Introduction to gauge gravity duality

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Chapter 1

Introduction

Let us start by giving the (rough) statement of gauge gravity duality:

- Some quantum field theories are equivalent to (quantum) gravity theories.
- In particular limits, the gravity theory becomes classical and the corresponding quantum field theory (QFT) strongly coupled.

The second point makes the duality particularly useful since by other methods, dynamical processes are inaccessible in the strongly coupled regime of QFTs: Normally, QFT calculations are done by means of perturbation theory, but this only works at weak coupling. Lattice gauge theory might be a powerful way out of this dilemma, but it is hard to use for capturing dynamics. Also, for technical reasons, it is problematic at high temperature or large density and chemical potential.
The purpose of this lecture is the following:

- explain how and for which QFTs the gauge gravity duality works
- work out the details of the subtle limit
- give nice examples and applications

Gauge gravity duality originates from string theory. (However, there is a limit of the duality in which string theory reduces to classical gravity, i.e. general relativity (GR).) The duality generalizes the so-called *AdS/CFT correspondence*,

- AdS $\equiv$ anti deSitter spacetime, a solution of Einstein's equations
- CFT $\equiv$ conformal field theory

a conjecture for equivalence between

- string theory on certain ten dimensional backgrounds involving AdS spacetime
- four dimensional QFT with conformal symmetry (supersymmetric $SU(N)$ Yang Mills)

There is no mathematical proof for the AdS/CFT correspondence but overwhelming evidence of its correctness. The conjecture states that these two theories are equivalent including observables, states, correlation functions and dynamics. It is interesting to ask in which way the conformal symmetry could be dropped in order to cover non-conformal theories such as QCD. The ten dimensional spacetime of the string theory side contains a five dimensional anti deSitter spacetime with a four dimensional boundary. The four dimensional QFT can be regarded as living on this four dimensional boundary. In analogy to conventional holograms (which encode three dimensional information on a lower dimensional surface), the AdS/CFT correspondence is said to realize the *holographic principle*.

As in any field theory, symmetrie are of central importance for gauge gravity duality. The two equivalent theories have the same symmetries. Moreover, the correspondence provides a one-to-one map between classical gravity fields and quantum operators of the field theory, i.e. some sort of holographic dictionary. This map then identifies representations of the common symmetry group.

As to the literature to this subject, there are (at present) no textbooks available. Let us instead refer to the original papers [1], [2], [3], [4] which marked the birth of the AdS/CFT correspondence. Several review articles followed [5], [6], [7] which assume lots of background knowledge and usually emphasize particular aspects of the duality. Finally, at later stages of this course, [10], [11] are helpful references for applications.

Chapter 2

Preparations

arations

In this section we elucidate the several subject areas which will be connected by the correspondence.

2.1 Conformal field theory in d dimensions

2.1.1 Conformal coordinate transformations

sec:CFT

Conformal coordinate transformations are defined as those local transformations $x^\mu \mapsto x'^\mu(x)$ that leave angles invariant. In a Euclidean d-dimensional space $\mathbb{R}^d$ we therefore can write

$$\mathrm{d}x_\mu \, \mathrm{d}x^\mu \;=\; \Omega^{-2}(x) \, \mathrm{d}x'_\mu \, \mathrm{d}x'^\mu. \tag{2.1.1}$$ eq:metri

The corresponding infinitesimal coordinate transformation from old coordinates x to new ones x' looks like

$$x'^\mu \;=\; x^\mu \;+\; v^\mu(x) \tag{2.1.2}$$

and we have

$$\Omega(x) \;=\; 1 \;-\; \sigma(x) \;, \qquad \sigma(x) \;=\; \frac{1}{d} \, \partial \cdot v(x) \;. \tag{2.1.3}$$ 2,3

Equivalently to (2.1.1) we can formulate an equation for the vector v, the *conformal Killing equation*,

$$\partial_\mu v_\nu \;+\; \partial_\nu v_\mu \;=\; 2 \, \sigma(x) \, \eta_{\mu\nu} \;, \tag{2.1.4}$$ eq:confK

taking its trace yields the expression (2.1.3) for $\sigma(x)$. We will work in d dimensional Euclidean space where $\eta_{\mu\nu} = \delta_{\mu\nu}$. Solutions v to (2.1.4) are referred to as *conformal Killing vectors*, the most general one reads

$$v_\mu \;=\; a_\mu \;+\; \omega_{\mu\nu} \, x^\nu \;+\; \lambda \, x_\mu \;+\; b_\mu x^2 \;-\; 2 \, (b \cdot x) \, x_\mu \;, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu} \;. \tag{2.1.5}$$ eq:solut

This Killing vector leads to the scale factor $\sigma(x) = \lambda - 2(b \cdot x)$. Equation (2.1.5) is valid for any d. Note that in the special case of $d = 2$ the conformal Killing equation (2.1.4) is nothing but the *Cauchy-Riemann equations*

$$\partial_1 v_1 \;=\; \partial_2 v_2 \;, \qquad \partial_1 v_2 \;=\; -\partial_2 v_1 \;. \tag{2.1.6}$$

Thus, in $d = 2$ all holomorphic functions $v(x)$ are solutions and generate conformal coordinate transformations. In this case we have an infinite number of functions solving (2.1.5), accompanied by an infinite number of associated conserved quantities.
However, we will mostly consider theories in $d = 4$ dimensions, for example in Minkowski space or on the boundary of Ads_5. Here we have a finite amount of conserved quantities. Counting the independent components of the factors in the solutions (2.1.5) amounts to a total number of 15:

a_μ	4
$\omega_{\mu\nu}$	+ 6
λ	+ 1
b_μ	+ 4
total	15

The general conformal Killing vector (2.1.5) may be viewed as the combination of elementary transformations. The group of "large" conformal transformation is generated by infinitesimal elements of the conformal algebra. We define locally orthogonal tranformations $\mathcal{R}$ corresponding to a group element g of the conformal group as

$$\mathcal{R}^g_{\mu\alpha}(x) \;:=\; \Omega^g(x)\,\frac{\partial x'_\mu}{\partial x^\alpha}\;. \tag{2.1.7}$$

One can easily show that $\mathcal{R} \in O(d)$, i.e. that $\mathcal{R}^g_{\mu\alpha}(x)\mathcal{R}^g_{\nu\alpha}(x) = \delta_{\mu\nu}$. The group multiplication and the inverse are given as follows:

$$\mathcal{R}^{g'}(gx)\,\mathcal{R}^g(x) \;=\; \mathcal{R}^{g'g}(x)\;, \qquad \big(\mathcal{R}^g(x)\big)^{-1} \;=\; \mathcal{R}^{g^{-1}}(gx) \tag{2.1.8}$$

With these we can construct translations and rotations as

$$x'_\mu \;=\; \mathcal{R}_{\mu\nu}\,x_\nu \,+\, a_\mu\;, \qquad \Omega(x) \;=\; 1\;. \tag{2.1.9}$$

Scale transformations ($\leftrightarrow \lambda$) and *special conformal transformations* ($\leftrightarrow b_\mu$) involve a non-trivial Ω factor:

$$x'_\mu \;=\; \lambda\,x_\mu\;, \qquad \Omega(x) \;=\; \lambda \tag{2.1.10}$$

$$x'_\mu \;=\; \frac{x_\mu + b_\mu x^2}{\Omega^g(x)} \,, \qquad \Omega^g(x) = 1 + 2b\cdot x + bx^2. \tag{2.1.11}$$

Together, these transformations form a group isomorphic to $SO(d+1,1)$ (or $SO(d,2)$ in Minkowski spacetime). All transformations belonging to this group can be constructed by performing translations, rotations, and *inversions*; the latter are given by

$$x'_\mu \;=:\; (\mathrm{i}x)_\mu \;=\; \frac{x_\mu}{x^2} \,, \qquad \Omega^{\mathrm{i}}(x) \;=\; x^2 \tag{2.1.12}$$

$$\mathcal{R}^{\mathrm{i}}_{\mu\nu}(x) \;=:\; I_{\mu\nu}(x) \;=\; \delta_{\mu\nu} \;-\; 2\,\frac{x_\mu x_\nu}{x^2} \,. \tag{2.1.13}$$

Special conformal transformations can be composed by concatenating inversion + translation + inversion.

2.1.2 Conformal fields and correlation functions

So far we examined coordinate transformations. Now we will investigate the behaviour of fields. For instance, the $\mathcal{N}=4$ *super Yang Mills theory* (SYM) mentioned in the introduction only contains fields transforming covariantly under the conformal group. In general QFTs (such as QED or QCD), conformal symmetry is generically broken by quantum effects (anomalies).
Necessary condition for a field theory to be conformally symmetric is a vanishing *β-function*. The latter describes the change of a coupling g with energy scales μ, i.e.

$$\beta(g) \;=\; \mu\,\frac{\partial g}{\partial \mu} \,, \tag{2.1.14}$$

so $\beta(g) = 0$ rephrases scale invariance.
A conformally covariant operator $\mathcal{O}$ of a conformal field theory (CFT) transforms as follows under infinitesimal conformal transformations (with Killing vector v and $\sigma = \partial\cdot v/d$):

$$\delta_v\mathcal{O} \;=\; -\,(L_v\,\mathcal{O}) \,, \qquad L_v \;=\; v(x)\cdot\partial \;+\; \Delta\,\sigma(x) - \frac{1}{2}\,\partial_{[\mu}v_{\nu]}(x)\,S_{\mu\nu} \tag{2.1.15}$$

Here, Δ denotes the *scaling dimension* of the operator $\mathcal{O}$ and $S_{\mu\nu}$ a generator of $O(d)$ in an appropriate representation. It only affects spinor-, vector- and tensor fields but no scalars φ:

$$\delta_v\varphi \;=\; -\big(v(x)\cdot\partial \;+\; \Delta\,\sigma(x)\big)\,\varphi \tag{2.1.16}$$

In general QFTs, correlation functions are defined as time ordered vacuum expectation values, e.g. a two point function of some field φ is given by

$$\langle\varphi(x)\,\varphi(y)\rangle \;:=\; \langle 0|\,\mathcal{T}\,\varphi(x)\,\varphi(y)\,|0\rangle \,, \tag{2.1.17}$$ 2,5

three-, four- and higher point functions by analogous expressions. Generically, their computation is quite involved and possible only in the framework of perturbation theory.

Let us also give the path integral analogue of the definition (2.1.17) in the operator approach. In a scalar field theory governed by action $\mathcal{S}[\varphi]$, the *partition function* $\mathcal{Z}$ and a general correlation function $\langle \mathcal{O} \rangle$ is defined by the path integrals

$$\mathcal{Z} \quad := \quad \int \mathcal{D}\varphi \; \mathrm{e}^{-\mathcal{S}[\varphi]} \; , \qquad \langle \mathcal{O} \rangle \quad := \quad \frac{1}{\mathcal{Z}} \int \mathcal{D}\varphi \; \mathcal{O} \, \mathrm{e}^{-\mathcal{S}[\varphi]} \; . \tag{2.1.18}$$

In CFTs, conformal symmetry is so strong that it determines the form of the two- and three point correlation functions up to a managable number of parameters. In the notation $(x-y)^2 = (x-y)_\mu (x-y)^\mu$, the two- and three point functions of scalars φ_i with scale dimensions Δ_i are given by

$$\langle \varphi_1(x) \, \varphi_2(y) \rangle \quad := \quad \frac{c \, \delta_{\Delta_1, \Delta_2}}{(x-y)^{2\Delta_1}} \tag{2.1.19}$$

$$\langle \varphi_1(x) \, \varphi_2(y) \, \varphi_3(z) \rangle \quad := \quad \frac{k}{(x-y)^{\Delta_1+\Delta_2-\Delta_3} \, (y-z)^{-\Delta_1+\Delta_2+\Delta_3} \, (x-z)^{\Delta_1-\Delta_2+\Delta_3}} \tag{2.1.20}$$

with constants c, k determined by the field content.

Four point correlators $\langle \varphi_1(x) \varphi_2(y) \varphi_3(z) \varphi_4(w) \rangle$ are less constraint by the symmetry since they involve dimensionless *cross ratios* $\frac{(x-y)^2}{(z-w)^2}$ and $\frac{(x-z)^2}{(y-w)^2}$.

2.1.3 The energy momentum tensor in a CFT

The symmetric *energy momentum tensor* $T_{\mu\nu}$ subject to the conservation law $\partial_\mu T^{\mu\nu} = 0$ (or rather $\nabla_\mu T^{\mu\nu} = 0$ in curved spacetime) generates the Noether currents associated with conformal symmetry. The infinitesimal transformations with conformal Killing vector v^μ gives rise to the conserved current

$$j^\mu \quad = \quad T^{\mu\nu} \, v_\nu \; . \tag{2.1.21}$$

In this subsection, we will now show an important property of the energy momentum tensor in a conformal field theory, namely its tracelessness $T^\mu{}_\mu = 0$.

It is a common method in QFT to introduce sources for operators in a QFT's action, and then express the operator (in correlation functions) as the functional derivative of the generating functional. To do so, the action $\mathcal{S}_0$ of our theory is modified by an additive term which couples the operator to its source. For instance consider some scalar operator φ and its source J,

$$\mathcal{S}[\varphi, J] \quad = \quad \mathcal{S}_0[\varphi] \; + \; \int \mathrm{d}^d x \; \varphi(x) \, J(x) \; . \tag{2.1.22}$$

Correlation function of that operator φ may now be calculated as the functional derivative of the generating functional $W[J] := -\ln \mathcal{Z}[J]$ of the theory with respect to the source J, e.g.

$$\langle \varphi(x) \rangle \quad \propto \quad \frac{\delta W[J]}{\delta J(x)} \; . \tag{2.1.23}$$

One can also apply this procedure to vector- and tensor operators,

$$\mathcal{S} \;=\; \mathcal{S}_0 \;+\; \int \mathrm{d}^d x \, \left(\varphi\, J \;+\; V_\mu\, A^\mu \;+\; T_{\mu\nu}\, g^{\mu\nu}\right) \,. \tag{2.1.24}$$

It can be shown that the source of the energy momentum tensor is exactly the quantity that has the properties of the metric. So the energy momentum tensor is obtained by calculating

$$T_{\mu\nu}(x) \;=\; -\frac{2}{\sqrt{|\det g|}}\, \frac{\delta W[g]}{\delta g^{\mu\nu}(x)} \,. \tag{2.1.25}$$

eq:defEM

The metric transforms under conformal coordinate transformations induced by a vector field v as $\delta_v g^{\mu\nu} = 2\sigma g^{\mu\nu}$, so requiring invariance of W implies

$$\begin{aligned} 0 \;=\; \delta_v W[g] \;&=\; \int \mathrm{d}^d x \, \frac{\delta W[g]}{\delta g^{\mu\nu}(x)}\, \delta_v g^{\mu\nu}(x) \;=\; \int \mathrm{d}^d x \, \left(-\frac{\sqrt{|\det g|}\, T_{\mu\nu}}{2}\right) \cdot \left(2\,\sigma\, g^{\mu\nu}\right) \\ &=\; -\int \mathrm{d}^d x \, \sqrt{|\det g|}\, T_\mu{}^\mu \cdot \sigma \,. \end{aligned} \tag{2.1.26}$$

Since $T_\mu{}^\mu$ vanishes upon integration against an arbitrary function σ, one can conclude the announced tracelessness of the energy momentum tensor

$$T_\mu{}^\mu \;=\; 0 \,. \tag{2.1.27}$$

2.2 $\mathcal{N}=4$ super Yang Mills theory

maxiSusy

In this section we want to develop the field theory side of the AdS/CFT correspondence – the maximally supersymmetric $SU(N)$ gauge theory. This $\mathcal{N}=4$ super Yang Mills theory is an example for a $d=4$ dimensional CFT. In the following, the ingredients will be introduced step by step.

2.2.1 Non-abelian gauge theories

Super Yang Mills theory is a non-abelian gauge theory, i.e. its fields take values in the algebra of a non-abelian gauge group. QED, on the other hand, is associated with the abelian gauge group $U(1)$. Let us take it as an introductory example for the necessity of a gauge field: Consider a complex scalar field $\varphi(x)$ transforming under local $U(1)$ transformations as

$$\varphi(x) \;\to\; e^{i\vartheta(x)}\, \varphi(x) \,, \qquad \partial_\mu\varphi(x) \;\to\; \partial_\mu\big(e^{i\vartheta(x)}\, \varphi(x)\big) \;\neq\; e^{i\vartheta(x)} \cdot \partial_\mu\varphi(x) \,. \tag{2.2.1}$$

The derivative $\partial_\mu\varphi$ obviously does not transform like the field φ itself, so a connection A_μ is required in order to define a gauge covariant derivative:

$$D_\mu\varphi(x) \;:=\; \left(\partial_\mu \,+\, iA_\mu\right) \varphi(x) \;\to\; e^{i\vartheta(x)} \cdot D_\mu\varphi(x) \quad\Leftrightarrow\quad A_\mu \;\to\; A_\mu \,-\, \partial_\mu\vartheta \tag{2.2.2}$$

With A_μ transforming like that, we can use the covariant derivative D_μ to construct gauge invariant objects (e.g. kinetic terms in the action). Furthermore, the field strength tensor

$$F_{\mu\nu} \;:=\; \partial_\mu A_\nu \;-\; \partial_\nu A_\mu \tag{2.2.3}$$

is unaffected by gauge transformations of A_μ since $\partial_{[\mu}\partial_{\nu]}\vartheta = 0$.

