

3.6 (a) Show that the category of pointed sets has all binary products and all binary sums.

(b) Consider the category of sets with a distinguish subset. Show that this has all binary products. Does it have all binary sums?

3.7 (a) Complete the proofs of Example 3.10. That is, show that ***Rng*** is cartesian but ***Idm*** is not.

(b) Show that ***Rng*** has all binary coproducts. [*Sort this out*]

3.8 Complete the proof of Theorem 3.12. That is, show that constructed morphism n is the only one with the required properties.

3.9 Show that in ***Abg*** the cartesian product of two objects implements both the product *and* the sum.

Does this work in ***Grp***?

3.2 Equalizers and coequalizers

In this section we look at the second example of the limit and colimit of a certain diagram. As mentioned earlier these used to be called the left limit and the right limit (and sometimes still are). This examples illustrates why this terminology was used.

We will follow the same general set-up of section 3.1. This will help with the comparison with products and coproducts, and with the general notions to be done in section 3.4. However, for this case we do not need to make such a meal of it. Nevertheless, you may find the section a little slow. If you do then you can jump straight to the succint Definition 3.16.

3.13 DEFINITION. Let $\mathbb{D}$ be the diagram

$$A \rightrightarrows B$$

consisting of two objects and a parallel pair of arrows. ■

Although these two arrows are parallel, they need not agree. We want to make them agree by modifying one end or the other.

3.14 DEFINITION. For the diagram $\mathbb{D}$ (of Definition 3.13) a

left right

solution is an object X together with a selected arrow

$$X \longrightarrow A \qquad B \longrightarrow X$$

such that the parallel pair of composites

$$X \longrightarrow A \rightrightarrows B \qquad\qquad A \rightrightarrows B \longrightarrow X$$

agree. ■

It is sometimes useful to say that this selected arrow (the solution) solve the

equalizing coequalizing

problem of the parallel pair (of the diagram). More succinctly we may say the selected arrow

equalizes coequalizes

the given parallel pair. However, sometime that can be misleading because of the terminology used to describe the universal solution (which is the more common usage of these words).

3.15 DEFINITION. A solution

$$S \longrightarrow A \qquad B \longrightarrow S$$

on the

left right

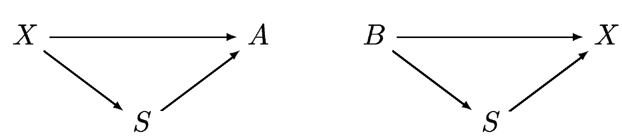
for the diagram $\mathbb{D}$ (of Definition 3.13) is universal (on that side) if for each solution

$$X \longrightarrow A \qquad B \longrightarrow X$$

there is a *unique* arrow

$$X \longrightarrow S \qquad S \longrightarrow X$$

such that the diagram



commutes. That is each solution factors uniquely through the universal solutions via a mediating arrow. ■

This is the general idea, but we can give a concise definition. Notice how this official terminology conflicts with the terminology suggested above.

3.16 DEFINITION. For a parallel pair

$$A \rightrightarrows B$$

of arrows

an equalizer | a coequalizer

is an arrow

$$E \longrightarrow A \qquad B \longrightarrow C$$

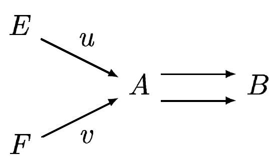
such that the parallel pair of composites

$$E \longrightarrow A \rightrightarrows B \qquad A \rightrightarrows B \longrightarrow C$$

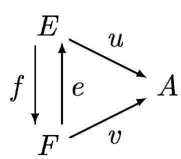
agree, and which is universal for all such arrows. ■

As with all universal solutions, equalizers and coequalizers are essentially unique. In the equalizer case this is made precise as follows.

3.17 LEMMA. *Suppose the left hand arrows u and v*



are equalizers for the parallel pair on the right. Then there are unique arrows f and e



such that the two triangles commute. Furthermore, f and e are an inverse pair of ismorphisms.

Proof. The arrow e exists since v is a solution and u is a universal solution of the equalizing problem. The arrow f exists for a similar reason. This ensures that the two triangles commute, that is

$$u \circ e = v \qquad v \circ f = u$$

hold.

For the various other parts it is useful to observe a consequence of the uniqueness of mediating arrows. Thus, for any parallel pair

$$G \underset{k}{\overset{h}{\rightrightarrows}} E$$

of arrows, if the two parallel composites

$$G \underset{k}{\overset{h}{\rightrightarrows}} E \xrightarrow{u} A \rightrightarrows B$$

agree then $h = k$.

Using this, since

$$u \circ e \circ f = v \circ f = u = u \circ 1_E$$

we have

$$e \circ f = 1_E \qquad f \circ e = 1_F$$

where the right hand equality follows be a similar argument.

Finally, mediations uniqueness give the uniqueness of f and g. ■

Often when they are first seen the categorical notions of equalizer and coequalizer are not immediately recognized as something already known. In fact, they are used in other places but usually described in a different way. Let's have a look at what they are in some familiar categories.

In ***Set***, the category of sets, one way to equalize a parallel pair of functions

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

is to extract the subset on which they agree

$$X = \{a \in A \,|\, fa = ga\}$$

and then the insertion

$$X \hookrightarrow A$$

solves the equalizing problem. In fact, by a simple argument, we see that this is the equalizer.

We can try the same idea in a category $\boldsymbol{C}$ of structured sets. Thus A, B are 'algebras' of some kind (carried by single sets) and f, g are morphisms. However, the extracted subset X may not have the right closure properties to be a 'subalgebra'. In this case we have to close off to the smallest 'subalgebra' generated by X. This can produce a much larger subset, and can even produce the whole of A even though X is quite small.

Coequalizers in some concrete situations can be produced in a similar kind of way.

Consider a pair f, g of ***Set***-arrows as above. To solve the coequalizing problem we consider the relation $\sim$ on B given by

$$x \sim y \iff (\exists a \in A)[x = fa, y = ga]$$

for $x, y \in B$. This doesn't have any special properties, but it can be used to generate an equivalence relation on B. We can also get at this from above. Consider the family of all equivalence relations $\cong$ on B such that

$$x \sim y \Longrightarrow x \cong y$$

holds for all $x, y \in B$. There is a least one such equivalence relation, namely the relation that holds for all x, y. It is easily to check that the intersection of all equivalence relations with this property itself has this property. Thus there is a unique least equivalence relation $\approx$ with this property.

Using the blocks (equivalence classes) we obtain a surjective function

$$\begin{array}{rcl} B & \longrightarrow & B/\approx \\ b & \longmapsto & [b] \end{array}$$

which certainly solves the coequalizer problem. In fact, by a simple argument, we see that this is the coequalizer.

When we deal with structured sets equivalence relations are not good enough. We need to use congruences, that is equivalence relations which respect the carried structure. With this modification the same construction works, thus we find the smallest congruence relation $\approx$ which extends the relation $\sim$, and then the set of blocks carries a natural structure for which the assignment $b \longmapsto [b]$ is a morphism. This produces the coequalizer in the ambient category.

Exercises

3.10 A quiver in a category is a collection of arrows

$$A \quad \begin{array}{c} \longrightarrow \\ \vdots \\ \longrightarrow \end{array} \quad B$$

with a common source and a common target. Such a diagram poses a left problem and a right problem.

(a) Show that if the ambient category has all equalizers then each finite quiver has a left universal solution.

(b0 Does this result extend to infinite quivers?

3.11 (a) Show that each equalizer is monic.

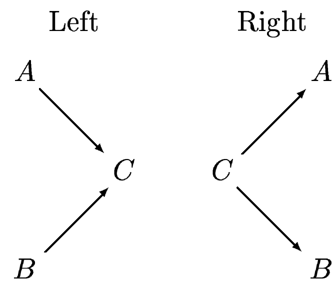
(b) State and prove the corresponding result for coequalizers.

3.3 Pullbacks and pushouts

In this section we look at the third example of limits and colimits. However, in this example the limit (the left limit) uses one diagram whereas the colimit (the right limit) use the mirror image diagram. This example is a kind of generalization of the example of section 3.2, and at first sight it looks a little odd until we see what it is doing. As with the previous section, you might find the going a little slow and repetitious, in which case you can jump straight to the Definition 3.21.

As usual we do the two versions in parallel. However, at a first reading you might want to do just one version and leave the other for later.

3.18 DEFINITION. Let $\mathbb{D}$ be the diagram



consisting of three objects and a pair of arrows. ■

Although the description of $\mathbb{D}$ does say three objects, in fact it could be that A and B are the same object. For that case the universal solution we produce is just the equalizer or coequalizer of the parallel pair of arrows. Also, if we let C be the final or initial object (assuming this exists) then the universal solution we produce is just the product or the sum of A and B. respectively.

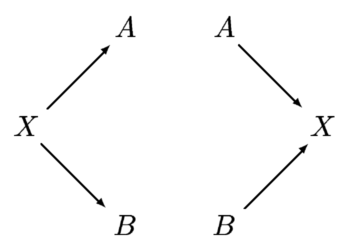
We choose one of the diagrams and look for the corresponding universal solution. If we choose the left diagram then we look for a left universal solution, which is a limit. (The left diagram has a trivial colimit, namely the identity arrow on C.) If we choose the right diagram then we look for a right universal solution, which is a colimit. (The right diagram has a trivial limit, namely the identity arrow on C.)

At a first reading you should pick one side or the other, left or right, and read only that case. Later you should read the other case, and then observe how the two cases could be done in parallel.

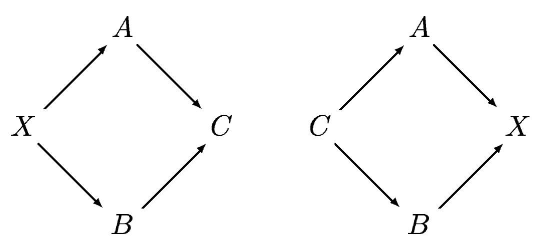
3.19 DEFINITION. For the diagram $\mathbb{D}$ (of Definition 3.18) a

left right

solution is an object X together with a pair of arrows



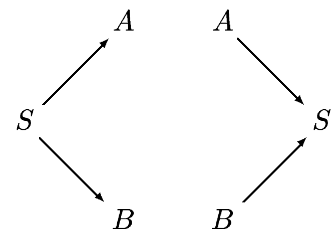
such that the square



commutes. ■

By now you should be able to write down the associated notion of a universal solution. If you find there are too many arrows around and you are not sure which is which, simply label all the arrows as they occur and keep track of them. This might help with the formal definition.

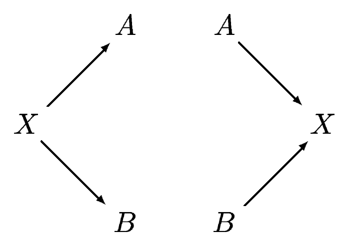
3.20 DEFINITION. A solution



on the

left right

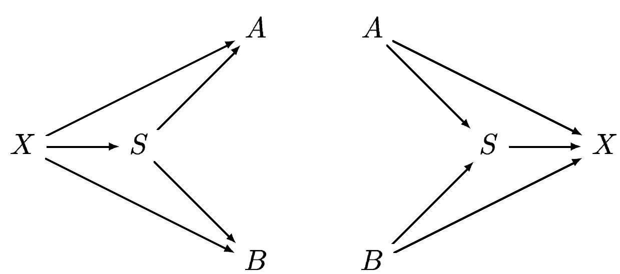
for the diagram $\mathbb{D}$ (of Definition 3.18) is universal (on that side) if for each solution



there is a *unique* arrow

$$X \longrightarrow S \qquad S \longrightarrow X$$

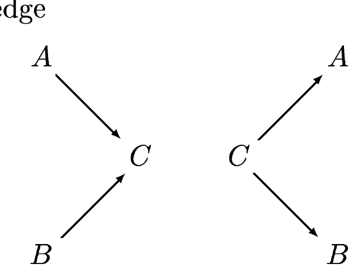
such that the diagram



commutes. That is each solution factors uniquely through the universal solutions via a mediating arrow. ■

As with the earlier examples this universal solution is usually defined in a more succinct way.

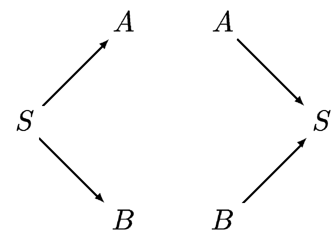
3.21 DEFINITION. For a wedge



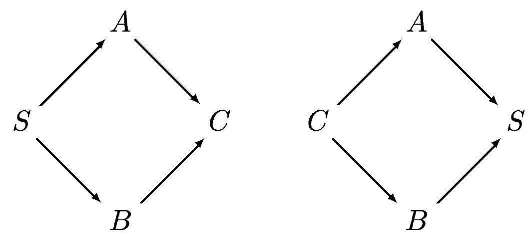
of arrows a

pullback pushout

is wedge



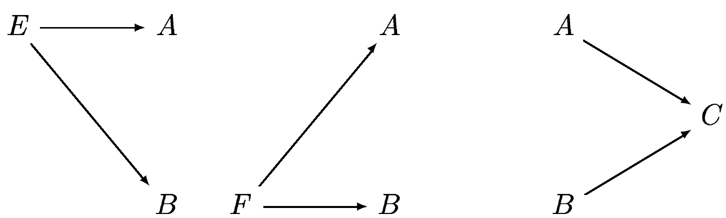
(with the opposite parity) such that the square



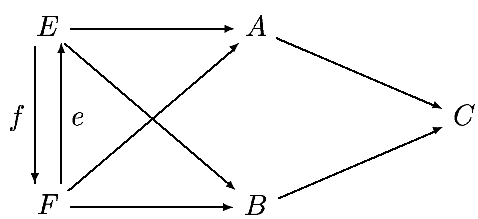
commutes and which is universal for all such wedges. ■

As with every universal solution, pullbacks and pushouts are essentially unique. In the pullback case this is made precise as follows.

3.22 LEMMA. *Suppose the two left hand wedges are pullbacks for the wedge on the right.*



Then there are unique arrows f and e



such that the four triangles on base e or f commute. Furthermore, f and e are an inverse pair of isomorphisms.

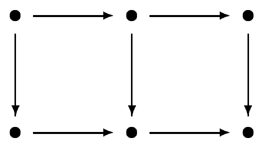
As pointed out at the beginning of this section, a pullback is a generalized equalizer (in the sense that each equalizer is a particular kind of pullback). Also, if the category has a final object then a pullback is a generalized product (in the sense that each product is a particular kind of pullback). In fact, every pullback can be produced by a combination of products and equalizers.

Similar remarks hold for pushouts, coequalizers, and sums.

Exercises

3.12 The category ***Set*** has all pullbacks and all pushouts. Describe how these are formed in terms of elements and functions.

3.13 Consider a commuting diagram



in some category.

(a) Show that if each of the two squares is a pullback, then so is the outer rectangle.

(b) Show that if the outer rectangle and the right hand square are pullbacks, then so is the left hand square.

(c) State the corresponding results for pushouts.

3.14 Show that if a cartesian category has all equalizers then it has all pullbacks.

3.15 Prove Lemma 3.22

3.4 Limits and colimits

We have seen three pairs of examples of the idea of a universal solution to a problem. In this section we look at the general notion which covers all the examples of this chapter. In a more ascetic development this is the place to start (and perhaps the only place ever seen), but that doesn't help the understanding.

We work in some arbitrary category $\boldsymbol{C}$.

3.23 DEFINITION. A diagram $\mathbb{D}$ (in the ambient category) is a collection of objects and a collection of arrows between these objects. ■

The two collections of this diagram $\mathbb{D}$ may be finite or infinite, or one of each. In the extreme they can both be empty. Of course, in the infinite cases there are some extra technicalities that have to be handled, but the general idea doesn't change.

Each arrow of $\mathbb{D}$ has the form

$$B \xrightarrow{\ \theta\ } A$$

where A, B are objects in $\mathbb{D}$. These can be the same object. Also, there may be one or more objects in $\mathbb{D}$ which is neither a source nor a target of an arrow of $\mathbb{D}$. By convention, we think of the arrows of $\mathbb{D}$ as moving from left to right. This explains some of the terminology. However, in practice we draw the arrows in any direction that is convenient.

Notice that we do not require the diagram $\mathbb{D}$ to commute. We could close off the diagram by adding in all the composite arrows to obtain a new commuting diagram $\mathbb{D}^*$. It turns out that $\mathbb{D}$ and $\mathbb{D}^*$ have the same limit and the same colimit, so closing off only helps to obscure the idea. There is a related notion in which the diagram is replaced by a functor from an indexing category to the ambient category. This deals with the problem posed by $\mathbb{D}^*$ rather than that posed by $\mathbb{D}$.

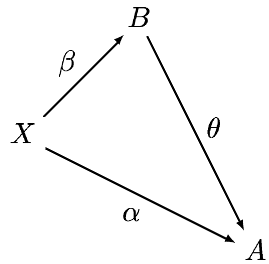
3.24 DEFINITION. For the diagram $\mathbb{D}$ (of Definition 3.23) a

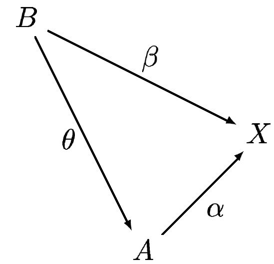
left right

solution is an object X together with a selected arrow

$$X \xrightarrow{\alpha} A \qquad A \xrightarrow{\alpha} X$$

to each object A of $\mathbb{D}$ where the triangle





commutes for each arrow θ of $\mathbb{D}$. ■

In other words a solution to the problem posed by the diagram is trying to make the diagram commute from one side or the other. Of course, this does *not* mean that if the diagram already commutes then there is a trivial solution, for there may be no left-most or right-most object. In fact, the three particular diagrams we have looked at so far (which produce products/coproducts or equalizers/coequalizers or pulbacks/pushouts) commute vacuously.

3.25 DEFINITION. A solution

$$S \xrightarrow{a} A \qquad A \xrightarrow{a} S$$

on the

left right

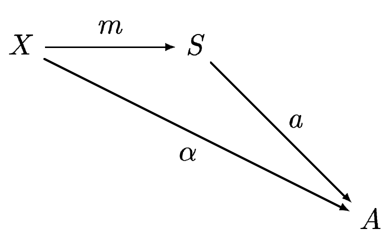
for the diagram $\mathbb{D}$ (of Definition 3.23) is universal (on that side) if for each solution

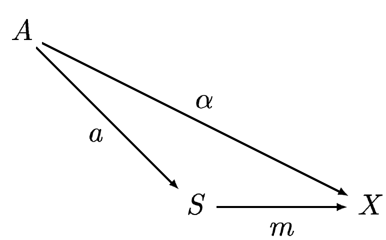
$$X \xrightarrow{\alpha} A \qquad A \xrightarrow{\alpha} X$$

there is a *unique* arrrow

$$X \xrightarrow{m} S \qquad S \xrightarrow{m} X$$

such that for each object A of $\mathbb{D}$ the triangle





commutes. This is the mediating arrow for that solution. ■

This section is quite short but could be even shorter. The essential content is the three definitions. Once these two dual notions are understood most of the previous section can be put to one side.

Exercises

3.16 Show that if a cartesian category has all equalizers then each finite diagram has a limit.

3.17 State and prove the result saying that the limit of a diagram is essentially unique.

3.5 Inverse and direct limits

In this section we consider a whole family of examples of universal solutions. All of these have a certain similarity and are not as general as they might be. However, these particular cases do occur in several concrete situations.

As usual work we work in an arbitrary category $\boldsymbol{C}$. We also use a partially order set $\mathbb{I}$ as an indexing gadget. We write $\leq$ and $<$ for the unstrict and strict comparison carried by $\mathbb{I}$. We let $i, j, k, \ldots$ range over $\mathbb{I}$, and think of these as indexes. We could also think of $\mathbb{I}$ as an indexing category, but in the end that doesn't help much and in any case we cover a more general situation here.

3.26 DEFINITION. A diagram $\mathbb{D}$ (over $\mathbb{I}$ in $\boldsymbol{C}$) consists of

- an object $A(i)$ for each $i \in \mathbb{I}$
- an arrow

  $$A(i) \xrightarrow{A(j,i)} A(j)$$

  for a selection of pairs $i < j$ for $\mathbb{I}$

with no hidden conditions. ■

Notice that there can be many diagrams with the same selection

$$\mathcal{A} = (A(i) \mid i \in \mathbb{I})$$

of objects, for the selection of arrows can vary. There are some extreme examples. For instance where there are no arrows, or where there is an arrow $A(j,i)$ for all $i < j$. Some of these might be silly and never used in practice, but that doesn't matter for there will be many sensible examples.

It can happen that for indexes $i < j < k$ there are selected arrows

$$\begin{array}{ccc} A(i) & \xrightarrow{A(k,i)} & A(k) \\ & {\scriptstyle A(j,i)}\searrow \quad \nearrow{\scriptstyle A(k,j)} & \\ & A(j) & \end{array}$$

but this triangle need not commute.

There are several variants of this idea which are useful at times, but there is little point in attempting to produce here the most general notion possible.

We assume we have some diagram $\mathbb{D}$ of the kind described.

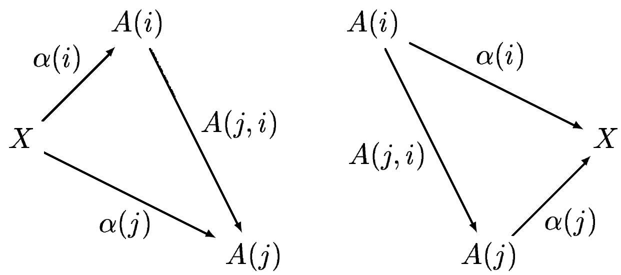
3.27 DEFINITION. For the diagram $\mathbb{D}$ a

left right

solution is an object X together with a selected arrow

$$X \xrightarrow{\alpha(i)} A(i) \qquad A(i) \xrightarrow{\alpha(i)} X$$

for each index $i \in \mathbb{I}$ where the triangle



commutes for each pair $i < j$ in $\mathbb{I}$ with a selected arrow. In other words

$$A(j,i) \circ \alpha(i) = \alpha(j) \qquad \alpha(j) \circ A(j,i) = \alpha(i)$$

for all such pairs $i < j$. ■

A solution to the problem posed by the diagram is trying to make the diagram commute as far as possible from one side or the other.

3.28 DEFINITION. A solution

$$S \xrightarrow{\sigma(i)} A(i) \qquad A(i) \xrightarrow{\sigma(i)} S$$

on the

left right

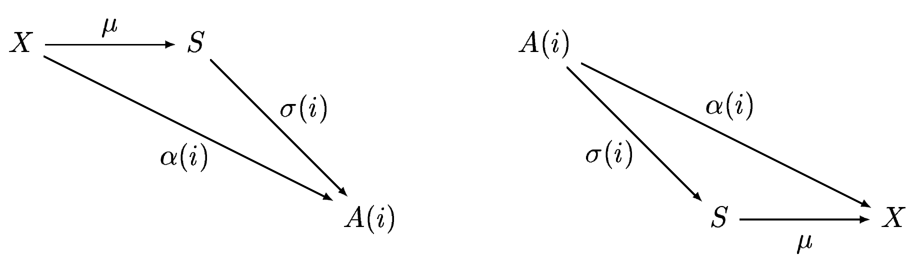
for the diagram $\mathbb{D}$ is universal (on that side) if for each solution

$$X \xrightarrow{\alpha(i)} A(i) \qquad A(i) \xrightarrow{\alpha(i)} X$$

there is a *unique* arrow

$$X \xrightarrow{\mu} S \qquad S \xrightarrow{\mu} X$$

such that the triangle



commutes for each index $i \in \mathbb{I}$. In other words

$$\sigma(i) \circ \mu = \alpha(i) \qquad \alpha(i) = \mu \circ \sigma(i)$$

for each $i \in \mathbb{I}$. This μ is the mediating arrow for that solution. ■

These limits and colimits go by various names. Projective limit, direct limit, and inverse limit are notions used in several particular disciplines, often with the restriction that the indexing poset is directed in one sense or the other. You should be aware that sometimes a gadget which is called a limit is technically a colimit. This terminology originated before the categorical notions were sorted out.

The construction of a limit (that is left limit) is worth looking at even in the simplest case where the ambient category $\boldsymbol{C}$ is $\boldsymbol{Set}$.

The diagram gives us an indexed family

$$\mathcal{A} = (A(i) \mid i \in \mathbb{I})$$

of sets. Recall that a choice function for this family is a function

$$a : \mathbb{I} \longrightarrow \bigcup \mathcal{A}$$

such that

$$a(i) \in A(i)$$

for each index $i \in \mathbb{I}$. Of course, for such a choice function to exists we need each $A(i)$ to be non-empty. We look at special choice functions.

A thread for the diagram $\mathbb{D}$ is a choice function a (as above) such that

$$A(j,i)a(i) = a(j)$$

for each pair $i < j$ in $\mathbb{I}$ with a selected arrow. In other words a thread is a choice function which passes through the diagram in a respectful manner. Of course, there may be no such threads.

Let S be the set of all such threads, and for each index i let

$$\begin{array}{rcl} S & \xrightarrow{\sigma(i)} & A(i) \\ a & \longmapsto & a(i) \end{array}$$

be the evaluation-at-i function, as indicated. Thus

$$\sigma(i)a = a(i)$$

for each index i and thread a.

An almost trivial calculation shows that the family of all these evaluation functions

$$S \xrightarrow{\sigma(i)} A(i)$$

is a solution to the problem. We show that it is a universal solution.

To this end suppose that

$$X \xrightarrow{\alpha(i)} A(i)$$

is any solution. For each $x \in X$ let the function

$$x(\cdot) : \mathbb{I} \longrightarrow \bigcup \mathcal{A}$$

be given by

$$x(i) = \alpha(i)x$$

for each $i \in \mathbb{I}$. By construction, this $x(\cdot)$ is a choice function, and a simple calculation shows that it is a thread. Thus we have a function

$$\begin{array}{ccc} X & \xrightarrow{\mu} & S \\ x & \longmapsto & x(\cdot) \end{array}$$

and, by another simple calculation, this satisfies

$$\sigma(i) \circ \mu = \alpha(i)$$

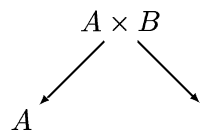
for each $i \in \mathbb{I}$. Finally, a third simple calculation shows that $x \longmapsto x(\cdot)$ is the only function satisfying these equalities.

Exercises

3.18 Consider a diagram (in the sense of this section) of groups and group morphisms or of monoids and monoid morphisms (whicheverr you prefer). Show that the construction given for ***Set*** carries over to produce the limit in ***Grp*** or ***Mon***.

Chapter 4
Cartesian closed categories

We have seen, in Chapter 3 that many categories have gadgets which capture the more familar idea of 'cartesian product' of structures. Thus, for two objects A, B of a category, a product (with selected projections) is an object, usually written $A \times B$, together with a pair of arrows



with a certain universal property, that is this wedge is the universal solution for all such wedges. Remember also that sometimes a wedge of this kind is called a cone. The two arrows are the projections selected for the product. In the more common cases where objects are structured sets, such products can be implemented using the cartesian product of the two carriers.

Accordingly we say a category is cartesian if each finite family of objects has a product, or equivalently if it has a terminal object and each pair of objects has a product. (This terminolgy is almost standard but can mean something slightly different, namely that the category has products *and* certain other limits.) We know that in a cartesian category the product construction gives two endofunctors and the projections are natural.

In this chapter we will work in an arbitrary cartesian category and ask when it has more facilities of a certain kind. There are two ways of motivating these extra properties.

For each pair of objects K, T of a category we have a set $[K, T]$ of arrows. In general, this is nothing more than a set. However, sometimes it can be viewed as an object of the category. In the more concrete cases the set can be furnished to become an object. We want to investigate the general consequences of such a situation.

The second motivation may seem a little odd, as a motivation, but in time you will see that it has a deeper significance.

Consider some logical system set up in judgemental form, such as the propositional calculus or the λ-calculus. This system manipulates judgements

$$\Gamma \vdash \phi$$

(where there may be extra information around). We know there is a strong resemblance between conjunction in such a system and products in a category. Let us write $\times$ for the conjunction operation (rather than the more common $\wedge$). Now consider judgements

$$\Gamma, \theta \times \psi \vdash \phi \qquad \Gamma, \theta \vdash (\psi \longrightarrow \phi)$$

where the arrow on the right indicates some kind of implication. In most natural systems [joke intended] these two judgements are inter-derivable, that is if we have one then we can get the other. (The passage from left to right is a version of the Deduction Theorem.)

Now remember the analogy between conjunction and product. Is there a categorical anologue of implication for which a version of the above equivalence holds? There is, and this takes place in a cartesian closed category.

4.1 Cartesian closedness

In this section we set up the definition of a cartesian closed category and look at one or two generalities. In later sections we look at some particular examples in more detail.

Let $\boldsymbol{C}$ be some fixed cartesian category. Let K be a fixed object and consider the induced functor

$$\boldsymbol{C} \xrightarrow{F = - \times K} \boldsymbol{Set}$$

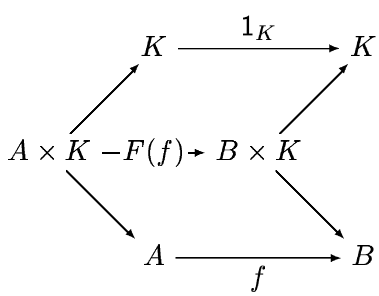
to $\boldsymbol{Set}$. Thus for each $\boldsymbol{C}$-object A we have

$$FA = A \times K$$

the selected product with the controlling object K, and for each $\boldsymbol{C}$-arrow

$$A \xrightarrow{f} B$$

the diagram



produces $F(f)$ as the unique mediating arrow. The unamed arrows are the selected projections. The common notation for the arrow $F(f)$ is

$$f \times K$$

(which is very common).

4.1 DEFINITION. An object K of a cartesian category $\boldsymbol{C}$ is exponentiable if the product functor $- \times K$ has a right adjoint. ■

This is the succinct definition, but we are going to expand on its meaning. However, there are two points to remember. In Section 2.7 (Subsection 2.7.3) we set down the basic properties of adjunctions but did not look at any of the details. This doesn't matter for even when the notion of an adjunction is understood, Definition 4.1 is not the best way to approach exponentiabilty.

We know that in an adjoint situation between functors there is lots of data and properties. Furthermore, various bits of these determine the rest. Remembering this we obtain the following (or rather, we would do if we filled in the missing details).

4.2 PROPOSITION. *An object K of a cartesian category $\boldsymbol{C}$ is exponentiable if (and only if) for each object A there is an object ${}^\bullet A$ and an arrow*

$$ {}^\bullet A \times K \xrightarrow{\ \epsilon_A\ } A $$

with the following universal property.

For each arrow

$$ B \times K \xrightarrow{\ g\ } A $$

there is a unique arrow

$$ B \xrightarrow{\ g_\flat\ } {}^\bullet A $$

such that the $\boldsymbol{C}$-triangle

$$ \begin{array}{ccc} B \times K & \xrightarrow{\ g\ } & A \\ & {\scriptstyle g_\flat \times K} \searrow \quad \nearrow {\scriptstyle \epsilon_B} & \\ & {}^\bullet A \times K & \end{array} $$

commutes.

Remember that on the left hand side of this triangle $g_\flat \times K$ is the standard, and not very helpful, notation for $g_\flat \times 1_K$.

We know that this selected family ϵ of arrows is natural for variation of A. Each component ϵ_A is called the evaluation arrow for A. Let's try to see why.

We know that, on general grounds, there is an inverse pair of bijections

$$ \begin{array}{ccc} f & \longmapsto & f^\sharp \\ \boldsymbol{C}[B, {}^\bullet A] & & \boldsymbol{C}[B \times K, A] \\ g_\flat & \longleftarrow\!\dashv & g \end{array} $$

between the indicated hom sets. Furthermore, these are natural and the transform $(\cdot)_\flat$ is the same as that used in the universal property given in Proposition 4.2. (This information is one way of characterizing an adjunction.) Now consider the case where $B = 1$, the final object of $\boldsymbol{C}$. Since $1 \times K \cong K$ we see that the two families

$$ \boldsymbol{C}[1, {}^\bullet A] \qquad \boldsymbol{C}[K, A] $$

are essentially the same. In other words the arrows

$$ K \longrightarrow A $$

are essentially the same as the global elements

$$ 1 \longrightarrow {}^\bullet A $$

of A. This suggests a better notation for ${}^\bullet A$. We write it as $(K \Rightarrow A)$.

Now go back to the general bijection. We have

$$\begin{array}{ccc} f & \longmapsto & f^\sharp \\ \boldsymbol{C}[B,(K\Rightarrow A)] & & \boldsymbol{C}[B\times K, A] \\ g_\flat & \longleftarrow\!\!\dashv & g \end{array}$$

which is nothing more than an abstract version of currying. (Remember that currying is the idea which underlies much of λ-calculus, and is the trick by which we can hide parameters in functions.)

Finally consider a concrete version of the commuting triangle. Think of a category where the objects are structured sets, so we may talk about elements of objects. Suppose the arrows are functions of some kind, and suppose that products in this category can be obtained via cartesian products of sets.

Given

$$B\times K \xrightarrow{\ g\ } A$$

and a pair $(b,x)\in B\times K$, how can we calculate $g(b,x)$? The curried version of g is the functions

$$g_\flat : B \longrightarrow (K\Rightarrow A)$$

where

$$g_\flat bx = g(b,x)$$

and so the value we want is the function $g_\flat b$ evaluated a x. We may write this as

$$g(b,x) = \mathrm{eval}(g_\flat b, x)$$

where

$$(K\Rightarrow A)\times K \xrightarrow{\ \mathrm{eval}\ } A$$

is the evaluation function. This is precisely what the selected arrow ϵ_A is doing.

We call $(K\Rightarrow A)$ the internal arrow object of K to A.

At this point there are several general abstract properties we could look at, but we won't. It is much more instructive to look at several examples. In the next three sections we look at four such examples. We know already that the categories ***Set*** of sets and ***Pos*** of posets are cartesian (with the obvious way of obtaining product objects). We will show that both of these are cartesian closed, and again for both of these there is an obvious way of obtaining internal arrow objects. These examples show that in the 'nice' cases things are what you expect them to be.

After that we will look at two more complicated examples. In these categories products are, more or less, what you expect them to be. However, the arrow objects are not at all obvious.

Exercises

4.1 Each poset is a category.

(a) When is this category cartesian?

(b) When is this category cartesian closed?

4.2 Some simple examples

In this section we will look at a simple examples of a cartesian closed category, and also indicate some cartesian categories which are not cartesian closed.

The simplest example, of course, is the category ***Set*** of sets. However, this is so simple it is hardly a useful illustration of the notion. At some time you should go through the relevant details, but to begin it is better to look at an example with slighly more content.

The category ***Pos*** is cartesian closed.

Consider the category ***Pos*** of posets. Thus an object $(A, \leq)$ is a set A furnished with a partial ordering. We will follow the usual convention and write 'A' for the object (that is, we will hide the furnishings). We also write '$\leq$' for any partial ordering that occurs, even when there is more than one around. Here this will never cause confusion. An arrow

$$B \xrightarrow{\ f\ } A$$

is a function f in the indicated direction which is monotone, that is

$$y \leq x \Longrightarrow fy \leq fx$$

holds for all $x, y \in B$.

We know that ***Pos*** is cartesian. For objects A, K the product object $A \times K$ is carried by the set of all ordered pairs

$$(a, x)$$

for $a \in A$ and $x \in K$. This is furnished with the pointwise comparison, that is

$$(b, y) \leq (a, x) \Longleftrightarrow b \leq a \text{ and } y \leq x$$

for all $a, b \in A$ and $x, y \in K$. The two projection arrrows

$$\begin{array}{ccc} & A \times K & \\ \swarrow & & \searrow \\ A & & K \end{array}$$

are given by

$$\begin{array}{rcl} A \times K & \longrightarrow & A \\ (a, x) & \longmapsto & a \end{array} \qquad \begin{array}{rcl} A \times K & \longrightarrow & K \\ (a, x) & \longmapsto & x \end{array}$$

respectively. Of course, we need to check that these projection functions are monotone, but the ordering of $A \times K$ is chosen precisely with this in mind.

On general grounds we know that for a fixed K this construction extends to give an endofunctor $- \times K$ of ***Pos***. In particular, each ***Pos***-arrow

$$A \xrightarrow{\ f\ } B$$

gives a a ***Pos***-arrow

$$\begin{array}{ccc} A \times K & \xrightarrow{f \times K} & B \times K \\ (a, x) & \longmapsto & (fa, x) \end{array}$$

with the indicated behaviour (for $a \in A, x \in K$).

To show that ***Pos*** is cartesian closed we must produce an internal arrow object $(K \Rightarrow A)$ together with suitable furnishings for each pair K, A of objects. For partially ordered sets this is easy.

4.3 DEFINITION. Let K, A be a pair of posets. The pointwise comparison on the set ***Pos***$[K, A]$ of monotone maps from K to A is given by

$$q \le p \iff (\forall x \in K)[qx \le px]$$

for $p, q \in$ ***Pos***$[K, A]$. ■

It is routine to check that this furnishes ***Pos***$[K, A]$ with a partial ordering, and so we can view this set as an object of ***Pos***.

4.4 DEFINITION. For each pair K, A of ***Pos***-objects, let $(K \Rightarrow A)$ be the set ***Pos***$[K, A]$ with the pointwise comparison. ■

Just producing this object isn't enough to show that ***Pos*** is cartesian closed, we need various other gadgets. We are looking for a right adjoint of the functor $(- \times K)$, and we know this can be done in various ways. In particular, once we have done this we know that $(K \Rightarrow -)$ will become an endofunctor of ***Pos***. However, here we illustrating the general notion, so it is instructive to verify this functorality directly.

4.5 LEMMA. *For each monotone map*

$$B \xrightarrow{f} A$$

between posets, the function

$$\begin{array}{ccc} (K \Rightarrow B) & \longrightarrow & (K \Rightarrow A) \\ p & \longmapsto & f \circ p \end{array}$$

is monotone.

This arrow assignment converts the object assigment into an endofunctor of ***Pos***.

Proof. Consider any comparison

$$q \le p$$

in $(K \Rightarrow B)$. We must show that

$$f \circ q \le f \circ p$$

holds in $(K \Rightarrow A)$, that is

$$f(qx) \le f(px)$$

holds A for each $x \in K$. But, for each $x \in K$ we have $qx \le px$ (since $q \le p$) and so the monotonicity of f gives the required result.

To show that we have an endofunctor $(K \Rightarrow -)$ consider a composable pair

$$C \xrightarrow{g} B \xrightarrow{f} A$$

of ***Pos***-arrows. We must show that the composite of the arrows

$$\begin{array}{ccc} (K \Rightarrow C) & \longrightarrow & (K \Rightarrow B) \\ q & \longmapsto & f \circ q \end{array} \qquad \begin{array}{ccc} (K \Rightarrow B) & \longrightarrow & (K \Rightarrow A) \\ p & \longmapsto & g \circ p \end{array}$$

is that induced by the composite $f \circ g$. But this is immediate since function composition is associative.

The required preservation of identity arrows is trivial. ■

For the most direct proof of the result we are after we don't use Lemma 4.5. We use some other associated gadets. In particular, we use an evaluation arrow.

4.6 LEMMA. *For each pair K, A of **Pos**-objects, the function*

$$\begin{array}{ccc} (K \Rightarrow A) \times K & \xrightarrow{\epsilon_A} & A \\ (p \quad , \quad x) & \longmapsto & px \end{array}$$

is monotone

Proof. Consider any comparison

$$(q, y) \le (p, x)$$

which holds in $(K \Rightarrow A) \times K$. We must show that

$$qy \le px$$

holds in A. From the given comparison we have

$$q \le p \qquad y \le x$$

where

$$(\forall z \in K)[qz \le pz]$$

is the unravelled meaning of the first of these. Using these together with the given monotonicity of either p or q we have one of

$$qy \le py \le px \qquad qy \le qx \le px$$

to give the required result. ■

As indicated by the notation, we use this arrow ϵ_A as the evaluation arrow for A. The crucial result is the factorization property of Proposition 4.2.

4.7 THEOREM. *For each **Pos**-arrow*

$$B \times K \xrightarrow{g} A$$

there is precisely one ***Pos****-arrow*

$$B \xrightarrow{\ h\ } (K \Rightarrow A)$$

such that the ***Pos****-triangle*

$$\begin{array}{ccc} B \times K & \xrightarrow{\quad g \quad} & A \\ & {\scriptstyle h \times K} \searrow \quad \nearrow {\scriptstyle \epsilon_A} & \\ & (K \Rightarrow A) \times K & \end{array}$$

commutes.

Proof. We do the two parts, existence and uniqueness, separately.

Since the given function g is monotone, for each $b_2 \le b_1$ in B and $y \le x$ in K we have

$$g(b_2, y) \le g(b_1, x)$$

in B. In particular

$$g(b, y) \le g(b, x)$$

for each $b \in B$ and $y \le x$ from K. This shows that for each $b \in B$ the function

$$g(b, \cdot) : K \longrightarrow A$$

is monotone, and hence is a ***Pos***-arrow. Thus we have an assignment

$$\begin{array}{rcl} B & \xrightarrow{\quad g_\flat \quad} & (K \Rightarrow A) \\ b & \longmapsto & g(b, \cdot) \end{array}$$

that is with

$$g_\flat b x = g(b, x)$$

for each $b \in B, x \in K$. We check that $g_\flat$ is monotone, and hence is a ***Pos***-arrow.

Consider a comparison $b_2 \le b_1$ in B. We require

$$g_\flat b_2 \le g_\flat b_1$$

in $(K \Rightarrow A)$, that is

$$g_\flat b_2 x \le g_\flat b_1 x$$

for each $x \in K$. But this is

$$g(b_2, x) \le g(b_1, x)$$

which is part of the given property of g.

To complete the proof of existence we must show that the composite of the two arrows

$$\begin{array}{rcl} B \times K & \xrightarrow{\ g_\flat \times K\ } & (K \Rightarrow A) \times K \\ (b, x) & \longmapsto & (g_\flat b, x) \end{array} \qquad \begin{array}{rcl} (K \Rightarrow A) \times K & \xrightarrow{\quad \epsilon_A \quad} & A \\ (p, x) & \longmapsto & px \end{array}$$

is just the given arrow g. But this composite is

$$(b, x) \longmapsto g_\flat bx = g(b, x)$$

to give the required result.

For the uniqueness consider any arrow h which makes the indicated triangle commute. Then for each $b \in B, x \in K$ we see that

$$hbx = \epsilon_A(ha, x) = (\epsilon_A \circ (h \times K))(b, x) = g(b, x) = g_\flat bx$$

holds. Since both x and b are arbitrary, this gives $h = g_\flat$, as required. ■

This result ensures that ***Pos*** is cartesian closed. In a more economical account this is more or less the only result that needs to be proved. However, it is always useful to verify or at least describe the details of the various associated results. In particuilar, we know there must be a bijective correspondence

$$\begin{array}{ccc} f & \longmapsto & f^\sharp \\ \boldsymbol{Pos}[B, (K \Rightarrow A)] & & \boldsymbol{Pos}[B \times K, A] \\ g_\flat & \longleftarrow & g \end{array}$$

which is natural for variations of A and B. We have seen above that the lower assignment $g \longmapsto g_\flat$ is just the currying of g. In a similar fashion we find that

$$f^\sharp(b, x) = fbx$$

for $b \in A, x \in K$. In other words this is the uncurrrying of f. Trivially, these two assigments form a bijective correspondence.

Other examples

Simple and instructive examples of cartesian closed categories are rather thin on the ground.

There are several examples which are refinements of ***Pos***. Thus we may give the objects extra properties such as having certain completeness properties, or we can give the objects extra structure such as certain infima. Not all such refinements are cartesian closed, but when one is the proof is straight forward, and we use the obvious pointwise structure.

Consider the category ***Abg*** of abelian groups. This is certainly cartesian, since the cartesian product of two such groups is an abelian group. Furthermore, the set of morphisms from one such group to another is itself an abelian group under the pointwise operation. However, this does not give an internal arrow object. In fact, ***Abg*** is not cartesian closed.

The category ***Top*** of topological spaces and continuous maps is cartesian but not cartesian closed. Imposing a suitable topology on the set of continuous maps from one space to another is one of the perenial problems of topology. It turns out that for a space to be exponentiable it must satisfy rather severe conditions.

Exercises

4.2 Verify directly (by calculation) that for ***Pos*** the evaluation arrows ϵ are natural.

4.3 By observing that the final object of ***Abg*** is also initial, show that ***Abg*** is not cartesian closed.

4.3 Monoid actions

In this section we look at a family of 'non-trivial' examples of cartesian closed categories. In fact, each of these categories has much stronger properties. Each is a topos. (We won't even attempt to explain what that word means, except to say it is an important notion that connects category theory with both higher order logic and algebraic geometry). We will look only at the cartesian closedness. We will go through a typical example quite slowly. This means there will be quite a lot to check. Much of this is routine (and some of it you have seen before). We will leave some of the checking as exercises, and use ♣ to indicate where these occur. Once you have done all these you can join the club. (Some of these jokes do take the biscuit.)

If at first you find some of these details a bit complicated then you might try Exercise 4.5 or 4.6.

Let R be a monoid. We use the category of right R-sets as explained in subsection 1.5.5. Thus each object is a set A with an action

$$\begin{array}{rcl} A, R & \longrightarrow & A \\ a, r & \longmapsto & ar \end{array}$$

satisfying

$$a(rs) = (ar)s \qquad a1 = a$$

for each $a \in A$ and $r, s \in R$. An arrow

$$A \xrightarrow{\ f\ } B$$

between two R-sets is a function $f : A \longrightarrow B$ satisfying

$$f(ar) = (fa)r$$

for each $a \in A, r \in R$. We sometimes call this an R-linear map. It is routine to check that these form a category $\widehat{R}$. ♣ (There is also a category of left R-sets, but we don't use that here.)

Before we begin to analyse the properties of $\widehat{R}$ it is instructive to obtain a different description of the category which puts it in a wider context. We show that it is a presheaf category.

The monoid R is itself a category. It has exactly one object, which we may write as $*$ for the time being. The arrows are just the elements of R, and these must have the form

$$* \longrightarrow *$$

(since there is just the one object). Composition is given by the carried operation of R. Thus $r \circ s = rs$, that is the triangle

$$\begin{array}{ccccc} * & & \xrightarrow{\quad rs \quad} & & * \\ & \searrow^{s} & & \nearrow_{r} & \\ & & * & & \end{array}$$

commutes. The neutral element of R is the identity arrow.

A presheaf on R is a contravariant functor from R (as a category) to $\mathbf{Set}$, the category of sets. Such a functor attaches a set A to the sole object and a function

$$A \xrightarrow{A(r)} A$$

to each arrow (element) of R. This assignment is contravariant which means that the commuting diagram above induces a commuting diagram

$$\begin{array}{ccc} A & \xleftarrow{A(rs)} & A \\ {\scriptstyle A(s)}\nwarrow & & \swarrow{\scriptstyle A(r)} \\ & A & \end{array}$$

with a reversal of the direction of the arrows. In other words we have

$$A(s) \circ A(r) = A(rs)$$

for all $r, s \in A$. At this point we use a notational trick, and write

$$\begin{array}{ccc} A & \xrightarrow{A(r)} & A \\ a & \longmapsto & ar \end{array}$$

for the behaviour of the function $A(r)$. Using this we find that a presheaf over R is nothing more than an R-set. Furthermore, the morphisms between R-sets are precisely the natural transformations between the corresponding presheaves. ♣

This description makes everything we are going to do a particular instance of a much more general construction. However, this does not mean the particular case is uninteresting.

The monoid R is itself an R-set, with the obvious action. It can be checked that, for an arbitrary R-set A, the arrows

$$R \longrightarrow A$$

are in bijective correspondence with the elements of the set A. ♣ (This object R of $\widehat{R}$ is an example of a separator or a generator of the category.)

The singleton set 1 (with just one element) is an R set with the only possible action it can have. This is the terminal object of $\widehat{R}$. For an arbitrary R-set A the arrows

$$1 \longrightarrow A$$

are in bijective correspondence with certain elements of of A. Perhaps you can sort out which ones. ♣ These are called the global elements of A. (This terminology comes from category theory, not from the study of R-sets.)

We know that $\widehat{R}$ is cartesian, with 'obvious' products. Thus given two R-sets A and B we take the cartesian product $A \times B$ of the two sets and impose the pointwise action

$$\begin{array}{ccc} A \times B, R & \longrightarrow & A \times B \\ (a, b), r & \longmapsto & (ar, br) \end{array}$$

to obtain an R-set. It is easy to check that this, together with the obvious projections, implements the categorical product in $\widehat{R}$. ♣

To show that $\widehat{R}$ is cartesian closed we do four things.

- We construct an R-set $(K \Rightarrow A)$ for each pair of R-sets K, A.
- We show that the construction $(K \Rightarrow -)$ is an endofunctor of $\widehat{R}$. (The properties for variation of K is not important.)
- We set up an inverse pair

$$\begin{array}{ccc} f & \longmapsto & f^\sharp \\ \widehat{R}[B, (K \Rightarrow A)] & & \widehat{R}[B \times K, A] \\ g_\flat & \longleftarrow\!\dashv & g \end{array}$$

  of bijections.
- We verify the naturality of these bijections for variation of A and B.

Of course, this is not the only approach, but it is the most convenient one here.

The construction of $(K \Rightarrow A)$ is not quite what you first expect.

As a set let

$$(K \Rightarrow A) = \widehat{R}[R \times K, A]$$

that is the set of all functions

$$\phi : R \times K \longrightarrow A$$

which satisfy

$$\phi(s, x)r = \phi(sr, xr)$$

for all $r, s \in R$ and $x \in K$. (Notice that we do *not* take the obvious carrying set $\widehat{R}[K, A]$.)

We impose an action

$$\begin{array}{ccc} (K \Rightarrow A), R & \longrightarrow & (K \Rightarrow A) \\ \phi, r & \longmapsto & \phi^r \end{array}$$

by

$$(\phi^r)(s, x) = \phi(rs, x)$$

for $\phi \in (K \Rightarrow A)$, and $r, s \in R, x \in K$. Of course we need to check that

$$\phi(rs) = (\phi^r)s \qquad \phi^1 = \phi$$

(for $r, s \in R$) but this is straight forward. (We have written the action as

$$\phi, r \longmapsto \phi^r$$

rather than ϕr to avoid confusion with 'evaluation of ϕ at r'. In this setting that evaluation doesn't make sense.) ♣

This constructs $(K \Rightarrow A)$ as an R-set. However, we want $(K \Rightarrow -)$ to be a functor. Surely this is easy for we have a hom-functor! This observation is not quite good enough. A hom-functor passes from $\widehat{R}$ to ***Set***, whereas we need an endofunctor of $\widehat{R}$. We need to show that the induced action (in a different sense) on an R-linear map produces an R-linear map. The rest of the required properties *do* follow since we have a hom-functor.

Thus let

$$A \xrightarrow{\ k\ } A'$$

be an R-linear map and consider the induced behaviour

$$\begin{array}{rcl} (K \Rightarrow A) & \longrightarrow & (K \Rightarrow A') \\ \phi & \longmapsto & k \circ \phi \end{array}$$

as a function. We need to check that this is R-linear. For this verification let us write $K(l)$ for the induced function. (We are thinking of $(K \Rightarrow -)$ as a functor K.) We must show that

$$K(k)(\phi r) = (K(k)\phi)r$$

for each $\phi \in (K \Rightarrow A)$ and $r \in R$. Remembering how $K(k)$ is defined we see that we require

$$k \circ \phi^r = (k \circ \phi)^r$$

for each $\phi \in (K \Rightarrow A)$ and $r \in R$. This may be checked by evaluating at an arbitrary $(s, x) \in R \times K$. As you do this you should note how the properties of ϕ are used. $\clubsuit$.

This gives us the functor. We now set up the inverse pair of bijections.