The most important examples of non-abelian gauge groups in these lectures are $SU(N)$ with $N \geq 2$. One has to distinguish two transformation properties of fields under the non-abelian $SU(N)$:

- Fields transforming in the *fundamental representation* of the gauge group are elements of an N dimensional vector space:

$$q_i(x) \;\;\rightarrow\;\; \left(e^{i\vartheta^a(x)T^a}\right)_i{}^j \, q_j(x) \;, \qquad i,j = 1,2,...,N \tag{2.2.4}$$

  The $SU(N)$ generators T^a are traceless hermitian $N \times N$ matrices and ensure that $e^{i\vartheta^a T^a}$ is unitary. If the parameters $\vartheta^a(x)$ are infinitesimal, the field q_i is shifted by an algebra element

$$q_i(x) \;\;\rightarrow\;\; q_i(x) \;+\; i\vartheta^a(x)\,(T^a)_i{}^j\,q_j(x) \tag{2.2.5}$$

- Fields transforming in the *adjoint representation* of the gauge group are aligned into the $N^2 - 1$ dimensional algebra $su(N)$,

$$\phi_i{}^j \;\;\equiv\;\; \phi^a\,(T^a)_i{}^j \;\;\rightarrow\;\; \left(e^{i\vartheta^b T^b}\right)_i{}^k \, \phi^a\,(T^a)_k{}^l\,\left(e^{-i\vartheta^c T^c}\right)_l{}^j \;. \tag{2.2.6}$$

  Infinitesimally, conjugation by a group element $e^{i\vartheta^a T^a}$ involves the commutator $[T^a, T^b] = if^{abc}T^c$ of the $su(N)$ generators:

$$\begin{aligned} \phi^a\,T^a \;\;&\rightarrow\;\; \phi^a\,T^a \;+\; i\left(\vartheta^b\,T^b\phi^a\,T^a \;-\; \phi^a\,T^a\,\vartheta^b\,T^b\right) \\ &=\;\; \phi^a\,T^a \;-\; i\,\vartheta^b\,\phi^a\left[T^a\,,\,T^b\right] \\ &=\;\; \phi^a\,T^a \;+\; f^{abc}\,\phi^a\,\vartheta^b\,T^c \end{aligned} \tag{2.2.7}$$

Non-abelian gauge fields $A_\mu = A^a_\mu T^a$ give rise to a *non-abelian field strength tensor* in the adjoint representation

$$\begin{aligned} F_{\mu\nu} \;\;&:=\;\; \partial_\mu A_\nu \;-\; \partial_\nu A_\mu \;+\; ig\left[A_\mu\,,\,A_\nu\right] \\ &=\;\; \left(\partial_\mu A^a_\nu \;-\; \partial_\nu A^a_\mu \;-\; g\,f^{abc}\,A^b_\mu\,A^c_\nu\right)T^a \;. \end{aligned} \tag{2.2.8}$$

The transformation properties of $F_{\mu\nu}$ can be deduced from its alternative definition as a commutator of (non-abelian) gauge covariant derivatives (with g denoting the gauge coupling)

$$(D_\mu)_i{}^j \;\; := \;\; \delta_i{}^j\,\partial_\mu \;+\; ig\,A^a_\mu\,(T^a)_i{}^j\,, \qquad F_{\mu\nu} \;\; = \;\; -\frac{i}{g}\,\big[D_\mu\,,\,D_\nu\big]\,. \tag{2.2.9}$$

One can thus form a gauge invariant action for the field strength by taking a trace over the i,j indices of the generators:

$$\mathcal{S}[A] \;\; \sim \;\; \int \mathrm{d}^4x\;\mathrm{Tr}\big\{F^{\mu\nu}\,F_{\mu\nu}\big\} \tag{2.2.10}$$

The non-linear contribution to $F_{\mu\nu}$ gives rise to interactions with the vertices

PICTURE: 3 vertex $\sim g$, 4 vertex $\sim g^2$

Later, we will discuss QCD, an $SU(3)$ Yang Mills theory of gluons together with fundamental quarks. It has two essential features following from the negative sign of the β function

- *asymptotic freedom*, attenuation of the coupling in the UV region, i.e. $\lim_{\mu\to\infty} g(\mu) = 0$
- *confinement*, the coupling g grows rapidly in the IR regime $\mu \to 0$

2.2.2 The $1/N$ expansion

It was suggested by Gerald t'Hooft that non-abelian gauge theories may simplify when $SU(N)$ is studied in the limit $N \to \infty$. The diagrammatic expansion of $SU(N)$ field theory suggests that it is a free string theory in the $N \to \infty$ limit with string coupling $1/N$.

To understand this, let us consider a toy model: let ϕ^a_λ denote a set of fields with an adjoint index ^{a} and a label $_\lambda$ for spin- or flavour degrees of freedom. We assume that the interaction vertices mimic Yang Mills theory – a three point vertex $\sim g$ and a four point vertex $\sim g^2$. The toy model's Lagrangian then reads

$$\mathcal{L} \;\; \sim \;\; \mathrm{Tr}\big\{\mathrm{d}\phi_\lambda\,\mathrm{d}\phi_\lambda\big\} \;+\; g\,c^{\mu\nu\lambda}\,\mathrm{Tr}\big\{\phi_\mu\,\phi_\nu\,\phi_\lambda\big\} \;+\; g^2\,d^{\mu\nu\lambda\rho}\,\mathrm{Tr}\big\{\phi_\mu\,\phi_\nu\,\phi_\lambda\,\phi_\rho\big\}\,. \tag{2.2.11}$$

A rescaling $g\phi_\lambda \mapsto \phi_\lambda$ turns it into

$$\mathcal{L} \;\; \sim \;\; \frac{1}{g^2}\,\Big(\mathrm{Tr}\big\{\mathrm{d}\phi_\lambda\,\mathrm{d}\phi_\lambda\big\} \;+\; c^{\mu\nu\lambda}\,\mathrm{Tr}\big\{\phi_\mu\,\phi_\nu\,\phi_\lambda\big\} \;+\; d^{\mu\nu\lambda\rho}\,\mathrm{Tr}\big\{\phi_\mu\,\phi_\nu\,\phi_\lambda\,\phi_\rho\big\}\Big)\,. \tag{2.2.12}$$ 2,20

To have a well-defined $N \to \infty$ limit, it is convenient to introduce the *t'Hooft coupling*

$$\lambda \;\; := \;\; g^2\,N\,. \tag{2.2.13}$$

If we send $N \to \infty$ at constant λ, the coefficient of (2.2.12) diverges but the number N^2-1 of components in the fields diverges as well. This point becomes clearer after an analysis of

Feynman graphs in the t'Hooft limit. The propagator will have the following $SU(N)$ index structure to ensure tracelessness,

$$\left\langle \phi_i{}^j(x)\, \phi_k{}^l(y) \right\rangle \;\sim\; \left(\delta_i{}^l\, \delta_k{}^j \;-\; \frac{1}{N}\, \delta_i{}^j\, \delta_k{}^l \right)\;, \tag{2.2.14}$$

regardless of the spacetime dependence. In the $N \to \infty$ limit, the second term can be safely ignored, this suggests *double line notation*

PICTURE of a i-k over j-l double line

Feynman diagrams then become networks of double lines. Vertices scale as $\frac{N}{\lambda}$, propagators as $\frac{\lambda}{N}$, and the sum over indices in a trace contributes a factor of N for each closed loop. If we introduce shorthands (V, E, F) for the numbers of vertices, propagators (edges) and loops (faces) respectively, diagrams are proportional to

$$\text{diagram}(V, E, F) \;\sim\; N^{V-E+F}\, \lambda^{E-V} \;=\; N^{\chi}\, \lambda^{E-V}\;. \tag{2.2.15}$$

The power of the expansion parameter N is precisely the *Euler characteristic*

$$\chi \;:=\; V \;-\; E \;+\; F \;=\; 2 \;-\; 2\,g\;, \tag{2.2.16}$$

related to the surface's number of handels (the *genus*) g.
Any physical quantity in this theory is given by a perturbative expansion of type

$$\sum_{g=0}^{\infty} N^{2-2g} \sum_{i=0}^{\infty} c_{g,i}\, \lambda^i \;=\; \sum_{g=0}^{\infty} N^{2-2g}\, f_g(\lambda) \tag{2.2.17}$$

with $f_g(\lambda)$ a polynomial in the t'Hooft coupling. For large N, the series is clearly dominated by surfaces of minimal genus, the so-called *planar diagrams*. As an example, compare the following vacuum amplitudes

(—) with N^2 and non-planar (X) with N^0

The form of this expansion is the same as in a perturbative theory of closed oriented strings with string coupling $\frac{1}{N}$. The propagator and the interaction vertex of a closed string is depicted below.

PIC: Zylinder und Hose

In this simple toy model, one cannot say which string theory fits to the perturbative series. For $\mathcal{N} = 4$ SYM, however, the AdS/CFT correspondence tells us which string theory leads to the correct expansion: ten dimensional type IIB superstring theory on $\text{AdS}_5 \times S^5$.

2.2.3 Supersymmetry

We know to have Poincaré symmetry in the flat Minkowski spacetime, which is equipped with a "mostly positive" metric of signature $\eta = \text{diag}(-, +, +, +)$. Generators of translations and

Lorentz transformations will be denoted as P_μ and $L_{\mu\nu}$ respectively. Supersymmetry now enlarges the Poincaré algebra

$$\begin{aligned} \left[L_{\mu\nu}\,,\,P_\lambda\right] &= -i\left(\eta_{\mu\lambda}\,P_\nu \;-\; \eta_{\nu\lambda}\,P_\mu\right) & (2.2.18) \quad \boxed{\texttt{L,P}} \\ \left[L_{\mu\nu}\,,\,L_{\lambda\rho}\right] &= -i\left(\eta_{\mu\lambda}\,L_{\nu\rho} \;-\; \eta_{\mu\rho}\,L_{\nu\lambda} \;+\; \eta_{\nu\rho}\,L_{\mu\lambda} \;-\; \eta_{\nu\lambda}\,L_{\mu\rho}\right) & (2.2.19) \quad \boxed{\texttt{L,L}} \end{aligned}$$

by including *spinor supercharges* Q. In so-called Weyl notation we have aleft-handed spinor Q^a_α and its right-handed counterpart $\bar{Q}_{a\dot{\alpha}} = (Q^a_\alpha)^\dagger$ where the $SL(2,\mathbb{C})$ indices $\alpha, \dot{\alpha}$ take values $1, 2$ and a counts the number of independent supersymmetries $a = 1, \ldots, \mathcal{N}$. The Q's transform as Weyl spinors of $SO(1,3) \cong SL(2,\mathbb{C})/\mathbb{Z}_2$.
The two-component Weyl spinor notation is related to the Dirac four-spinor notation by

$$Q^a_{\mathrm{D}} \;=\; \begin{pmatrix} Q^a_\alpha \\ \bar{Q}^{a\dot{\alpha}} \end{pmatrix}\,, \qquad \gamma^\mu \;=\; \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot{\beta}} \\ \bar{\sigma}^{\mu\dot{\alpha}\beta} & 0 \end{pmatrix}\,, \tag{2.2.20}$$

where $\sigma^\mu = (-\mathbb{1}, \sigma^i)$ and $\bar{\sigma}^\mu = (-\mathbb{1}, -\sigma^i)$ are four vectors of 2×2 matrices with the standard Pauli matrices σ^i as their spatial entries.
The supercharges commute with the generators of translations but otherwise obey the algebra

$$\left\{Q^a_\alpha\,,\,\bar{Q}_{b\dot{\beta}}\right\} \;=\; -\,2\,\sigma^\mu_{\alpha\dot{\beta}}\,P_\mu\,\delta^a{}_b\,, \qquad \left\{Q^a_\alpha\,,\,Q^b_\beta\right\} \;=\; 2\,\varepsilon_{\alpha\beta}\,Z^{ab}\,. \qquad (2.2.21) \quad \boxed{\texttt{eq:susyA}}$$

Here the operators Z^{ab} are referred to as *central charges*. They commute with all the Poincaré- and supersymmetry generators Q^a and need to by antisymmetric $Z^{ab} = -Z^{ba}$ in order to respect the anticommutator's symmetry. Therefore, for $\mathcal{N} = 1$ supersymmetry, we have $Z = 0$.
The supersymmetry algebra (2.2.21) is invariant under global phase rotations of the supercharges $Q^a_{1,2}$ into each other. This forms an *R symmetry group* denoted as $U(1)_{\mathrm{R}}$. In addition, when $\mathcal{N} > 1$, the different supercharges may be rotated into one another under the unitary group $SU(N)_{\mathrm{R}}$ which extends the R symmetry.
The field theory in the AdS/CFT dictionary has $\mathcal{N} = 4$ supersymmetries. Let us briefly explain why this is the maximal supersymmetry for a pure gauge theory without gravity: Each supercharge $Q^a_\alpha, \bar{Q}_{a\dot{\alpha}}$ changes the spin of the state it acts on by $1/2$. In absence of gravity, helicities between -1 and +1 occur, hence no spin modification greater than $2 = \mathcal{N}_{\max} \cdot 1/2$ is allowed.
In the $\mathcal{N} = 4$ theory we have R symmetry $SU(4) \cong SO(6)$. Exactly this is the isometry group of the sphere in the $\mathrm{AdS}_5 \times S^5$ background of the string theory side of the correspondence. The AdS_5 factor has the symmetries encoded by $SO(4,2)$ in Minkowski space or $SO(5,1)$ in a Euclidean formulation. These groups are isomorphic to the conformal group in $d = 4$ dimensions according to our analysis in subsection 2.1.1.

	field	range	representation of $SU(4)_{\mathrm{R}}$
vector	A_μ		$(\underline{1})$ singlet
Weyl fermions	λ^a_α, $\bar\lambda^a_\alpha$,	$a = 1, 2, 3, 4$	$(\underline{4})$ fundamental
real scalars	X^i,	$i = 1, 2, \ldots, 6$	$(\underline{6})$ adjoint

Table 2.1: The field content of the $\mathcal{N} = 4$ supersymmetry multiplet and the representation in which these fields transform with respect to the R symmetry group $SU(4)_{\mathrm{R}} \cong SO(6)_{\mathrm{R}}$ tab:n4co

2.2.4 Field content of $\mathcal{N} = 4$ supersymmetric field theory

Representations of the supersymmetry algebra make up the SUSY multiplets. Their components are spin 1 vector fields, spin $\frac{1}{2}$ fermion fields and spin 0 scalar fields. In $\mathcal{N} = 4$ supersymmetry we encounter *maximal supersymmetry* if $s = 1$ is the highest spin in a SUSY-multiplet. This implies that we cannot describe gravity with this theory, because the graviton is supposed to have spin 2.

For any $\mathcal{N}$ with $1 \leq \mathcal{N} \leq 4$ we encounter one gauge multiplet, which is a multiplet transforming in the adjoint representation of the gauge group (while we are used to have matter fields in the fundamental representation in non-supersymmetric theories). For $\mathcal{N} = 4$ this is the only possible multiplet.

Lower symmetry $\mathcal{N} = 1$ and $\mathcal{N} = 2$ also admits matter multiplets which we will not discuss here, though. (But to make you familiar with the names, the multiplet in the fundamental representation in $\mathcal{N} = 1$ SUSY is called *chiral multiplet*, and the multiplet in the fundamental representation in $\mathcal{N} = 2$ SUSY is called the *hypermultiplet*). The content of the $\mathcal{N} = 4$ multiplet is given in table 2.1. Note that this theory is non-chiral. The Lagrangian may be written as

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}\Big\{ & -\frac{1}{2\,g^2}\,F_{\mu\nu}\,F^{\mu\nu} \;+\; \frac{\theta^I}{8\pi^2}\,F_{\mu\nu}\,\tilde F^{\mu\nu} \;-\; i\sum_a \bar\lambda^a\,\bar\sigma^\mu\,D_\mu\lambda_a \\ & -\sum_i D_\mu X^i\,D^\mu X^i \;+\; g\sum_{a,b,i} C^{ab}{}_i\,\lambda_a\left[X^i\,,\,\lambda_b\right] \\ & +\; g\sum_{a,b,i}\bar C_{iab}\,\bar\lambda^a\left[X^i\,,\,\bar\lambda^b\right] \;+\; \frac{g^2}{2}\sum_{i,j}\left[X^i\,,\,X^j\right]^2\Big\}\,.\end{aligned} \tag{2.2.22}$$
eq:N4Lag

Here the trace is summing over gauge indices $\tilde\alpha, \tilde\beta$ which are suppressed in the expression above. They appear if we rewrite the adjoint fields correctly as linear combinations of the generators T^A of the gauge group, e.g. $X^i = X^{iA}T^A{}_{\tilde\alpha}{}^{\tilde\beta}$. The symbol θ^I denotes the instanton number and $\tilde F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}$.

The $C^{ab}{}_i$ are the structure constants of $SU(4)_{\mathrm{R}}$. Note that there is only one coupling constant g. On the classical level this theory is conformal with engeneering dimensions of the fields as

$[A_\mu] = 1$, $[\lambda] = 3/2$, $[X] = 1$ and therefore $[g] = 0$. The dimensionless coupling and absence of any mass term are necessary for conformal invariance.
The Lagrangian (2.2.22) is invariant under SUSY-transformations given by

$$\begin{aligned}
(\delta X^i)^a{}_\alpha &= \left[Q^a{}_\alpha\,,\, X^i\right] &&= C^{iab}\,\lambda_{\alpha b}, \\
(\delta \lambda_{\beta b})^a{}_\alpha &= \left\{Q^a{}_\alpha\,,\, \lambda_{\beta b}\right\} &&= F^+_{\mu\nu}\,(\sigma^{\mu\nu}\,\varepsilon)_{\alpha\beta}\,\delta^a{}_b \;+\; \left[X^i\,,\, X^j\right]\varepsilon_{\alpha\beta}\,(C_{ij})^a{}_b \\
(\delta \bar{\lambda}_{\dot\beta}{}^b)^a{}_\alpha &= \left\{Q^a{}_\alpha\,,\, \bar{\lambda}^b_{\dot\beta}\right\} &&= C_i{}^{ab}\,\sigma^\mu_{\alpha\dot\beta}\,D_\mu X^i \\
(\delta A^\mu)^a{}_\alpha &= \left[Q^a{}_\alpha\,,\, A^\mu\right] &&= \sigma^\mu_{\alpha\dot\beta}\,\bar{\lambda}^{\dot\beta a}\,.
\end{aligned} \tag{2.2.23}$$

Note that $F^+_{\mu\nu}$ is the self-dual part $\frac{1}{2}(F_{\mu\nu} + \tilde{F}_{\mu\nu})$ of the field strength, and the constants $(C_{ij})^a{}_b$ are related to bilinears in Clifford Dirac matrices of $SO(6)_{\rm R}$.
Upon quantization of this theory, one finds that the β-function vanishes to all orders of pertubation theory (and even non-perturbatively), therefore we are left with a CFT even at quantum level.

2.2.5 The superconformal algebra and its representations

The concept of supersymmetry together with the conformal group form the *superconformal group* $SU(2,2|4)$. The $SU(2,2)$ part represents the symmetry of the Weyl spinors while the $SU(4)$ refers to the R symmetry group $SU(4)_{\rm R}$ of the $\mathcal{N} = 4$ supersymmetry.
The AdS/CFT map will provide a direct one to one mapping between operators on both sides of the correspondence. This relies heavily on the fact that on both sides the operators fall into representations of the same symmetry groups.
The generators of the superconformal group are given by

- Conformal symmetry with generators P_μ, $L_{\mu\nu}$, D, K_μ: In addition to the Poincaré algebra (2.2.18) and (2.2.19), the conformal algebra involves commutators

$$\begin{aligned}
\left[D\,,\, P_\mu\right] &= -i\,P_\mu \\
\left[D\,,\, K_\mu\right] &= i\,K_\mu \\
\left[L_{\mu\nu}\,,\, D\right] &= 0 \\
\left[L_{\mu\nu}\,,\, K_\rho\right] &= -i\,(\eta_{\mu\rho}\,K_\nu \;-\; \eta_{\nu\rho}\,K_\mu) \\
\left[P_\mu\,,\, K_\nu\right] &= 2i\,L_{\mu\nu} \;-\; 2i\,\eta_{\mu\nu}\,D
\end{aligned} \tag{2.2.24}$$

- R symmetry $SO(6)_{\rm R} \cong SU(4)_{\rm R}$ with generators T^A, $A = 1, 2, \ldots, 15$. The $SO(4,2)$- and $SU(4)_{\rm R}$ subgroups commute.