Consider next an R-linear map

$$B \xrightarrow{\ f\ } (K \Rightarrow A)$$

from an arbitrary R-set B. By a simple calculation we see that the assignment

$$\begin{array}{rcl} B \times K & \xrightarrow{\ f^\sharp\ } & A \\ (b, x) & \longmapsto & (fb)(1, x) \end{array}$$

is R-linear. $\clubsuit$ To go the other way consider a R-linear map

$$B \times K \xrightarrow{\ g\ } A$$

again for an arbitrary R-set B. The assignment

$$B \xrightarrow{\ g_\flat\ } (K \Rightarrow A)$$

given by

$$(g_\flat b)(r, x) = g(br, x)$$

for $b \in B, r \in R, x \in K$ is R-linear. $\clubsuit$ Furthermore the two assignments $(\cdot)^\sharp$ and $(\cdot)_\flat$ are an inverse pair. $\clubsuit$

There are several claims here and all require some justification, but none of these is difficult.

This doesn't quite show that $\widehat{R}$ is cartesian closed. We still need to show that the two assignments $(\cdot)^\sharp$ and $(\cdot)_\flat$ are natural.

To this end consider a pair

$$B' \xrightarrow{\ l\ } B \qquad\qquad A \xrightarrow{\ k\ } A'$$

of R-linear maps, as indicated. These induce a diagram

$$\begin{array}{ccccc} f & \widehat{R}[B, (K \Rightarrow A)] & \underset{(\cdot)_\flat}{\overset{(\cdot)^\sharp}{\rightleftarrows}} & \widehat{R}[B \times K, A] & g \\ \big\downarrow & \big\downarrow & & \big\downarrow & \big\downarrow \\ k \circ f \circ l & \widehat{R}[B', (K \Rightarrow A')] & \underset{(\cdot)_\flat}{\overset{(\cdot)^\sharp}{\rightleftarrows}} & \widehat{R}[B' \times K, A'] & k \circ g \circ (l \times id) \end{array}$$

where the two hom-actions are indicated. We must show that two separate squares commute. One square goes from top left to bottom right and uses the two $(\cdot)^\sharp$ as horizontal arrows. The other square goes from top right to bottom left and uses the two $(\cdot)_\flat$ as horizontal arrows. Thus we require

$$(k \circ f \circ l)^\sharp = k \circ f^\sharp \circ (l \times id) \qquad k \circ g_\flat \circ l = (k \circ g \circ (l \times id))_\flat$$

where f and g are R-linear maps of the indicated type. (Actually one of these will suffices, since it implies the other, and you shoud find out why. ♣)

To verify the left hand equality we evaluate both composites at an arbitrary pair $(b', x) \in B' \times K$, use the definition of $(\cdot)^\sharp$, and some of the properties of the given k, f, l. ♣. To verify the right hand equality we consider arbitrary $b' \in B', r \in R, x \in K$ and evaluate both sides at b' and then evaluate both resulting functions at (r, x). ♣

This is enough to show that $\widehat{R}$ is cartesian closed.

Exercises

4.4 Fill in all the missing details indicated by ♣.

Describe the evaluation map ϵ and verify (by calculation) the required factorization property.

> Although it is not immediately obvious, the following two exercises are particular cases of $\widehat{R}$ for two different but quite simple monoids R. You can check the details directly.

4.5 An involution algebra is a structure

$$(A, (\cdot)^*)$$

where A is a set and $(\cdot)^*)$ is an involution on A, that is an assignment

$$\begin{array}{ccc} A & \longrightarrow & A \\ a & \longmapsto & a^* \end{array}$$

such that

$$a^{**} = a$$

for each $a \in A$.

In the usual way we identify an algebra with its carrier.

A morphism

$$(B, (\cdot)^*) \xrightarrow{f} (A, (\cdot)^*)$$

of such algebras is a function $f : B \longrightarrow A$ such that

$$(fb)^* = f(b^*)$$

for each $b \in B$.

Given a pair K, A of algebras let $(K \Rightarrow A)$ be the set of all *functions* $\phi : K \longrightarrow A$, not just morphisms.

(a) Show that idempotent algebras and morphisms form a category.

(b) Show that this category is cartesian (using products obtained pointwise).
(c) Show that for a pair K, A of algebras the assignment

$$\begin{array}{ccc} (K \Rightarrow A) & \longrightarrow & (K \Rightarrow A) \\ \phi & \longmapsto & \phi^* \end{array}$$

given by

$$\phi^*(x) = \phi(x^*)^*$$

(for $x \in K$) converts $(K \Rightarrow A)$ into an algebra.
(d) Show that for each morphism

$$A \xrightarrow{\quad k \quad} A$$

the assignment

$$\begin{array}{ccc} (K \Rightarrow A) & \longrightarrow & (K \Rightarrow A') \\ \phi & \longmapsto & k \circ \phi \end{array}$$

is a morphism.
(e) Show that each morphism

$$B \xrightarrow{\ f\ } (K \Rightarrow A)$$

induces a morphism

$$B \times K \xrightarrow{\ f^\sharp\ } A$$

given by

$$f^\sharp(b, \cdot) = fb$$

for each $b \in B$.
(f) Show that each morphism

$$B \times K \xrightarrow{\ g\ } A$$

induces a morphism

$$B \xrightarrow{\ g_\flat\ } (K \Rightarrow A)$$

given by

$$g_\flat b = g(b, \cdot)$$

for each $b \in B$.
(g) Show the category is cartesian closed.
(h) Show the category is $\widehat{R}$ for a particular monoid R.

4.6 An idempotent algebra is a structure

$$(A, (\cdot)^\bullet)$$

where A is a set and $(\cdot)^\bullet)$ is an idempotent operation on A, that is an assignment

$$\begin{array}{ccc} A & \longrightarrow & A \\ a & \longmapsto & a^\bullet \end{array}$$

such that

$$a^{\bullet\bullet} = a^\bullet$$

for each $a \in A$.

In the usual way we identify an algebra with its carrier.

A morphism

$$(B, (\cdot)^\bullet) \xrightarrow{f} (A, (\cdot)^\bullet)$$

of such algebras is a function $f : B \longrightarrow A$ such that

$$(fb)^\bullet = f(b^\bullet)$$

for each $b \in B$.

Given a pair K, A of algebras let $(K \Rightarrow A)$ be the set of all pairs (ϕ_0, ϕ_1) of functions $K \longrightarrow A$ where ϕ_0 is a morphism and

$$(\phi_0 x)^\bullet = (\phi_1 x)^\bullet$$

for all $x \in K$.

(a) Show that idempotent algebras and morphisms form a category.
(b) Show that this category is cartesian (using products obtained pointwise).
(c) Show that for a pair K, A of algebras setting

$$(\phi_0, \phi_1)^\bullet = (\phi_0, \phi_0)$$

converts $(K \Rightarrow A)$ into an algebra.

(d) Show that for each morphism

$$A \xrightarrow{\quad k \quad} A'$$

the assignment

$$\begin{array}{ccc} (K \Rightarrow A) & \longrightarrow & (K \Rightarrow A') \\ (\phi_0, \phi_1) & \longmapsto & (k \circ \phi_0, k \circ \phi_1) \end{array}$$

is a morphism.

(e) Show that each morphism

$$B \xrightarrow{\quad f \quad} (K \Rightarrow A)$$

induces a morphism

$$B \times K \xrightarrow{\quad f^\sharp \quad} A$$

given by

$$f^\sharp(b, \cdot) = (fb)_1$$

for each $b \in B$.

(f) Show that each morphism

$$B \times K \xrightarrow{\quad g \quad} A$$

induces a morphism

$$B \xrightarrow{\quad g_\flat \quad} (K \Rightarrow A)$$

given by

$$(g_\flat b)_0 = g(b^\bullet, \cdot) \qquad (g_\flat b)_1 = g(b, \cdot)$$

for each $b \in B$.

(g) Show the category is cartesian closed.

(h) Show the category is $\widehat{R}$ for a particular monoid R.

4.4 Developing sets

In this section we will describe, in some detail, the most complicated example of a cartesian closed category that we will look at. In fact, we will describe a typical member of a whole family of examples. As with R-sets (described in section 4.3) each category has several other properties, and is a topos. However, we will not go into these extra details.

The inner workings of the example are rather complicated, as we will see when we start to verify its properties. This illustrates one of the main attributes of category theory. For many purposes the inner details of objects and arrows are a distraction. By making use of categorical notions and gadgets these can be hidden. Of course, there will be times when these innards have to be uncovered, but that should not be done routinely.

We will go through a typical example in this family quite slowly, but we will not set down all the routine details. Many of these will be left as an exercise, and we use ♣ to indicate the occurrence of one of these.

Let S be an arbitrary poset with $\leq$ as the carried comparison. We first construct a category $\widehat{S}$ and then analyse some of its properties. In fact, we have seen $\widehat{S}$ before, but let's approach it from a different direction.

The poset S is a rather simple category. The objects are the elements $r, s, t, \ldots$ of S. Give a pair of objects $r, s \in S$ there is at most one arrow

$$s \longrightarrow r$$

and there is such an arrow precisely when $s \leq r$. Thus the arrows of the category are merely a way of indicating which comparisons hold. Since the comparison is reflexive and transitive it is routine to check that we do have a category. ♣ (We do not need the antisymmetry of the comparison, and everything we do here also works with a pre-ordered set. However, that extra generality illustrates nothing useful.)

We think of S as a category. A presheaf on S is a contravariant functor from S to ***Set***, the category of sets. We need the details of these gadgets.

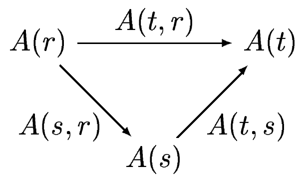
Thus a presheaf A is an S-indexed family of sets

$$(A(s) \mid s \in S)$$

together with a family of functions indexed by the arrows of S. Thus for each pair $t \leq s$ from S there is a function

$$A(s) \xrightarrow{A(t,s)} A(t)$$

where the contravariance causes a reversal of direction. These two families must form a functor, so the functions must fit together in a certain way. For each $r \in A$ the function $A(r,r)$ must be the identity function on $A(r)$. More importantly, for each triple $t \leq s \leq r$ in S, the triangle



must commute, that is

$$A(t,s) \circ A(s,r) = A(t,r)$$

holds.

In other words, a presheaf over S is nothing more than a developing set in the sense of subsection 1.5.6. These are the objects of $\widehat{S}$. The arrows are just the natural transformations between the functors. We will deal with them shortly after we have condensed the notation.

I suggest you now read the previous but one paragraph again and observe how the gadgets have been indexed. The seemingly perverse notation has been chosen to make various manipulations a bit smoother. This is part of a trick for handling contravariance, and we are going to take it a bit further.

Consider the function $A(t,s)$ for a pair $t \leq s$ from S. The function sends each element $a \in A(s)$ to some element $A(t,s)a \in A(t)$. This notation will become a bit cumbersome, so we write

$$\begin{array}{ccc} A(s) & \longrightarrow & A(t) \\ a & \longmapsto & a|t \end{array}$$

for the behaviour of $A(t,s)$. We think of '$a|t$' as a generalized restriction of $a \in A(s)$ to $A(t)$ (via the connecting function). In particular, for each $t \leq s \leq r$ and $a \in A(r)$ we have

$$a|r = a \qquad (a|s)|t = a|t$$

by the functorial properties. You should check this. ♣

Of course, this condensed notation has to be used with some care, especially when there is more than one presheaf around (which there soon will be). But in the end it does make some calculations less cluttered. (An even neater notation is to write 'ar' for '$a|r$' and think of this behaviour as an action. You should try that sometime.)

The objects of $\widehat{S}$ are just the presheaves over S. The arrows of $\widehat{S}$ are the natural transformation between these presheaves. Thus given two objects A, B an arrow

$$B \xrightarrow{f} A$$

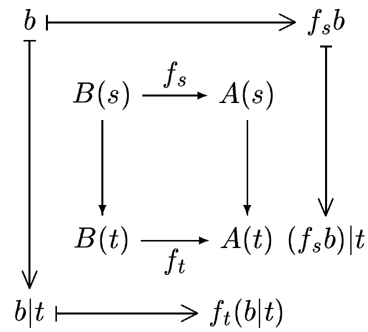
is an S-indexed family of functions

$$B(s) \xrightarrow{f_s} A(s)$$

where for each pair $t \leq s$ the square

$$\begin{array}{ccc} B(s) & \xrightarrow{f_s} & A(s) \\ {\scriptstyle B(t,s)}\downarrow & & \downarrow{\scriptstyle A(t,s)} \\ B(t) & \xrightarrow[f_t]{} & A(t) \end{array}$$

commutes. We need to express this in terms of the condensed notation. We do this by tracking an arbitrary $b \in B(s)$ around the two paths to $A(t)$. Doing this we get



from which we see that

$$f_t(b|t) = (f_s b)|t$$

is the required condition (for all $t \leq s$ and $b \in B(s)$). ♣ This equality must be read with some care for there are two different restrictions here, one for A and one for B.

This sets up the basic mechanics of $\widehat{S}$. We are going to first describe the internal product objects (which will be routine) and then the internal arrow objects (which look a bit weird at first).

The product of two objects A and B is obtained pointwise using cartesian products. Thus we set

$$(A \times B)(s) = A(s) \times B(s)$$

for each $s \in S$. Here the '$\times$' of the left is the defined construction and the one on the right is taken from the category ***Set***. Thus $(A \times B)(s)$ is the set of all pairs (a, b) where $a \in A(s)$ ad $b \in B(s)$. For $t \leq s$ two restrictions in parallel

$$\begin{array}{rcl} A(s) \times B(s) & \longrightarrow & A(t) \times B(t) \\ (a, b) & \longmapsto & (a|t, b|t) \end{array}$$

gives the required connecting functions. Thus

$$(a, b)|t = (a|t, b|t)$$

gives the interaction between these three restrictions. It is routine to check that this furnishes $\widehat{S}$ with products. ♣

Our main job is to set up the internal arrow objects and verify the required properties. We will do this slowly, so this may take some time.

Let K and A be an arbitrary pair of objects.

For each $s \in S$ let $(K \Rightarrow A)(s)$ be the family

$$\alpha = (\alpha_i \mid i \leq s)$$

of functions

$$K(i) \xrightarrow{\alpha_i} B(i)$$

indexed by the principal lower section $\downarrow s$ of S, and which is natural 'as far as it goes', that is for each $j \le i \le s$ the square

$$\begin{array}{ccc} K(i) & \xrightarrow{\alpha_i} & A(i) \\ \downarrow & & \downarrow \\ K(j) & \xrightarrow[\alpha_j]{} & A(j) \end{array} \qquad \alpha_j(x|j) = (\alpha_i x)|j$$

commutes. By tracking $x \in K(i)$ around the square we see that the equality holds.

For $t \le s$ we require a connection function

$$\begin{array}{ccc} (K \Rightarrow A)(s) & \longrightarrow & (K \Rightarrow A)(t) \\ \alpha & \longmapsto & \alpha|t \end{array}$$

which we can write as a restriction. But in this case it *is* a restriction. We take

$$\alpha = (\alpha_i \mid i \le s) \longmapsto \alpha|t = (\alpha_i \mid i \le t)$$

by merely restricting to those $i \le t \le s$.

Notice that there is two kinds of naturality in this construction. There is the naturality required by the connection maps of $(K \Rightarrow -)$, that is for moving between 'outer' indexes $t \le s$. There is also the inner naturality required of each family $\alpha \in (K \Rightarrow A)(s)$ for moving between 'inner' indexes $j \le i \le s$. The use of different symbols will help to keep these apart.

Remembering this it is straight forward to check that this produces $(K \Rightarrow A)$ as a presheaf over S. ♣

Why does this construction ensure that $\widehat{S}$ is cartesian closed? We could show that the construction $(K \Rightarrow -)$ is functorial and set up an inverse pair

$$\begin{array}{ccc} f & \longmapsto & f^\sharp \\ \widehat{S}[B, (K \Rightarrow A)] & & \widehat{S}[B \times K, A] \\ g_\flat & \longleftarrow\!\dashv & g \end{array}$$

of bijections with the required naturality; but we won't. We will take the co-free approach. Thus we will set up an 'evaluation' arrow

$$(K \Rightarrow A) \times K \xrightarrow{\epsilon_A} A$$

with the appropriate factorization property. The beauty of this approach is that we don't need to verify any functorality or naturality. These properties are consequences of the universal factorization.

Since we are thinking of K as fixed throughout, the evaluation arrow depends on B, as indicated. However, the subscript A will become a distraction, so we will drop this and write

$$(K \Rightarrow A) \times K \xrightarrow{\epsilon} A$$

for the arrow. This, of course, is a family of functions

$$(K \Rightarrow A)(s) \times K(s) \xrightarrow{\epsilon_s} A(s)$$

indexed by S. (You can see how an extra subscript could get under the feet.)

The function ϵ_s requires an input (α, x) where $\alpha \in (K \Rightarrow A)(s)$ and $x \in K(s)$. This α is an indexed family of functions which has a principal component

$$K(s) \xrightarrow{\alpha_s} A(s)$$

so we may take

$$\begin{array}{ccc} (K \Rightarrow A)(s) \times K(s) & \xrightarrow{\epsilon_s} & A(s) \\ (\alpha \quad , \quad x) & \longmapsto & \alpha_s x \end{array}$$

as the behaviour of ϵ_s. Of course, we need to check that ϵ is natural for variation of s.

There is a point here that is worth expanding on. The arrow $\epsilon = \epsilon_A$ must be a natural transformation between two functors. This naturality is for variation of the index $s \in S$, and it is this we must check. However, the whole construction must be natural for variation of A, and it is this naturality which will be an automatic consequence of the other properties.

To show the naturality of ϵ for variation of the index consider two indexes $t \leq s$ in S. We must show that the inner square commutes.

$$\begin{array}{ccccc} (\alpha, x) & & \longmapsto & & \alpha_s x \\ \Big\downarrow & (K \Rightarrow A)(s) \times K(s) & \xrightarrow{\epsilon_s} & A(s) & \Big\downarrow \\ & \Big\downarrow & & \Big\downarrow & \\ & (K \Rightarrow A)(t) \times K(t) & \xrightarrow[\epsilon_t]{} & A(t) & (\alpha_s x)|t \\ (\alpha|t, x|t) & & \longmapsto & \alpha_t(x|t) & \end{array}$$

To do that we take arbitrary $\alpha \in (K \Rightarrow A)(s)$ and $x \in K(s)$ and track the pair (α, x) around the two paths. Doing that we see that

$$\alpha_t(x|t) = (\alpha_s x)|t$$

is required. But this is a consequence of the naturality of α 'as far as it goes'. ♣

The next step is to verify the factorization property (of Definition 2.15. Thus starting from an arbitrary arrow

$$B \times K \xrightarrow{g} A$$

we must produce an arrow

$$B \xrightarrow{g_\flat} (K \Rightarrow A)$$

such that the $\widehat{S}$-triangle

$$\begin{array}{ccc} B \times K & \xrightarrow{\quad g \quad} & A \\ & {\scriptstyle g_\flat \times K} \searrow \quad \nearrow {\scriptstyle \epsilon_A} & \\ & (K \Rightarrow A) \times A & \end{array}$$

commutes. Furthermore, we must check that we produce the only possible arrow $g_\flat$ which makes the triangle commute.

The arrow g is an S-indexed family of functions which is natural for variation of the index. We need a consequence of that. For indexes $j \leq i$ the square

$$\begin{array}{ccc} A(i) \times B(i) & \xrightarrow{g_i} & B(i) \\ \downarrow & & \downarrow \\ A(j) \times B(j) & \xrightarrow[g_j]{} & B(j) \end{array}$$

commutes where the two vertical arrows are restriction maps to j. Thus, for each $a \in A(i), x \in K(i)$ the equality

$$g_j(a|j, x|j) = g_i(a, x)|j$$

holds. We will need a slight generalization of this. Thus, for each $j \leq i \leq s$ and $a \in A(s), x \in K(i)$ the equality

$$g_j(a|j, x|j) = g_i(a|i, x)|j$$

holds. You should make sure you understand this. ♣

For each index $s \in S$ let $g_{\flat s}$ be the function

$$\begin{array}{ccc} B(s) & \xrightarrow{g_{\flat s}} & (K \Rightarrow A)(s) \\ b & \longmapsto & \alpha \end{array}$$

where α is the $\downarrow s$-indexed family of functions given by

$$\alpha_i x = g_i(b|i, x)$$

for each $i \leq s$ and $x \in K(i)$. Let's spell this out again. By construction $g_\flat$ is a certain S-indexed family h. For each index s and $b \in B(s)$ the component h_s produces is an $\downarrow s$-indexed family $h_s b$ of functions. For each $i \leq s$ the component $(h_s b)_i$ is given by

$$(h_s b)_i x = g_i(b|i, x)$$

for $x \in K(i)$. This double indexing is perfect for causing confusion.

We need to check a couple of things.

The family α, which depends on $b \in B(s)$, must be natural 'as far as it goes'. Thus, for indexes $j \leq i \leq s$ we require that the inner square

$$
\begin{array}{ccccccc}
x & \longmapsto & & & & \longrightarrow & g_i(b|i,x) \\
\big\downarrow & & K(i) & \xrightarrow{\alpha_i} & A(i) & & \big\downarrow \\
\big\downarrow & & \downarrow & & \downarrow & & \\
\big\downarrow & & K(j) & \xrightarrow[\alpha_j]{} & A(j) & & g_i(b|i,x)|j \\
x|j & \longmapsto & & \longrightarrow & g_j(b|j,x|j) & &
\end{array}
$$

commutes. To show this we track an arbitrary $x \in K(i)$ around the square, and hence

$$g_j(b|j, x|j) = g_i(b|i, x)|j$$

is required. But this is nothing more than the given naturality of g. ♣

Next we must check that the family $g_\flat$ is natural for variation of the 'outer' index. Let $h = g_\flat$ (to hide the $\flat$). For each pair $t \leq s$ from S we require that the inner square

$$
\begin{array}{ccccccc}
b & \longmapsto & & & & \longrightarrow & h_s b \\
\big\downarrow & & B(s) & \xrightarrow{g_{\flat s}} & (K \Rightarrow A)(s) & & \big\downarrow \\
\big\downarrow & & \downarrow & & \downarrow & & \\
\big\downarrow & & B(t) & \xrightarrow[g_{\flat t}]{} & (K \Rightarrow A)(t) & & (h_s b)|t \\
b|t & \longmapsto & & \longrightarrow & h_t(b|t) & &
\end{array}
$$

commutes. To show this we track an arbitrary $b \in B(s)$ around the square.

Both

$$h_t(b|t) \qquad (h_s b)|t$$

are $\downarrow t$-indexed families of functions

$$K(j) \longrightarrow A(j)$$

for $j \leq t$. Thus we require

$$(h_t(b|t))_j x = ((h_s b)|t)_j x$$

for each $j \leq t$ and $x \in K(j)$. Using the definition of these families and various other properties we have

$$(h_t(b|t))_j x = g_j((b|t)|j, x) = g_j(b|j, x) = (h_s b)_j x = ((h_s b)|t)_j x$$

as required. You should work out why each of these steps holds. ♣

This shows that $g_\flat$ is an arrow.

We now come to the factorization property. Thus, for $s \in S$, we must show that the triangle

$$
\begin{array}{ccc}
B(s) \times K(s) & \xrightarrow{\quad g_s \quad} & A(s) \\
& {}_{g_{\flat s} \times id} \searrow \qquad \nearrow {}_{\epsilon_s} & \\
& (K \Rightarrow A)(s) \times K(s) &
\end{array}
$$

commutes. To do this we start from arbitrary $b \in B(s)$ and $x \in K(s)$. Going across the top produces a value $g_s(b,x)$. Going down the left hand side we obtain a pair (α, x) where α is a $\downarrow s$-indexed family with

$$\alpha_i y = g_i(b|i, y)$$

for each $i \leq s$ and $y \in K(i)$. Taking this up the right hand side produces

$$\epsilon_s(\alpha, x) = \alpha_s x = g_s(b|s, x) = g_s(b, x)$$

to give the required result. ♣

To complete the proof that $\widehat{S}$ is cartesian closed we must show that the constructed arrow $g_\flat$ is the only one that makes the triangle commute.

Suppose, for the given arrow g as above, we have an arrow

$$B \xrightarrow{\ h\ } (K \Rightarrow A)$$

such that the $\widehat{S}$-triangle

$$\begin{array}{ccc} B \times K & \xrightarrow{\quad g \quad} & A \\ & {}_{h \times K} \searrow \quad \nearrow_{\epsilon_A} & \\ & (K \Rightarrow A) \times A & \end{array}$$

commutes. We must show that $h = g_\flat$. In other words we must show

$$(h_s b)_i x = g_i(b|i, x)$$

for each $i \leq s$ and $b \in B(s), x \in K(i)$. To do this we use two properties of h, one of which is easily forgotten.

Remember that h is an $\widehat{S}$-arrow, and hence

$$\begin{CD} B(s) @>{h_s}>> (K \Rightarrow A)(s) \\ @VVV @VVV \\ B(i) @>>{h_i}> (K \Rightarrow A)(i) \end{CD}$$

commutes for each $i \leq s$. Thus for each $b \in B(s), x \in K(i)$ the equalities

$$h_i(b|i) = (h_s a)|i \qquad (h_i(b|i))_i x = (h_s b)_i x$$

hold. Here the first comes from chasing round the square, and the second by evaluation at x and remembering how the 'actual' restriction works. ♣

Next, for $i \leq s$, we use the i-component of the factorization triangle. For $b \in B(s)$ and $x \in K(i)$ we have $(b|i, x) \in B(i) \times K(i)$ and we traipse this the long way round to see that

$$g_i(b|i, x) = \epsilon_i(h_i(b|i), x) = (h_i(b|i))x$$

holds. ♣

Putting these together completes the proof. ♣

There seems to be a lot going on here, and at first sight the example seems to be quite complicated. Well, there is some good news and some bad news.

To show that $\widehat{S}$ is cartesian closed we have go through a lot of calculations. However, each one of these are straight forward. It's just that there is a lot of them.

In fact, what we have done here is a rather simple case of a much more general result (which also covers the result of section 4.3 on $\widehat{R}$). Given any category $\boldsymbol{C}$ whatsoever (such as $\widehat{R}$ or $\widehat{S}$ or the most complicated category you have ever seen), a presheaf on $\boldsymbol{C}$ is a contravariant functor from $\boldsymbol{C}$ to $\boldsymbol{Set}$. These form the objects of a category $\widehat{\boldsymbol{C}}$ where the arrows are the natural transformations between the functors. It is fairly easy to see that this is cartesian (with the obvious products). Furthermore, it is cartesian closed. The proof of this is a generalization of the proof given in this section.

Parts of the more general proof are more complicated, but we also get a better picture of what is going on. An important tool in the general proof is the Yoneda embedding (which embeds $\boldsymbol{C}$ into $\widehat{\boldsymbol{C}}$). This helps us to separate two aspects of the proof.

The yoneda aspects of the proof of this section are hidden. Roughly speaking it is concerned with the stuff about being natural 'as far as it goes', and the places where a double indexing occurred. The 'inner' naturality can be separated from the 'outer' naturality, so the two don't interfere with each other. In the more general situation this makes the proof cleaner, if not shorter.

Each presheaf category $\widehat{\boldsymbol{C}}$ is an example of a topos. More generally, given some extra data, we can determine whether or not a presheaf is a sheaf (relative to the data). These sheaves form a category which again is a topos. These toposes (not topoi as some ex-colonials seem to think) form a connection between several parts of mathematics. They first arose in algebraic geometry where they are used to organize some of the complicated machinery needed. They seem to be the appropriate sites for analysing the semantics of higher order languages. As we have not quite seen here they provide many examples of cartesian closed categories. More generally, via the Yoneda embedding, they enable categories to be 'rounded out' or 'completed' in a certain sense.

Exercises

4.7 Fill in all the missing details indicated by ♣.

4.8 *Is it worth asking them to sort out* $(\cdot)^\sharp$ *and* $(\cdot)_\flat$.

4.9 *Do* $\boldsymbol{Set}^{\rightarrow}$

Part II

Solutions

These are the soloutions that are available at the moment. Others we be included the next time the course is taught (in 2004).

A
The solutions

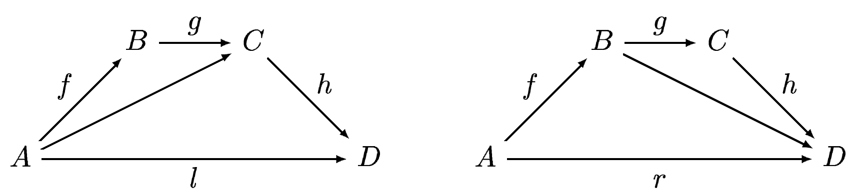
A.1 For chapter 1

A.1.1 For section 1.1

Are solutions necessary?

A.1.2 For section 1.2

1.2 (a) Think of the diagram as a hat which opens along the two parallel arrrows at the bottom. Open the hat and look at the two sides (front and back or left and right depending on what rank you are). Label the two bottom adges as shown.



These four triangles must commute. The two diagonals are

$$g \circ f \qquad\qquad h \circ g$$

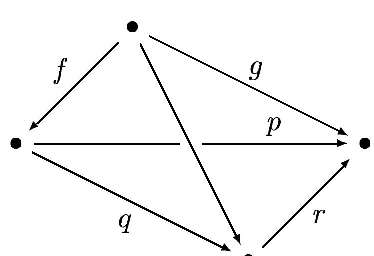
so that

$$l = h \circ (g \circ f) \qquad\qquad r = (h \circ g) \circ f$$

are the two bottom edges.

(b) This really is trivial.

(c) The pyramid isn't very well drawn. Here is a better version



with all except one edge labelled. Because the left hand face commutes, the unlabelled edge is

$$q \circ f$$

and hence

$$g = r \circ q \circ f$$

since the right hand face commutes. But the base commutes, so that

$$p = r \circ q$$

and hence

$$g = p \circ f$$

to show that the back face commutes. ■

1.3 (m) Using the faces Back, Left, Top, Front, Right in that order we have

$$m \circ l \circ e = j \circ f \circ e = j \circ g \circ b = k \circ c \circ b = k \circ d \circ a = m \circ h \circ a$$

that is

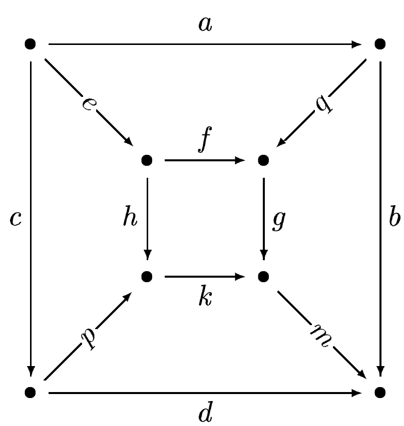
$$m \circ l \circ e = m \circ h \circ a$$

and hence

$$l \circ e = h \circ a$$

by the the cancellation property on the monic m. ■

1.4 Let us label the edges as follows



so that

$$q \circ a = f \circ e$$
$$p \circ c = h \circ e \qquad b = m \circ g \circ q$$
$$m \circ k \circ p = d$$

are the four given conditions.

When the inner square commutes we have

$$g \circ f = k \circ h$$

and hence

$$b \circ a = m \circ g \circ q \circ a = m \circ g \circ f \circ e = m \circ k \circ h \circ e = m \circ k \circ p \circ c = d \circ c$$

to show that the outer square commutes.

When the outer square commutes we have

$$m \circ g \circ f \circ e = m \circ g \circ q \circ a = m \circ g \circ q \circ a = b \circ a = d \circ c = m \circ k \circ p \circ c = m \circ k \circ h \circ e$$

that is

$$m \circ g \circ f \circ e = m \circ k \circ h \circ e$$

holds. Thus if m is monic and e is epic, then we may cancel to get

$$g \circ f = k \circ h$$

to show that the inner square commutes. ■

A.1.3 For section 1.3

For subsection 1.3.1

1.5 Recall that the notions of monic and epic are defioned in subsection 1.6.3.
(m) Trivially, each injective function is a monic arrow.
Conversely, suppose

$$B \xrightarrow{\ m\ } A$$

is a monic arrow and consider $b_1, b_2 \in B$ with $mb_1 = mb_2$. Let 1 be the 1-element set, say $1 = \{\star\}$. Consider the two arrows

$$1 \underset{g_2}{\overset{g_1}{\rightrightarrows}} A$$

given by $g_i\star = b_i$ for $i = 1, 2$. Then

$$g_1 \circ m = g_2 \circ m$$

so that $g_1 = g_2$ (since m is monic) and hence

$$b_1 = g_1\star = g_2\star = b_2$$

to show that m is injective.

(e) Trivially, each surjective function is a monic arrow.
Conversely, suppose

$$B \xrightarrow{\ e\ } A$$

is an epic arrow and consider the range $e[B]$ of values of e. Let $2 = \{0, 1\}$ be the 2-element set, and consider the two arrows

$$A \underset{h}{\overset{g}{\rightrightarrows}} 2$$

given by

$$gx = 1 \iff x \in e[B] \qquad gx = 1$$

for $x \in A$. Thus g is the characteristic functions of $e[B]$ and h is a constant function. Then

$$e \circ g = e \circ h$$

so that $g = h$ (since e is epic) and hence

$$gx = hx = 1$$

for each $x \in A$. This shows that $e[B] = a$, and hence e is surjective. ■

1.6 Consider a compatible pair

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

of functions. Then

$$c\Gamma(g)b \iff c = gb \qquad b\Gamma(g)a \iff b = fa \qquad c\Gamma(g \circ f)a \iff c = (g \circ f)a$$

for $a \in A, b \in B, c \in C$. Using these, for $a \in A, c \in C$, we have

$$\begin{aligned} c(\Gamma(g) \circ \Gamma(f))a &\iff (\exists b \in B)[c\Gamma(g)b\Gamma(f)a] \\ &\iff (\exists b \in B)[c = gb \text{ and } b = fa] \\ &\iff c = g(fa) \qquad \iff c\Gamma(g \circ f)a \end{aligned}$$

to give the required result. ■

For subsection 1.3.2

1.7 (b) Consider any pair A, B of monoids where B is commutative and contains an element b with $b^2 = b \neq 1$. It is easy to check that the assignment

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ a & \longmapsto & ba \end{array}$$

is a semigroup morphism. It is not a monoid morphism since $f1 = b \neq 1$. ■

1.9 (i) For each set A we let $F(A)$ be the monoid of words

$$x, = a_1 a_2 \cdots a_n$$

on A under concatenation. The unit of the monoid is the empty word $\perp$. For each $a \in A$, $\eta(a)$ is the word of length 1 whose only letter is a. (In some areas it is a common notation to write 'ϵ' for the empty word. However, here this will cause confusion with an arrow in part (iii).)

For each function

$$A \xrightarrow{\ f\ } B$$

and word $x \in FA$ (as above) we set

$$F(f)x = f(a_1)f(a_2) \cdots f(a_n)$$

to obtain a word over B.

Clearly $F(f)$ is a ***Mon***-arrow

$$FA \longrightarrow FB$$

that is we have $F(f)\bot = \bot$ and

$$(F(f)x)(F(f)y) = F(f)(xy)$$

for words $x, y \in FA$.

The ***Set***-square

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & FA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F(f)} \\ B & \xrightarrow[\eta_B]{} & FB \end{array}$$

commutes, since both paths send the element $a \in A$ to the sigleton word $fa \in FB$.

(ii) For each function

$$A \xrightarrow{f} S$$

where S is a monoid let

$$FA \xrightarrow{f^\sharp} S$$

be given by

$$f^\sharp x = (fa_1)(fa_2)\cdots(fa_n)$$

for each word x as above. Note that the monoid operation of S is used on the right hand side of this definition. It is routine to check that $f^\sharp$ is a ***Mon***-arrow and that $f = f^\sharp \circ \eta$.

To show that $f^\sharp$ is the unique fill in for the triangle we check that η is '***Mon***-epic' in the sense that if

$$FA \underset{\psi}{\overset{\phi}{\rightrightarrows}} S$$

is a parallel pair of ***Mon***-arrows with $\phi \circ \eta = \psi \circ \eta$, then $\phi = \psi$.

(iii) For each monoid S consider the function

$$FS \xrightarrow{\epsilon} S$$

given by

$$\epsilon(s_1 s_2 \cdots s_n) = s_1 s_2 \cdots s_n$$

for each word in FS. Here on the left concatenation is used to form a word over S, but on the right the monoid operation is used to obtain a new element of S. Then, for each f and word $x \in FA$ (as above) we have

$$\epsilon(F(f)x) = \epsilon(fa_1)(fa_2)\cdots(fa_n) = (fa_1)(fa_2)\cdots(fa_n) = f^\sharp x$$

as required. ■

For subsection 1.3.3

1.10 (b) Consider the function

$$fr = \alpha^r a$$

which sends each $r \in \mathbb{N}$ to the r^{th} iterate of α applied to a. A rountine calculation show that this is a ***Pno***-arrow. A simple proof by induction shows that it is the only possible arrow. (The result can be rephrased to give a categorical characterization of iteration and induction.) ■

A.1.4 For section 1.4

For subsection 1.4.1

1.11 (a) The definition of $\sqsubseteq$ on $A/\approx$ has a hidden problem (of well definedness). Letting $\alpha, \beta, \gamma, \ldots$ range over $A/\approx$ (i.e. the equivalence classes) the definition of $\sqsubseteq$ is

$$\alpha \sqsubseteq \beta \Longleftrightarrow (\exists a \in \alpha, b \in \beta)[a \leq b].$$

To show this is transitive suppose

$$\alpha \sqsubseteq \beta \sqsubseteq \gamma.$$

Then there are

$$a \in \alpha \qquad b_1, b_2 \in \beta \qquad c \in \gamma$$

with

$$a \leq b_1 \approx b_2 \leq c$$

and hence $a \leq c$, which witnesses $\alpha \leq \gamma$.

For anti-symmetry suppose

$$\alpha \sqsubseteq \beta\,,\ \beta \sqsubseteq \gamma.$$

Then there are

$$a_1, a_2 \in \alpha \qquad b_1, b_2 \in \beta$$

with

$$a_1 \leq b_1 \quad , \quad b_2 \leq a_2.$$

But then $a_i \approx b_j$, so that $\alpha = \beta$. ■

1.12 Consider a ***Pre***-arrow

$$A \xrightarrow{\ f\ } B$$

where B is a poset. Observe that for $x, y \in A$

$$x \approx y \Longrightarrow f(x) = f(y)$$

and so we may define

$$FA \xrightarrow{\ f^\sharp\ } B$$

by

$$f^\sharp(\alpha) = f(a)$$

for $a \in \alpha \in FA$. You should check that you understand that this is well-defined. The verification that $f = f^\sharp \circ \eta$ is now straight forward.

To show that this fill in is unique observe that $A \longrightarrow FA$ is surjective and hence epic (in ***Set***).

(b) For a ***Pre***-arrow

$$A \xrightarrow{\ f\ } B$$

the definition of $F(f)$ is

$$F(f)(\alpha) = \langle f(a) \rangle$$

for any $a \in \alpha$. This is well defined since

$$x \approx y \Longrightarrow f(x) = f(y)$$

for all $x, y \in A$. ■

For subsection 1.4.2

1.13 (a) Everything follows from the relationship

$$fx \leq y \Longleftrightarrow x \leq gy$$

(for $x \in A, y \in B$) and the mononicity of f ands g.

Thus, since $f(x) \leq f(x)$, we have $x \leq g(f(x))$, to show that $g \circ f$ is inflationary. A similar argument shows that $g \circ f$ is deflationary.

Both $g \circ f$ and $f \circ g$ are monotone since they are composites of monotone maps.

Since $x \leq (g \circ f)(x)$, an application of f gives $f(x) \leq (f \circ g \circ f)(x)$. Also, $(f \circ g)(y) \leq y$, so setting $y = f(x)$ gives $(f \circ g \circ f)(x) \leq f(x)$. This shows that $f \circ g \circ f = f$, and a similar argument shows that $g \circ f \circ g = g$.

Using these we have

$$(g \circ f)^2 = g \circ f \circ g \circ f = g \circ f \qquad (f \circ g)^2 = f \circ g \circ f \circ g = f \circ g$$

so that both $g \circ f$ and $f \circ g$ are idempotent, and hence are a closure operation and an interior operation, respectively.

(b) Suppose f and g are monotone maps (as given) with $g \circ f$ inflationary and $f \circ g$ deflationary. Then, for $x \in A, y \in B$

$$f(x) \leq y \Longrightarrow x \leq (g \circ f)(x) \Longrightarrow g(y) \qquad x \leq g(y) \Longleftrightarrow, f(x) \leq (f \circ g)(y) \Longleftrightarrow y$$

and hence $f \dashv g$. ■

1.14 Consider

$$\lambda = \lceil \cdot \rceil \qquad \rho = \lfloor \cdot \rfloor$$

the ceiling and floor function, respectively. For each $m \in \mathbb{Z}$ and $x \in \mathbb{R}$ we have

$$x \leq i(m) = m \iff \lceil x \rceil \leq m$$

to show that $\lambda \dashv i$. A similar argument shows $i \dashv \rho$. ■

1.15 For each each $X \in \mathcal{L}S$ we set

$$f^\sharp(X) = \Downarrow\phi[X] \qquad\qquad f_\flat(X)' = \Uparrow\phi[X']$$

so that, for each $Y \in \mathcal{L}T$ both

$$\begin{array}{lllllll} f^\sharp(X) \subseteq Y & \Longleftrightarrow & \phi[X] \subseteq Y & \Longleftrightarrow & X \subseteq \phi^{\leftarrow}(Y) & \Longleftrightarrow & X \subseteq f(Y) \\ f_\flat(X)' \subseteq Y' & \Longleftrightarrow & \phi[X'] \subseteq Y' & \Longleftrightarrow & X' \subseteq \phi^{\leftarrow}(Y') = \phi^{\leftarrow}(Y)' & \Longleftrightarrow & X' \subseteq f(Y)' \end{array}$$

hold. Thus

$$f^\sharp \dashv f \dashv f_\flat$$

as required.

Some simple experiments with monotone maps between small posets shows that, in general, $f^\sharp$ and $f_\flat$ are not related in any other way. ■

1.16 (a) We show that

$$f = g^\sharp \qquad\qquad f_\flat = g$$

and hence obtain a triple adjunction

$$\begin{array}{ccc} & \xrightarrow{\hspace{3em}} & \\ & f^\sharp \dashv f & \\ \mathcal{L}S & \xleftarrow{\hspace{3em}} & \mathcal{L}T \\ & f \dashv g & \\ & \xrightarrow{\hspace{3em}} & \\ & g \dashv g_\flat & \\ & \xleftarrow{\hspace{3em}} & \end{array}$$

as indicated. It suffices to show that $f = g^\sharp$ (for then $f_\flat = g$ follows by general properties of adjunctions).

Recall that

$$1_S \leq \psi \circ \phi \qquad\qquad \phi \circ \psi \leq 1_T$$

(using the pointwise comparison). Then, for each $Y \in \mathcal{L}T$ and $x \in S$, we have

$$x \in f(Y) \Longrightarrow \phi(x) \in Y \Longrightarrow x \leq \psi(\phi(x)) \in \psi[Y] \Longrightarrow x \in \Downarrow\psi[V] = g^\sharp(Y)$$

so that $f(Y) \subseteq g^\sharp(Y)$. Similarly

$$\begin{array}{lll} x \in g^\sharp(Y) & \Longrightarrow & x \in \Downarrow\psi[Y] \\ & \Longrightarrow & (\exists y \in Y)[x \leq \psi(y)] \\ & \Longrightarrow & (\exists y \in Y)[\phi(x) \leq \phi(\psi(y)) \leq y] \\ & \Longrightarrow & \phi(x) \in Y \qquad\qquad \Longrightarrow \quad x \in \phi^{\leftarrow}(Y) = f(Y) \end{array}$$

and hence $g^\sharp(Y) \subseteq f(Y)$. This gives $f = g^\sharp$, as required.

(b) From Exercise 1.15 we can find a double adjunction

$$\begin{array}{ccc} & \xrightarrow{\hspace{3em}} & \\ & \phi \dashv \psi & \\ S & \xleftarrow{\hspace{3em}} & T \\ & \psi \dashv \theta & \\ & \xrightarrow{\hspace{3em}} & \end{array}$$

with $\phi \neq \theta$. From part (a) each adjunction generates a triple adjunction

$$\mathcal{L}S \begin{array}{c} \xrightarrow{\;f^\sharp\;} \\ \xleftarrow{\quad} \\ \xrightarrow[f \dashv g]{} \\ \xleftarrow[g_\flat]{} \end{array} \mathcal{L}T \begin{array}{c} \xleftarrow{\;g^\sharp\;} \\ \xrightarrow{\quad} \\ \xleftarrow[g \dashv h]{} \\ \xrightarrow[h_\flat]{} \end{array} \mathcal{L}S$$

where

$$f = \phi^{\leftarrow} \qquad g = \psi^{\leftarrow} \qquad h = \theta^{\leftarrow}$$

and

$$f^\sharp \dashv f \dashv g \dashv g_\flat \qquad g^\sharp \dashv g \dashv h \dashv h_\flat$$

hold. Also

$$f = g^\sharp \qquad f_\flat = g = h^\sharp \qquad g_\flat = h$$

so we obtain a quadruple stack

$$f^\sharp \dashv f \dashv g \dashv h \dashv h_\flat$$

as required. ■

A.1.5 For section 1.5

For subsection 1.5.1

1.19 The only real problem is to show that the defined composition is associative. To do this consider three arrows

$$A \xrightarrow{\;\;f\;\;} B \xrightarrow{\;\;g\;\;} C \xrightarrow{\;\;h\;\;} D$$

for which both compounds

$$h \circ (g \circ f) \qquad (h \circ g) \circ f$$

may be formed. Consider $a \in A, d \in D$. We show that for the input (a, d) both compounds produce

$$\sum \{ f(a,y)g(y,z)h(z,d) \mid y \in B, z \in C \}$$

as the output. The secret is to remember that both the sets B, C are finite, so that the manipulation of the double summation is unproblematic. ■

1.20 This is almost trivial, and becomes even easier one a decent notation is sorted out. Let us write

$$\begin{array}{cccc} A_0 & B_0 & C_0 & D_0 \\ \Big\downarrow{\scriptstyle\alpha} & \Big\downarrow{\scriptstyle\beta} & \Big\downarrow{\scriptstyle\gamma} & \Big\downarrow{\scriptstyle\delta} \\ A_1 & B_1 & C_1 & D_1 \end{array}$$

and so on (!) for the object of $\boldsymbol{C}^\frown$. Similarly, let us write

$$\begin{array}{ccccccc}
A_0 & & B_0 & & C_0 & & D_0 \\
\alpha\downarrow & \overset{f_0}{\underset{f_1}{\Longrightarrow}} & \downarrow\beta & \overset{g_0}{\underset{g_1}{\Longrightarrow}} & \downarrow\gamma & \overset{h_0}{\underset{h_1}{\Longrightarrow}} & \downarrow\delta \\
A_1 & & B_1 & & C_1 & & D_1
\end{array}$$

for the arrows of $\boldsymbol{C}^\frown$. Thus each one of these expands to a pair of arrows in $\boldsymbol{C}$

$$\begin{array}{ccccccc}
A_0 & \xrightarrow{f_0} & B_0 & \xrightarrow{g_0} & C_0 & \xrightarrow{h_0} & D_0 \\
\alpha\downarrow & & \downarrow\beta & & \downarrow\gamma & & \downarrow\delta \\
A_1 & \xrightarrow[f_1]{} & B_1 & \xrightarrow[g_1]{} & C_1 & \xrightarrow[h_1]{} & D_1
\end{array}$$

where each of the small rectangles commutes. We must do (at least) two things. We must define the composition of $\boldsymbol{C}^\frown$-arrows, and show that this composition is associative.

We define the composite

$$\begin{array}{ccccc}
A_0 & & B_0 & & C_0 \\
\alpha\downarrow & \overset{f_0}{\underset{f_1}{\Longrightarrow}} & \downarrow\beta & \overset{g_0}{\underset{g_1}{\Longrightarrow}} & \downarrow\gamma \\
A_1 & & B_1 & & C_1
\end{array}$$

to be the pair of composites

$$\begin{array}{ccc}
A_0 & \xrightarrow{g_0 \circ f_0} & C_0 \\
\alpha\downarrow & & \downarrow\gamma \\
A_1 & \xrightarrow[g_1 \circ f_1]{} & C_1
\end{array}$$

that is, the composites of the correspondeing components. Thus we find that either way of forming the triple composite in $\boldsymbol{C}^\frown$unravels to

$$\begin{array}{ccc}
A_0 & \xrightarrow{h_0 \circ g_0 \circ f_0} & D_0 \\
\alpha\downarrow & & \downarrow\delta \\
A_1 & \xrightarrow[h_2 \circ g_1 \circ f_1]{} & D_1
\end{array}$$

to give the required associativity. ■

1.21 This can be done in the rather tedious manner of Solution 1.20, that is by first sorting out a decent notation and the looking at various commuting diagrams.

There is a neater way of doing this, but for that we need the notion of a functor. This will be explained in [`Make sure this is done`]. ■

1.22 As usual, the main problem is to show that the arrow composition in the constructed category is associative. For these constructions this is immediate (since the appropriate composites of commuting diagrams produce commuting diagrams.)

A function $1 \xrightarrow{f} A$ is essentially a set A with a distinguished element $a = f(\bullet)$. Thus $1\backslash\boldsymbol{Set}$ is essentially the category of pointed sets.

A function $A \longrightarrow 2$ is essentially a set A with a distinguished subset

$$X = \{a \in A \mid f(a) = 1\}$$

(the set characterized by f). These are the objects of $\boldsymbol{Set}/2$, and the arrows preserve this structure. ■

For subsection 1.5.2

1.24 This is linear algebra. Consider the category of finite dimensional vector spaces over $\mathbb{R}$. Consider such a vector space V of diminsion m, say, and suppose we select a base. This sets up a bijection

$$\begin{array}{rcl} \mathbb{R}^m & \longrightarrow & V \\ x & \longmapsto & \underline{x} \end{array}$$

where here we have written 'x' for an arbitrary (concrete) vector in $\mathbb{R}^m$, and '$\underline{x}$' for the corresponding (abstract) vector in V. Consider two such coordinatization assignments

$$\begin{array}{rcl} \mathbb{R}^m & \longrightarrow & V \\ x & \longmapsto & \underline{x} \end{array} \qquad \begin{array}{rcl} \mathbb{R}^n & \longrightarrow & W \\ y & \longmapsto & \underline{y} \end{array}$$

form arbitrary spaces V, W of dimensions m, n, respectively. There is an associated representation

$$\begin{array}{rcl} \mathbb{R}^{m\times n} & \longrightarrow & [W, V] \\ A & \longmapsto & \underline{A} \end{array}$$

giving a bijection from the $m \times n$ matrices A over $\mathbb{R}$ to the linear transformation from W to V. We find that

$$\underline{Ax} = \underline{A}(\underline{x})$$

that is evaluation on the abstract category is tracked by multiplication in the concrete category.

Finally, consider a pair of linear transformations with their tracking matrices.

$$\begin{array}{ccccc} \mathbb{R}^n & \xrightarrow{B} & \mathbb{R}^k & \xrightarrow{A} & \mathbb{R}^m \\ \downarrow & & \downarrow & & \downarrow \\ W & \xrightarrow[\underline{B}]{} & U & \xrightarrow[\underline{A}]{} & V \end{array}$$

We find that

$$\underline{A} \circ \underline{B} = \underline{AB}$$

that is arrow composition is tracked by matrix multiplication.

(This is the basis of an example of a functor and a natural transformation.) ■

For subsection 1.5.3

1.25 This is the same as Exercise 1.5. ■

1.26 (a) This is essentially the same as part (m) of Exercise 1.5.

(b) consider an epic arrow

$$A \xrightarrow{\ e\ } B$$

and, by way of contradiction, suppose that e is not surjective. Consider the pair of arrows

$$B \underset{g}{\overset{f}{\rightrightarrows}} C$$

as suggested. These arrows agree on the range $e[A]$ of e but one sends b to b and the other sends b to c. By construcrtion, the two composites $f \circ e$ and $g \circ e$ agree which, since e is epic, gives $f = g$, the contradiction. ■

1.27 (a) Let A be an arbitrary monoid and consider any $a \in A$. It is routine to check that

$$\begin{array}{ccc} \mathsf{N} & \longrightarrow & A \\ r & \longmapsto & a^r \end{array}$$

is a monoid morphism, and hence the elements of A are in bijective correspondence with the arrows $\mathsf{N} \longrightarrow A$. With this the usual separating argument shows that monics are injective.