- Poincaré supersymmetry with generators Q^a_α, $\bar{Q}^a_{\dot\alpha}$, $a = 1, 2, 3, 4$ subject to (2.2.21).

- Conformal supersymmetry generators $S_{\alpha a}$ and $\bar{S}^{\dot{\alpha} a}$ which introduce the following anti-commutation relations:

$$\begin{aligned}
\left\{Q^a_\alpha \,,\, Q^b_\beta\right\} &= \left\{S_{\alpha a} \,,\, S_{\beta b}\right\} \;=\; \left\{Q^a_\alpha \,,\, \bar{S}^b_{\dot{\beta}}\right\} \;=\; 0 \\
\left\{Q^a_\alpha \,,\, \bar{Q}_{\dot{\beta} b}\right\} &= 2\,(\sigma^\mu)_{\alpha\dot{\beta}}\, P_\mu\, \delta^a_b \\
\left\{S^a_\alpha \,,\, \bar{S}_{\dot{\beta} b}\right\} &= 2\,(\sigma^\mu)_{\alpha\dot{\beta}}\, K_\mu\, \delta^a_b \\
\left\{Q^a_\alpha \,,\, S_{\beta b}\right\} &= \varepsilon_{\alpha\beta}\, \delta^a_b\, D \;+\; \frac{1}{2}\, \delta^a_b\, L_{\mu\nu}\, (\sigma^{\mu\nu}\, \varepsilon)_{\alpha\beta}
\end{aligned} \tag{2.2.25}$$

Central charges are assumed to vanish throughout the rest of these lectures.
The fields $A_\mu(x)$, $\lambda^a_\alpha(x)$, $\bar{\lambda}^a_{\dot{\alpha}}(x)$ and $X^i(x)$ of the SUSY multiplet ($a = 1, 2, 3, 4$ and $i = 1, 2, ..., 6$) can be used to construct composite operators of $\mathcal{N} = 4$ SYM. Some regularization prescription is needed when multiplying fields at the same spacetime point.
We define a *superconformal primary operator* $\mathcal{O}$ by

$$\left[S \,,\, \mathcal{O}\right] \;=\; 0 \,, \tag{2.2.26}$$ 2,40

i.e. the $\mathcal{O}$'s are the lowest dimensional operators in a representation of $SU(2,2|4)$. This is the generalization of the primary operator condition $[K_\mu, \mathcal{O}] = 0$ in bosonic conformal field theory (which is in fact implied by (2.2.26) since two S generators anticommute to K's).
An operator $\mathcal{O}'$ is a *superconformal descendant* of $\mathcal{O}$ if

$$\mathcal{O}' \;=\; \left[Q \,,\, \mathcal{O}\right] \,, \tag{2.2.27}$$

$\mathcal{O}$ and $\mathcal{O}'$ then belong to the same superconformal multiplet, i.e. the same representation of $SU(2,2|4)$. The scale dimension is shifted as $\Delta_{\mathcal{O}'} = \Delta_{\mathcal{O}} + \frac{1}{2}$.
Of central importance are single trace operators (taking a trace is necessary to ensure gauge invariance)

$$\mathcal{O} \;=\; \mathrm{Tr}\left\{X^{(i_1}\, X^{i_2} \ldots X^{i_n)}\right\} \;=\; \mathrm{sTr}\left\{X^{i_1}\, X^{i_2} \ldots X^{i_n}\right\} \tag{2.2.28}$$

They are also referred to as *half BPS states* since they are annihilated by half the spinorial generators S (but not by the other half Q).

2.3 Anti-de Sitter space

In this section we will examine the Anti-de Sitter spacetime and compare it to flat Minkowski spacetime. As mentioned earlier, one side of the AdS/CFT correspondence is so-called type IIB string theory formulated on the spacetime $\mathrm{AdS}_5 \times S^5$. We will not discuss string theory now. Instead we want to get familiar with the spacetime and see how it may be connected to the more familiar Minkowski spacetime $\mathbb{R}^{1,3}$.

The most important facts about $\mathrm{AdS}_5 \times S^5$ spacetime for us are of geometrical nature. We already stated that the isometry group of this spacetime is the same as the symmetry group of the quantum field theory on the other side of the correspondence.

The key result of this section will be that the boundary of the Euclidian compactification of AdS_5 spacetime is equal to compactified $\mathbb{R}^4$, which is the Euclidean compactification of the Minkowski spacetime we live in. To see this equivalence we will make use of so called *conformal diagrams* which enable us to draw an image of the entire spacetime on a single sheet of paper making the causal structure of the spacetime visible. A short introduction to conformal diagrams is for example given in appendix H of [8].

The $(p+2)$-dimensional version AdS_{p+2} of this spacetime can be defined as the embedding of a hyperboloid (with *AdS radius* L)

$$X_0^2 \; + \; X_{p+2}^2 \; - \; \sum_{i=1}^{p+1} X_i^2 \; = \; L^2 \tag{2.3.1}$$

into a flat $(p+3)$-dimensional space $\mathbb{R}^{p+3}$ with metric

$$\mathrm{d}s^2 \;\; = \;\; -\,\mathrm{d}X_0^2 \; - \; \mathrm{d}X_{p+2}^2 \; + \; \sum_{i=1}^{p+1} \mathrm{d}X_i^2 \; . \tag{2.3.2}$$

eq:adsEm

The AdS radius is a measure for the constant curvature: Riemann tensor and cosmological constant are given by

$$R_{\mu\nu\lambda\rho} \;\; = \;\; -\frac{1}{L^2}\left(g_{\mu\lambda}\, g_{\nu\rho} \; - \; g_{\mu\rho}\, g_{\nu\lambda}\right) , \qquad \Lambda \;\; = \;\; -\,\frac{d\,(d-1)}{L^2} \;\; < \;\; 0 \tag{2.3.3}$$

where d is the dimension of the boundary.

One possible parametrization of this spacetime is given by

$$\begin{aligned} X_0 \;\; &= \;\; L\, \cosh\rho\, \cos\tau \\ X_{p+2} \;\; &= \;\; L\, \cosh\rho\, \sin\tau \\ X_i \;\; &= \;\; L\, \Omega_i\, \sinh\rho \end{aligned} \tag{2.3.4}$$

eq:adsPa

with angular coordinates $\Omega_i,\; i = 1, \ldots, p+1$ such that $\sum_i \Omega_i^2 = 1$ and ranges $0 \leq \rho,\; 0 \leq \tau < 2\pi$ for the remaining coordinates.

Inserted into (2.3.2), this yields the metric

$$\mathrm{d}s^2 \;\; = \;\; L^2\left(-\,\cosh^2\rho\, \mathrm{d}\tau^2 \; + \; \mathrm{d}\rho^2 \; + \; \sinh^2\rho\; \mathrm{d}\Omega_p^2\right) . \tag{2.3.5}$$

eq:adsMe

It features a timelike killing vector ∂_τ on the whole manifold, so τ may be called the global time coordinate. The isometry group $SO(2,p+1)$ of AdS_{p+2} has a maximal compact subgroup $SO(2) \times SO(p+1)$, the former generating translations in τ, the latter rotating the X_i's.

picture of AdS as hyperboloid?

Near $\rho = 0$ we have $\cosh\rho \approx 1$ and $\sinh\rho \approx \rho$, so in this environment the metric of AdS_5 looks like

$$\mathrm{d}s^2 \approx L^2\big(-\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\Omega_3^2\big) \tag{2.3.6}$$

and thus is seen to be topologically $S^1 \times \mathbb{R}^4$. The S^1 parametrized by the time coordinate τ represents closed timelike curves. To prevent inconsistencies concerning causality, AdS_5 is therefore regarded as the causal spacetime obtained by unwrapping these circles, taking $-\infty < \tau < \infty$ without any identification.
Introducing a new coordinate θ, the metric (2.3.5) becomes that of the *Einstein static universe* $\mathbb{R} \times S^p$:

$$\tan\theta = \sinh\rho \quad\Rightarrow\quad \mathrm{d}s^2 = \frac{L^2}{\cos^2\theta}\big(-\mathrm{d}\tau^2 + \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\Omega_3^2\big)\ . \tag{2.3.7}$$

However, since $0 \leq \theta < \frac{\pi}{2}$, only half of $\mathbb{R} \times S^p$. The causal structure remains unchanged when scaling this metric to get rid of the overall factor. Further, adding the point $\theta = \frac{\pi}{2}$ corresponding to spatial infinity results in the compactified spacetime

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\Omega_3^2\ , \qquad 0 \leq \theta \leq \frac{\pi}{2}\ , \qquad -\infty < \tau < \infty\ . \tag{2.3.8}$$

eq:adsCo

If we specify boundary conditions on $\mathbb{R} \times S^p$ at $\theta = \frac{\pi}{2}$, then the Cauchy problem is well-posed. As one can easily read off from (2.3.8), the $\theta = \frac{\pi}{2}$ boundary of conformally compactified AdS_{p+2} is identical to the conformal compactification of $(p+1)$ dimensional Minkowski spacetime.
Let us take a quick look at the special case of conformally compactified $(1+1)$ dimensional Minkowski spacetime. It is convenient to introduce light cone coordinates,

$$u_\pm := t \pm x \quad\Rightarrow\quad \mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 = -\mathrm{d}u_+\,\mathrm{d}u_-\ . \tag{2.3.9}$$

If we furthermore restrict the coordinates to a finite range, a useful choice is

$$u_\pm =: \tan\tilde{u}_\pm\ , \qquad \tilde{u}_\pm =: \frac{\tau \pm \vartheta}{2} \quad\Rightarrow\quad \mathrm{d}s^2 = \frac{-\mathrm{d}\tau^2 + \mathrm{d}\vartheta^2}{4\cos^2\tilde{u}_+\cos^2\tilde{u}_-}\ . \tag{2.3.10}$$

Another neat parametrization of AdS_{p+2} are the *Poincaré coordinates* which cover half of the hyperboloid. Introduce $(y, t, \vec{x})$ such that $y > 0$ and $\vec{x} \in \mathbb{R}^p$, then:

$$\begin{aligned} X_0 &= \frac{1}{2y}\left(1 + y^2\left(L^2 + \vec{x}^2 - t^2\right)\right) \\ X_{p+1} &= \frac{1}{2y}\left(1 - y^2\left(L^2 - \vec{x}^2 + t^2\right)\right) \\ X_{p+2} &= L\,y\,t \\ X_i &= L\,y\,x_i \end{aligned} \tag{2.3.11}$$

The boundary at $y \to \infty$ can be better analyzed in terms of a new variable u

$$u := \frac{1}{y} \quad \Rightarrow \quad \mathrm{d}s^2 = L^2\left(\frac{\mathrm{d}u^2}{u^2} + \frac{1}{u^2}\,\eta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\right). \tag{2.3.12}$$ 2,last

After a conformal rescaling by u^2, we obtain the Minkowski metric by freezing $u = 0$.

Chapter 3

Introduction to superstring theory

This chapter aims to give a brief introduction to selected aspects of superstring theory. Of course, we cannot provide a self-contained course about this topic, the following sections will shed light only on those aspects which are relevant for the AdS/CFT correspondence.

As we have emphasized before, the AdS/CFT map relates the $\mathcal{N} = 4$ SYM field theory to string theory. Relations of that type have been known for some time, in fact the original motivation to study string theory in the 1960's was to describe mesons and hadrons (bound states of quarks). This picture gives a relation between mass m and spin J of hadrons, $m^2 = J/\alpha' + const.$ The $m^2(J)$ plot is known as *Regge trajectory* and the parameter α' as *Regge slope*. Mass and angular momentum are assumed to come from a rotating relativistic string.

3.1 Bosonic strings in Minkowski spacetime

The basic idea behind the bosonic string is to take one-dimensional strings as the fundamental objects rather than point particles. Such a string sweeps out a 1+1 dimensional *worldsheet* (cf. worldline of point particles). Strings can be closed or open – closed string will represent the gravity side of the correspondence whereas open strings will cover the gauge sector.

The worldsheet is parametrized by two coordinates, proper time τ and the spatial extent σ of the string. The embedding of the worldsheet of the fundamental string into the *target spacetime* is defined by functions $X^\mu(\tau, \sigma)$.

The string action is simply given by the worldsheet area (similar to the length of a point particle's worldline),

$$\mathcal{S} \;=\; \frac{1}{2\pi\alpha'} \int \mathrm{d}^2\sigma \, \sqrt{-\det \partial_\alpha X^\mu \, \partial_\beta X_\mu} \;, \tag{3.1.1}$$ 3,1

where $(\sigma^0, \sigma^1) \equiv (\tau, \sigma)$ and α' is the inverse string tension.

In order to get rid of the square root in view of quatization, a worldsheet metric $h_{\alpha\beta}(\sigma)$ is introduced as an auxiliary field subject to certain constraints. This gives rise to the *Polyakov action*

$$\mathcal{S} \;=\; -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\ \sqrt{-h}\,h^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X_\mu\ . \tag{3.1.2}$$ 3,2

The equation of motion for X^μ is a relativistic wave equation. In the gauge with $h_{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{diag}(-1,1)$, it takes the particularly simple form

$$(\partial_\tau^2 \;-\; \partial_\sigma^2)\,X^\mu(\tau,\sigma) \;=\; 0\ . \tag{3.1.3}$$ 3,3

This is supplemented by the *Virasoro constraints*

$$\partial_\tau X^\mu\,\partial_\sigma X_\mu \;=\; \partial_\tau X^\mu\,\partial_\tau X_\mu \;-\; \partial_\sigma X^\mu\,\partial_\sigma X_\mu \;=\; 0\ . \tag{3.1.4}$$ 3,4

3.1.1 Closed strings in Minkowski spacetime

For AdS/CFT, it is essential to have two different types of strings, closed and open ones, depending on the boundary conditions they satisfy. Closed strings are equivalent to a circle, they do not have endpoints and satisfy periodic boundary conditions. With $\sigma \in [0,2\pi[$ we have

$$X^\mu(\tau,0) \;=\; X^\mu(\tau,2\pi)\ , \qquad \partial_\sigma X^\mu(\tau,0) \;=\; \partial_\sigma X^\mu(\tau,2\pi) \tag{3.1.5}$$ 3,5

and also $h_{\alpha\beta}(\tau,0) = h_{\alpha\beta}(\tau,2\pi)$. The mode expansion for the closed string is governed by the solutions to the wave equation (3.1.3) which split into *left- and right movers*

$$X^\mu(\tau,\sigma) \;=\; X^\mu_{\mathrm{L}}(\tau+\sigma) \;+\; X^\mu_{\mathrm{R}}(\tau-\sigma)\ . \tag{3.1.6}$$ 3,6

The periodic boundary conditions give rise to a discrete Fourier expansion

$$\begin{aligned} X^\mu_{\mathrm{L}}(\tau+\sigma) \;&=\; \frac{x_0^\mu}{2} \;+\; \alpha'\,p^\mu_{\mathrm{L}}\,(\tau+\sigma) \;+\; i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde\alpha^\mu_n}{n}\,e^{-in(\tau+\sigma)} \\ X^\mu_{\mathrm{R}}(\tau-\sigma) \;&=\; \frac{x_0^\mu}{2} \;+\; \alpha'\,p^\mu_{\mathrm{R}}\,(\tau-\sigma) \;+\; i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}\,e^{-in(\tau-\sigma)} \end{aligned} \tag{3.1.7}$$ 3,7

The x_0^μ and $p^\mu = p^\mu_{\mathrm{L}} = p^\mu_{\mathrm{R}}$ are center of mass positions and -momenta, the latter can be viewed as the zero modes of the expansion via $\alpha_0^\mu = \tilde\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$. The constraint $p^\mu = p^\mu_{\mathrm{L}} = p^\mu_{\mathrm{R}}$ is enforced by periodicity. Reality of X^μ requires $\alpha^\mu_{-n} = (\alpha^\mu_n)^*$ and $\tilde\alpha^\mu_{-n} = (\tilde\alpha^\mu_n)^*$.

In the quantization procedure, the α_n modes become creation- and annihilation operators, e.g. the graviton as the lowest closed string excitation corresponds to the massless spin 2 state $\alpha^\mu_{-1}\tilde\alpha^\nu_{-1}|0,0,k\rangle$.

3.1.2 Open strings in Minkowski spacetime

Open strings have two endpoints. The usual convention is to delimit $\sigma \in [0, \pi[$ in this sector. In each direction μ of spacetime, either *Neumann- or Dirichlet boundary conditions* are possible:

- Neumann boundary conditions

$$\partial_\sigma X_\mu(\tau, 0) \;=\; \partial_\sigma X_\mu(\tau, \pi) \;=\; 0 \tag{3.1.8}$$ 3,8

Momentum flow through the endpoints of the string is forbidden by Neumann boundary conditions. This is reflected in the Neumann mode expansion

$$X^\mu(\tau, \sigma) \;=\; x_0^\mu \;+\; 2\,\alpha'\, p^\mu\, \tau \;+\; i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n}\, e^{-in\tau} \cos(n\sigma)\ . \tag{3.1.9}$$ 3,9

Left- and right movers are reflected into each other. Again, $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$.

- Dirichlet boundary conditions

Now the endpoints of the string are fixed

$$X^\mu(\tau, 0) \;=\; X^\mu(\tau, \pi) \;=\; x_0^\mu\ , \tag{3.1.10}$$ 3,10

which gives rise to the mode expansion

$$X^\mu(\tau, \sigma) \;=\; x_0^\mu \;+\; \sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n}\, e^{-in\tau} \sin(n\sigma)\ . \tag{3.1.11}$$ 3,11

Open string boundary conditions (b.c.) can be interpreted as follows: Dirichlet b.c. define a hyperplane in target space, so-called *Dp branes*, on which open strings can end. In p spatial dimensions and in the time direction, Dirichlet b.c. are used whereas in the other directions Neumann b.c. are imposed.
The quantization procedure naturally leads to a massless spin 1 state $\alpha_{-1}^\mu|0, k\rangle$, the photon.

3.2 Bosonic string theory in background fields

Up to now, we have considered the propagation of open and closed strings in Minkowski spacetime. By coupling the fundamental string to the massless closed string excitations (which involve the graviton), strings propagating through curved background spacetime (such as $\mathrm{AdS}_5 \times S^5$) can be described. In particular, the symmetric traceless part of the state $\alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu |0, 0, k\rangle \leftrightarrow g_{\mu\nu}$ can be identified with the metric of the target spacetime.