(b) Suppose

$$\mathsf{Z} \underset{g}{\overset{f}{\rightrightarrows}} A$$

is a parallel pair of ***Mon***-arrows which agree on N, that is $fm = gm$ for all $m \in \mathsf{N}$. Consider any $n \in \mathsf{Z} - \mathsf{N}$ and let $m = -n$. Then (since $f(0) - g(0)$ is the neutral element of A, and $fm = gm$ we have

$$\begin{array}{lllllll} fn & = & (fn)(g0) & & & & \\ & = & (fn)(g(m+n)) & & & & \\ & = & (fn)(gm)(gn) & = & (fn)(fm)(gn) & & \\ & & & = & (f(n+m))(gn) & & \\ & & & = & (f0)(gn) & = & gn \end{array}$$

as required. ■

1.28 It is more convenient to write abelian groups addditively.

(m) Let $\mathbb{Z}$ be the groups of integers under addition. A routine argument shows that for each abelian group A the morphims

$$\mathbb{Z} \longrightarrow A$$

are in bijective correspondence with the elements of A. Thus $\mathbb{Z}$ is a separator for $\boldsymbol{Abg}$, and the usual argument shows that each monic is injective.

(e) Suppose

$$A \xrightarrow{\ e\ } B$$

is epic in $\boldsymbol{Abg}$ and consider the range $e[A]$ of e in B. This is a subgroup, and so may be factored out. Consider the parallel pair

$$B \overset{f}{\underset{g}{\rightrightarrows}} B/e[A]$$

of arrows, where f is the canonically associated morphism and g sends everything to 0 (the neutral element of the quotient). The two composites $f \circ f$ and $g \circ e$ agree (since each sends each elelemnt of A to 0). But e is epic, and hence $f = g$, so that $B/e[A]$ is trivial, to show that $e[A] = B$. ■

For subsection 1.5.4

[*Are solutions necesssary?*]

For subsection 1.5.5

1.31 (b) Given and ideal I, for each $r, s, t \in R$ we have

$$\begin{aligned} s \in I : t &\Longrightarrow ts \in I \\ &\Longrightarrow (ts)r \in I \\ &\Longrightarrow t(sr) \in I \quad \Longrightarrow sr \in I : t \end{aligned}$$

to show that $I : t$ is an ideal.

(c) Suppose that $I : t = R$. Then $1 \in I : t$ so that $t = 1t \in I$. Conversely, suppose that $t \in I$. Then for each $s \in R$ we have $ts \in I$ so that $s \in I : t$ to show $I : t = R$. ■

1.32 We have to check that

$$(I : r) : s = I : rs \qquad I : 1 = I$$

for each ideal I and $r, s \in R$. These are straight forward. ■

1.33 (a) The ideals.

(b) For each $a \in A$ and $r, s \in R$ we have

$$\begin{aligned} s \in B : a &\Longrightarrow as \in B \\ &\Longrightarrow (as)r \in B \\ &\Longrightarrow a(sr) \in B \quad \Longrightarrow sr \in B : a \end{aligned}$$

to show that $B : a$ is an ideal.

The second part is similar to Solution 1.31(c).

(c) Let β be the given assignment. We must show that

$$\beta(ar) = (\beta a) : r$$

for each $a \in A$a nd $r \in R$. This is

$$B : ar = (B : a) : r$$

and unravelling

$$s \in B : ar \qquad s \in (B : a) : r$$

almost gets to the bottom of the matter. ■

1.34 We are given that

$$\beta(ar) = (\beta a) : r$$

for each $a \in A$a nd $r \in R$. We must show that there is a unique sub-R-set B of A such that

$$\beta a = B : a$$

for each $a \in A$. Note that if there is scu a B then

$$a \in B \Longleftrightarrow 1 \in B : a \Longleftrightarrow 1 \in \beta a$$

so it suffices to take this as the definition of a subset of A, show it is a sub-R-set, and show that β is its character.

For this subset we have

$$a \in B \Longrightarrow 1 \in \beta a \Longrightarrow 1 \in (\beta a) : r = \beta(ar) \Longrightarrow ar \in B$$

for each $a \in A$ and $r \in R$, to show that B is a sub-R-set.

Similarly, for each $a \in A$ and $r \in R$ we have

$$r \in \beta a \Longleftrightarrow 1 \in (\beta a) : r = \beta(ar) \Longleftrightarrow ar \in B \Longleftrightarrow r \in B : a$$

to show that

$$\beta a = B : a$$

as required. ■

For subsection 1.5.6

1.35 (a) For indexes $r \leq s \leq t$ we require

$$(a|s)|r = a|r \qquad a|t = a$$

for each $a \in A(t)$.

(b) For indexes $r \leq s$ we require

$$f_s(a|r) = (f_s a)|r$$

for each $a \in A(s)$. ■

A.1.6 For section 1.6

For subsection 1.6.1

1.36 The opposite of a poset is the poset turned upside down.

Let $(M, \cdot, 1)$ be a monoid. We can form a new monoid $(M, *, 1)$ by setting

$$a * b = b \cdot a$$

for each $a, b \in M$. When we view the original monoid as a category this gives its opposite. ■

For subsection 1.6.2

1.37 Suppose I is an initial object. Thus for each object A there is a unique arrow

$$I \longrightarrow A$$

and, in particular, there is a unique arrow

$$I \longrightarrow I$$

from I to itself. But we know one such arrow, namely id_I, and hence any arrow from I to I must be this one.

Now suppose that I and J are initial objects. Then there are unique arrows

$$I \xrightarrow{f} J \qquad J \xrightarrow{g} I$$

and these combine to give composites

$$I \xrightarrow{g \circ f} I \qquad J \xrightarrow{g \circ f} J$$

between the separate objects. By the observation above these must be

$$id_I \qquad id_J$$

respectively. Thus f and g are an inverse pair of isomorphisms.

A continuation of this argument shows that these are the only arrows between I, J.

Any two final object F and G are uniquely isomorphic. ■

1.38 We are given an arrow

$$A \xrightarrow{r} I$$

and there is a unique arrow

$$I \xrightarrow{s} A$$

which combine to give an arrow

$$I \xrightarrow{r \circ s} I$$

which we know must be id_I. Thus r is a retraction and s is a section.

The remaining arguments are similar. ■

1.39 The final object of ***Pno*** is $(1,\ \iota,\ \bullet)$, the singleton set $1 = \{\bullet\}$ with the only possible furnishings.

The initial object is $(\mathbb{N},\ S,\ 0)$ where S is the successor function. The initiality of $\mathbb{N}$ is the essence of recursion (induction). ■

1.40 In the category of groups the 1-element group is both initial and final.

In the category of unital rings the integers $(\mathbb{Z}, +, 0, \times, 1)$ is initial. The 1-element ring (in which 1=0) is final.

Certainly, in the category of integral domains the integers $\mathbb{Z}$ are initial. The existence of a final object depends on whether or not '$1 \neq 0$' is part of the axioms. If it is then there is no final object. If it isn't then the trivial ring is final.

Usually for a field we insist that $1 \neq 0$. This means that all morphisms are embeddings, and there is neither an initial nor a final object. However, for the category of fields of a specified characteristic, then there is an initial object. ■

1.41 Each element $a \in A$ gives a function

$$1 \xrightarrow{\ a^\frown\ } A$$

where $a^\frown(\bullet) = a$. Conversely, for each

$$1 \xrightarrow{\ p\ } A$$

we have $p = a^\frown$ where $a = {}^\frown p = p(\bullet)$. This sets up the inverse pair of bijections

$$\begin{array}{ccc} a & \longmapsto & a^\frown \\ A & & [1, A] \\ {}^\frown p & \longleftarrow\!\shortmid & p \end{array}$$

as required.

Now consider the composite

$$1 \xrightarrow{\ a^\frown\ } A \xrightarrow{\ f\ } B$$

for an arbitrary function f, as indicated. By the above correspondence (applied to B) this is $b^\frown$ where

$$b = {}^\frown(f \circ a^\frown) = (f \circ a^\frown)\bullet = f(a^\frown\bullet) = fa$$

as required. ■

1.42 (a) The singleton set $1 = \{\bullet\}$ with the only possible action is the final element.

(b) Consider any global element

$$1 \xrightarrow{\ \beta\ } A$$

of the R-set A. Thus, with $b = \beta\bullet$, we have

$$br = (\beta\bullet)r = \beta(\bullet r) = \beta(\bullet) = b$$

for each $r \in R$. In other words, β picks out a special kind of element b, namely one that satisfies $br = b$ for each $r \in R$.

(c) We may view the monoid R as an R-set. Consider any linear map

$$1 \xrightarrow{\alpha} A$$

to an R-set A. Thus

$$\alpha(sr) = (\alpha s)r$$

for each $r, s \in R$. In particular, with $a = \alpha 1 \in A$ we have

$$\alpha r = \alpha(1r) = ar$$

for each $r \in R$. This shows that α picks out an element of A and is determined by that element.

Conversely, for each $a \in A$, setting

$$\alpha r = ar$$

(for $r \in R$) produces a linear map which picks out a.

Thus R is the required separator. ■

1.43 (a) For each $s \in \mathbb{S}$ let

$$1(s) = \{\bullet\}$$

(the 1-element set). For each $r \leq s$ taken from $\mathbb{S}$ let

$$1(s) \xrightarrow{1(r,s)} 1(r)$$

be the identity function.

It is easy to check that these form a presheaf 1 over $\mathbb{S}$ and, in fact, this is the final element.

(b) Let $A = (\mathcal{A}, \mathsf{A})$ be an arbitrary presheaf and consider a global element

$$1 \longrightarrow A$$

of A. For each index $s \in S$ the function

$$1(s) \longrightarrow A(s)$$

selects some element $a(s) \in A(s)$. Thus the global element is a choice function for the family $\mathcal{A}$ of sets.

However, the selected family

$$(a(s) \mid s \in \mathbb{S})$$

must have some compatibility, since for the global element certain squares must commute. In this case we must have

$$A(r,s)a(s) = a(r)$$

for all $r \leq s$ from $\mathbb{S}$. This is

$$a(s)|r = a(r)$$

is the notation of Solution 1.35. ■

For subsection 1.6.3

1.44 (a) Consider a retraction, section pair, that is arrows

$$A \underset{s}{\overset{r}{\rightleftarrows}} B$$

with $r \circ s = id_B$.

To show that r is epic consider a parallel pair

$$B \underset{g}{\overset{f}{\rightrightarrows}} C$$

of arrows with $f \circ r = g \circ s$. Then

$$f = f \circ id_B = f \circ r \circ s = g \circ r \circ s = g \circ id_B = g$$

to give the required result.

Suppose the retraction r is also monic. For the parallel pair

$$A \underset{id_A}{\overset{s \circ r}{\rightrightarrows}} A \xrightarrow{\ r\ } B$$

we have

$$r \circ (s \circ r) = (r \circ s) \circ r = id_B \circ r = r = r \circ id_A$$

and hence

$$s \circ r = id_A$$

since r is monic. But now we see that r, s are an inverse pair of isomorphisms.

(b)Using the given identities we have

$$g = id_A \circ g = h \circ f \circ g = h \circ id_B = h$$

so that now f and $g = h$ and an inverse pair of isomorphisms. ■

1.46 An element is monic or epic if it is cancellable on the appropriate side.

An element is a retraction or a section if it has a one sided inverse on the appropriate side.

An element is an isomorphism if it has a two sided inverse

A monoid is balanced precisely when it is embeddable in a group. ■

1.47 Consider a pair of arrows

$$X \underset{l}{\overset{k}{\rightrightarrows}} A$$

to A.

Assuming both f, g are monic we have

$$g \circ f \circ k = g \circ f \circ l \Longrightarrow f \circ k = f \circ l \Longrightarrow k = l$$

which more or less shows that $g \circ f$ is monic.

Assuming both $g \circ f$ is monic we have

$$f \circ k = f \circ l \Longrightarrow g \circ f \circ k = g \circ f \circ l \Longrightarrow k = l$$

which more or less shows that f is monic. ■

A.2 For chapter 2

A.2.1 For section 2.1

2.1 (a) A covariant functor

$$S \xrightarrow{\quad f \quad} T$$

between posets is a monotone map

$$f : S \longrightarrow T$$

that is a function that satisfies

$$x \leq y \Longrightarrow fx \leq fy$$

for $x, y \in S$. In other words a covariant functor between posets viewed as categories is an arrow on $\boldsymbol{Pos}$.

The opposites of a poset is the poset turned upside down. Thus a contravariant functor

$$S \xrightarrow{\quad f \quad} T$$

between posets is an antitone map

$$f : S \longrightarrow T$$

that is a function that satisfies

$$x \leq y \Longrightarrow fy \leq fx$$

for $x, y \in S$.

(b) A covariant functor between monoids viewed as categories is a monoid morphism between the monoids, that is an arrrow in $\boldsymbol{Mon}$.

The opposite of a monoid $(A, \star, a)$ is the monoid $(A, \bullet, a)$ where

$$x \bullet y = y \star x$$

for $x, y \in A$. A contravariant functor between monoids is a morphism betweeen the two where one of them is viewed as its opposite. ■

2.2 (a) This is not true. For a category $\boldsymbol{C}$ a contravariant endofunctor F on $\boldsymbol{C}$ would have to send each arrow

$$A \xrightarrow{\quad f \quad} B$$

to an arrow

$$FB \xrightarrow{F(f)} FA$$

between the selected objects. Remember that the object assignment $A \longmapsto FA$ does not depend on the arrow. For the opposite construction we would have to have $FA = A$, and there may not be an arrow

$$B \xrightarrow{F(f)} A$$

that can be assigned to f.

(b) A covariant functor

$$\boldsymbol{S}^{op} \xrightarrow{F} \boldsymbol{T} \qquad \text{or} \qquad \boldsymbol{S} \xrightarrow{F} \boldsymbol{T}^{op}$$

consists of an object assignment and an arrow assignment

$$A \longmapsto FA \qquad f \longmapsto F(f)$$

with certain source and target compatibility. Look what happens in the two cases.

Each source arrow

$$A \xrightarrow{f} B$$

of $\boldsymbol{S}^{op}$, which in fact is an arrow

$$B \xrightarrow{f} A$$

of $\boldsymbol{S}$, is sent to a target arrow

$$FB \xrightarrow{F(f)} FA$$

in $\boldsymbol{T}$

Each source arrow

$$A \xrightarrow{f} B$$

of $\boldsymbol{S}$ is sent to a target arrow

$$FA \xrightarrow{F(f)} FB$$

of $\boldsymbol{T}^{op}$ which in fact is an arrow

$$FB \xrightarrow{F(f)} FA$$

in $\boldsymbol{T}$

It doesn't matter if we flip before or after. ■

2.3 Left selection is covariant and right selection is contravariant. ■

2.4 [If you don't know any group theory then this won't mean much to you.]

Recall that a comutator of a group A is an element

$$[x, y] = x^{-1}y^{-1}xy$$

for some $x, y \in A$. Since

$$[x, y]^{-1} = [y, x]$$

we see that the inverse of a comutator is a comutator. In particular, the set of all products of commutators is a subgroup δA (not just a subsemigroup).

To show that $A \longmapsto \delta A$ is the object part of a functor, we must check that for each morphism

$$A \xrightarrow{f} B$$

we have

$$f[\delta A] \subseteq \delta B$$

(where $f[\cdot]$ indicates direct image). This is immediate since

$$f[x, y] = [fx, fy]$$

for each $x, y \in A$.

To form $A/\delta A$ we must first show that δA is a normal subgroup of A. This follows since

$$a[x, y] = [axa^{-1}, aya^{-1}]a$$

for each $a, x, y \in A$.

To show that $A \longmapsto A/\delta A$ is the object part of a functor, we must check that each morphism

$$A \xrightarrow{\ f\ } B$$

induces a morphism

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ \downarrow & & \downarrow \\ A/\delta A & & B/\delta B \end{array}$$

between the factor groups. Remembering how kernels work this boils down to

$$f[\delta A] \subseteq \delta B$$

which, as we have seen, it true. ■

2.5 Contravariant functors are closed under composition. ■

A.2.2 For section 2.2

2.6 (a) We have

$$\begin{aligned} b \in (\exists f)(X) &\iff (\exists a \in A)[b = fa \wedge a \in X] \\ b \in (\forall f)(X) &\iff b \notin (\exists f)(X') \\ &\iff \neg(\exists a \in A)[b = fa \wedge a \in X'] \\ &\iff (\forall a \in A)[b \neq fa \vee a \notin X'] \qquad \iff (\forall a \in A)[b = fa \Rightarrow a \in X] \end{aligned}$$

for $b \in B$.

(b) Suppose f is not surjective and consider $b \in B - f[A]$. Then

$$b \notin \exists(f)(X) \qquad b \in \forall(f)(X)$$

for each $X \subseteq A$.

(c) The monotone properties are straight forward.

For the adjunction properties we require

$$\exists(f)(X) \subseteq Y \iff X \subseteq f^{\leftarrow}(Y)$$
$$f^{\leftarrow}(Y) \subseteq X \iff Y \subseteq \forall(f)(X)$$

for $X \subseteq A$ and $Y \subseteq B$.

For the first of these we have

$$\begin{aligned}\exists(f)(X) \subseteq Y &\iff (\forall a \in A)[a \in X \Rightarrow fa \in Y]\\ &\iff (\forall a \in A)[a \in X \Rightarrow a \in f^{\leftarrow}(Y)] \quad \iff X \subseteq f^{\leftarrow}(Y)\end{aligned}$$

as required.

For the second we have

$$\begin{aligned}Y \subseteq \forall(f)(X) &\iff Y \subseteq f[X']'\\ &\iff f[X'] \subseteq Y'\\ &\iff (\forall a \in A)[a \notin X \Rightarrow fa \notin Y]\\ &\iff (\forall a \in A)[fa \in Y \Rightarrow a \in X]\\ &\iff (\forall a \in A)[a \in f^{\leftarrow}(Y) \Rightarrow a \in X] \quad \iff f^{\leftarrow}(Y) \subseteq X\end{aligned}$$

as required. ■

2.7 The inverse image part is immediate.

For the existential part observe that

$$(g \circ f)[X] = g[f[X]]$$

for each $X \subseteq A$.

The universal part follows in the same way and various complements cancel out. ■

2.8 (a) We have

$$\mathcal{P}(f) = f^{\leftarrow} \qquad \Pi(f) = \mathcal{P}(f)^{\leftarrow}$$

so that

$$Y \in \Pi(f)(\mathcal{X}) \iff Y \in \mathcal{P}(f)^{\leftarrow}(\mathcal{X}) \iff \mathcal{P}(f)(Y) \in \mathcal{X} \iff f^{\leftarrow}(Y) \in \mathcal{X}$$

as required.

(b) In general, the composite of two contravariant functors is a covariant functor. As an instance of this $\Pi = \mathcal{P} \circ \mathcal{P}$ is a covariant functor. However, it is probably more instructive to do the particular calculation.

Using part (a) we have

$$\begin{aligned}Y \in \Pi(g \circ f)(\mathcal{X}) &\iff (g \circ f)^{\leftarrow}(Y) \in \mathcal{X}\\ &\iff (f^{\leftarrow} \circ g^{\leftarrow})(Y) \in \mathcal{X}\\ &\iff f^{\leftarrow}(g^{\leftarrow}(Y)) \in \mathcal{X}\\ &\iff g^{\leftarrow}(Y) \in \Pi(f)(\mathcal{X})\\ &\iff Y \in \Pi(g)(\Pi(f)(\mathcal{X})) \quad \iff Y \in (\Pi(g) \circ \Pi(f))(\mathcal{X})\end{aligned}$$

to give

$$\Pi(g \circ f) = \Pi(g) \circ (\Pi(f)$$

as required. The identity requirement is immediate. ■

A.2.3 For section 2.3

2.9 (a) The first part, concerning upper and lower sections is straight forward.

For the second part we show first that $\between H$ is convex. To this end consider $x \le z \le y$ with $x, y \in \between H$. Then

$$z \le y \in \between H \subseteq \downarrow H \qquad z \ge x \in \between H \subseteq \uparrow H$$

so that

$$z \in \downarrow H \qquad z \in \uparrow H$$

and hence

$$z \in \downarrow H \cap \uparrow H = \between H$$

as required.

Next consider any convex part K with $H \subseteq K$. We required $\between H \subseteq K$. Consider any

$$z \in \between H = \downarrow H \cap \uparrow H$$

so that

$$z \le y \in H \qquad z \ge x \in H$$

for some x, y. But then $x \le z \le y$ and

$$x, y \in H \subseteq K$$

to give $z \in K$ (since K is convex), as required.

(b) It is routine to check that the complement of a lower section is an upper section and the complement of an upper section is a lower section. In particular

$$(\uparrow H')' \qquad (\downarrow H')'$$

is a

lower section upper section

respectively.

Since $H' \subseteq \uparrow H'$ we have $(\uparrow H')' \subseteq H'' = H$, to show one of the required inclusions. The other is similar.

Finally, suppose L is a lower section $L \subseteq H$. Then $H' \subseteq L'$ with L' upper to give $\uparrow H' \subseteq L'$, and hence $L \subseteq (\uparrow H')'$, to gives the required maximality. ■

2.10 The top two sets are

$$\uparrow H \qquad \downarrow H$$

respectively.

The bottom two are the sets of

lower upper

bounds of H, that is the sets given by

$$l \in L \iff (\forall a \in H)[l \le a] \qquad u \in U \iff (\forall a \in H)[a \le u]$$

respectively. ■

2.11 It is easier if we do the second part first. We find that

$$X \leq^\sharp Y \iff \quad \uparrow Y \subseteq \uparrow X$$

$$X \leq^\natural Y \iff \begin{cases} \uparrow Y \subseteq \uparrow X \\ \text{and} \\ \downarrow X \subseteq \downarrow Y \end{cases}$$

$$X \leq^\flat Y \iff \quad \downarrow X \subseteq \downarrow Y$$

for $X, Y \in \mathcal{P}A$. With these the preordering properties are immediate. For instance

$$X \leq^\natural Y \leq^\natural Z \Longrightarrow \left\{ \begin{matrix} \uparrow Z \subseteq \uparrow Y \subseteq \uparrow X \\ \text{and} \\ \downarrow X \subseteq \downarrow Y \subseteq \uparrow Z \end{matrix} \right\} \Longrightarrow \left\{ \begin{matrix} \uparrow Z \subseteq \uparrow X \\ \text{and} \\ \downarrow X \subseteq \uparrow Z \end{matrix} \right\} \Longrightarrow X \leq^\natural Z$$

to show that $\leq^\natural$ is transitive.

Each preordered set can be converted into a partially ordered set by factoring out an approriate equivalence relation. If we do that here we get the

$\flat$	$\natural$	$\sharp$
lower	convex	upper

sections under the appropriate comparison. ■

2.12 (a) Your experiance with Exercise 2.6 should help here.

[*I think some of the notation in Solution 1.15 has gone missing or got screwed up.*]

Given a monotone map

$$S \xrightarrow{\ \phi\ } T$$

between posets the inverse image function

$$\mathcal{L}S \xleftarrow{\ f = \phi^{\leftarrow}\ } \mathcal{L}T$$

is monotone between the posets of lower sections. For $X \in \mathcal{L}S$ set

$$\exists(\phi)(X) = \downarrow\phi[X] \qquad \forall(\phi)(X) = (\uparrow\phi[X'])'$$

to produce two lower section in $\mathcal{L}T$. In more detail we have

$$t \in \exists(\phi)(X) \iff (\exists s \in S)[t \leq \phi s \wedge s \in X]$$
$$t \in \forall(\phi)(X) \iff (\forall s \in S)[\phi s \leq t \Rightarrow s \in X]$$

for $t \in T$. We can now show

$$\exists(\phi) \dashv \phi^{\leftarrow} \dashv \forall(\phi)$$

as in Solution 2.6. For instance

$$\begin{aligned} Y \subseteq \forall(\phi)(X) &\iff Y \subseteq (\uparrow\phi[X'])' \\ &\iff \uparrow\phi[X'] \subseteq Y' \\ &\iff \phi[X'] \subseteq Y' \\ &\quad\vdots \\ &\iff (\forall s \in S)[s \in \phi^{\leftarrow}(Y) \Rightarrow s \in X] \quad \iff \phi^{\leftarrow}(Y) \subseteq X \end{aligned}$$

as required. Here the third equivalence holds since Y is a lower section and hence Y' is an upper section.

(b) This is more or less the same as part (a) with certain comparisons reversed. ■

A.2.4 For section 2.4

For subsection 2.4.1

2.13 The main problem is to show that both constructions pass across composition in the required manner. It is instructive to do these two proofs in parallel.

Thus let

$$F = \boldsymbol{C}[K, \cdot] \qquad F = \boldsymbol{C}[\cdot, K]$$

and consider a composite

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

of arrows.

We require

$$F(g \circ f) = F(g) \circ F(f)$$

for the covariant case. Consider how these three constructions behave. We have

$$\begin{array}{ccccc} \boldsymbol{C}[K,A] & \xrightarrow{F(f)} & \boldsymbol{C}[K,B] & \xrightarrow{F(g)} & \boldsymbol{C}[K,C] \\ p & \longmapsto & f \circ p & & \\ & & q & \longmapsto & g \circ q \\ p & \longmapsto & & & g \circ f \circ p \\ \boldsymbol{C}[K,A] & & \xrightarrow[F(g \circ f)]{} & & \boldsymbol{C}[K,C] \end{array}$$

and hence

$$\begin{aligned} (F(g) \circ F(f))(p) &= F(g)(F(f)(p)) \\ &= F(g)(f \circ p) \\ &= g \circ f \circ p \ = F(g \circ f)(p) \end{aligned}$$

as required.

We require

$$F(g \circ f) = F(f) \circ F(g)$$

for the contravariant case. Consider how these three constructions behave. We have

$$\begin{array}{ccccc} \boldsymbol{C}[A,K] & \xleftarrow{F(f)} & \boldsymbol{C}[B,K] & \xleftarrow{F(g)} & \boldsymbol{C}[C,K] \\ & & r \circ g & \longleftarrow\!\shortmid & r \\ q \circ f & \longleftarrow\!\shortmid & q & & \\ r \circ g \circ f & \longleftarrow\!\shortmid & & & r \\ \boldsymbol{C}[K,A] & & \xleftarrow[F(g \circ f)]{} & & \boldsymbol{C}[K,C] \end{array}$$

and hence

$$\begin{aligned} (F(f) \circ F(g))(r) &= F(f)(F(g)(p)) \\ &= F(f)(r \circ g) \\ &= r \circ g \circ f \ = F(g \circ f)(r) \end{aligned}$$

as required.

Observe the similarites and differences here. ■

2.14 The pointwise comparison between to monotone maps

$$S \underset{p}{\overset{q}{\rightrightarrows}} T$$

on posets S, T is given by

$$p \leq q \iff (\forall s \in S)[ps \leq qs]$$

and it is straight forward to show that this turns $\boldsymbol{Pos}[S, T]$ into a poset.

Consider the covariant hom-functor

$$\boldsymbol{Pos} \xrightarrow{F = \boldsymbol{Pos}[K, \cdot]} \boldsymbol{Set}$$

for some fixed poset K. We have just seen that for each poset A the set $FS] = \boldsymbol{Pos}[K, S]$ is a poset. To show that F becomes a functor

$$\boldsymbol{Pos} \xrightarrow{F = \boldsymbol{Pos}[K, \cdot]} \boldsymbol{Pos}$$

we must show that for each monotone map ($\boldsymbol{Pos}$-arrow)

$$S \xrightarrow{f} T$$

the result

$$\begin{array}{ccc} \boldsymbol{Pos}[K, S] & \longrightarrow & \boldsymbol{Pos}[K, T] \\ p & \longmapsto & f \circ p \end{array}$$

is monotone (not just a function). But, for $p, q \in \boldsymbol{Pos}[K, S]$ we have

$$\begin{aligned} p \leq q &\Longrightarrow (\forall s \in S)[ps \leq qs] \\ &\Longrightarrow (\forall s \in S)[f(ps) \leq f(qs)] \\ &\Longrightarrow (\forall s \in S)[(f \circ p)s \leq (f \circ q)s] \quad \Longrightarrow (f \circ p) \leq (f \circ q) \end{aligned}$$

to give the required result.

The contravariant case is similar. ■

2.15 Let k be an arbitrary element of the poset S and consider the two hom-functors given by

$$fs = S[k, s] \qquad fs = S[s, k]$$

(for $s \in S$). Technically each value fs is either the empty set or a singleton, We may represent these by 0 and 1, respectivley. Let

$$2 = \{0, 1\}$$

thought of as a poset. Then the two hom functors are the maps

$$S \xrightarrow{f} 2$$

given by

$$fs = \begin{cases} 1 \text{ if } k \le s \\ 0 \text{ if } k \not\le s \end{cases} \qquad fs = \begin{cases} 1 \text{ if } s \le k \\ 0 \text{ if } s \not\le k \end{cases}$$

respectively. It is show that

$$s \le t \Longrightarrow fs \le ft \qquad s \le t \Longrightarrow ft \le fs$$

and hence f is

monotone antitone

respectively, for the two cases. ■

For subsection 2.4.2

2.16 (a) We use the more sensible notation of Solution 1.20. Consider a composable pair of arrows

$$\begin{array}{ccccc} A_0 & & B_0 & & C_0 \\ \alpha\downarrow & \overset{f_0}{\underset{f_1}{\Longrightarrow}} & \beta\downarrow & \overset{g_0}{\underset{g_1}{\Longrightarrow}} & \gamma\downarrow \\ A_1 & & B_1 & & C_1 \end{array}$$

of $\boldsymbol{C}^{\to}$. These unravel to

$$\begin{array}{ccccc} A_0 & \xrightarrow{f_0} & B_0 & \xrightarrow{g_0} & C_0 \\ \alpha\downarrow & & \downarrow\beta & & \downarrow\gamma \\ A_1 & \xrightarrow[f_1]{} & B_1 & \xrightarrow[g_1]{} & C_1 \end{array}$$

where each square commutes. Then

$$\begin{array}{lll} S(f) = f_0 & S(g) = g_0 & S(g \circ f) = (g \circ f)_0 = g_0 \circ f_0 \\ T(f) = f_1 & T(g) = g_1 & T(g \circ f) = (g \circ f)_1 = g_0 \circ f_1 \end{array}$$

which gives the required result.

(b) We have

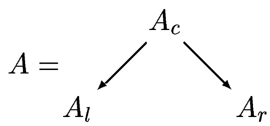
$$\Delta A = \begin{array}{c} A \\ 1_A\big\downarrow \\ A \end{array}$$

$$\Delta(A \xrightarrow{f} B) = \begin{array}{ccc} A & \xrightarrow{f} & B \\ 1_A\big\downarrow & & \big\downarrow 1_B \\ A & \xrightarrow[f]{} & B \end{array}$$

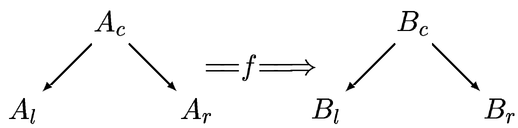
$$\Delta(A \xrightarrow{f} B \xrightarrow{g} C) = \begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ 1_A\big\downarrow & & \big\downarrow 1_B & & \big\downarrow 1_C \\ A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C \end{array} = \Delta g \circ \Delta(f)$$

to give the required results. ■

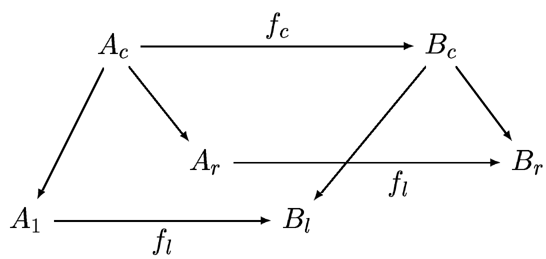
2.17 As with other cases it makes life easier if we choose a decent notation. Thus an object of $\boldsymbol{C}^\wedge$ is pair of arrows of $\boldsymbol{C}$



and an arrow of $\boldsymbol{C}^\wedge$



is a triple of arrows of $\boldsymbol{C}$



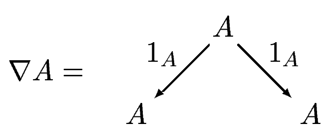
for which the two squares commute.

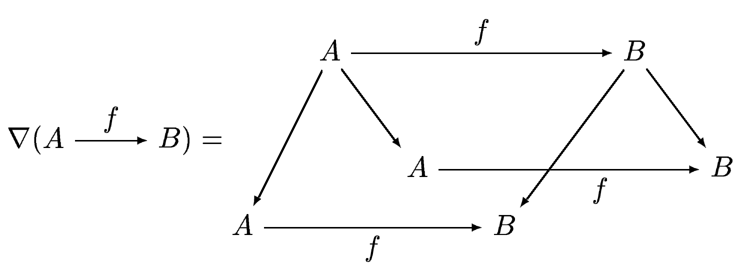
(a) In the above notation, setting

$$\begin{array}{lll} LA \;= A_l & CA \;= A_c & RA \;= A_r \\ L(f) = f_l & C(f) = f_c & R(f) = f_r \end{array}$$

produces three functors $\boldsymbol{C}^\wedge \longrightarrow \boldsymbol{C}$.

(b) Setting





for each object A and arrow f of $\boldsymbol{C}$ produces a functor $\boldsymbol{C} \longrightarrow \boldsymbol{C}^\wedge$. ■

For subsection 2.4.3

2.19 (a) The comma category

$$(1 \xrightarrow{L} \boldsymbol{C} \;,\; \boldsymbol{C} \;,\; \boldsymbol{C} \xleftarrow{Id} \boldsymbol{C})$$

is the slice $K\backslash\boldsymbol{C}$ where L selects the object K.

(b) The slice $\boldsymbol{C}/K$ is the comma category

$$(\boldsymbol{C} \xrightarrow{Id} \boldsymbol{C} \;,\; \boldsymbol{C} \;,\; \boldsymbol{C} \xleftarrow{R} \boldsymbol{C})$$

where R selects the object K.

(c) The comma category

$$(\boldsymbol{C} \xrightarrow{Id} \boldsymbol{C} \;,\; \boldsymbol{C} \;,\; \boldsymbol{C} \xleftarrow{Id} \boldsymbol{C})$$

is the arrow category $\boldsymbol{C}^{\rightarrow}$. ■

2.20 When applied to

$$\big(A, L(A) \xrightarrow{f} R(B), B\big)$$

the three object assignments pick out the three components

$$A \qquad L(A) \xrightarrow{\quad f \quad} R(B) \qquad B$$

which are objects of

$$\boldsymbol{A} \qquad \boldsymbol{C}^{\rightarrow} \qquad \boldsymbol{B}$$

respectively. The arrow assignments then do the obvious thing. ■

A.2.5 For section 2.5

2.21 By definition, a natural transformation

$$F \xrightarrow{\quad \eta \quad} G$$

is a natural isomorphism if there is a natural transformation

$$F \xleftarrow[\quad \zeta \quad]{} G$$

in the opposite direction such that for each source object A the two composites

$$FA \underset{\zeta_A}{\overset{\eta_A}{\rightleftarrows}} GA$$

are the identities of FA and GA. In particular, each η_A is an isomorphism in the target category.

Conversely, suppose that η is a natural transformation such that each component η_A is an isomorphism. Then each component η_A has a unique inverse arrow ζ_A, as above. We must show that this family ζ is natural. To this end consider an arrow

$$A \xrightarrow{\ f\ } B$$

in the source category. We must show that

$$\begin{array}{ccc} GA & \xrightarrow{\zeta_A} & FA \\ {\scriptstyle G(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ GB & \xrightarrow[\zeta_B]{} & FB \end{array}$$

commutes. Consider the diagram

$$\begin{array}{ccccc} GA & \xrightarrow{\zeta_A} & FA & \xrightarrow{\eta_A} & GA \\ {\scriptstyle G(f)}\downarrow & & \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ GB & \xrightarrow[\zeta_B]{} & FB & \xrightarrow[\eta_B]{} & GB \end{array}$$

where we know that the right hand square does commute (since η is natural). But now

$$\eta_B \circ F(f) \circ \zeta_A = G(f) \circ \eta_A \circ \zeta_A = G(f)$$

and hnce

$$\zeta_B \circ G(f) = \zeta_B \circ \eta_B \circ F(f) \circ \zeta_A = F(f) \circ \zeta_A$$

as required. ■

A.2.6 For section 2.6

For subsection 2.6.1

2.23 For each set A consider the function

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \mathcal{P}_\forall \\ a & \longmapsto & \{a\}' \end{array}$$

which send an element to the complement of its singleton. We show this is natural, that is for each function

$$A \xrightarrow{\ f\ } B$$

the square

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \mathcal{P}_\forall A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{P}_\forall(f)} \\ B & \xrightarrow[\eta_B]{} & \mathcal{P}_\forall B \end{array}$$

commutes. Remembering that

$$\mathcal{P}_\forall(f)(X) = f[X']'$$

for each $X \subseteq A$ we see that each $a \in A$ is sent to $\{fa\}'$ by both routes. ∎

2.24 Remembering that

$$Y \in \Pi(f)(\mathcal{X}) \Longleftrightarrow f^{\leftarrow}(Y) \in \mathcal{X}$$

(for $Y \subseteq B$ and $\mathcal{X} \subseteq \mathcal{P}A$) for each $a \in A$ we have

$$\begin{aligned} Y \in (\Pi(f) \circ \eta_A)a &\Longleftrightarrow Y \in \Pi(f)(\circ\eta_A a) \\ &\Longleftrightarrow f^{\leftarrow}(Y) \in \eta_A a \\ &\Longleftrightarrow a \in f^{\leftarrow}(Y) \\ &\Longleftrightarrow fa \in Y \\ &\Longleftrightarrow Y \in \eta_B(fa) \qquad \Longleftrightarrow Y \in (\eta_B \circ f)a \end{aligned}$$

to give $\Pi(f) \circ \eta_A = \eta_b \circ f$, as required. ∎

[***Has the notation for the next solution been set up***]

2.26 For each poset S the poset ΥS is the family of all upper section of S under inclusion. Each monotone map

$$S \xrightarrow{\ \phi\ } T$$

gives a monotone map

$$\Upsilon S \xrightarrow{\ \phi^{\leftarrow}\ } \Upsilon T$$

by taking inverse images. This map $\phi^{\leftarrow}$ has a left and a right adjoint

$$\exists(\phi) \dashv \phi^{\leftarrow} \dashv \forall(\phi)$$

given by

$$\exists(\phi(U) = {\uparrow}\phi[U] \qquad \forall(\phi)(U) = ({\downarrow}\phi[U'])'$$

for each $U \in \Upsilon S$. These are the endofuntors called $\Upsilon^{\exists}$ and $\Upsilon^{\forall}$ in the question.

It is routine to check that for each poset S the two assignments

$$\begin{array}{ccc} S & \xrightarrow{\ \eta^{\exists}_S\ } & \Upsilon S \\ s & \longmapsto & {\uparrow}s \end{array} \qquad\qquad \begin{array}{ccc} S & \xrightarrow{\ \eta^{\forall}_S\ } & \Upsilon S \\ s & \longmapsto & ({\downarrow}s)' \end{array}$$

are monotone. We show they are natural.

We must check that for each monotone map

$$S \xrightarrow{\ \phi\ } T$$

the two squares

$$\begin{array}{ccc} S & \xrightarrow{\ \eta^{\exists}_S\ } & \Upsilon S \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\exists(\phi)} \\ T & \xrightarrow[\ \eta^{\exists}_T\ ]{} & \Upsilon T \end{array} \qquad\qquad \begin{array}{ccc} S & \xrightarrow{\ \eta^{\forall}_S\ } & \Upsilon S \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\forall(\phi)} \\ T & \xrightarrow[\ \eta^{\forall}_T\ ]{} & \Upsilon T \end{array}$$

commute. This is done by tracking and element $s \in S$ from the top left to bottom right via both routes. We get

$$\uparrow(fs) = \uparrow(\phi[\uparrow s]) \qquad (\downarrow\phi s)' = (\downarrow\phi[\downarrow s])'$$

in the two cases. ■

2.27 (a) This is a case where a more detailed notation helps. Thus we write $(S, \leq)$ for as poset, that is a set S furnished with a comparison $\leq$ which is a partial ordering.

The opposite is given by

$$\mathcal{O}(S, \leq) = (S, \sqsubseteq)$$

where

$$s \sqsubseteq t \Longleftrightarrow t \leq t$$

for $s, t \in S$. In other words the opposite is the same set furnished with the reverse of the given comparison.

This is the object assignment of the functor.

For the arrow assignment consider a monotone map

$$(S, \leq) \xrightarrow{\ \phi\ } (T, \leq)$$

between two posets. Notice that ϕ is still monotone relative to the opposite comparisons of $\leq$. Thus we set

$$\mathcal{O}\big((S, \leq) \xrightarrow{\ \phi\ } (T, \leq)\big) = \big((S, \sqsubseteq) \xrightarrow{\ \phi\ } (T, \sqsubseteq)\big)$$

that is we take the same function but view it differently.

We need to check that this gives a functor. However, this is almost immediate since under both views arrow composition is just function composition.

(b) Consider a poset $(S, \leq)$. Then $\mathcal{O}(S, \leq)$ is the poset $(S, \sqsubseteq)$ using the opposite comparsion. From this

$$(\mathcal{L}^{\exists} \circ \mathcal{O})(S, \leq)$$

is the set of lower sections of $(S, \sqsubseteq)$ under inclusion. This is just the set of upper sections of $(S, \leq)$ under inclusion.

Similarly $\mathcal{L}^{\forall}(S, \leq)$ takes the set of lower section of $(S, \leq)$ under inclusion. But now

$$(\mathcal{O} \circ \mathcal{L}^{\forall})(S, \leq)$$

takes the same lower section and compares them by *reverse* inclusion. This is equivalent to comparing the lower sections by inclusion of their complements.

In this way we find that

$$\begin{array}{ccc} (\mathcal{L}^{\exists} \circ \mathcal{O})(S, \leq) & \longrightarrow & (\mathcal{O} \circ \mathcal{L}^{\forall})(S, \leq) \\ U & \longmapsto & U' \end{array}$$

is an isomorphism of posets (where $U \in \Upsilon(S, \leq)$).

With a little more effort we find that this isomorphism is natural for variation of $(S, \leq)$.

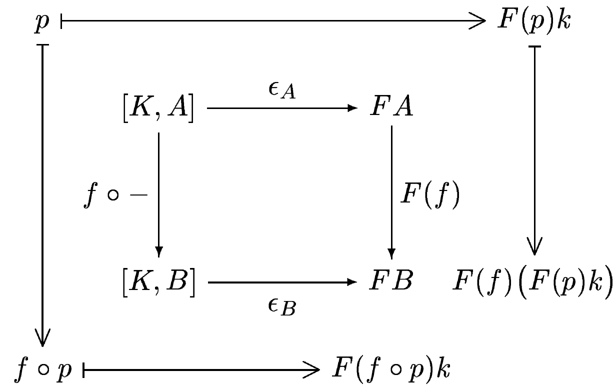
Finally, we can show that $\mathcal{L}^{\exists} \circ \mathcal{O}$ is just $\Upsilon^{\exists}$. ■

For subsection 2.6.2

2.28 The first main problem is to show the naturality of the constructed ϵ. To this end consider a $\boldsymbol{C}$-arrow

$$A \xrightarrow{\ f\ } B$$

and the associated square in $\boldsymbol{Set}$



as on the inside of the diagram. To show this square commutes we track and abitrary arrow

$$K \xrightarrow{\ p\ } A$$

via both routes to the opposite corner, as on the oputside of the diagram. But now, since F is a functor, we have

$$F(f \circ p)k = \big(F(f) \circ F(p)\big)k = F(f)\big(F(p)k\big)$$

to give the required result.

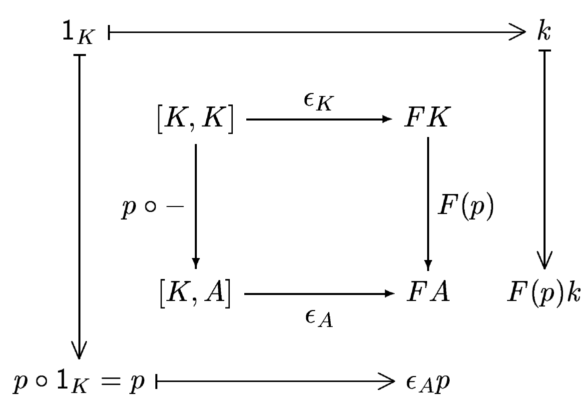
The second main problem is to show that an arbitrary natural transformation

$$[K,-] \xrightarrow{\ \epsilon\ } F$$

is just evaluation at $k = \epsilon_K(1_K)$. For this we use an aribtrary arrrow

$$K \xrightarrow{\ p\ } A$$

to produce a diagram



together with a tracking of 1_K. The naturality ensures that the inner square commutes, and hence

$$\epsilon_A p = F(p)k$$

as required. ■

2.29 For each object A we have a pair of bijections

$$[K, A] \xrightarrow{\epsilon_A} FA \xleftarrow{\eta_A} [L, A]$$

where these are natural for variation of A. Furthermore, these are given by evaluation at

$$k = \epsilon_K(1_K) \qquad l = \epsilon_L(1_L)$$

respectively.

Since ϵ_A is surjective we have

$$(\forall m \in FA)(\exists K \xrightarrow{p} A)[F(p)k = \epsilon_A(p) = m]$$

and hence

$$(\forall L \xrightarrow{q} A)(\exists K \xrightarrow{p} A)[F(p)k = \eta_A q]$$

as a particular case. Specializing to $A = L$ with $q = 1_L$ we get some

$$K \xrightarrow{p} L$$

with

$$F(p)k = \eta_A 1_L = l$$

as required. ■

[***Relate the next solution to the use of separators***]

2.30 (a) This is similar but easier than part (b) using a 1-element set $\{\bullet\}$ in place of $\mathbb{N}$.

(b) Let

$$\boldsymbol{Mon} \xrightarrow{F} \boldsymbol{Set}$$

be the forgetful functor. Thus for each monoid A the set FA is the carrier of A. (Usually we do not distinguish between A and FA, but here we need to.) Let $\mathbb{N}$ be the additive monoid of natural numbers. For each monoid consider the function

$$\begin{array}{ccc} [\mathbb{N}, A] & \xrightarrow{\epsilon_A} & FA \\ p & \longmapsto & p1 \end{array}$$

that is each monoid morphism

$$\mathbb{N} \xrightarrow{p} A$$

is sent to its value at 1. First observe that this is a bijection of sets. Then show that the family ϵ is natural for variation of A. Thus the pair $(\mathbb{N}, 1)$ provide a representation of F. ■

2.33 Lemma 2.9 is essentially about the use of characteristic functions to locate subsets. For each *set* A there are bijective correspondences

$$\begin{array}{ccccc} \mathcal{P}A & & \boldsymbol{Set}[A,2] & & \mathcal{P}A \\ X & \longleftrightarrow & p & \longleftrightarrow & U \end{array}$$

given by

$$a \in X \iff pa = 0 \qquad pa = 1 \iff a \in U$$

for each $a \in A$. In other words we can either use 'value is 0' or 'value is 1' to locate the subset. Once we have this Lemma 2.9 needs just a little bit more work.

Suppose A is a poset and consider corresponding

$$X \longleftrightarrow p \longleftrightarrow U$$

as above. Here X, U are subsets and p is a function. We now check that

$$X \in \mathcal{L}A \iff p \text{ is monotone } \iff U \in \Upsilon A$$

to prove the Lemma.

For instance, suppose $X \in \mathcal{L}A$ and consider arbitrary $a \le b$ in A. If $pb = 1$ then certainly $pa \le pb$. If $pb = 0$ then $a \le b \in X$ so that $a \in X$ and hence $pa = 0 \le pb$. Thus in all cases $pa \le pb$, to show that p is monotone.

The other five required implications follow by variants of this idea.

To prove Lemma 2.10 we must show that for an arbitrary monoid morphism

$$B \xrightarrow{\ \phi\ } A$$

we must show that the induced squares

$$\begin{array}{ccc} \Upsilon A & \xrightarrow{\ \phi^{\leftarrow}\ } & \Upsilon B \\ \updownarrow & & \updownarrow \\ \boldsymbol{Pos}[A,2] & \xrightarrow{\ (-\circ\phi)\ } & \boldsymbol{Pos}[B,2] \\ \updownarrow & & \updownarrow \\ \mathcal{L}A & \xrightarrow[\ \phi^{\leftarrow}\ ]{} & \mathcal{L}B \end{array}$$

all commute. For instance, suppose $U \in \Upsilon$ corresponds to the character $p : A \longrightarrow 2$. By a simple calculation we find that $\phi^{\leftarrow}(U)$ corresponds to the character $p \circ \phi : B \longrightarrow 2$.

Much of the proof of Theorem 2.11 has alrteady been done. What remains is to show that several functions are, in fact, monoid morphism.

For instance, we must show that for each monoid morphism

$$B \xrightarrow{\ \phi\ } A$$

the induced function

$$\boldsymbol{Pos}[A,2] \xrightarrow{-\circ\phi} \boldsymbol{Pos}[B,2]$$

is monotone, that is

$$q \leq p \Longrightarrow q\circ\phi \leq p\circ\phi$$

for $\boldsymbol{Pos}$-characters p, q on A. But if

$$q \leq p$$

then (by definition)

$$(\forall x : A)[qx \leq px]$$

so that (by specialization)

$$(\forall y : B)[q(\phi y) \leq p(\phi y)]$$

to give the required result.

The other requirements are just as easy. ■

2.34 (a) There are many things to be checked here, but none of them is very difficult. The main problem is making sure that nothing is overlooked. Let's list what has to be done, and look at the proofs of a few of these.

(i) Fix a monotone map $f : A \longrightarrow \Upsilon S$. For a given $s \in S$ define $\phi s \subseteq A$ by

$$a \in \phi s \Longleftrightarrow s \in fa$$

(for $a \in A$). We must check that $\phi s \in \Upsilon A$. To this end consider $a \leq b$ in A. Then, since f is monotone, we have

$$a \in \phi s \Longrightarrow s \in fa \subseteq fb \Longrightarrow s \in fb \Longrightarrow b \in \phi s$$

to give the required result.

(ii) This converts the given f into a function $\phi : S \longrightarrow \Upsilon A$. We must check that ϕ is monotone, that is

$$s \leq t \Longrightarrow \phi s \subseteq \phi t$$

for $s, t \in S$. This follows since each value of f is an upper section of S.

(iii) This produces an assignment

$$\begin{array}{c} \boldsymbol{Pos}[A,\Upsilon S] \longrightarrow \boldsymbol{Pos}[S,\Upsilon A] \\ f \longmapsto \phi \end{array}$$

which we must show is monotone. To this end suppose

$$f \longmapsto \phi \qquad g \longmapsto \psi$$

and suppose $f \leq g$. Consider any $s \in S$, so we require $\phi s \subseteq \psi s$. But for each $a \in A$ we have

$$a \in \psi s \Longrightarrow s \in fa \subseteq ga \Longrightarrow s \in ga \Longrightarrow a \in \psi s$$

to give the required result.

(iv) This produces a monotone map

$$\begin{array}{ccc} \boldsymbol{Pos}[A,\Upsilon S] & \longrightarrow & \boldsymbol{Pos}[S,\Upsilon A] \\ f & \longmapsto & \phi \end{array}$$

and in the same way we obtain a monotone map

$$\begin{array}{ccc} \boldsymbol{Pos}[A,\Upsilon S] & \longleftarrow & \boldsymbol{Pos}[S,\Upsilon A] \\ f & \longleftarrow\!\dashv & \phi \end{array}$$

in the opposite direction. Almost trivially these form an inverse pair of assignments, and hence an inverse pair of monoid isomorphisms.

(v) Finally, we must show this bijection is natural for varaition of A and S. In fact, this can be checked for separate variation of A and S.

Let

$$B \xrightarrow{\ h\ } A \qquad\qquad T \xrightarrow{\ \theta\ } S$$

be arbitraray monotone maps, and consider the induced squares

$$\begin{array}{ccc} [A,\Upsilon S] & \longleftrightarrow & [S,\Upsilon A] \\ {\scriptstyle -\circ h}\downarrow & & \downarrow{\scriptstyle h^{\leftarrow}\circ -} \\ [B,\Upsilon S] & \longleftrightarrow & [S,\Upsilon B] \end{array} \qquad \begin{array}{ccc} [A,\Upsilon S] & \longleftrightarrow & [S,\Upsilon A] \\ {\scriptstyle \theta^{\leftarrow}\circ -}\downarrow & & \downarrow{\scriptstyle -\circ\theta} \\ [A,\Upsilon T] & \longleftrightarrow & [T,\Upsilon A] \end{array}$$

where the left hand one is concerende with variation across h and the right hand one is concerened with variation across θ. We must show that both squares commute. (Strictly speaking there are four squares here, but we need deal only with two of them.)