3.2.1 Background fields of the closed string sector

sec:3,1

Weyl invariance of string theory implies that spacetime has to satisfy the vacuum Einstein equations. The Polyakov action becomes

$$\mathcal{S} \;=\; -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\,\sqrt{-h}\,h^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\,g_{\mu\nu}(X)\;. \tag{3.2.1}$$

3,12

In addition, we have a *Kalb Ramond field* $B_{[\mu\nu]}$ and a dilaton φ associated with the remaining irreducibles $\alpha^{[\mu}_{-1}\tilde{\alpha}^{\nu]}_{-1}|0,0,k\rangle$ and $\alpha^{\mu}_{-1}\tilde{\alpha}_{-1\mu}|0,0,k\rangle$. Their action reads

$$\mathcal{S}_{B,\varphi} \;=\; \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\,\sqrt{-h}\,\big(i\,\varepsilon^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\,B_{\mu\nu}(X)\;+\;\alpha'\,R_h\,\varphi(X)\big)\;. \tag{3.2.2}$$

3,13

By comparison with the string theory perturbative expansion, we find a string coupling $g_s = e^\varphi$. To ensure Weyl invariance of the quantized theory, we have to impose tracelessness of the worldsheet energy momentum tensor. In bosonic string theory, this is possible in $D = 26$ spacetime dimensions only. The critical dimension of superstring theory is $D = 10$. The worldsheet energy momentum trace reads

$$T^\alpha{}_\alpha \;=\; -\frac{1}{2\alpha'}\,\beta^g_{\mu\nu}\,h^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu \;-\; \frac{i}{2\alpha'}\,\beta^B_{\mu\nu}\,\varepsilon^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu \;-\; \frac{1}{2}\,\beta^\varphi\,R_h \tag{3.2.3}$$

where R_h denotes the Ricci scalar on the worldsheet (with respect to the metric $h_{\alpha\beta}$) and the β functions are given as follows (to order α'):

$$\begin{aligned}
\beta^g_{\mu\nu} &= -\,\alpha'\left((\mathcal{R}_g)_{\mu\nu} \;+\; \nabla_\mu\nabla_\nu\varphi \;-\; \frac{1}{4}\,H_{\mu\lambda\rho}\,H_\nu{}^{\lambda\rho}\right)\\
\beta^B_{\mu\nu} &= \alpha'\left(-\frac{1}{2}\,\nabla^\lambda H_{\lambda\mu\nu} \;+\; \nabla^\lambda\varphi\,H_{\lambda\mu\nu}\right)\\
\beta^\varphi_{\mu\nu} &= \alpha'\left(-\frac{1}{2}\,\nabla^2\varphi \;+\; \nabla_\mu\varphi\,\nabla^\mu\varphi \;-\; \frac{1}{2\varphi}\,H_{\mu\nu\lambda}\,H^{\mu\nu\lambda}\right)\;.
\end{aligned} \tag{3.2.4}$$

3,16

(By the usual method of differential forms, one defines a field strength $H = \mathrm{d}B$ for the Kalb Ramond field,

$$H_{\mu\nu\lambda} \;:=\; \partial_\mu B_{\nu\lambda} \;+\; \partial_\nu B_{\lambda\mu} \;+\; \partial_\lambda B_{\mu\nu}\;.\,) \tag{3.2.5}$$

3,14

The theory is Weyl invariant if $\beta^g_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\varphi = 0$. Remarkably, the vanishing of the β functions (3.2.4) may be derived as equations of motion from the target spacetime action

$$\mathcal{S} \;=\; \frac{1}{2\kappa_0}\int \mathrm{d}^{26}X\;\sqrt{|\det g|}\,e^{-2\varphi}\left(\mathcal{R}_g \;+\; 4\,\nabla_\mu\varphi\,\nabla^\mu\varphi \;-\; \frac{1}{12}\,H_{\mu\nu\lambda}\,H^{\mu\nu\lambda}\right)\;. \tag{3.2.6}$$

3,15

with spacetime Ricci scalar $\mathcal{R}_g$ built from the $g_{\mu\nu}$ metric. This is the effective action for the massless string states $\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|0,0,k\rangle \leftrightarrow g_{\mu\nu}, B_{\mu\nu}, \varphi$ of the closed string sector. From the form of $\mathcal{S}$, the $g_{\mu\nu}$ field may be identified with the target spacetime metric.

3.2.2 Background fields of the open string sector

Similarly, we may couple the open string to an abelian gauge field living on a D brane. This is achieved via worldsheet action

$$\mathcal{S}_A \;=\; \int_{\partial\Sigma} \mathrm{d}\tau \; A_\mu(X)\,\partial_\tau X^\mu \tag{3.2.7}$$

3,17

where $\partial\Sigma$ is the boundary of the worldsheet. The effective spacetime action for the open string sector (to leading order in α') is given by

$$\mathcal{S} \;=\; -\,C \int \mathrm{d}^{26}X \; e^{-\varphi}\, F_{\mu\nu}\, F^{\mu\nu} \; . \tag{3.2.8}$$

3,18

Therefore, the tree level open string physics is described by Yang Mills theory. Recall that α' can be interpreted as the squared string length, so the $\alpha' \to 0$ limit extracts the point particle-like behaviour.

A single D brane gives rise to the gauge group $U(1)$, but this can be generalized to non-abelian symmetry by taking a stack of coinciding D branes. Superposition of branes introduces non-dynamical degrees of freedom (from the worldsheet point of view) called *Chan Paton factors*. They arise on a stack of N Dp branes and are therefore assigned to the endpoints of the string. The Chan Paton factor λ_{ij} labels strings stretching from brane i to j where $i, j = 1, 2, ..., N$. The matrix λ is an element of some Lie algebra. It turns out that the only Lie algebras consistent with open string scattering amplitudes is $U(N)$. Note that the Chan Paton degrees of freedom parametrize a global symmetry on the worldsheet but a local symmetry in target spacetime. The theory of open strings ending on coincident Dp branes can effectively by described by a non-abelian gauge theory.

3.3 Superstring theory

Bosonic string theory which we have described so far has two major shortcomings. Firstly, it contains tachyons in both the open string- and the closed string sector which are states of negative mass square. Secondly, the bosonic string lacks fermionic degrees of freedom necessary to model particles observed in nature.

Let us give the supersymmetrized Polyakov action for the string position X^μ and its worldsheet superpartner Ψ^μ in conformal gauge $h_{\alpha\beta}(\tau,\sigma) = e^{\omega(\tau,\sigma)}\eta_{\alpha\beta}$:

$$\mathcal{S} \;=\; -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \; \left(\partial_\alpha X_\mu\, \partial^\alpha X^\mu \;+\; i\bar{\Psi}^\mu\, \gamma^\alpha\, \partial_\alpha \Psi_\mu\right) \tag{3.3.1}$$

3,19

The Ψ^μ are spacetime vectors of two components spinors on the worldsheet, $\Psi^\mu = \left(\psi^\mu_-, \psi^\mu_+\right)^T$ with real entries $\psi^\mu_\pm$. The γ^α denote worldsheet γ matrices for which one possible representation is

$$\gamma^0 \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \, , \qquad \gamma^1 \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \, . \qquad (3.3.2) \quad \boxed{\texttt{3,20}}$$

In terms of lightcone derivatives $\partial_\pm := \frac{\partial}{\partial \sigma^\pm}$ with $\sigma^\pm = \tau \pm \sigma$, the fermionic part of the action (3.3.1) may be rewritten as

$$\mathcal{S}_f \;=\; \frac{i}{2\pi\alpha'} \int \mathrm{d}^2\sigma \, \left(\psi^\mu_- \, \partial_+ \psi_{-\mu} \;+\; \psi^\mu_+ \, \partial_- \psi_{+\mu} \right) \, , \qquad (3.3.3) \quad \boxed{\texttt{3,21}}$$

the equations of motion describe left- and right moving waves just like in the bosonic sector,

$$\partial_+ \psi^\mu_- \;=\; \partial_- \psi^\mu_+ \;=\; 0 \, . \qquad (3.3.4) \quad \boxed{\texttt{3,22}}$$

The total action is invariant under the worldsheet supersymmetry transformations $\delta_\varepsilon X^\mu = \bar{\varepsilon} \Psi^\mu$ and $\delta_\varepsilon \Psi^\mu = \gamma^\alpha \partial_\alpha X^\mu \varepsilon$ where the parameter ε is an infinitesimal constant Majorana spinor.

3.3.1 Open superstrings

Upon integrating the action (3.3.3) by parts, one encounters the boundary term

$$\delta \mathcal{S}_f \;=\; \frac{i}{4\pi\alpha'} \int \mathrm{d}\tau \, \left(\psi^\mu_- \, \delta \psi_{-\mu} \;-\; \psi^\mu_+ \, \delta \psi_{+\mu} \right) \Big|_{\sigma=0}^{\sigma=\pi} \, . \qquad (3.3.5) \quad \boxed{\texttt{3,23}}$$

In the open string sector, we have to impose that the contributions from $\sigma = 0$ and $\sigma = \pi$ vanish separately. This is equivalent to

$$\psi^\mu_- \, \delta \psi_{-\mu} \;-\; \psi^\mu_+ \, \delta \psi_{+\mu} \Big|_{\sigma=0,\pi} \;=\; 0 \quad \Leftrightarrow \quad \delta \big(\psi_{+\mu} \big)^2 \Big|_{\sigma=0,\pi} \;=\; \delta \big(\psi_{+\mu} \big)^2 \Big|_{\sigma=0,\pi} \;=\; 0 \, . \qquad (3.3.6) \quad \boxed{\texttt{3,24}}$$

Since the overall sign of the spinor components can be chosen arbitrarily, we impose $\psi^\mu_+(\tau, 0) = \psi^\mu_-(\tau, 0)$, then the boundary condition at $\sigma = \pi$ leaves two options corresponding to the *Neveu Schwarz-* and the *Ramond sector* of the theory:

$$\begin{aligned} \mathrm{R} : \quad \psi^\mu_+(\tau, \pi) &\;=\; +\, \psi^\mu_-(\tau, \pi) \\ \mathrm{NS} : \quad \psi^\mu_+(\tau, \pi) &\;=\; -\, \psi^\mu_-(\tau, \pi) \end{aligned} \qquad (3.3.7) \quad \boxed{\texttt{3,25}}$$

These boundary conditions give rise to the Fourier expansions

$$\begin{aligned} \mathrm{R} : \quad \psi^\mu_\mp(\tau, \pi) &\;=\; \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} d^\mu_n \, e^{-in\sigma_\mp} \\ \mathrm{NS} : \quad \psi^\mu_\mp(\tau, \pi) &\;=\; \frac{1}{\sqrt{2}} \sum_{r \in \mathbb{Z} - \frac{1}{2}} b^\mu_r \, e^{-ir\sigma_\mp} \end{aligned} \qquad (3.3.8) \quad \boxed{\texttt{3,26}}$$

with Grassmann valued modes d_n, b_r. The string states are created by acting on the ground state of the NS- and R sectors with creation operators. The NS ground state is tachyonic and will be removed from the spectrum. The spectrum of both NS- and R sector can be truncated in a specific way which eliminates the tachyons. This truncation prescription is called *GSO projection* due to Gliozzi, Scherk and Olive. This projection leaves an equal number of fermions and bosons at each mass level and therefore paves the way for spacetime supersymmetry.

3.3.2 Closed superstrings

The closed sector of superstring theory can be constructed in four different ways. Each of left- and right movers may be taken from open string NS- or R sectors. From spacetime point of view, we find the following statistics for the states:

- NS-NS, R-R sectors $\leftrightarrow$ spacetime bosons
- NS-R, R-NS sectors $\leftrightarrow$ spacetime fermions

The NS-NS sector contains the fields $g_{\mu\nu}, B_{\mu\nu}, \varphi$ which we had already discussed in bosonic string theory whereas the "mixed" NS-R, R-NS sectors contain SUSY superpartners such as gravitino and dilatino.
The R-R sector is more complicated due to the degenerate ground state. There are two possible inequivalent R-R ground states (which differ by chirality), corresponding to type IIA- and type II B superstring theory. In type IIB, left- and right moving sectors have the same chirality, this leads to a scalar C_0 and antisymmetric tensor fields C_2 and C_4 of rank 2 and 4 at the massless level. Type IIA (with R-R ground states of opposite chiralities) gives rise to C_1, C_3 tensor fields.
The Dp branes to which the C_p forms couple of course also differ between the two theories

$$\begin{aligned} \text{type II B} \quad &\leftrightarrow \quad \text{D1, D3, D5, D7, D9 branes} \\ \text{type II A} \quad &\leftrightarrow \quad \text{D0, D2, D4, D6, D8 branes} \end{aligned}$$

where the D3 branes play a major role in the AdS/CFT correspondence. Although types IIA/B are inequivalent theories, they are related by dualities.
As it was shown in the 90's by Witten, there in fact three further consistent superstring theories known as type I and heterotic string theories (with gauge groups $SO(32)$ and $E_8 \times E_8$). They are connected with each other and the type II models by a web of dualities. For AdS/CFT purposes, however, it is sufficient to focus on type II.

3.4 D branes

D branes have a dual interpretation which is crucial for the AdS/CFT correspondence:

- hyperplanes where open strings can end

  To lowest order in α', massless excitations of D branes are described by supersymmetric Yang Mills theory (with gauge group $SU(N)$ in presence of N branes).

- solitonic solutions of type IIB supergravity in $D = 10$ dimensions

  D branes are very massive and curve spacetime around them. The lowest energy closed string excitations are gravitons.

The AdS/CFT is based on the identification of these two pictures in a particular limit!

3.4.1 Effective actions for D branes

Just as fundamental strings, D branes can couple to background fields, in particular to gravity. We aim to find a world volume action describing their dynamics as a generalization of the worldsheet action for strings. The background fields act as generalized couplings.
Let ξ^a denote the coordinates for the world volume of a Dp brane (which reduces to $\xi^0 = \tau$ and $\xi^1 = \sigma$ in case of the fundamental string). In direct analogy to the string worldsheet area action, the bosonic part of the D brane action is given by

$$\mathcal{S}_{\mathrm{DBI}}^{(p)} \;=\; -\,\mu_p \int \mathrm{d}^{p+1}\xi \; e^{-\varphi} \sqrt{\det\bigl(g^*_{ab} \,+\, B^*_{ab} \,+\, 2\pi\,\alpha'\,F_{ab}\bigr)} \;. \tag{3.4.1}$$ 3,27

The action (3.4.1) is known as *Dirac Born Infeld action*, or in short, *DBI action*. Its prefactor $\mu_p = (2\pi)^{-p}\alpha'^{-(p+1)/2}$ relates to the (genuinely non-perturbative) brane tension $T_p = \mu_p/g_{\mathrm{s}}$, and g^* is the induced metric on the brane obtained via pullback of the spacetime metric to the brane worldvolume,

$$g^*_{ab} \;=\; \frac{\partial X^\mu}{\partial \xi^a}\,\frac{\partial X^\nu}{\partial \xi^b}\; g_{\mu\nu} \;. \tag{3.4.2}$$ 3,28

The same applies to the B field.
Expanding the DBI action in flat spacetime (with $g^*_{ab} = \eta_{ab}$) by means of $\det(1+M) = 1 - \frac{1}{4}\mathrm{Tr}\{M^2\}$ for antisymmetric matrices M, we see that the DBI action for D3 branes is a generalization of Yang Mills theory

$$\mathcal{S}_{\mathrm{DBI}}^{(p=3)} \;\sim\; \alpha'^{-2} \int \mathrm{d}^4\xi \;\mathrm{Tr}\bigl\{\mathcal{F}_{ab}\,\mathcal{F}^{ab}\bigr\} \;, \qquad \mathcal{F}_{ab} \;=\; B^*_{ab} \,+\, 2\pi\,\alpha'\,F_{ab} \;. \tag{3.4.3}$$ 3,29

D branes also carry some charge under the R-R p form fields C_p. The full action describing a charged *BPS brane* (named after Bogomolnyi, Prasad and Sommerfeld) involves a *Chern Simons term*, $\mathcal{S} = \mathcal{S}_{\mathrm{DBI}} \pm \mathcal{S}_{\mathrm{CS}}$,

$$\mathcal{S}_{\mathrm{CS}} \;=\; \mu_p \int \mathrm{d}^{p+1}\xi \;\sum_q C^*_{q+1} \wedge \mathrm{Tr}\bigl\{e^{\mathcal{F}}\bigr\} \;, \tag{3.4.4}$$ 3,30

it describes the interaction of the R-R fields C_{q+1} with the NS-NS field B. The exponential of the two form $\mathcal{F}$ has to be understood in terms of the wedge product.
BPS branes are stable due to charge conservation. In type IIA/B superstring theory, Dp branes with p even/odd are BPS stable since R-R gauge potentials C_{p+1} are present to which Dp branes can couple. Unlike fundamental strings, D branes are non-perturbative objects since the tension and therefore their energy scales as $1/g_s$, i.e. with the inverse string coupling.

3.4.2 D branes in supergravity

c:Dsugra

We have discussed in subsection 3.2.1 that to leading order in α' (i.e. at low energies when only massless excitations contribute), Weyl invariance of the string worldsheet action in curved background is equivalent to certain field equations which can be derived from a gravity action. In superstring theory, this effective target space action is precisely that of *supergravity*. For this reason, the supergravity theories are referred to as type IIA/B although they can be motivated independent of string theory.
In type IIB supergravity, the bosonic field consists of the massless closed string states, $g_{\mu\nu}, B_{\mu\nu}$ and φ from the NS-NS sector and the form the R-R form fields C_0, C_2 and C_4. In addition, there are fermions with an equal number of degrees of freedom as in the bosonic part.
Moreover, we define the *axio-dilaton* τ and a complex 3 form G_3 by

$$\tau \;:=\; C_0 \;+\; i\,e^{-\varphi} \,, \qquad G_3 \;:=\; F_3 \;-\; \tau\, H_3 \tag{3.4.5}$$ 3,31

where F_3, H_3 are the field strengths of C_2 and B_2 (in differential form notation $F_3 = \mathrm{d}C_2$ and $H_3 = \mathrm{d}B_2$). The C_4 potential is more conveniently represented by the field strength

$$\tilde{F}_5 \;=\; \mathrm{d}C_4 \;+\; \frac{1}{2}\, B_2 \wedge F_3 \;-\; \frac{1}{2}\, C_2 \wedge H_3 \;. \tag{3.4.6}$$ 3,32

Let us finally introduce the rescalings $\tilde{g}_{\mu\nu} = e^{(\varphi_0 - \varphi)/6}$ and $\kappa = \kappa_0 e^{\varphi_0} = \sqrt{8\pi G_\mathrm{N}}$ into the *Einstein frame*, then the type IIB supergravity action is given by

$$\mathcal{S}_\mathrm{IIB} \;=\; \frac{1}{2\,\kappa^2} \int \mathrm{d}^{10}x\, \sqrt{-\tilde{g}} \left(\mathcal{R}_{\tilde{g}} \;-\; \frac{|\partial_\mu \tau|^2}{2\,(\mathrm{Im}\,\tau)^2} \;-\; \frac{|G_3|^2}{12\,\mathrm{Im}\,\tau} \;-\; \frac{|\tilde{F}_5|^2}{4\cdot 5!} \right) \;+\; \frac{1}{8i\,\kappa^2} \int \frac{C_4 \wedge G_3 \wedge \bar{G}_3}{\mathrm{Im}\,\tau} \;. \tag{3.4.7}$$ 3,33

The field strength $\tilde{F}_5$ has to be self-dual in the sense that

$$(\star F)_{\mu_1 \ldots \mu_5} \;=\; F_{\mu_1 \ldots \mu_5} \tag{3.4.8}$$ 3,34

where the Hodge dual $(\star\omega)_k$ of a k form ω in D dimensions is defined by

$$(\star\omega)_{\mu_1 \ldots \mu_{D-k}} \;=\; \frac{|\det g|}{k!}\, \varepsilon_{\nu_1 \ldots \nu_k \mu_1 \ldots \mu_{D-k}}\, \omega^{\nu_1 \ldots \nu_k} \;, \tag{3.4.9}$$ 3,35

e.g. $\star F_{\mu\nu} = \frac{|\det g|}{2} \varepsilon_{\mu\nu\lambda\rho} F^{\lambda\rho}$ in $D = 4$ dimensions.
Now let us look for solitonic solutions of the equations of motion due to (3.4.7). A Dp brane is a BPS solution of 10 dimensional supergravity, i.e. it is annihilated by half the Poincaré supercharges Q_α. It has a $p+1$ dimensional flat hypersurface with Poincaré invariance group $\mathbb{R}^{p+1} \times SO(1,p)$. The transverse space is then of dimension $D-p-1$.