Consider any monotone map

$$A \xrightarrow{\ f\ } \Upsilon S$$

which lives in the top right hand corner of each square. Tracking this produces

$$\begin{array}{ccc} f & \longmapsto & \phi \\ \downarrow & \begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array} & \downarrow \\ & & h^{\leftarrow}\circ\phi \\ f\circ h & \longmapsto & \psi \end{array} \qquad \begin{array}{ccc} f & \longmapsto & \phi \\ \downarrow & \begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array} & \downarrow \\ & & \phi\circ\theta \\ \theta^{\leftarrow}\circ f & \longmapsto & \chi \end{array}$$

where ϕ is obtained from f, as above, with ψ obtained from $f \circ h$ and χ obtained from $\theta^{\leftarrow} \circ f$ in an analogous way. We must show that both

$$\psi = h^{\leftarrow} \circ \phi \qquad \chi = \phi \circ \theta$$

hold. For each $a \in A, b \in B, s \in S, t \in T$ we have

$$\begin{aligned} b \in \psi s &\iff s \in (f \circ h)b &&\iff hb \in \phi s \iff b \in (h^{\leftarrow} \circ \phi)s \\ a \in \chi t &\iff t \in (\theta^{\leftarrow} \circ f)a &&\iff \theta t \in fa \iff a \in (\phi \circ \theta)t \end{aligned}$$

to give the required results.

(b) Let us write write

$$A \xrightarrow{F} [S,2] \quad A \xrightarrow{f} \Upsilon S \quad S \xrightarrow{\phi} \Upsilon \quad S \xrightarrow{\Phi} [A,2]$$

for the arrows in the various places. The outer correspondences

$$\begin{array}{ccc} [A,[S,2]] & \longleftrightarrow & [A,\Upsilon S] \\ F & \Longleftrightarrow & f \end{array} \qquad \begin{array}{ccc} [S,\Upsilon A] & \longleftrightarrow & [S,[A,2]] \\ \phi & \Longleftrightarrow & \Phi \end{array}$$

are given by

$$Fas = 1 \iff s \in fa \qquad a \in \phi s \iff \Phi sa = 1$$

for $a \in A, s \in S$. The inner correspondence is described above. Thus

$$Fas = 1 \iff s \in fa \iff a \in \phi s \iff \Phi sa = 1$$

to give

$$Fas = \Phi sa$$

that is we pass from one side to the other by merely inerting the order of the inputs.

(c) What we have here is a simple example of a contravariant adjunction, or more precisely, a schizophrenically induced contravariant adjunction. There are many examples of this in mathematics, and often the details can be a bit hard to verify.

There are several kinds of structures around and the secret is to encode in terms of 'characters', arrows to some special object S. This is combined with the currying of 2-placed functions, and the chipping, interchanging, of the order of two inputs. See the next solution for a simple example of this. ■

2.35 **Do not read this solution until you are quite sure you understand the previous solution**

The obvious thing to try is to use the 'lower' part Lemma 2.10 to produce a modification of Example 2.12

$$\begin{array}{ccc} \boldsymbol{Pos}[A,\mathcal{L}S] & & \boldsymbol{Pos}[S,\mathcal{L}A] \\ f & \Longleftrightarrow & \phi \end{array}$$

which deals with the lower sections $\mathcal{L}(\cdot)$ of a poset. We can try the same assignments

$$s \in fa \iff a \in \phi s$$

(for $a \in A, s \in S$) to produce the correspondence. After all, many of the requirements have been verified in Solution 2.34. Try that before you read the next sentence.

Are you sure you managed that. If you are sure then you are wrong. You can certainly set up a bijection of sorts at the *set* level, but not between monotone maps. For the assignments top produce lower section we need f and ϕ to be antitone, not monotone. In other words these assignments set up a bijective correspondence

$$\begin{array}{ccc} \boldsymbol{Pos}[A^{op},\mathcal{L}S] & & \boldsymbol{Pos}[S^{op},\mathcal{L}A] \\ f & \Longleftrightarrow & \phi \end{array}$$

using the opposites of the posets.

Try this.

If you want to set up a correspondence

$$\begin{array}{ccc} \boldsymbol{Pos}[A, \mathcal{L}S] & & \boldsymbol{Pos}[S, \mathcal{L}A] \\ f & \longleftrightarrow & \phi \end{array}$$

then you must use the assignments

$$s \in fa \Longleftrightarrow a \notin \phi s$$

(for $a \in A, s \in S$). This negation seems strange, but it works. Try it. ■

[*For the next solution it would be covenient to have some earlier stuff on composition of natural transformations. Is it worth doing the horizontal and vertical stuff.*]

2.36 (a) We use the bijection

$$\begin{array}{ccc} \mathcal{P}A & & \boldsymbol{Set}[A, 2] \\ X & \longleftrightarrow & p \end{array}$$

given by

$$a \in X \Longleftrightarrow pa = 1$$

(for $a \in A$). We find that for each function

$$A \xrightarrow{f} B$$

the square

$$\begin{array}{ccc} \mathcal{P}B & \xrightarrow{f^{\leftarrow}} & \mathcal{P}A \\ \downarrow & & \downarrow \\ [B, 2] & \xrightarrow[- \circ f]{} & [A, 2] \end{array}$$

commutes. In other words, the two contravariant functors

$$\boldsymbol{Set} \underset{[\cdot, 2]}{\overset{\mathcal{P}}{\rightrightarrows}} \boldsymbol{Set}$$

are naturally isomorphic.

The functor Π is just $\mathcal{P} \circ \mathcal{P}$ (and hence is covariant). In the same way we can form the composite

$$\boldsymbol{Set}[\boldsymbol{Set}[\cdot, 2], 2]$$

to obtain a covariant functor which is naturally isomorphism to Π.

Remember that the natural transformation

$$A \xrightarrow{\eta_A} \Pi A$$

is given by

$$X \in \eta_A a \Longleftrightarrow a \in X$$

for $a \in A$ and $X \subseteq A$. We viewed in terms of characters this is quite neat.

For each $a \in A$ let

$$a^\star : [A, 2] \longrightarrow 2$$

be the function given by

$$a^\star p = pa$$

for each $p \in [A, 2]$. You should check that this assignment $(\cdot)^\star$ is natural for variation of A. In other words, it produces a a natural transformation

$$I \longrightarrow \boldsymbol{Set}[\boldsymbol{Set}[\cdot, 2], 2]$$

form the identity functor.

Finally, you should check that if

$$\begin{array}{ccc} \mathcal{P}A & & \boldsymbol{Set}[A, 2] \\ X & \longleftrightarrow & p \end{array}$$

then

$$X \in \eta_A a \iff a^\star p = 1$$

for each $a \in A$.

(b) Let $\boldsymbol{Vec}_K$ be the category of vector spaces over the field K. Remember that the arrows of $\boldsymbol{Vec}_K$ are the linear transformations between the spaces.

The first dual V^* of a vector space (over a field K) is the set

$$\boldsymbol{Vec}_K[V, K]$$

of all linear transformations from V to K. In other words, it is the set of K-characters on V. It is a standard result that if V has finite dimension then so does V^* and the two spaces have the same dimension. Thus if V is finite dimensional then V and V^* are isomorphic. However, there appears to be no 'canonical' or 'natural' isomorphism between the two.

The second dual V^{**} is the set of all K-characters of V^*, the set of all linear transformations

$$V^* \longrightarrow K$$

to K. For each $a \in V$ and $\alpha \in V^*$ let

$$a^\star \alpha = \alpha a$$

to attach a function

$$a^\star : V^* \longrightarrow K$$

to each $a \in V$. It can be checked that $a^\star$ is a linear transformation, and so is a member of V^*. This gives an assignment

$$\begin{array}{ccc} V & \longrightarrow & V^{**} \\ a & \longrightarrow & a^\star \end{array}$$

which we may check is linear (an arrow in $\boldsymbol{Vec}_K$), and is natural for variation of V.

When V is finite dimensional this assignment is an isomorphism. Thus in this case V and V^* are isomorphic in some fashion or other, but V and V^{**} are naturally isomorphic. This is one of the examples which prompted an investigation of what 'natural' ought to mean and eventually lead to category theory. ■

2.37 These characters pick out precisely the open sets on the carrying space S. ■

A.2.7 For section 2.7

2.38 For each group G let

$$LG = G/\delta G$$

the quotient of G by its normalizer subgroup, and let

$$G \xrightarrow{\ \eta\ } LG$$

be the associated quotient morphism which sends $a \in G$ to the coset of all elements

$$a[x_1, y_1] \cdots [x_m, y_m]$$

for $x_1, y_1, \ldots, x_m, y_m \in G$. It is a standard result that LG is abelian and has the following universal property.

For each morphism

$$G \xrightarrow{\ f\ } A$$

to an abelain group there is a unique morphism

$$LG \xrightarrow{\ f^\sharp\ } A$$

such that the triangle

$$\begin{array}{ccc} G & \xrightarrow{\ f\ } & A \\ & \searrow_{\eta_A} \quad \nearrow_{f^\sharp} & \\ & LG & \end{array}$$

commutes.

This says that L is left adjoint to the forgetful functor. ■

2.39 The functors D and I convert a set S into the discrete poset and the indiscrete preset on S, respectively. ■

A.3 For chapter 3

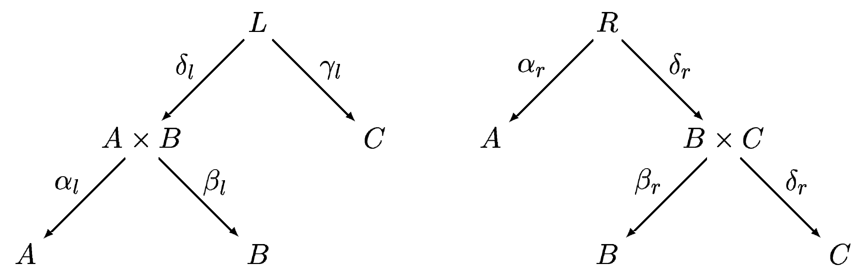
A.3.1 For section 3.1

3.1 For elements a, b of a poset the product and the sum are the meet $a \wedge b$ and the join $a \vee b$, respectively. ■

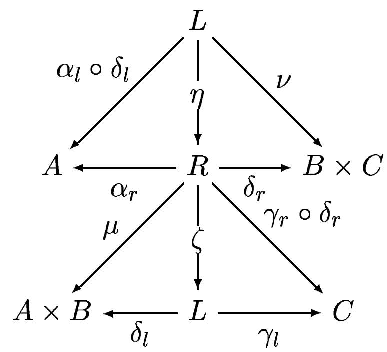
3.2 (a) Let

$$L = (A \times B) \times C \qquad R = A \times (B \times C)$$

and consider the projections



attached to these various products. Using these we build up various commuting triangles



where

$$\mu = \langle \alpha_r, \beta_r \circ \delta_r \rangle \qquad \nu = \langle \beta_l \circ \delta_l, \gamma_l \rangle$$

and η, ζ are the corresponding mediating arrows. We show that

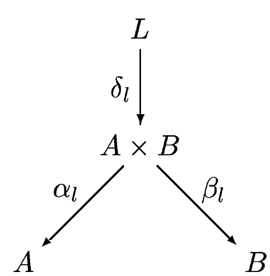
$$\zeta \circ \eta = id_L \qquad \eta \circ \zeta = id_R$$

using the uniqueness of these various mediators. Notice that

$$\begin{array}{ll} \alpha_l \circ \mu = \alpha_r & \beta_r \circ \nu = \beta_l \circ \delta_l \\ \beta_l \circ \mu = \beta_r \circ \delta_r & \gamma_r \circ \nu = \gamma_l \end{array}$$

hold (by certain mediating properties).

Consider now the diagram



where the bottom part is a product wedge. Because of this the only arrow

$$L \xrightarrow{\theta} A \times B$$

with

$$\alpha_l \circ \theta = \alpha_l \circ \delta_l \qquad \beta_l \circ \theta = \beta_l \circ \delta_l$$

is the given δ_l. We use this to get at the composite $\zeta \circ \eta$.

Using various commuting triangles and equalities from above we have

$$\alpha_l \circ \delta_l \circ \zeta \circ \eta = \alpha_l \circ \mu \circ \eta = \alpha_r \circ \eta = \alpha_l \circ \delta_l$$

and

$$\beta_l \circ \delta_l \circ \zeta \circ \eta = \beta_l \circ \mu \circ \eta = \beta_r \circ \delta_r \circ \eta = \beta_r \circ \nu = \beta_l \circ \delta_l$$

so that

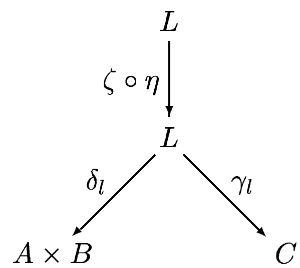
$$\delta_l \circ \zeta \circ \eta = \delta_l$$

by the above remark.

A similar calculation gives

$$\gamma_l \circ \zeta \circ \eta = \gamma_l$$

and then the same trick with the diagram



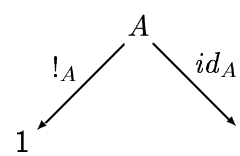
gives

$$\zeta \circ \eta = id_L$$

which is one of the required equalities.

A repeat of this argument starting from a different diagram give the other required equality.

(b) We show that

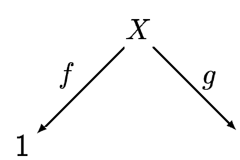


is a product wedge, and the use the fact that different selected products of the same pair are isomorphic.

[*Has that been proved anywhere*]

Here $!_A$ is the unique arrow from A to the terminal object.

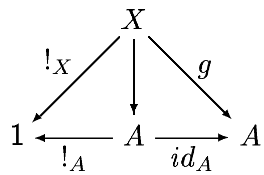
Consider any wedge



from an arbitrary object X. Here f must be the unique arrow $!_X$. Furthermore, there is precisely one arrow

$$X \longrightarrow A$$

such that



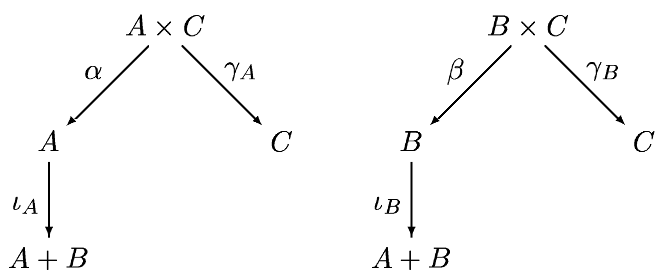
commutes, namely g, and so we do have a product wedge, as required. ■

3.5 (a) For the time being fix A, B, C and let

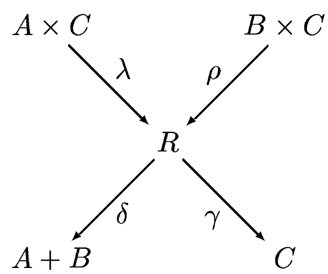
$$L = A \times C + B \times C \qquad R = (A + B) \times C$$

be the two compounds. There are various product or sum wedges we can look at.

We have



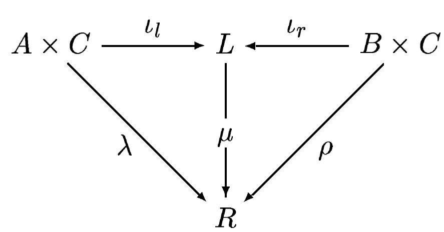
where $\alpha, \beta, \gamma_A, \gamma_B$ are projections from a product, and ι_A, ι_B are insertions to a sum. Since R is a product this gives us arrows



where δ, γ are projections from the product, and λ, ρ are unique mediating arrows such that

$$\delta \circ \lambda = \iota_A \circ \alpha \quad \gamma \circ \lambda = \gamma_A \qquad \delta \circ \rho = \iota_B \circ \beta \quad \gamma \circ \rho = \gamma_B$$

hold. Finally, since L is a sum we have a unique mediator such that



commutes. Here ι_l, ι_r are the associated sum insertions.

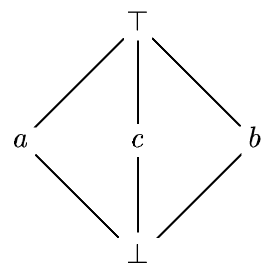
This gives us the existence of the required arrow μ. To verify naturality we have to see how μ changes as A, B, C vary along arrows. This is a bit tedious, but because of all the uniqueness we generate many commutinh squares.

(b) We may view the compounds L, R as functors

$$\boldsymbol{C}^3 \longrightarrow \boldsymbol{C}$$

and then we find that μ is a natural transformation in the technical sense.

(c) A counter example can be found in a suitable poset. Consider the lantern poset



as shown. Then

$$a \wedge c = \bot = b \wedge c \qquad a \vee b = \top$$

to give

$$l = (a \wedge c) \vee (b \wedge c) = \bot \qquad r = (a \vee b) \wedge c = c$$

and hence $r \not\leq l$. ■

3.6 (a) A pointed set (A, a) is a set with a distinguished element. The associated morphisms are functions that preserve the distinguished element.

Binary products can be obtained using the obvious cartesian product construction.

Binary sums are a bit more interesting, and tedious. Consider

$$(A, a) \qquad (B, b)$$

to pointed sets. Consider two disjoint sets L, R together with bijections

$$L \xrightarrow{\ l^-\ } A - \{a\} \qquad B - \{b\} \xleftarrow{\ r^-\ } R$$

where in each case the distinguished element has been removed. (Such sets L, R can be obtained from $A - \{a\}, B - \{b\}$ by the usual kind of tagging trick. However, the details of this are not important and, in fact, a distraction.)

Let

$$S = L \cup R \cup \{*\}$$

where $*$ is a new element. This gives a pointed set $(S, *)$ and a pair of morphisms

$$S \xrightarrow{\ l\ } A - \{a\} \qquad B - \{b\} \xleftarrow{\ r\ } R$$

where l, r are the extensions of l^-, r^- which send the distinguished element to $*$. A tedious argument now shows that this is a sum wedge.

(b) An object in this category is a pair (A, X) where A, X are sets with $X \subseteq A$. A morphism

$$(A, X) \xrightarrow{\ f\ } (B, Y)$$

in this category is a function $f : A \longrightarrow B$ with $f[X] \subseteq Y$, that is f must send each element of the distinguished subset X to an element of the distinguished subset Y.

It is routine to check that the cartesian product

$$(A \times B, X \times Y)$$

with the obvious projections gives the categorical product.

To produce the sum first obtain isomorphisms

$$(A, X) \xrightarrow{\alpha} (A', X') \qquad (B, Y) \xrightarrow{\beta} (B', Y')$$

where $A' \cap B' = \emptyset$. (This can be done using the standard tagging trick.) We now show that

$$\begin{array}{ccc}
(A, X) & & \\
\alpha \downarrow & & \\
(A', X') & \hookrightarrow & \\
 & & (A' \cup B', X' \cup Y') \\
(B', Y') & \hookrightarrow & \\
\beta \uparrow & & \\
(B, Y) & &
\end{array}$$

is a sum wedge in the category. To this end consider a diagram

$$\begin{array}{cccc}
(A, X) & & \searrow^{f} & \\
\alpha \downarrow & & & \\
(A', X') & \hookrightarrow & & \\
 & (A' \cup B', X' \cup Y') & & (C, Z) \\
(B', Y') & \hookrightarrow & & \\
\beta \uparrow & & & \\
(B, Y) & & \nearrow_{g} &
\end{array}$$

which requires a unique arrow

$$(A' \cup B', X' \cup Y') \xrightarrow{h} (C, Z)$$

to mediate between the given f and g. Because of the disjointness this arrow can only be given by

$$hs = \begin{cases} fa \text{ if } s = \alpha a \\ gb \text{ if } s = \beta b \end{cases}$$

(for $s \in A' \cup B'$). It is routine to check that this does, in fact, give a morphism. ■

3.8 Consider any morphism

$$A \amalg B \xrightarrow{\ n\ } G$$

such that

$$n \circ u = f \qquad n \circ v = g$$

holds. We know the morphism $h^\sharp$ is special, so it suffices to show

$$n \circ m = h^\sharp$$

and then apply Lemma 3.11 to obtain the required uniqueness of n.

Expanding u and v we have

$$n \circ m \circ k \circ i = f = h \circ i \qquad n \circ m \circ k \circ j = g = h \circ j$$

and hence

$$n \circ m \circ k = h = h^\sharp \circ k$$

since i and j are insertions into a sum in ***Set***.

Now both

$$n \circ m \qquad h^\sharp$$

are morphisms from a freely generated group K which agree on the generators (via k). Hence the freeness gives the required equality. ■

3.9 Let A, B be a pair of abelain groups (written multiplicatively). We define the cartesian product $A \times B$ in the usual way, and then check that this is the categorical product.

The proof that $A \times B$ is the categorical sum in ***Abg*** is less routine.

First of all we observe that the functions

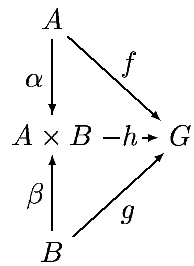
$$\begin{array}{rcl} A & \xrightarrow{\ \alpha\ } & A \times B \\ a & \longmapsto & (a, 1) \end{array} \qquad \begin{array}{rcl} B & \xrightarrow{\ \beta\ } & A \times B \\ b & \longmapsto & (1, b) \end{array}$$

are morphisms. We show that these provide the sum insertions.

Consider any pair of morphisms

$$A \xrightarrow{\ f\ } G \xleftarrow{\ g\ } B$$

to an abelian group G. We require a commuting diagram



for some unique morphism h.

Suppose first that there is such a morphism h. For each $a \in A, b \in B$ we have

$$(a, b) = (a, 1) \cdot (1, b) = (\alpha a) \cdot (\beta b)$$

in $A \times B$. Hence

$$h(a, b) = h((\alpha a) \cdot (\beta b)) = h(\alpha a) \cdot h(\beta b)$$

to give

$$(?) \quad h(a, b) = (fa) \cdot (gb)$$

in G. This shows there can be at most one mediating morphism h, and it must be given by this rule. Thus it suffices to show that (?) does give a morphism.

Consider $a_1, a_2 \in A, b_1, b_2 \in B$. Then

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1a_2, b_1b_2)$$

so that

$$\begin{aligned} h((a_1, b_1) \cdot (a_2, b_2)) &= h(a_1a_2, b_1, b_2) \\ &= f(a_1a_2)g(b_1, b_2) \\ &= (fa_1)(fa_2)(gb_1)(gb_2) \\ &= (fa_1)(gb_1)(fa_2)(gb_2) \quad = h(a_1, b_1)h(a_2, b_2) \end{aligned}$$

as required. Here the crucial step is the fourth equality, for this uses the comutativity of G.

This proof does not work if G is not abelian. In fact, by a slight reworking we can show that for abelian groups A, B the product $A \times B$ is not the sum in ***Grp*** (unless one of A or B is trivial).

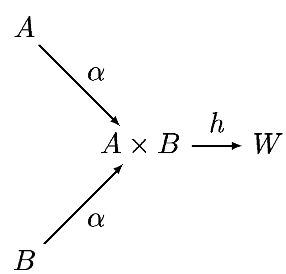
To see this observe that the set

$$W(A, B)$$

of 'collapsed' words gives a group in the obvious way, with embeddings

$$A \longrightarrow W \qquad\qquad B \longrightarrow W$$

via the singleton words. (A word is 'collapsed' if not two consecutive letters come from the same component group.) However, there is no mediating morphism



which gives these embeddings. For suppose there is some h.

For $a \in A, b \in B$ we have

$$(a, b) = (\alpha a) \cdot (\beta b)$$

so that

$$h(a, b) = h(\alpha a)h(\beta b) = ab$$

in W. In particular

$$h(a, 1) = a \qquad h(1, b) = b$$

for arbitrary a, b. But

$$(a, 1) \cdot (1, b) = (a, b) = (1, b) \cdot (a, 1)$$

and hence

$$ab = h(a, 1) \cdot h(1, b) = h((a, 1) \cdot (1, b)) = h((1, b) \cdot (a, 1)) = h(1, b) \cdot h(a, 1) = ba$$

which is false (unless either a or b is 1). ■

A.3.2 For section 3.2

3.10 (a) If the quiver is just one arrow

$$A \longrightarrow B$$

then the limit (left limit) is the identity arrow on A.

If the quiver has precisely two arrows

$$A \rightrightarrows B$$

then the limit is the equalizer of this pair.

For more arrows we move through the quiver taking equaklizers as we go. Suppose we have a quiver

$$A \begin{array}{c} \longrightarrow \\ \xrightarrow{\mathsf{f}} \\ \xrightarrow[g]{} \end{array} B$$

which consists of a bunch of arrows f and another arrow g. We first find the limit of the bunch f

$$L \xrightarrow{l} A \begin{array}{c} \longrightarrow \\ \mathsf{f} \\ \longrightarrow \end{array} B$$

to give a common arrow

$$L \xrightarrow{\text{‘}\mathsf{f} \circ l\text{’}} B$$

(with a suggestive notation). Now consider the parallel pair (small quiver)

$$L \begin{array}{c} \xrightarrow{\text{‘}\mathsf{f} \circ l\text{’}} \\ \xrightarrow[g \circ l]{} \end{array} B$$

using a genuine compsite for the other component. Let

$$E \xrightarrow{\;m\;} L \underset{g\circ l}{\overset{\text{‘f}\circ l\text{’}}{\rightrightarrows}} B$$

be the equalizer of this pair. A simple argument show that the composite $l \circ m$

$$E \xrightarrow{\;l\circ m\;} A \begin{array}{c} \longrightarrow \\ \xrightarrow{\;\mathsf{f}\;} \\ \underset{g}{\longrightarrow} \end{array} B$$

is the limit of the original quiver.