Here, it would be nice to have a few more words about the 'BPS' term and about the relation between SUSY conditions and eq. of motion...
A p brane in 10 dimensions has symmetries $\mathbb{R}^{p+1} \times SO(1,p) \times SO(9-p)$. An ansatz which solves the equations of motion of type IIB supergravity is

$$\mathrm{d}s^2 \;=\; \frac{1}{\sqrt{H(\vec{y})}} \,\mathrm{d}x^\mu \,\mathrm{d}x_\mu \;+\; \sqrt{H(\vec{y})}\,\mathrm{d}\vec{y}\cdot\mathrm{d}\vec{y} \tag{3.4.10}$$ 3,36

where x^μ are the coordinates on the brane world volume and $\vec{y}$ denote the coordinates perpendicular to the brane. It turns out by means of the supergravity equations of motion that

$$e^{\varphi(\vec{y})} \;=\; \big[H(\vec{y})\big]^{\frac{3-p}{4}} \;, \qquad H \;\equiv\; \text{harmonic function of } y = \sqrt{\vec{y}\cdot\vec{y}}\;. \tag{3.4.11}$$ 3,37

Far away from the brane, i.e. at $y \to \infty$, flat space has to be recovered, this boundary condition uniquely fixes H to be

$$H(\vec{y}) \;=\; 1 \;+\; \left(\frac{L}{y}\right)^{D-p-3} \;. \tag{3.4.12}$$ 3,38

L is a length scale related to the only dimensionful parameter α'. For a stack of N coincident Dp branes, one finds

$$L^{D-p-3} \;=\; N\, g_s\, (4\pi)^{(5-p)/2}\, \Gamma\left(\tfrac{7-p}{2}\right)\, \alpha'^{(D-p-3)/2}\;. \tag{3.4.13}$$ 3,39

Let us finally summarize the special features of D3 branes:

- its worldvolume has 1+3 dimensional Poincaré invariance
- axion- and dilaton fields (C_0, φ) are constant with relation to the coupling $g_{\mathrm{YM}}^2 = g_{\mathrm{s}} = e^\varphi$
- it is a regular supergravity solution for $y \to 0$
- it couples to a self-dual five form $\mathrm{d}C_4 = F_5 = \star F_5$
- string theory implies (since $g_{\mathrm{YM}}^2 = g_s$) that

$$L^4 \;=\; 4\pi\, g_{\mathrm{s}}\, N\, \alpha'^2 \;=\; 4\pi\,\lambda\,\alpha'^2\;, \qquad \lambda \;=\; g_{\mathrm{s}}\, N \;\equiv\; \text{t'Hooft coupling} \tag{3.4.14}$$ 3,42

Chapter 4

The AdS/CFT correspondence

4.1 Maldacena's original argument

Following the arguments of [5], let us consider type IIB string theory in 9+1 dimensional spacetime with a stack of N D3 branes. There are two kinds of excitations:

- closed strings: excitations of empty space with the graviton as the massless mode
- open strings ending on the D3 branes: $\exists$ excitations of D branes

At energies below the string mass scale $(\alpha')^{-1/2}$, only massless string states are excited:

- massless closed string states $\leftrightarrow$ gravity multiplet of type IIB supergravity
- massless open strings states $\leftrightarrow$ $\mathcal{N} = 4$ vector multiplet in 3+1 dimensions, $SU(N)$ SYM

4.1.1 D3 branes from the open string point of view

The low energy effective action for the massless excitations of N D3 branes in flat ten dimensional space has the schematic form

$$\begin{aligned}
\mathcal{S} \;&=\; \mathcal{S}_{\text{bulk}} \;+\; \mathcal{S}_{\text{brane}} \;+\; \mathcal{S}_{\text{int}} \\
\mathcal{S}_{\text{bulk}} \;&\equiv\; D = 10 \text{ supergravity including higher derivative terms, i.e. } \alpha' \text{ corrections} \\
\mathcal{S}_{\text{brane}} \;&\equiv\; \text{DBI- and CS action defined on 3+1 dimensional brane world volume:} \\
&\quad\;\; \text{for small } \alpha'\text{, we get SYM} \sim \mathrm{Tr}\{F_{\mu\nu}F^{\mu\nu}\} \text{ plus interactions} \sim \alpha'\mathrm{Tr}\{F^4\} + ... \\
\mathcal{S}_{\text{int}} \;&\equiv\; \text{bulk-brane interaction: leading term is appearance of the background} \\
&\quad\;\; \text{metric } g \text{ in the brane action}
\end{aligned}$$

For $\alpha' \to 0$, the bulk action becomes the Einstein Hilbert action with coupling $\kappa \sim g_s \alpha'^2$. In the expansion $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ about flat space (with Minkowski metric η), the leading terms are

$$\mathcal{S}_{\text{bulk}} \;=\; \frac{1}{2\,\kappa^2} \int \mathrm{d}^{10}x \,\sqrt{|g|}\,\mathcal{R}_g \;\sim\; \int \mathrm{d}^{10}x \,\Big((\partial h)^2 \;+\; \kappa\,(\partial h)^2\,h \;+\; ... \Big)\,. \qquad (4.1.1) \quad \boxed{\texttt{3,43}}$$

In the low energy limit $\kappa \sim g_s \alpha'^2 \to 0$, the interaction terms $\mathcal{O}(\kappa)$ drop out, so gravity becomes free at long distances. Similar behaviour can be observed in the $\mathcal{S}_{\text{int}}$ sector. The term "low energy limit" should not be taken too literally: the relevant energies E are certainly kept fixed but we send the dimensionful parameter $\alpha' \to 0$, therefore various dimensionless quantities such as $\alpha' E^2$ are suppressed.

4.1.2 D3 branes from the closed string point of view

In their solitonic interpretation, D branes are viewed as massive charged objects which act as sources for the various supergravity fields. Specializing (3.4.12) to D3 branes in ten dimensions ($D = 10$ and $p = 3$) yields the metric

$$\begin{aligned}
\mathrm{d}s^2 \;&=\; \frac{1}{\sqrt{H(\vec{y})}}\,\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \;+\; \sqrt{H(\vec{y})}\,\big(\mathrm{d}y^2 \;+\; y^2\,\mathrm{d}\Omega_5^2\big) \\
H(\vec{y}) \;&=\; 1 \;+\; \left(\frac{L}{y}\right)^4 \,.
\end{aligned} \qquad (4.1.2) \quad \boxed{\texttt{3,44}}$$

Let us discuss the limits of this metric: When $y^4 \gg L^4 = 4\pi g_s N \alpha'^2$, one recovers flat 10 D space. When $y < L$, on the other hand, the metric appears to be singular as $y \to 0$. To examine this limit more carefully, let us define a new coordinate $u := L^2/y$. In the limit of large u (where $H = 1 + u^4/L^4 \to u^4/L^4$), the metric takes the asymptotic form

$$\mathrm{d}s^2\Big|_{u\to\infty} \;=\; L^2\left(\frac{1}{u^2}\,\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \;+\; \frac{\mathrm{d}u^2}{u^2} \;+\; \mathrm{d}\Omega_5^2\right)\,. \qquad (4.1.3) \quad \boxed{\texttt{3,41}}$$

In this *near horizon limit* $y \to 0 \Leftrightarrow u \gg L$, the geometry close to the brane is regular and highly symmetrical (with isometry group $SO(4,2) \times SO(6)$). Apart from the S^5 sphere represented by $\mathrm{d}\Omega_5^2$, we rediscover the AdS$_5$ metric (2.3.12).
An important property of the metric (4.1.2) is its non-constant *redshift factor* $\big(H(\vec{y})\big)^{-1/4} = g_{tt}$ with an interesting near horizon limit:

$$\big(H(\vec{y})\big)^{-1/4} \;=\; \big(1 \;+\; L^4/y^4\big)^{-1/4} \;=\; \begin{cases} \sim \;\; 1 & : \text{ large } y \\ \sim \;\; y/L & : \text{ small } y \end{cases} \qquad (4.1.4) \quad \boxed{\texttt{3,44a}}$$

The energy E_p of an object measured by an observer at constant position y differs from the energy E_i of the same object, this time measured by an observer at infinity,

$$\big(H(\vec{y})\big)^{-1/4}\,E_p \;=\; E_i\,. \qquad (4.1.5) \quad \boxed{\texttt{3,45}}$$

When the object approaches $y \to 0$, it appears to have lower and lower energy to the observer at infinity. This gives another, geometric notion of low energy regime. We have to distinguish two kinds of low energy excitations:

- particles approaching $y \to 0$
- massless particles propagating in the bulk (away from $y = 0$)

Their excitations decouple from each other in the low energy limit: Bulk massless particles decouple from the near horizon region around $y \to 0$. Excitations close to $y = 0$ are trapped by the gravitational potential to the $\mathrm{AdS}_5 \times S^5$ region.

4.1.3 Different forms of the AdS/CFT correspondence

Both from the point of view of open strings' field theory limit and from the supergravity point of view, there are two decoupled theories in the low energy regime. One of them is free supergravity in flat space, and we are led to identify it with the supersymmetric gauge theory which appears in both descriptions:

$$\boxed{\mathcal{N} = 4 \text{ SYM with gauge group } SU(N) \quad \overset{(*)}{\Longleftrightarrow} \quad \text{type IIB supergravity}}$$

The $(*)$ above the arrow indicates that the correspondence claimed in this *AdS/CFT conjecture* holds in the $N \to \infty$ limit at large and fixed t'Hooft coupling $\lambda = g_s N$. Maldacena generalized this idea to conjecture that the duality goes beyond the supergravity approximation.

The *strongest form of the AdS/CFT correspondence* conjectures that the duality between the supersymmetric $SU(N)$ gauge theory and type IIB supergravity holds for *any* value of N and g_s. This implies that $\mathcal{N} = 4$ SYM is exactly equivalent to the full type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$. However, it is at present not possible to test the strongest form since there is no consistent non-perturbative quantization of string theory yet, in particular not in curved spacetime.

In the (modestly) *strong form of the AdS/CFT conjecture*, one keeps $\lambda = g_s N$ fixed while sending $N \to \infty$. In this case the ground state is classical type IIB string theory on $\mathrm{AdS}_5 \times S^5$. The perturbative expansion parameter is $g_s = \lambda/N \ll 1$ on the string theory side, this corresponds to a perturbative $1/N$ expansion on the field theory side.

Finally, there is the *weak form of the AdS/CFT conjecture* described above. It states that the correspondence is only valid in the *Maldacena limit* $N \to \infty$ and λ very large. It relates $\mathcal{N} = 4$ SYM at strong coupling and $N \to \infty$ with classical supergravity. In contrast to previous forms,

α' is assumed to be small now, and the α' expansion of supergravity is dual to a field theory expansion in $\lambda^{-1/2}$ powers around the strong coupling limit.

$$\text{weak form of AdS/CFT correspondence} \;:\quad \begin{pmatrix} \lambda \to \infty \\ N \to \infty \end{pmatrix} \quad \leftrightarrow \quad \begin{pmatrix} g_{\mathrm{s}} \to 0 \\ \alpha' \to 0 \end{pmatrix}$$

The AdS/CFT map provides a weak/strong coupling duality:

- more complicated to test: only direct tests based on objects which are independent of the coupling
- interesting predictive power: non-trivial prediction for strongly coupled gauge theories

4.2 Field operator map

The aim of this section is to work out the precise dictionary between objects of the two equivalent theories,

$$\begin{pmatrix} \mathcal{N} = 4 \text{ SYM} \\ N, \lambda \to \infty \end{pmatrix} \quad \leftrightarrow \quad \begin{pmatrix} \text{type IIB supergravity} \\ \text{on AdS}_5 \times S^5 \end{pmatrix}$$

in particular between representations of the common symmetry groups. We will relate field theory operators to supergravity fields which transform in the same representation of the superconformal group $SU(2,2|4)$ or its bosonic subgroup $SO(6) \times SO(4,2)$. This provides a one-to-one map between gauge invariant operators in $\mathcal{N} = 4$ SYM and classical fields in IIB supergravity on $\text{AdS}_5 \times S^5$.

4.2.1 CFT correlation functions

A crucial role in testing the AdS/CFT correspondence is played by the computation and comparison of correlation functions. Correlators which obey *non-renormalization theorems* (i.e. which are λ independent) will be of particular interest. Let us give a brief review of correlation functions in QFT.

Composite operators with coinciding arguments such as $\big(\varphi(x)\big)^2$ require regularization, the regularized version will be denoted by $\big[\varphi^2(x)\big]$. Consider an n point function of composite regularized gauge invariant operators $\mathcal{O}_k(x)$,

$$\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \dots \mathcal{O}_n(x_n) \rangle \; .$$

An important tool to compute this correlator is the generating functional $\mathcal{Z}[J]$ (and its analogue $W[J]$ for connected diagrams) defined by

$$\mathcal{Z}[J] \quad := \quad \left\langle \exp\left(- \int \mathrm{d}^D x \; \mathcal{L}_J \right) \right\rangle \quad = \quad e^{-W[J]} \tag{4.2.1}$$
4,1

where $\mathcal{L}_J$ is the Lagrangian of a given QFT with added source term coupled to a basis $\{\mathcal{O}_i\}$ of gauge invariant local operators:

$$\mathcal{L}_J \quad = \quad \mathcal{L} \; + \; \sum_i J_i \, \mathcal{O}_i \tag{4.2.2}$$
4,2

The n point function is then given by

$$\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \ldots \mathcal{O}_n(x_n) \rangle \quad = \quad \frac{\delta^n \ln \mathcal{Z}[J]}{\delta J_1(x_1)\, \delta J_2(x_2) \ldots \delta J_n(x_n)} \bigg|_{J_i = 0} \, . \tag{4.2.3}$$
4,3

To calculate correlation functions in $\mathrm{AdS}_5 \times S^5$, it is convenient to work in Euclidean AdS_5 with Poincaré coordinates

$$H \quad := \quad \left\{ (z_0, \vec{z}), \;\; z_0 > 0, \;\; \vec{z} \in \mathbb{R}^4 \right\} , \qquad \partial H \quad = \quad \mathbb{R}^4 \, . \tag{4.2.4}$$
4,4

The metric

$$\mathrm{d}s^2 \quad = \quad \frac{1}{z_0^2} \left(\mathrm{d}z_0^2 \; + \; \mathrm{d}\vec{z}^2 \right) \tag{4.2.5}$$
4,5

diverges at the boundary $z_0 \to 0$, but it is merely a coordinate singularity, not a curvature singularity. The divergence may be removed by a Weyl rescaling. As we will see later, however, sometimes it is necessary (and useful) to consider a cutoff at fixed $z_0 = \varepsilon$. The UV cutoff $\Lambda = \frac{1}{\varepsilon}$ is mapped to an IR cutoff ε in AdS. It is natural to assume that $\mathcal{N} = 4$ SYM lives on the boundary of AdS_5.

Typical gauge invariant operators in $SU(N)$ SYM with $\mathcal{N} = 4$ in $D = 4$ are

$$\mathcal{O}_\Delta(x) \quad := \quad \mathrm{sTr}\big\{ X^{i_1} \, X^{i_2} \ldots X^{i_\Delta} \big\} \quad = \quad N^{(1-\Delta)/2} \, C_{i_1 \ldots i_\Delta} \, \mathrm{Tr}\big\{ X^{i_1} \, X^{i_2} \ldots X^{i_\Delta} \big\} \, . \tag{4.2.6}$$
4,6

Here, Δ denotes the conformal dimension of the operators, X^i are the elementary scalar fields of $\mathcal{N} = 4$ SYM transforming in the representation $(\underline{6})$ of $SO(6) \cong SU(4)$ and $C_{i_1 \ldots i_\Delta}$ fall into the totally symmetric rank Δ tensor representation of $SO(6)$. The trace is taken over color indices (recall that all the fields transform in the adjoint representation of $SU(N)$). The normalization is chosen such that all planar graphs scale with N^2.

4.2.2 The dual fields of supergravity

On the AdS side, we decompose all fields into Kaluza Klein towers on S^5, i.e. we expand the fields in spherical harmonics $Y_\Delta(\vec{y})$ of S^5:

$$\varphi(z, \vec{y}) \quad = \quad \sum_{\Delta=0}^{\infty} \varphi_\Delta(z) \, Y_\Delta(\vec{y}) \tag{4.2.7}$$
4,7

The ten dimensional Klein Gordon equation implies a massive wave equation in the five dimensional AdS sector,

$$\left(\Box_5 \,+\, m_\Delta^2\right)\varphi_\Delta(z) \;=\; 0\,, \qquad m_\Delta^2 \;=\; \Delta\,(\Delta\,-\,4)\,. \tag{4.2.8}$$ $\boxed{\texttt{4,8}}$

It has two independent solutions which can be characterized by their asymptotics as $z_0 \to 0$:

$$\varphi_\Delta(z_0,\vec{z}) \;\sim\; \begin{cases} z_0^\Delta & : \text{ normalizable} \\ z_0^{4-\Delta} & : \text{ non-normalizable} \end{cases} \tag{4.2.9}$$ $\boxed{\texttt{4,9}}$

The non-normalizable fields define associated boundary fields [3] by virtue of

$$\bar{\varphi}_\Delta(\vec{z}) \;:=\; \lim_{z_0\to 0}\varphi_\Delta(z_0,\vec{z})\,z_0^{\Delta-4}\,. \tag{4.2.10}$$ $\boxed{\texttt{4,10}}$

We may identify the normalizable AdS modes φ_Δ as vacuum expectation values of the field theory operators $\mathcal{O}_\Delta$ and the non-normalizable modes $\bar{\varphi}_\Delta$ as sources for these operators:

$$\varphi_\Delta(z_0,\vec{z}) \;\sim\; \langle\mathcal{O}_\Delta\rangle\,z_0^\Delta \;+\; \bar{\varphi}_\Delta\,z_0^{4-\Delta} \tag{4.2.11}$$ $\boxed{\texttt{4,11}}$

The mapping between correlation functions in SYM theory and the supergravity dynamics is given as follows: The generating functional $W[\bar{\varphi}_\Delta]$ for all correlators of single trace operators $\mathcal{O}_\Delta$ in SYM is given in terms of the source fields $\bar{\varphi}_\Delta$. The boundary values of these supergravity fields become the sources for the QFT. In other words, on the field theory side we have

$$e^{-W[\bar{\varphi}_\Delta]} \;=\; \left\langle \exp\left(-\int_{\partial H} \mathrm{d}^4 z\;\bar{\varphi}_\Delta\,\mathcal{O}_\Delta\right)\right\rangle\,. \tag{4.2.12}$$ $\boxed{\texttt{4,12}}$

The AdS side is governed by an action in terms of the bulk fields $\mathcal{S}[\varphi_\Delta]$ in the framework of type IIB supergravity on $\mathrm{AdS}_5 \times S^5$. The AdS/CFT conjecture for correlation functions says that precisely this classical gravity action enters the generating functional for the subclass $\{\mathcal{O}_\Delta\}$ of operators in the $\mathcal{N} = 4$ QFT. The AdS boundary conditions have to be adjusted to meet the field theory values of the source fields:

$$W[\bar{\varphi}_\Delta] \;=\; \mathcal{S}[\varphi_\Delta]\Big|_{\lim_{z_0\to 0}\varphi_\Delta(z_0,\vec{z})\,z_0^{\Delta-4}=\bar{\varphi}_\Delta(\vec{z})} \tag{4.2.13}$$ $\boxed{\texttt{4,13}}$

The action $\mathcal{S}$ is the generating functional for tree diagrams on AdS space, i.e. for the classical expansion of correlators. These tree level graphs in AdS are referred to as *Witten diagrams* [2], let us give the corresponding Feynman rules:

- Each external source $\bar{\varphi}_\Delta(\vec{z})$ is located at the boundary.
- Propagators depart from the external sources either to another boundary point or to an interior interaction point (in which case they are called *bulk-to-boundary propagators*)

- The structure of the interior interaction points is governed by the interaction vertices of the supergravity action. These are obtained from the Kaluza Klein reduction on S^5.

- Two interior interaction points may be connected by *bulk-to-bulk propagators*.

picture of 2pt, 3pt and 2 times 4pt (1 or 2 vertices)

4.2.3 AdS propagators

:adsprop

In this section we will derive the scalar propagator in Euclidean AdS spacetime H as defined in (4.2.4). For simplicity, the AdS radius is set to $L = 1$. The four vector $\vec{z}$ in the metric $\mathrm{d}s^2 = \frac{1}{z_0^2}(\mathrm{d}z_0^2 + \mathrm{d}\vec{z}^2)$ parametrizes the boundary ∂H. The geodesic distance is obtained by solving the geodesic equation (where the parameter ξ is called *chordal distance*):

$$d(z,w) = \int_z^w \mathrm{d}s = \ln\left(\frac{1 + \sqrt{1-\xi^2}}{\xi}\right), \qquad \xi = \frac{2\, z_0\, w_0}{z_0^2 + w_0^2 + (\vec{z}-\vec{w})^2} \tag{4.2.14}$$ 4,15

Let us start from the scalar part of the action which we obtain by Kaluza Klein reduction of the ten dimensional IIB supergravity on S^5. Schematically we get

$$\mathcal{S}[\varphi_\Delta] = \int \mathrm{d}^5 z\, \sqrt{|g|}\left(\frac{1}{2}\, g^{\mu\nu}\, \partial_\mu\varphi_\Delta\, \partial_\nu\varphi_\Delta + \frac{m_\Delta^2}{2}\, \varphi_\Delta + \mathcal{L}_{\text{int}}\right) \tag{4.2.15}$$ 4,16

where $\mathcal{L}_{\text{int}}$ denotes higher order interaction terms from KK reduction. Now the propagators are represented by integral kernels K_Δ, G_Δ subject to

$$\varphi_\Delta(z) = \int_{\partial H} \mathrm{d}^4\vec{x}\; K_\Delta(z,\vec{x})\, \bar{\varphi}_\Delta(\vec{x}) \equiv \text{bulk-to-boundary propagator} \tag{4.2.16}$$ 4,17

$$\varphi_\Delta(z) = \int_H \mathrm{d}^5 x\; G_\Delta(z,x)\, J(x) \equiv \text{bulk-to-bulk propagator} \tag{4.2.17}$$ 4,18

The scalar Green function satisfies

$$\left(\Box_g + m_\Delta^2\right) G_\Delta(z,x) = \delta^5(z,x) \equiv \frac{\prod_{i=1}^5 \delta(z_i - x_i)}{\sqrt{|g|}}, \qquad m_\Delta^2 = \Delta\,(\Delta - 4) \tag{4.2.18}$$ 4,19

where the action of the Laplacian $\Box_g$ on scalar fields is in general given by

$$\Box_g\, \varphi = -\frac{1}{\sqrt{|g|}}\, \partial_\mu \sqrt{|g|}\, g^{\mu\nu}\, \partial_\nu\varphi \tag{4.2.19}$$

and reduces to the following expression in the AdS metric (**??**):

$$\Box_g\Big|_{\text{AdS}} = -\,z_0^2\,\partial_0^2 + (d-1)\, z_0\, \partial_0 - z_0^2 \sum_{i=1}^{d} \partial_i^2 \tag{4.2.20}$$ 4,20

This turns (4.2.18) into a hypergeometric equation. The Green function which solves it is thus given by a hypergeometric function in the argument ξ from (4.2.14):

$$\begin{aligned} G_\Delta(z,w) \;=\; G_\Delta(\xi) \;&=\; \frac{C_\Delta}{2^\Delta\,(2\Delta-d)}\,\xi^\Delta\,F_{2,1}\left(\tfrac{\Delta}{2},\tfrac{\Delta+1}{2},\Delta-1;\xi^2\right) \\ C_\Delta \;&=\; \frac{\Gamma(\Delta)}{\pi^2\,\Gamma(\Delta-2)} \end{aligned} \qquad (4.2.21) \quad \boxed{\texttt{4,21}}$$

When x is located at the boundary, G_Δ reduces to the bulk-to-boundary propagator

$$K_\Delta(z,\vec{x}) \;=\; C_\Delta\left(\frac{z_0}{z_0^2\,+\,(\vec{z}-\vec{x})^2}\right)^\Delta . \qquad (4.2.22) \quad \boxed{\texttt{4,22}}$$

Calculation of the two point function requires careful treatment of potential divergences at the boundary. We Fourier transform the boundary coordinates to momentum space. The $d+1$ dimensional bulk action

$$\mathcal{S}[\varphi] \;=\; \int \mathrm{d}^{d+1}z\,\sqrt{|g|}\,\left(\frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi\;+\;\frac{m^2}{2}\,\varphi^2\right) \qquad (4.2.23) \quad \boxed{\texttt{4,23}}$$

gives rise to a boundary term after integration by parts,

$$\mathcal{S}[\bar\varphi] \;=\; \frac{1}{2\,\varepsilon^{d-1}}\int \mathrm{d}^d\vec{z}\;\bar\varphi(\vec{z})\,\partial_0\varphi(\varepsilon,\vec{z})\;, \qquad (4.2.24) \quad \boxed{\texttt{4,24}}$$

it is regularized by cutting off the z_0 integral at $z_0=\varepsilon$. In the notation $\varphi(\varepsilon,\vec{p})\equiv\bar\varphi(\vec{p})\varepsilon^{d-\Delta}$, we Fourier transform

$$\varphi(z_0,\vec{z}) \;=\; \int \mathrm{d}^d\vec{p}\;e^{i\vec{p}\cdot\vec{z}}\,\varphi(z_0,\vec{p}) \qquad (4.2.25) \quad \boxed{\texttt{4,25}}$$

in order to simplify the equations of motion:

$$\left(z_0^2\,\partial_0^2\;-\;(d-1)\,z_0\,\partial_0\;-\;(\vec{p}^2\,z_0^2\;+\;m_\Delta^2)\right)\varphi(z_0,\vec{p}) \;=\; 0 \qquad (4.2.26) \quad \boxed{\texttt{4,26}}$$

This is a *Bessel equation* with solutions $z_0^{d/2}K_\nu(z_0p)$ (where $\nu=\Delta-d/2$ and $p=\sqrt{\vec{p}\cdot\vec{p}}$). The asymptotics is governed by $\lim_{z_0\to\infty} z_0^{d/2}K_\nu(z_0p)=0$ and $K_\nu(z_0\to 0)\sim z_0^{d-\Delta}$.
The normalized solutions of the boundary problem read

$$\varphi(z_0,\vec{p}) \;=\; \frac{z_0^{d/2}\,K_\nu(z_0p)}{\varepsilon^{d/2}\,K_\nu(\varepsilon p)}\;\bar\varphi(\vec{p})\,\varepsilon^{d-\Delta}\;. \qquad (4.2.27) \quad \boxed{\texttt{4,27}}$$

The first term of the expansion of the supergravity action in correlation functions is

$$\mathcal{S}_p[\bar\varphi] \;=\; \int \mathrm{d}^p\vec{p}\,\mathrm{d}^d\vec{q}\;(2\pi)^d\,\delta^d(\vec{p}+\vec{q})\,\varphi(\varepsilon,\vec{p})\,\partial_0\varphi(\varepsilon,\vec{q})\;, \qquad (4.2.28)$$

This yields the following two point functions for the dual CFT operators:

$$\langle\mathcal{O}_\Delta(\vec{p})\,\mathcal{O}_\Delta(\vec{q})\rangle_\varepsilon \;=\; \frac{\delta^2\mathcal{S}_p[\bar\varphi]}{\delta\bar\varphi(\vec{p})\,\delta\bar\varphi(\vec{q})} \;=\; -\,\frac{(2\pi)^d\,\delta^d(\vec{p}+\vec{q})}{\varepsilon^{2\Delta-d-1}}\,\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\,\ln\!\left(\varepsilon^{d/2}\,K_\nu(\varepsilon\vec{p})\right) \qquad (4.2.29)$$

The Bessel index ν is a positive integer whenever the associated CFT operator $\mathcal{O}_\Delta$ with $\Delta = \nu + d/2$ is a chiral primary. Bessel functions have an asymptotic $u \to 0$ expansion of the schematic form

$$K_\nu(u) \quad \to \quad u^{-\nu}\,(a_0 \,+\, a_1\,u^2 \,+\, a_2\,u^4 \,+\, ...) \,+\, u^\nu\, \ln u\,(b_0 \,+\, b_1\,u^2 \,+\, b_2\,u^4 \,+\, ...)\;, \qquad (4.2.30)$$ 4,30

this translates as follows to the level of two point functions:

$$\langle \mathcal{O}_\Delta(\vec{p})\,\mathcal{O}_\Delta(\vec{q})\rangle_\varepsilon \;=\; \frac{(2\pi)^d\,\delta^d(\vec{p}+\vec{q})}{\varepsilon^{2\Delta-d}}\,\Bigg(-\frac{d}{2} \,+\, \nu\,(1 \,+\, c_2 \,+\, \varepsilon^2\,p^2 \,+\, c_4\,\varepsilon^4\,p^4 \,+\, ...)$$
$$-\,\frac{2\,\nu\,b_0}{a_0}\,\varepsilon^{2\nu}\,p^{2\nu}\,\ln(\varepsilon p)\,(1 \,+\, a_2\,\varepsilon^2\,p^2 \,+\, ...)\Bigg) \qquad (4.2.31)$$ 4,31

Explicitly, we have $\frac{2\nu b_0}{a_0} = \frac{(-1)^{\nu+1}}{2^{2(\nu-1)}\Gamma(\nu)^2}$ and $\varepsilon^{2\nu} = \varepsilon^{2\Delta-d}$ such that

$$\langle \mathcal{O}_\Delta(\vec{p})\,\mathcal{O}_\Delta(-\vec{p})\rangle_\varepsilon \;=\; \frac{\beta_0 \,+\, \beta_1\,\varepsilon^2\,p^2 \,+\, ... \,+\, \beta_\nu\,(\varepsilon p)^{2(\nu-1)}}{\varepsilon^{2\Delta-d}} \,-\, \frac{2\,\nu\,b_0}{a_0}\,p^{2\nu}\,\ln(\varepsilon p) \,+\, \mathcal{O}(\varepsilon^2)\;. \qquad (4.2.32)$$ 4,32

The field theory of the first terms is governed by scheme dependent contact terms $\sim \Box^m\delta^d(\vec{x}-\vec{y})$ and the second term gives the correct non-local result

$$\langle \mathcal{O}_\Delta(\vec{p})\,\mathcal{O}_\Delta(-\vec{p})\rangle \;=\; -\,\frac{2\,\nu\,b_0}{a_0}\,p^{2\nu}\,\ln(\varepsilon p) \qquad (4.2.33)$$ 4,33

independent on ε. Transforming back to position space yields

$$\langle \mathcal{O}_\Delta(\vec{p})\,\mathcal{O}_\Delta(-\vec{p})\rangle \;=\; \frac{\Gamma(\Delta)}{\Gamma(\Delta-d/2)}\,\frac{2\Delta \,-\, d}{\pi^{d/2}\,|x-y|^{2\Delta}}\;. \qquad (4.2.34)$$ 4,34

Chapter 5

Tests of the correspondence

5.1 Three point function of 1/2 BPS operators

An impressive test of the AdS/CFT correspondence is the agreement of the three point functions of 1/2 BPS operators in $\mathcal{N} = 4$ SYM at large N with the corresponding fields in supergravity. To demonstrate this result, we will proceed as follows:

- look at two point functions to fix the normalization
- calculate three point function in SYM to zeroth order in the coupling
- check that this is not normalized at higher orders, i.e. prove a non-renormalization theorem to show independence of the correlator on the coupling
- calculate the correlation function on the gravity side (spacetime dependence from the Green function and couplings from KK reduction)

5.1.1 Correlation functions of 1/2 BPS operators

For the purpose of this section, it is convenient to modify the notation: An 1/2 BPS operator of $\mathcal{N} = 4$ SYM will be denoted by

$$\mathcal{O}_k^I \;=\; C^I_{i_1 \dots i_k} \, \mathrm{Tr}\big\{ X^{i_1} \dots X^{i_k} \big\} \tag{5.1.1}$$

$\boxed{\texttt{5,1}}$

where $k \equiv \Delta$ and the C^I are totally symmetric traceless rank k tensors of $SO(6)$.

The SYM action is normalized such that $g^2_{\mathrm{YM}} = 4\pi g_{\mathrm{s}}$, and the normalization $\mathrm{Tr}\{T^a T^b\} = \delta^{ab}/2$ of the $SU(N)$ generators T^a allows to recast it into the form

$$\mathcal{S} \;=\; -\frac{1}{2\, g^2_{\mathrm{YM}}} \int \mathrm{d}^4 x \;\mathrm{Tr}\big\{ F_{\mu\nu} F^{\mu\nu} \big\} \;+\; \text{SUSY completion}$$

$$= \; -\frac{1}{4\,g_{\mathrm{YM}}^2}\int \mathrm{d}^4x\; F^a_{\mu\nu}\,F^{a\mu\nu} \;+\; \text{SUSY completion} \tag{5.1.2}$$ 5,2

this gives rise to the following scalar propagators:

$$\langle X^{ia}(x)\,X^{jb}(y)\rangle \;=\; \frac{g_{\mathrm{YM}}^2\,\delta^{ij}\,\delta^{ab}}{(2\pi)^2\,|x-y|^2} \tag{5.1.3}$$ 5,3

The two point function on the field theory side to lowest order in perturbation theory is therefore given by

$$\begin{aligned}\langle \mathcal{O}^I_k(x)\,\mathcal{O}^J_k(y)\rangle \;&=\; C^I_{i_1\ldots i_k}\,C^J_{j_1\ldots j_k}\,\langle \mathrm{Tr}\big\{X^{i_1}(x)\ldots X^{i_k}(x)\big\}\,\mathrm{Tr}\big\{X^{j_1}(y)\ldots X^{j_k}(y)\big\}\rangle\\ &=\; C^I_{i_1\ldots i_k}\,C^J_{j_1\ldots j_k}\,\frac{N^k\,g_{\mathrm{YM}}^{2k}\big(\delta^{i_1j_1}\,\delta^{i_2j_2}\ldots\delta^{i_kj_k}\;+\;\text{cyclic permutations}\big)}{(2\pi)^{2k}\,|x-y|^{2k}}\\ &=\; \frac{k\,\lambda^k\,\delta^{IJ}}{(2\pi)^{2k}\,|x-y|^{2k}}\end{aligned} \tag{5.1.4}$$ 5,4

line notation diagram

The last equality only holds to leading order in N.

By an appropriate generalization, one can obtain a nice result for the three point function to lowest order in perturbation theory and in the limit of large N:

$$\langle \mathcal{O}^I_{k_1}(x)\,\mathcal{O}^J_{k_2}(y)\,\mathcal{O}^K_{k_3}(z)\rangle \;=\; \frac{\lambda^{\Sigma/2}\,k_1\,k_2\,k_3\,\langle C^I\,C^J\,C^K\rangle}{N\,(2\pi)^{\Sigma}\,|x-y|^{2\alpha_3}\,|y-z|^{2\alpha_1}\,|x-z|^{2\alpha_2}} \tag{5.1.5}$$ 5,5

Note that the spacetime dependence is completely determined by conformal invariance. We have used shorthands

$$\Sigma \;=\; k_1 \;+\; k_2 \;+\; k_3\;, \qquad \alpha_i \;=\; \frac{\Sigma}{2} \;-\; k_i \tag{5.1.6}$$ 5,6

(such that e.g. $\alpha_1 = \frac{k_2+k_3-k_1}{2}$) and $\langle C^I C^J C^K\rangle$ denotes a uniquely defined $SO(6)$ tensor contraction of indices determined by the Feynman graph.

anotherlinepicture

The notation can be streamlined by defining normalized operators $\tilde{\mathcal{O}}^I := \frac{(2\pi)^k}{\lambda^{k/2}\sqrt{k}}\,\mathcal{O}^I$. Their two point function is normalized to one,

$$\langle \tilde{\mathcal{O}}^I_k(x)\,\tilde{\mathcal{O}}^J_k(y)\rangle \;=\; \frac{\delta^{IJ}}{|x-y|^{2k}}\;, \tag{5.1.7}$$ 5,7

and the three point function reads

$$\langle \tilde{\mathcal{O}}^I_k(x)\,\tilde{\mathcal{O}}^J_k(y)\,\tilde{\mathcal{O}}^K_k(z)\rangle \;=\; \frac{\sqrt{k_1\,k_2\,k_3}\,\langle C^I\,C^J\,C^K\rangle}{N\,|x-y|^{2\alpha_3}\,|y-z|^{2\alpha_1}\,|x-z|^{2\alpha_2}}\;. \tag{5.1.8}$$ 5,8

This holds for large values of N, otherwise non-planar corrections of order $\frac{1}{N^2}$ arise.