(b) This proof doesn't generalize to infinite quivers. In fact, the result isn't true. ■

3.11 (a) Consider an equalizer

$$E \xrightarrow{\;m\;} A \rightrightarrows B$$

of a pair of arrows. Consider also a parallel pair

$$S \underset{g}{\overset{f}{\rightrightarrows}} E$$

such that

$$m \circ f = m \circ g$$

holds. We require $f = g$. But this composite makes equal the origianl two arrows, and so must factor uniquely through the mediating arrow m. In other words $f = g$, as erquired.

(b) The dual result says that each coequalizer is epic. The proof is the same with the arrows turned around. ■

A.3.3 For section 3.3

3.12 The pullback of

$$\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow[\alpha]{} & C \end{array}$$

is easy. we first take the product $A \times B$ and then take the subset $E \subseteq A \times B$ of all pairs (a, b) for which $fa = gb$.

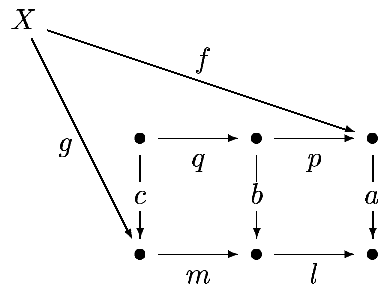
This can be generalized, as in Exercise 3.14.

[*sort out pushouts*] ■

3.13 We need to label the arrows (but not the objects). Consider a commuting diagram

$$\begin{array}{ccccc} \bullet & \xrightarrow{\;q\;} & \bullet & \xrightarrow{\;p\;} & \bullet \\ {\scriptstyle c}\downarrow & & {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ \bullet & \xrightarrow[\;m\;]{} & \bullet & \xrightarrow[\;l\;]{} & \bullet \end{array}$$

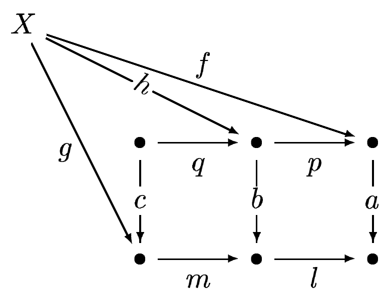
(a) Suppose the two squares are pulbacks and consider arrows f, g



for which

$$l \circ m \circ g = a \circ f$$

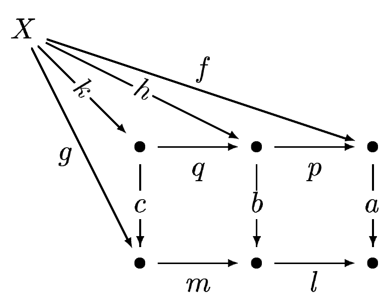
holds. From the right hand pullback there is a unique arrow h



such that

$$m \circ g = b \circ h \qquad f = p \circ h$$

hold. But now, from the left hand pullback there is a unique arrow k



such that

$$g = c \circ k \qquad h = q \circ k$$

hold. Since

$$f = p \circ h = p \circ q \circ k$$

this gives a required common factorization of f and g.

We must show that this factorization is unique. To this end suppose we have arrow k_1, k_2 such that

$$g = c \circ k_i \qquad f = p \circ q \circ k_i$$

for $i = 1, 2$. With

$$h_i = q \circ k_i$$

we have

$$m \circ g = m \circ c \circ k_i = b \circ q \circ k_i = b \circ h_i \qquad f = p \circ q \circ k_i = p \circ h_i$$

and hence

$$q \circ k_1 = h_1 = h_2 = q \circ k_2$$

by the mediating property of the right hand pullback. With

$$h = h_1 = h_2$$

we have

$$g = c \circ k_i \qquad h = q \circ k_i$$

and hence

$$k_2 = k_2$$

by the mediating property of the left hand pullback. ■

3.14 Starting from a wedge The pullback of

$$\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow[\alpha]{} & C \end{array}$$

we first form the product $A \times B$ to obtain a square

$$\begin{array}{ccc} A \times B & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow[\alpha]{} & C \end{array}$$

where the new arrows are the projections. This need not commute so we form the equalizer

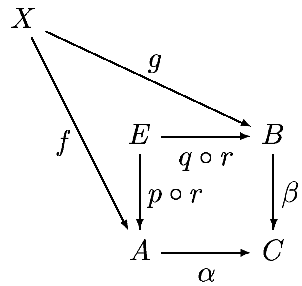
$$E \xrightarrow{\ r\ } A \times B \overset{\beta \circ q}{\underset{\alpha \circ p}{\rightrightarrows}} C$$

of the two compsite arrows. We show that

$$\begin{array}{ccc} E & \xrightarrow{q \circ r} & B \\ {\scriptstyle p \circ r}\downarrow & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow[\alpha]{} & C \end{array}$$

is a pullback.

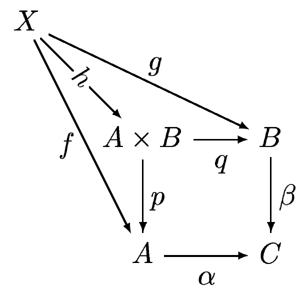
To this end consider arrows f, g



such that

$$\alpha \circ f = \beta \circ g$$

holds. From the product property we obtain an arrow h



such that

$$f = p \circ h \qquad g = q \circ h$$

hold. But now

$$\alpha \circ p \circ h = \alpha \circ f = \beta \circ g = \beta \circ q \circ h$$

to show that h makes equal the two parallel arrows $\alpha \circ p, \beta \circ q$. From the equalizer property we obtain an arrow k

$$X \xrightarrow{\;k\;} E \xrightarrow{\;r\;} A \times B$$

such that

$$h = r \circ k$$

holds. With this we have

$$f = p \circ h = p \circ r \circ k \qquad g = q \circ h = q \circ r \circ k$$

which is the required factorization of f and g.

Of course, we must show that k is the only possible arrow through which f and g factorize in this way.

To this end consider any arrow k such that

$$f = p \circ r \circ k \qquad g = q \circ r \circ k$$

hold. We show first that the composite

$$h = r \circ k$$

is uniquely determined and independent of k. For this we observe that

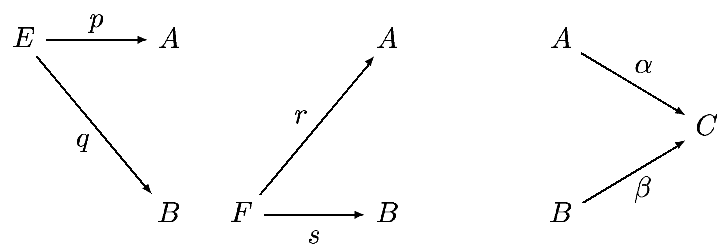
$$p \circ h = p \circ r \circ k = f \qquad q \circ h = q \circ r \circ k = g$$

and hence h is uniqely determined by thwe product property. Finally we check that

$$\alpha \circ p \circ f = \beta \circ q \circ h$$

and hence, by the equalizer property, this unique h factorizes uniquely through r. Since $h = r \circ k$, this shows that k is uniquely determined. ■

3.15 We need to label the arrows.



We will use the uniqueness of mediating arrows several times. Here is the basic trick.

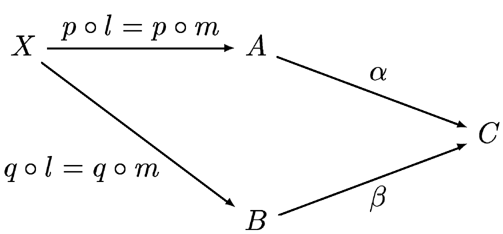
Suppose we have a parallel pair of arrows

$$X \overset{l}{\underset{m}{\rightrightarrows}} E$$

such that both

$$p \circ l = p \circ m \qquad q \circ l = q \circ m$$

hold. Then, since both l and m mediate for the commuting square



we have

$$l = m$$

(by the uniqueness of mediators).

As a particular case of this, the only arrow

$$E \xrightarrow{h} E$$

for which

$$p \circ h = p \qquad q \circ h = q$$

is $h = id_E$.

There are similar properties of F.

(In fact, every limit object has similar properties.)

Now, since

$$\alpha \circ r = \beta \circ s$$

and E is the limit corner of a pullback square, we have

$$r = p \circ e \qquad s = q \circ e$$

for some unique arrow e in the position shown. Similarly, swithching the roles of E and F, we have

$$p = r \circ f \qquad q = s \circ f$$

for some unique arrow f in the position shown. Using these equalities we have

$$p \circ e \circ f = r \circ f = p \qquad q \circ e \circ f = s \circ f = q$$

and hence

$$e \circ f = id_E$$

by the remarks above. Similar

$$f \circ e = id_F$$

by a dual argument. ■

A.3.4 For section 3.4

3.16 The idea of the proof is simple. A (finite) digram $\mathbb{D}$ is a (finite) collections of objects and a (finite) collection of arrows between these objects to form a certain configuration. To obtain the limit we first take the product P of all the family of objects. This produces an apex object P and a projection arrow to each object of $\mathbb{D}$. However, these projections may not 'solve' $\mathbb{D}$, that is they may not combine with the arrows of $\mathbb{D}$ to produce commuting triangles. To cure this SD start to refine P by taking equalizer, one for each triangle that does not commute.

That is the idea of one proof (and one that can be used in practise for particular cases). However, the general details can be a bit messy. Here is another proof that works for arbitrary diagrams (provided the category allows arbitrary products).

We need to set up the notation for a diagram in more detail.

[*Perhaps this should be put into the section.*]

The diagram $\mathbb{D}$ consists of two indexed families.

$$\text{Objects} \quad \mathbb{D}(Ob) = \big(A(v) \mid v \in V\big)$$

$$\text{Arrows} \quad \mathbb{D}(Ar) = \big(S(e) \xrightarrow{\delta(e)} T(e) \mid e \in E\big)$$

Think of V as the set of vertexes and E as the set of edges of a graph. This graph gives a template for the diagram in the category. Thus $\mathbb{D}$ assigns an object $A(v)$ to each vertex v and an arrow $\delta(e)$ to each edge e. (This is like a functor except there are no required commuting conditions.)

For each edge index $e \in E$ the arrow

$$S(e) \xrightarrow{\delta(e)} T(e)$$

passes between two object of $\mathbb{D}$. Thus for each $e \in E$ there are associated vertex indexes

$$s(e) \in V \qquad t(e) \in V$$

such that

$$S(e) = A(s(e)) \qquad T(e) = A(t(e))$$

are the source and target objects of $\delta(e)$.

How can we obtain the limit of $\mathbb{D}$?

We start by forming the products

$$P = \prod \big(A(v) \,|\, v \in V\big) \qquad Q = \prod \big(T(e) \,|\, e \in E\big)$$

indexes by V and E, respectively. This gives us two families

$$P \xrightarrow{p(v)} A(v) \qquad\qquad Q \xrightarrow{q(e)} T(e)$$

again indexed by V and E.

Next we set up a parallel pair

$$P \underset{\tau}{\overset{\sigma}{\rightrightarrows}} Q$$

for P to Q. To do this we use the universal product property of Q in two ways.

For each $e \in E$ we have a composite arrow

$$P \xrightarrow{p(s(e))} A(s(e)) = S(e) \xrightarrow{\delta(e)} T(e)$$

and this whole family must factor through some unique arrow

$$P \xrightarrow{\sigma} Q$$

by the product property of Q. Thus we have

$$(s) \quad \delta(e) \circ p(s(e)) = q(e) \circ \sigma$$

for each $e \in E$.

For each $e \in E$ we have an arrow

$$P \xrightarrow{p(t(e))} A(t(e)) = T(e))$$

and again this whole family must factor through some unique arrow

$$P \xrightarrow{\tau} Q$$

by the product property of Q. Thus we have

$$(t) \quad p(t(e)) = q(e) \circ \tau$$

for each $e \in E$.

Next we let

$$L \xrightarrow{\ \rho\ } P \underset{\tau}{\overset{\sigma}{\rightrightarrows}} Q$$

be the equalizer of the parallel pair. In particular

$$(=) \quad \sigma \circ \rho = \tau \circ \sigma$$

hold.

Finally we take the composite

$$\begin{array}{ccc} L & \xrightarrow{\ \alpha(v) = p(v) \circ \rho\ } & A(v) \\ & {\scriptstyle \rho} \searrow \quad \nearrow {\scriptstyle p(v)} & \\ & P & \end{array}$$

for each $v \in V$. We show that the family

$$\boldsymbol{\alpha} = (\alpha(v) \mid v \in V)$$

structure L as the limit of $\mathbb{D}$. This is done in a series of steps.

First we show that $\boldsymbol{\alpha}$ solves $\mathbb{D}$, that is

$$\begin{array}{ccc} & & S(e) = A(s(e)) \\ & \nearrow {\scriptstyle \alpha(s(e))} & \Big\downarrow {\scriptstyle \delta(e)} \\ L & & \\ & \searrow {\scriptstyle \alpha(t(e))} & \\ & & T(e) = A(t(e)) \end{array}$$

commutes for each $e \in E$. But

$$\delta(e) \circ \alpha(s(e)) = \delta(e) \circ p(s(e)) \circ \rho = q(e) \circ \sigma \circ \rho = q(e) \circ \tau \circ \rho = p(t(e)) \circ \rho = \alpha(t(e))$$

are required. Here the central equalities follow by (s),(=),(t), respectively.

Secondly we show that $\boldsymbol{\alpha}$ provides a mediator for an arbitrary solution to $\mathbb{D}$. To this end let

$$\big(X \xrightarrow{\ \xi(v)\ } A(v) \mid v \in V\big)$$

by any solution. Thus

$$(\text{sol}) \quad \delta(e) \circ \xi(s(e)) = \xi(t(e))$$

for each $e \in E$. We require some arrow

$$X \xrightarrow{\ \ \xi\ \ } L$$

such that

$$\xi(v) = \alpha \circ \xi$$

for each $v \in V$.

The product property of P gives an arrow

$$X \xrightarrow{\ \pi\ } P$$

with

$$(\text{proj}) \quad \xi(v) = p(v) \circ \pi$$

for each $v \in V$. We show that π makes equal σ and τ.

Consider the family of arrows

$$X \xrightarrow{\ \pi\ } P \underset{\tau}{\overset{\sigma}{\rightrightarrows}} Q \xrightarrow{\ q(e)\ } T(e)$$

indexed by $e \in E$. We have

$$q(e) \circ \sigma \circ \pi = \delta(e) \circ p(s(e)) \circ \pi = \delta(e) \circ \xi(s(e)) = \xi(t(e)) = p(t(e)) \circ \pi = q(e) \circ \tau \circ \pi$$

using (s), (proj), (sol), (proj), (t) in that order. The product property of Q now gives

$$\sigma \circ \pi = \tau \circ \pi$$

and hence π does make equal σ, τ, as claimed.

The equalizer property of ρ now gives an arrow

$$X \xrightarrow{\ \xi\ } L$$

such that

$$\pi = \rho \circ \xi$$

holds.

With this, for each $v \in V$ we have

$$\alpha(v) \circ \xi = p(v) \circ \rho \circ \xi = p(v) \circ \pi = \xi(v)$$

to show that ξ mediates between the special solution on L and the arbitrary solution on X.

This almost completes the proof. It remains to show that ξ is the only possible mediator for this arbitrary solution on X. To this end suppose

$$\alpha(v) \circ \chi = \xi(v)$$

for each $v \in V$. Then, since $\alpha(v) = p(v) \circ \rho$, the parallel composites

$$X \underset{\xi}{\overset{\chi}{\rightrightarrows}} L \xrightarrow{\ \rho\ } P \xrightarrow{\ p(v)\ } A(v)$$

agree for each $v \in V$. Thus th e product property of P gives

$$\rho \circ \chi = \rho \circ \xi$$

and hence

$$\chi = \xi$$

by the uniqueness of mediators for the equalizer. ■

3.17 This is a generalization of Lemma 3.22 which was dealt with in Exercise 3.15.

Suppose

$$\mathbb{D}(Ob) = \big(A(v) \mid v \in V\big)$$

is the indexed family of objects of a diagram $\mathbb{D}$*, and suppose the two families of arrows*

$$\boldsymbol{l} = \big(L \xrightarrow{l(v)} A(v) \mid v \in V\big) \qquad \boldsymbol{m} = \big(M \xrightarrow{m(v)} A(v) \mid v \in V\big)$$

are limits for $\mathbb{D}$*. Then there are unique arrows*

$$L \xrightarrow{m} M \qquad M \xrightarrow{l} L$$

such that

$$l(v) = m(v) \circ m \qquad m(v) = l(v) \circ l$$

for all $v \in V$*. Furthermore,* l *and* m *are an inverse pair of ismorphisms.*

The existence of l and m are immediate consequences of the limiting properties of $\boldsymbol{l}$ and $\boldsymbol{m}$. So is the isomorphism properties, but that need a preliminary remark.

Suppose we have a parallel pair

$$X \underset{g}{\overset{f}{\rightleftarrows}} L$$

such that

$$l(v) \circ f = l(v) \circ g$$

for all $v \in V$. Then, because mediators are unique, we have $f = g$. In particular, if an arrow

$$L \xrightarrow{h} L$$

satisfies

$$l(v) \circ h = l(v)$$

for all $v \in V$, then $h = id_L$ (since id_L does satifies this family of equalities). A similar remark holds for M.

Now consider the mediators l and m. For each $v \in V$ we have

$$l(v) \circ l \circ m = m(v) \circ m = l(v)$$

and hence

$$l \circ m = id_L$$

by the observation for L. Similarly

$$m \circ l = id_M$$

by the observation for M. ■

A.3.5 For section 3.5

[*to be done*]

A.4 For chapter 4

[*Some of these solutions may have 'A' and 'B' the wrong way round.*]

A.4.1 For section 4.1

4.1 (a) A poset S is cartesian exactly when it is a $\wedge$-semilattice. That is it has a top element $\top$ and carries a binary operation

$$\wedge : S \times S \longrightarrow S$$

such that

$$x \leq a \wedge b \Longleftrightarrow x \wedge a \text{ and } x \leq b$$

for all $a, b, x \in S$.

(b) A poset S is cartesian closed exactly when it is cartesian and carries an implication, that is a binary operation

$$\supset : S \times S \longrightarrow S$$

such that

$$x \leq (a \supset b) \Longleftrightarrow x \wedge a \leq b$$

for all $a, b, x \in S$.

A.4.2 For section 4.2

4.2 Consider any monotone map

$$B \xrightarrow{\ h\ } C$$

between posets B, C. We must show that the induced ***Set***-square

$$\begin{array}{ccccc}
(p, x) & & (K \Rightarrow B) \times K & \xrightarrow{\epsilon_B} & B \\
\big\downarrow & & \big\downarrow & & \big\downarrow h \\
(h \circ p, x) & & (K \Rightarrow C) \times K & \xrightarrow[\epsilon_C]{} & C
\end{array}$$

commutes. The left hand arrow has not been named but its behaviour has been given. The proof is not trivial. We track (p, x) round both paths to the opposite corner. ■

4.3 By way of contradiction suppose ***Abg*** is cartesian closed. Thus for each abelian groups K, A, B the three sets of arrows

$$\boldsymbol{Abg}[A, (K \Rightarrow B)] \qquad \boldsymbol{Abg}[A \times K, B]$$

are in bijective correspondence, and hence have the same size. Now consider the 1-element group 1. This is both the initial and the final object of ***Abg***. Setting $A = 1$ we see that

$$\boldsymbol{Abg}[1, (K \Rightarrow B)] \qquad \boldsymbol{Abg}[1 \times K, B]$$

have the same size. The left hand set has just one member (since 1 is initial). Also

$$1 \times K \cong K$$

(since 1 is final). Thus, if ***Abg*** is cartesian closed, then for each pair K, B of abelian groups there is precisely one arrow

$$K \longrightarrow B$$

which is nonsense. ■

A.4.3 For section 4.3

4.5 (a,b) These are routine.

(c) By direct calculation, for each $\phi \in (K \Rightarrow B)$ and $x \in K$ we have

$$\phi^{**}(x) = \phi^{*}(x^{*})^{*} = \phi(x^{**})^{**} = \phi(x)$$

to show $\phi^{**} = \phi$, as required.

(d) We are given

$$(lb)^{*} = l(b^{*})$$

for all $b \in B$, and we require

$$L(\phi)^{*} = L(\phi^{*})$$

for $\phi \in (K \Rightarrow B)$. For each $x \in K$ we have

$$L(\phi)^{*} = (L(\phi)(x^{*}))^{*} = ((l \circ \phi)(x^{*}))^{*} = ((l(\phi)(x^{*})))^{*} = l(\phi(x^{*})^{*}) = l(\phi^{*}(x)) = L(\phi^{*})x$$

to give the required result. Here the fourth equality follows by the morphism property of l.

(e) Since in $A \times K$ we have

$$(a, x)^{*} = (a,^{*}, x^{*})$$

for each $a \in A$ and $x \in K$, we must show

$$f^{\sharp}(a, x)^{*} = f^{\sharp}(a^{*}, x^{*})$$

for such a, x.

Since the given f is a morphism we have

$$(fa)^{*} = f(a^{*})$$

and hence

$$fa^{a}stx = (fa)^{*}x = (fax^{*})^{a}st$$

using the construction of the involution on $(K \Rightarrow B)$. In particular

$$fax = fa^{ast*}x = (fa^{a}stx^{*})^{*}$$

so that

$$(fax)^{*} = fa^{a}stx^{*}$$

for each $a \in A$ and $x \in K$. With this we have

$$f^\sharp(a,x)^* = (fax)^* = fa^*x^*f^\sharp(a^*, x^ast)$$

as required.

(f) We must show

$$(g_\flat a)^* = g_\flat a^ast$$

for $a \in A$. but, remembering the construction of the involution on $(K \Rightarrow B)$, for each $x \in K$ we have

$$(g_\flat a)^*x = (g_\flat ax^*)^* = g(a,x^*)^* = g(a^*,x) = g_\flat a^ast$$

to give the required result. here the third equality follows since g is a morphism.

(g) This is almost trivial.

(h) The monoid $R = \{-1, 1\}$ under multiplication gives these algebras. ■

4.6 (a,b) These are routine.

(c) Observe that for $(\phi_0, \phi_1) \in (K \Rightarrow B)$ we have

$$(\phi_0, \phi_1)^\bullet = (\phi_0, \phi_1) \in (K \Rightarrow B)$$

since ϕ_0 is a morphism and for for each $x \in K$ we have $(\phi_0 x)^\bullet = (\phi_0 x)^\bullet$ (trivially). Also

$$(\phi_0, \phi_1)^{\bullet\bullet} = (\phi_0, \phi_0)^\bullet = (\phi_0, \phi_0) = (\phi_0, \phi_1)^\bullet$$

to show that the constructed operation $(\cdot)^\bullet$ is idempotent.

(d) We are given

$$(lb)^\bullet = l(b^\bullet)$$

for all $b \in B$. For given $\phi_0, \phi_1) \in (K \Rightarrow B)$ we must show

$$\begin{array}{ll} (0) & l \circ \phi_0 \text{ is a morphism} \\ (\bullet) & ((l \circ \phi_0)x)^\bullet = ((l \circ \phi_1)x)^\bullet \end{array}$$

for each $x \in K$. Property (0) is immediate since both l and ϕ_0 are morphisms. For $(\bullet)$ we have

$$((l \circ \phi_0)x)^\bullet = (l(\phi_0 x))^\bullet = (l(\phi_0 x)^\bullet) = (l(\phi_1 x)^\bullet) = (l(\phi_1 x))^\bullet = ((l \circ \phi_0)x)^\bullet$$

as required. Here the second and fourth equality follows from the morphism property of l, and the third equality uses the condition on the pair (ϕ_0, ϕ_1).

(e) Since in $A \times K$ we have

$$(a,x)^\bullet = (a,^\bullet, x^\bullet)$$

for each $a \in A$ and $x \in K$, we must show

$$f^\sharp(a,x)^\bullet = f^\sharp(a^\bullet, x^\bullet)$$

for such a, x. To prove that we make an observation.

Since the given f is a morphism we have

$$(fa)^\bullet = f(a^\bullet)$$

for each $a \in A$. But

$$fa = ((fa)_0, (fa)_1)$$

where each $(fa)_i$ is a certain function $K \longrightarrow B$. In particular

$$(fa)^\bullet = \big((fa)_0, (fa)_0\big) \qquad f(a^\bullet) = \big((f(a^\bullet))_0, (f(a^\bullet))_1\big)$$

and hence

$$(f(a^\bullet))_0 = (fa)_0 = (f(a^\bullet))_1$$

for each $a \in A$. We also remember that

$$fa \in (K \Rightarrow B)$$

and $(fa)_0$ is a morphism so that

$$((fa)_0 x)^\bullet = ((fa)_1 x)^\bullet \qquad ((fa)_0 x)^\bullet = (fa)_0(x^\bullet)$$

for each $x \in K$.

With these we have

$$f^\sharp(a, x)^\bullet = ((fa)_1 x)^\bullet = ((fa)_0 x)^\bullet = (fa)_0(x^\bullet) = (f(a^\bullet))_1(x^\bullet) = f^\sharp(a^\bullet, x^\bullet)$$

as required. [`Check this`]

(f) We must show

$$\begin{array}{ll} (0) & g(a^\bullet, \cdot) \text{ is a morphism} \\ (\bullet) & g(a^\bullet, x)^\bullet = g(a, x)^\bullet \end{array}$$

for each $a \in A, x \in K$. But g is a morphism so

$$g(a, x)^\bullet = g(a^\bullet, x^\bullet)$$

and hence

$$g(a^\bullet, x)^\bullet = g(a^{\bullet\bullet}, x^\bullet) = g(a^\bullet, x^\bullet)$$

to give (0). Using this we have

$$g(a^\bullet, x)^\bullet = g(a^\bullet, x^\bullet) = g(a, x)^\bullet$$

to give $(\bullet)$.

(g) We must show

$$f^\sharp{}_\flat = f \qquad g_\flat{}^\sharp = g$$

for f and g as above. But for $a \in A$ we have

$$(f^\sharp a)_0 = f^\sharp(a^\bullet, \cdot) = (fa^\bullet)_1 = (fa)_0$$

by the property of f used in part (e). Also

$$(f^\sharp a)_1 = f^\sharp(a, \cdot) = (fa)_1$$

to give the left hand requirement. The right hand requirement is almost trivial.

(h) The monoid $R = \{0, 1\}$ under multiplication gives these algebras. ■

A.4.4 For section 4.4

[`to be done`]