5.1.2 The non-renormalization theorem

Next we demonstrate the absence of $\mathcal{O}(\lambda)$ terms both in $\langle\mathcal{O}\mathcal{O}\rangle$ and in $\langle\mathcal{O}\mathcal{O}\mathcal{O}\rangle$. The argument will hold for any N [9].

Define complex scalar fields $Z^i := X^i + iX^{i+3}$ making use of the embedding $SU(3) \subset SU(4)$. The Euclidean version of the $\mathcal{N}=4, SU(N)$ SYM Lagrangian then reads

$$
\begin{aligned}
\mathcal{L} \;=\; & \mathrm{Tr}\Big\{ \frac{1}{4}\, F_{\mu\nu}\, F^{\mu\nu} \;+\; \frac{1}{2}\, \bar{\lambda}\, \not{D}\lambda \;+\; D_\mu Z^i\, D^\mu \bar{Z}^i \;+\; \frac{1}{2}\, \bar{\psi}^i\, \not{D}\psi^i \\
& + i\sqrt{2}\, g\, f^{abc} \left(\bar{\lambda}_a\, \bar{Z}^i_b\, L\psi^i_c \;-\; \bar{\psi}^i_a R\, Z^i_b\, \lambda_c\right) \;-\; \frac{g}{\sqrt{2}}\, f^{abc}\, \varepsilon_{ijk} \left(\bar{\psi}^i_a L\, Z^j_a\, \psi^k_c \;+\; \bar{\psi}^i_a R\, \bar{Z}^i_b\, \psi^k_c\right) \\
& - \frac{g^2}{2}\, f^{abc}\, \bar{Z}^i_b\, Z^i_c\, f_{ade}\, \bar{Z}^{jd}\, Z^{je} \;+\; \frac{g^2}{2}\, f^{abc}\, f^{ade}\, \varepsilon_{ijk}\, \varepsilon_{ilm}\, Z^i_b\, Z^k_c\, \bar{Z}^l_d\, \bar{Z}^m_e \Big\}
\end{aligned}
\tag{5.1.9}
$$
5,9

where L, R denote the left- and right handed chirality projectors.

Due to supersymmetry, it is sufficient to consider

$$
\langle \mathrm{Tr}\big\{(Z^1)^k(x)\big\}\, \mathrm{Tr}\big\{(\bar{Z}^1)^k(y)\big\}\rangle \;=\; \frac{P_{k,k,0}(N)}{\big(4\pi^2\, |x-y|\big)^k}
\tag{5.1.10}
$$
5,10

with the following polynomial in N

$$
\begin{aligned}
P_{k,k,0}(N) \;&=\; \sum_{\sigma\in S_k} \mathrm{Tr}\big\{T^{a_1}\, T^{a_2} \ldots T^{a_k}\big\}\, \mathrm{Tr}\big\{T^{a_{\sigma(1)}}\, T^{a_{\sigma(2)}} \ldots T^{a_{\sigma(k)}}\big\} \\
&=\; k \left(\frac{N}{2}\right)^k \;+\; \text{lower order in } N\ .
\end{aligned}
\tag{5.1.11}
$$
5,11

There are various effects to consider at leading order in the coupling:

- self energy corrections

$$
a - -blob - -a' \;=\; \delta^{aa'}\, N\, A(x,y)\, G(x,y)
\tag{5.1.12}
$$
5,12

  with $A(x,y) = a_0 + a_1 \ln\big(\mu^2(x-y)^2\big)$ and the scalar propagator $G(x,y) = \frac{1}{4\pi^2|x-y|^2}$

- two particle exchange interactions

$$
(a\ to\ b)\ and\ (a'\ to\ b')\ +\ quartic\ vertex \;=\; \big(f^{pab}\, f^{pa'b'} \;+\; f^{pab'}\, f^{pa'b}\big)\, B(x,y)\, G(x,y)^2
\tag{5.1.13}
$$
5,13

  $B(x,y) = b_0 + b_1 \ln\big(\mu^2(x-y)^2\big)$

The possible corrections to the *rainbow graph* at order g^2_{YM} schematically look like

sum up graphs

and it turns out that these three graphs cancel each other for all N and for all k. The proof goes as follows:

- use a trace identity valid for any matrices N and M_i:

$$\sum_{i=1}^{n} \mathrm{Tr}\Big\{ M_1 \ldots M_{i-1} \, \big[M_i \, , \, N \big] \, M_{i+1} \ldots M_n \Big\} \tag{5.1.14}$$ 5,14

- combinatorics for color indices

- insert (5.1.12) between all pairs of adjacent lines using $[T^a, T^b] = i f^{abc} T^c$

- result for all exchange graphs (with S_k permutation σ):

$$\frac{1}{4} \, (-2\,B) \, \mathrm{Tr}\big\{ T^{a_1} \ldots T^{a_k} \big\} \sum_{i \neq j = 1}^{k} \mathrm{Tr}\Big\{ T^{a_{\sigma(1)}} \ldots \big[T^{a_{\sigma(i)}} \, , \, T^p \big] \ldots \big[T^{a_{\sigma(j)}} \, , \, T^p \big] \ldots T^{a_{\sigma(k)}} \Big\}$$

- apply (5.1.14) to one of the two commutators to find

$$\begin{aligned} & \frac{B}{2} \, \mathrm{Tr}\big\{ T^{a_1} \ldots T^{a_k} \big\} \sum_{i=1}^{k} \mathrm{Tr}\Big\{ T^{a_{\sigma(1)}} \ldots \Big[\big[T^{a_{\sigma(i)}} \, , \, T^p \big] \, , \, T^p \Big] \ldots T^{a_{\sigma(k)}} \Big\} \\ = \ & \frac{N\,B}{2} \, \mathrm{Tr}\big\{ T^{a_1} \ldots T^{a_k} \big\} \sum_{i=1}^{k} \mathrm{Tr}\Big\{ T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(i)}} \ldots T^{a_{\sigma(k)}} \Big\} \end{aligned} \tag{5.1.15}$$ 5,16

The last step follows from the fact that $\big[[\cdot, T^p], T^p\big]$ is the Casimir operator of the adjoint representation of $SU(N)$ such that $\big[[T^a, T^p] T^p\big] = N T^a$ and the sum over i yields k identical terms. In the self energy corrections, one also gets a factor of k by similar argument such that the overall contribution is

$$\frac{k\,N\,(B + 2A)}{2} \sum_{\sigma \in S_k} \mathrm{Tr}\big\{ T^{a_1} \ldots T^{a_k} \big\} \, \mathrm{Tr}\big\{ T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(k)}} \big\} \quad = \quad \frac{k\,N\,(B + 2A) \, P_{k,k,0}(N)}{2} \; . \tag{5.1.16}$$ 5,17

It follows from the non-renormalization theorem that $B + 2A = 0$. The reason for that is the following: The two point function $\mathrm{Tr}\{X^2\}$ falls into the same SUSY multiplet as the energy momentum tensor $T_{\mu\nu}$. It can be shown that the latter is not renormalized (in agreement with momentum conservation), so by supersymmetry, $\mathrm{Tr}\{X^2\}$ is protected as well.

On the other hand, it suffices to consider $k = 2$, i.e. to check explicitly that there are no quantum corrections to $\langle \mathcal{O}_k(x) \mathcal{O}_k(y) \rangle$ at order $\mathcal{O}(g_{\mathrm{YM}}^2) = \mathcal{O}(\lambda)$.

5.1.3 The three point function on the gravity side

Having obtained an exact result for the three point function of 1/2 BPS operators on the field theory side, we are ready to compare with a gravity counterpart. Let us consider three

point functions of scalar fields in AdS spacetimes. Their Feynman diagram has the structure of a Mercedes star. It is specified by three edge points $\vec{x}$, $\vec{y}$, $\vec{z}$, by three bulk-to-boundary propagators and a coupling in the center determined by Kaluza-Klein reduction of S^5.
Recall from section 4.2.3 that the bulk-to-boundary Green functions in AdS_{d+1} is given by

$$K_\Delta(z_0, \vec{z}, \vec{x}) \;=\; \frac{\Gamma(\Delta)}{\pi^{d/2}\,\Gamma(\Delta-2)} \left(\frac{z_0}{z_0^2 \,+\, (\vec{z}-\vec{x})^2} \right)^{\Delta} \tag{5.1.17}$$

Because of its defining property $\lim_{z_0\to 0}\left[z_0^{\Delta-d} K_\Delta(z_0,\vec{z},\vec{x})\right] = \delta^d(\vec{x}-\vec{z})$ we can express a bulk field ϕ in terms of its values at the boundary

$$\phi(z_0, \vec{z}) \;=\; \frac{\Gamma(\Delta)}{\pi^{d/2}\,\Gamma(\Delta-2)} \int \mathrm{d}^d\vec{x} \left(\frac{z_0}{z_0^2 \,+\, (\vec{z}-\vec{x})^2} \right)^{\Delta} \phi_0(\vec{x}) \;. \tag{5.1.18}$$

Now the Mercedes diagram of the gravity three point functions is evaluated as

$$A(\vec{x},\vec{y},\vec{z}) \;:=\; \int \mathrm{d}w_0\, \mathrm{d}^d\vec{w}\; \frac{1}{w_0^{d+1}} \left(\frac{w_0}{(w-\vec{x})^2} \right)^{\Delta_1} \left(\frac{w_0}{(w-\vec{y})^2} \right)^{\Delta_2} \left(\frac{w_0}{(w-\vec{z})^2} \right)^{\Delta_3} \tag{5.1.19}$$ 5,20

Here, we use the notation $(w-\vec{x})^2 := w_0^2 + (\vec{w}-\vec{x})^2$.
The number of functions in denominator can be reduced using the trick of inversion: Reexpress integration variable as $w_\mu = \frac{w'_\mu}{(w')^2}$ and similarly set $\vec{x} = \frac{\vec{x}'}{|\vec{x}'|^2}$, $\vec{y} = \frac{\vec{y}'}{|\vec{y}'|^2}$ and $\vec{z} = \frac{\vec{z}'}{|\vec{z}'|^2}$. Consequently, the propagators are affected as

$$K_\Delta(w, \vec{x}) \;=\; |\vec{x}'|^{2\Delta}\, K_\Delta(w', \vec{x}') \;. \tag{5.1.20}$$

The factor $|\vec{x}'|^{2\Delta}$ is a first parallel to field theory since $|\vec{x}'|^{2\Delta} = \frac{1}{|\vec{x}|^{2\Delta}}$. Note that inversion is an isometry of AdS, so its volume element is invariant $\frac{\mathrm{d}^{d+1}w}{w_0^{d+1}} = \frac{\mathrm{d}^{d+1}w'}{(w'_0)^{d+1}}$. This causes the Mercedes integral to transform as

$$A(\vec{x},\vec{y},\vec{z}) \;=\; |\vec{x}'|^{2\Delta_1}\, |\vec{y}'|^{2\Delta_2}\, |\vec{z}'|^{2\Delta_3}\, A(\vec{x}',\vec{y}',\vec{z}') \;. \tag{5.1.21}$$

To reduce the number of functions in the denominator of (5.1.19) from three to two, proceed as follows:

- set one argument to zero $\vec{z}\to 0$ using translation invariance,

  $$A(\vec{x},\vec{y},\vec{z}) \;=\; A(\vec{x}-\vec{z},\vec{y}-\vec{z},0) \;=:\; A(\vec{u},\vec{v},0) \tag{5.1.22}$$

  This brings the third terms into the nice form $\left(\frac{w_0}{(w-\vec{z})^2}\right)^{\Delta_3} = \left(\frac{w_0}{w^2}\right)^{\Delta_3} = (w'_0)^{\Delta_3}$.

- apply an inversion to find

$$A(\vec{u},\vec{v},0) \;=\; \frac{1}{|\vec{u}|^{2\Delta_1}|\vec{v}|^{2\Delta_2}} \int \frac{\mathrm{d}^{d+1}w'}{(w'_0)^{d+1}} \left(\frac{w'_0}{(w'-\vec{u}')^2} \right)^{\Delta_1} \left(\frac{w'_0}{(w'-\vec{v}')^2} \right)^{\Delta_2} (w'_0)^{\Delta_3} \tag{5.1.23}$$ 5,21

By translation invariance of the $\vec{w}$ integration variable, the integral can only depend on the difference $\vec{u}' - \vec{v}'$, and dimensional analysis fixes the power to be $|\vec{u}' - \vec{v}'|^{\Delta_3 - \Delta_1 - \Delta_2}$. Hence, we have already found the spacetime dependence:

$$
\begin{aligned}
A(\vec{u}, \vec{v}, 0) \quad &\sim \quad \frac{|\vec{u}' - \vec{v}'|^{\Delta_3 - \Delta_1 - \Delta_2}}{|\vec{u}|^{2\Delta_1} |\vec{v}|^{2\Delta_2}} \\
&= \quad \frac{1}{|\vec{x} - \vec{y}|^{\Delta_1 + \Delta_2 - \Delta_3} \, |\vec{y} - \vec{z}|^{\Delta_2 + \Delta_3 - \Delta_1} \, |\vec{z} - \vec{x}|^{\Delta_3 + \Delta_1 - \Delta_2}} \\
&=: \quad f(\vec{x}, \vec{y}, \vec{z})
\end{aligned}
\qquad (5.1.24) \quad \boxed{\texttt{Af1}}
$$

(Note that good care has to be taken to restore the old variables before the inversion transformation. A useful formula is: $(\vec{u}' - \vec{v}')^2 = \frac{(\vec{x}-\vec{y})^2}{(\vec{x}-\vec{z})^2 (\vec{y}-\vec{z})^2}$.)
An exact calculation of $A(\vec{u}, \vec{v}, 0)$ can be done using Feynman parameter methods [4], the prefactor in $A(\vec{x}, \vec{y}, \vec{z}) = a \cdot f(\vec{x}, \vec{y}, \vec{z})$ is found to be

$$
a \quad = \quad - \frac{\Gamma[\frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_3)] \, \Gamma[\frac{1}{2}(\Delta_2 + \Delta_3 - \Delta_1)] \, \Gamma[\frac{1}{2}(\Delta_3 + \Delta_1 - \Delta_2)] \, \Gamma[\frac{1}{2}(\sum_i \Delta_i - d)]}{2\pi^d \, \Gamma[\Delta_1 - \frac{d}{2}] \, \Gamma[\Delta_2 - \frac{d}{2}] \, \Gamma[\Delta_3 - \frac{d}{2}]} \, .
\qquad (5.1.25) \quad \boxed{\texttt{Af2}}
$$

The Gamma functions due to the Feynman parameter method have a number of poles.
Now need to consider coupling with which the Mercedes integral (5.1.19) enters the three point function:

$$
\left\langle \mathcal{O}^I(\vec{x}) \, \mathcal{O}^J(\vec{y}) \, \mathcal{O}^K(\vec{z}) \right\rangle \quad = \quad \lambda^{IJK} \, A(\vec{x}, \vec{y}, \vec{z}) \qquad (5.1.26)
$$

The A part was just calculated, we will next treat the cubic coupling λ^{IJK} coming from KK reduction in supergravity [12].
Recall from section 3.4.2 that type IIB supergravity contains a self dual five form field F. It enters the equations of motion for the graviton via

$$
R_{mn} \quad = \quad \frac{1}{3!} \, F_{mijkl} \, F_n{}^{ijkl} \, . \qquad (5.1.27)
$$

In the flat $\mathrm{AdS}_5 \times S^5$ background solution, the five form takes particularly simple values. Denote the AdS_5 indices by $\mu_i, \ i = 1, 2, ..., 5$ and the S^5 indices by $\alpha_i, \ i = 1, 2, ..., 5$, then the solution reads

$$
\begin{aligned}
\mathrm{d}s^2 \quad &= \quad \frac{1}{z_0^2} \left(\mathrm{d}\vec{z}^2 \ + \ \mathrm{d}z_0^2 \ + \ \mathrm{d}\Omega_5^2 \right) \quad =: \quad g_{mn} \, \mathrm{d}x^m \, \mathrm{d}x^n \\
\bar{F}_{\mu_1 \ldots \mu_5} \quad &= \quad \varepsilon_{\mu_1 \ldots \mu_5} \ , \qquad \bar{F}_{\alpha_1 \ldots \alpha_5} \quad = \quad \varepsilon_{\alpha_1 \ldots \alpha_5}
\end{aligned}
\qquad (5.1.28)
$$

Note that the curvatures of the AdS_5 and S^5 factors cancel:

$$
\mathrm{AdS}_5 \ : \quad R_{\mu\lambda\nu\sigma} \quad = \quad - \left(g_{\mu\nu} \, g_{\lambda\sigma} \ - \ g_{\mu\sigma} \, g_{\lambda\nu} \right) \ , \qquad R_{\mu\nu} \quad = \quad -4 \, g_{\mu\nu} \ , \qquad \mathcal{R}_{\mathrm{AdS}_5} \quad = \quad -20
$$

$$S^5: \quad R_{\alpha\gamma\beta\delta} \;=\; +(g_{\alpha\beta}\,g_{\gamma\delta} \;-\; g_{\alpha\delta}\,g_{\gamma\beta})\,, \qquad R_{\alpha\beta} \;=\; +4\,g_{\alpha\beta}\,, \qquad \mathcal{R}_{S^5} \;=\; +20 \tag{5.1.29}$$

Observe that $\mathcal{R} = \mathcal{R}_{\mathrm{AdS}_5} + \mathcal{R}_{S^5} = 0$.

Next we need to look at fluctuations ϕ_0 about this background which couple to operators $\mathcal{O}$ in the dual field theory via interaction terms $\mathcal{S}_{\mathrm{int}} = \int \mathrm{d}^d x\, \phi_0(x)\mathcal{O}(x)$. It was investigated in [13] how to decompose the supergravity equations of motion and how to decouple them from the fluctuations.

Starting point is the ansatz

$$G_{mn} \;=\; g_{mn} \;+\; h_{mn}\,, \qquad F \;=\; \bar{F} \;+\; \delta F \tag{5.1.30}$$

where the fluctiations h, δF are organized as

$$\begin{aligned} h_{\alpha\beta} &= h_{(\alpha\beta)} + \frac{h_2}{5}\, g_{\alpha\beta}\,, & g^{\alpha\beta}\, h_{(\alpha\beta)} &= 0 \\ h_{\mu\nu} &= h'_{(\mu\nu)} + \frac{h'}{5}\, g_{\mu\nu} - \frac{h_2}{3}\, g_{\mu\nu}\,, & g^{\mu\nu}\, h'_{(\mu\nu)} &= 0 \end{aligned} \tag{5.1.31}$$

$$\delta F_{ijklm} \;=\; 5\,\nabla_{[i}\, a_{jklm]} \tag{5.1.32}$$

It is convenient to work in de-Donder gauge (with respect to S^5) where

$$\nabla^\alpha h_{\alpha\beta} \;=\; \nabla^\alpha h_{\mu\alpha} \;=\; \nabla^\alpha a_{\alpha\mu_1\mu_2\mu_3} \;=\; 0\,. \tag{5.1.33}$$

The KK programme requires to expand this ansatz in spherical harmonics Y^I on S^5:

$$\begin{aligned} h'_{\mu\nu} &= \sum_I Y^I\, (h'_{\mu\nu})^I\,, \qquad h_2 \;=\; \sum_I Y^I\, h_2^I \\ a_{\alpha_1\ldots\alpha_4} &= \sum_I \nabla^\alpha\, Y^I\, \varepsilon_{\alpha\alpha_1\alpha_2\alpha_3\alpha_4}\, b^I \\ a_{\mu_1\ldots\mu_4} &= \sum_I Y^I\, a^I_{\mu_1\ldots\mu_4} \end{aligned} \tag{5.1.34}$$

Inserting this ansatz into the ten dimensional equations of motion leads to diagonalization and decoupling. The modes which couple to the field theory 1/2 BPS operators $\mathcal{O}^I$ are given by

$$S^I \;=\; \frac{1}{20\,(k+2)}\left(h_2^I \;-\; 10\,(k+4)\,b^I\right)\,. \tag{5.1.35}$$

Note that $k = \Delta$ in the different notations of the original papers. These S^5 modes satisfy a five dimensional equation of motion in AdS space

$$\left(\nabla_\mu \nabla^\mu \;-\; k\,(k-4)\right) S^I \;=\; \lambda^{IJK}\, S_J\, S_K \tag{5.1.36}$$

where λ^{IJK} is given by

$$\lambda^{IJK} \;=\; a(k_1,k_2,k_3)\,\frac{128\,\Sigma\,\big((\Sigma/2)^2\,-\,1\big)\,\big((\Sigma/2)^2\,-\,4\big)\,\alpha_1\alpha_2\alpha_3\,\big\langle C^I C^J C^K\big\rangle}{(k_1+1)(k_2+1)(k_3+1)}\;. \tag{5.1.37}$$

We are using the usual shorthands $\Sigma = k_1 + k_2 + k_3$ and $\alpha_1 = \frac{k_2+k_3-k_1}{2}$ (as well as cyclic variations thereof) and the numbers $a(k_1, k_2, k_3)$ relate S^5 integrals of spherical harmonics with the $SO(6)$ tensors $\langle C^I C^J C^K \rangle$ of (5.1.5),

$$\begin{aligned}\int_{S^5} \mathrm{d}\Omega\; Y^I(\Omega)\,Y^J(\Omega)\,Y^K(\Omega) \;&=\; a(k_1,k_2,k_3)\,\big\langle C^I\,C^J\,C^K\big\rangle\\ a(k_1,k_2,k_3) \;&=\; \frac{\pi^3}{(\Sigma/2)!\,2^{d/2(\Sigma-2)}}\,\frac{k_1!\,k_2!\,k_3!}{\alpha_1!\,\alpha_2!\,\alpha_3!}\end{aligned} \tag{5.1.38}$$

This gives rise to the following dimensionally reduced supergravity action for the S^I modes:

$$\mathcal{S} \;=\; \frac{4\,N^2}{(2\pi)^5}\int \mathrm{d}^5x\,\sqrt{g}\,\left[\frac{A_I}{2}\,(-\nabla S_I)^2\;-\;k\,(k-4)\,(S_I)^2\;+\;\frac{1}{3}\,\lambda_{IJK}\,S^I\,S^J\,S^K\right] \tag{5.1.39}$$

We can identify the lower dimensional gravitation coupling and the AdS radius as

$$\frac{1}{2\,\kappa^2} \;=\; \frac{4\,N^2}{(2\pi)^5}\;, \qquad L_{\mathrm{AdS}} \;=\; 1\;, \tag{5.1.40}$$

and the constant A_I is determined from IIB 10d SUGRA action to be

$$A_I \;=\; 32\,\frac{k\,(k-1)\,(k-2)}{k+1}\,Z(k)\;, \qquad \int_{S^5}\mathrm{d}\Omega\;Y^I(\Omega)\,Y^J(\Omega) \;=:\; Z(k)\,\delta^{IJ}\;. \tag{5.1.41}$$

Let us now use the action $\mathcal{S}$ as given above to calculate the the two point function,

$$\big\langle S^I(x)\,S^J(y)\big\rangle \;=\; \frac{4\,N^2}{(2\pi)^5}\,\frac{\pi}{2^{k-7}}\,\frac{k\,(k-1)^2\,(k-2)^2}{(k+1)^2}\,\frac{\delta^{IJ}}{(x-y)^{2k}} \tag{5.1.42}$$

then define normalized operators $\tilde{\mathcal{O}}^I(x) \sim S^I(x)$ such that $\big\langle \tilde{\mathcal{O}}^I(x)\tilde{\mathcal{O}}^J(y)\big\rangle = \frac{\delta^{IJ}}{(x-y)^{2k}}$. The three point function is computed on the basis λ^{IJK}, the operators' normalization as given above and the result (5.1.24), (5.1.25) for $A(x, y, z)$:

$$\Big\langle \tilde{\mathcal{O}}^I(x)\tilde{\mathcal{O}}^J(y)\tilde{\mathcal{O}}^K(z)\Big\rangle \;=\; \frac{1}{N}\,\frac{\sqrt{k_1\,k_2\,k_3}\,\big\langle C^I\,C^J\,C^K\big\rangle}{|x-y|^{2\alpha_3}\,|y-z|^{2\alpha_1}\,|z-x|^{2\alpha_2}} \tag{5.1.43}$$

Remarkably, this gravitational correlator coincides with the field theory result!

5.2 The conformal anomaly

As a second example of astonishing agreement between computations in AdS gravity and $\mathcal{N} = 4$ SYM, we will now compute the *conformal anomaly* using both approaches. The conformal anomaly parametrizes the failure of the energy momentum tensor to remain traceless under quantum corrections in a clasically conformal field theory. Recall that the energy momentum tensor's expectation value can be derived from the effective action via

$$\langle T_{\mu\nu}(x)\rangle \quad = \quad -\frac{2}{\sqrt{g}}\,\frac{\delta W}{\delta g^{\mu\nu}(x)}\,. \qquad (5.2.1) \quad \boxed{\texttt{eq:}}$$

In this definition, we can regard $g^{\mu\nu}$ as a classical background field, it does not propagate, but is a source for $T_{\mu\nu}$.

5.2.1 The conformal anomaly on the field theory side

Let us consider a classical field theory with conformal symmetry with action functional for the matter fields ϕ

$$S_{\text{mat}} \quad = \quad \int \mathrm{d}^4x\,\sqrt{g}\,\mathcal{L}_{\text{mat}} \qquad (5.2.2) \quad \boxed{\texttt{eq:1}}$$

Under variation of the metric $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ the chain rule implies

$$\delta\mathcal{S}_{\text{mat}} \quad = \quad -\frac{1}{2}\int \mathrm{d}^4x\,\sqrt{g}\,T^{\mu\nu}\,\delta g_{\mu\nu} \qquad (5.2.3) \quad \boxed{\texttt{eq:2}}$$

by definition of the energy momentum tensor. In Poincaré invariant theories, it is symmetric $T^{\mu\nu} = T^{\nu\mu}$ and satisfies the conservation law $\nabla_\mu T^{\mu\nu} = 0$. In conformal theories, it is also traceless $T_\mu{}^\mu = 0$. This can be seen from a Weyl transformation $\delta g_{\mu\nu} = -2\sigma g_{\mu\nu}$ in (5.2.3).
Next we proceed to the quantized theory. Matter fields ϕ will be promoted to quantum fields whereas the metric is still regarded as an external, classical field. The generating functional is given by

$$\mathcal{Z}[g] \quad = \quad e^{-W[g]} \quad = \quad \int \mathcal{D}\phi\,\exp\left\{-\int \mathrm{d}^4x\,\sqrt{g}\,\mathcal{L}_{\text{mat}}\right\}. \qquad (5.2.4) \quad \boxed{\texttt{eq:}}$$

Then the Weyl transformation gives rise to the following variation in the effective action:

$$\delta_\sigma W[g] \quad = \quad -\frac{1}{2}\int \mathrm{d}^4x\,\sqrt{g}\,\langle T^{\mu\nu}\rangle\,\delta_\sigma g_{\mu\nu} \quad = \quad \int \mathrm{d}^4x\,\sqrt{g}\,\langle T_\mu{}^\mu\rangle\,\sigma \qquad (5.2.5)$$

Generically, one has $\langle T_\mu{}^\mu\rangle \neq 0$ at the quantum level since counterterms needed for regularization give finite contribution to $\langle T_\mu{}^\mu\rangle$. This phenomenon is referred to as conformal anomaly.
Generically in a $d = 4$ QFT, the conformal anomaly is of the form:

$$\langle T_\mu{}^\mu(x)\rangle \quad = \quad \frac{c}{16\pi^2}\,C^{\mu\sigma\rho\nu}\,C_{\mu\sigma\rho\nu} \quad - \quad \frac{a}{16\pi^2}\,\varepsilon_{\alpha\beta\gamma\delta}\,\varepsilon_{\mu\nu\sigma\rho}\,R^{\alpha\beta\mu\nu}\,R^{\gamma\delta\sigma\rho} \qquad (5.2.6) \quad \boxed{\texttt{anomaly}}$$

Here, C is the Weyl tensor, obtained from the Riemann tensor by subtracting the traces such that $C_\mu{}^{\sigma\rho\mu} = 0$. The second contribution $\varepsilon_{\alpha\beta\gamma\delta}\varepsilon_{\mu\nu\sigma\rho}R^{\alpha\beta\mu\nu}R^{\gamma\delta\sigma\rho}$ is called the Euler density and gives a topological term (Gauss-Bonnet-term) $\int \mathrm{d}^4x\,\sqrt{g}\,\varepsilon\varepsilon RR = 4\pi\chi$ proportional to the Euler characteristic χ of the manifold. The coefficients c and a are model-dependent.

Note that in two dimensions, the conformal anomaly is of purely topological nature, $\langle T_\mu{}^\mu(x)\rangle = \frac{c}{24\pi}R$. The prefactor c is the Virasoro central charge. This affects the energy momentum tensor's two point functions with a trace involved: Taking $\frac{\delta}{\delta g^{\sigma\rho}(y)}$ gives $\langle T_\mu{}^\mu(x) T_{\sigma\rho}(y)\rangle = \frac{c}{24\pi}\partial_\sigma\partial_\rho\delta^{(2)}(x-y)$ in $d = 2$ two dimensions. In $d = 4$, however, taking first $g_{\mu\nu}$ derivatives of (5.2.6) we get zero in flat space (one curvature term remains in the derivative of the quadratic expression). Nonzero results arise from second derivatives, i.e. the $d = 4$ conformal anomaly manifests itself in the three point function $\langle T_\mu{}^\mu(x) T_{\sigma\rho}(y) T_{\alpha\beta}(z)\rangle$.

We are particularly interested in the case of $\mathcal{N} = 4$ SYM. Many explicit calculations methods (for instance heat-kernel) have been developed prior to AdS/CFT. To lowest order in λ one finds that c and a depend on the number of degrees of freedom (vectors, fermions, scalars) but not on λ:

$$c \;=\; a \;=\; \frac{1}{4}\,(N^2-1) \;\stackrel{N\to\infty}{\to}\; \frac{1}{4}\,N^2 \tag{5.2.7}$$

In total, the agreement of c and a yields the following conformal anomaly in $\mathcal{N} = 4$:

$$\langle T^\mu_\mu(x)\rangle \;=\; \frac{c}{8\pi^2}\left(R^{\mu\nu}\,R_{\mu\nu} \;-\; \frac{1}{3}\,R^2\right) \;\stackrel{N\to\infty}{\to}\; \frac{N^2}{32\pi^2}\left(R^{\mu\nu}\,R_{\mu\nu} \;-\; \frac{1}{3}\,R^2\right) \tag{5.2.8}$$

YManomal

5.2.2 The conformal anomaly on the gravity side

The gravity counterpart, i.e. the conformal anomaly from AdS space, was computed in [14] for the first time. Starting point is the action of $d = 5$ AdS gravity,

$$\mathcal{S} \;=\; -\frac{1}{16\pi\,G}\int \mathrm{d}^5 z\,\sqrt{g}\,\left(R \;+\; \frac{12}{L^2}\right)\,. \tag{5.2.9}$$

AdSactio

Recall that the metric for AdS_5 is given by $\mathrm{d}s^2 = L^2\left[\frac{\mathrm{d}\rho^2}{4\rho^2} + \frac{1}{\rho}\delta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\right]$ as long as the $\rho = 0$ boundary remains flat. If we allow for curvature terms at the boundary, this generalizes to

$$\mathrm{d}s^2 \;=\; L^2\left[\frac{\mathrm{d}\rho^2}{4\,\rho^2} \;+\; \frac{1}{\rho}\,g_{\mu\nu}(x,\rho)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\right]\,, \qquad \lim_{\rho\to 0} g_{\mu\nu}(x,\rho) \;=\; \overline{g}_{\mu\nu}(x)\,. \tag{5.2.10}$$

eq:Equat

Fluctuations about the flat case are parametrized as $\overline{g}_{\mu\nu}(x) = \delta_{\mu\nu} + h_{\mu\nu}(x)$, the energy momentum is then obtained by $\langle T_{\mu\nu}(x)\rangle = -\frac{2}{\sqrt{g}}\frac{\delta W}{\delta h^{\mu\nu}}$.

The metric's coordinate singularty at $\rho \to 0$ can be avoided by means of a cutoff at $\rho = \epsilon$. The integration region in the action is then restricted to $\rho \geq \epsilon$. The induced metric at $\rho = \epsilon$ will be denoted by

$$\gamma_{\mu\nu}(x) \;:=\; \frac{g_{\mu\nu}(x,\rho=\epsilon)}{\epsilon}\,. \tag{5.2.11}$$

As usual, Weyl transformation of the metric gives the trace of the energy-momentum tensor. Therefore, we need to translate a Weyl transformation in the boundary theory into a transformation in the bulk, i.e. in $d = 5$ AdS. The task is to find a $d = 5$ diffeomorphism which reduces to a Weyl transformation on the boundary. The desired diffeomorphism is known as the *Penrose-Brown-Henneaux transformation*:

$$\rho \;=\; \rho' \big(1 \;-\; 2\,\sigma(x')\big)\;, \qquad x^\mu \;=\; (x')^\mu \;+\; a^\mu(x',\rho') \tag{5.2.12}$$

One has to make sure that the form of the $d = 5$ metric (5.2.10) is covariant under this transformation, i.e. that $g'_{55} = g_{55}$ and $g'_{5\mu} = g_{5\mu}$ (where 5 denotes the ρ index). This imposes the constraints

$$\partial_5 a^\mu \;=\; \frac{L^2}{2}\, g^{\mu\nu}\,\partial_\nu \sigma \tag{5.2.13}$$ $\boxed{\texttt{diff}}$

on the functions a^μ and σ of (5.2.13). It follows that

$$a^\mu(x,\rho) \;=\; \frac{L^2}{2}\int_0^\rho \mathrm{d}\hat\rho\; g^{\mu\nu}(x,\hat\rho)\,\partial_\nu\sigma(x)\;. \tag{5.2.14}$$ $\boxed{\texttt{diff1}}$

Under this diffeomorphism, the $d = 4$ part $g_{\mu\nu}(x,\rho)$ of the metric transforms as

$$g_{\mu\nu} \;\mapsto\; g_{\mu\nu} \;+\; 2\sigma\left(1 \;-\; \rho\frac{\partial}{\partial\rho}\right) g_{\mu\nu} \;+\; \nabla_\mu a_\nu \;+\; \nabla_\nu a_\mu \tag{5.2.15}$$

such that at the boundary (where $\rho \to 0$) we have $a_\mu \to 0$ and $\rho\frac{\partial}{\partial\rho}g_{\mu\nu} \to 0$ and therefore $\delta g_{\mu\nu}(x) = 2\sigma(x)\overline{g_{\mu\nu}}(x)$
Inserting all that into action (5.2.9) yields

$$\delta W \;=\; \int \mathrm{d}^4x\;\langle T^{\mu\nu}\rangle\,\delta g_{\mu\nu}\;, \qquad \delta g_{\mu\nu} \;=\; 2\sigma\, g_{\mu\nu}\;. \tag{5.2.16}$$

The divergence of $\mathcal{S}$ at the boundary is regularized by introducing the cutoff $\rho \geq \epsilon$, subtracting counterterms make $\mathcal{S}_{\text{ren}} = \mathcal{S} - \mathcal{S}_{\text{ct}}$ finite. To get the explicit form of $\mathcal{S}_{\text{ct}}$, we need some more information about the form of $g_{\mu\nu}(x,\rho)$. This is provided by the *Fefferman-Graham theorem*: If a metric of the form (5.2.10) satisfies the Einstein equations, then $g_{\mu\nu}(x,\rho)$ may be expanded as:

$$g_{\mu\nu}(x,\rho) \;=\; \overline{g_{\mu\nu}}(x) \;+\; \rho\, g^{(2)}_{\mu\nu}(x) \;+\; \rho^2\, g^{(4)}_{\mu\nu}(x) \;+\; \rho^2\,\ln(\rho)\, h^{(4)}_{\mu\nu} \;+\; \ldots \tag{5.2.17}$$

The coefficients $g^{(n)}_{\mu\nu}$ are built out of the curvature for the boundary metric $\overline{g_{\mu\nu}}(x)$. They are calculated by inserting the expansion into the vacuum Einstein equation $\big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\big) = \Lambda g_{\mu\nu}$. For example, the linear coefficient is found to be $g^{(2)}_{\mu\nu}(x) = \frac{1}{2}\big(R_{\mu\nu} - \frac{1}{6}Rg_{\mu\nu}\big)$. The lowest order divergent terms in $\mathcal{S}$ then lead to

$$S_{\text{ct}} \;=\; \frac{1}{4\pi\,G}\int \mathrm{d}^4x\,\sqrt{\gamma}\,\left(\frac{3}{2L} \;-\; \frac{LR}{8} \;-\; \frac{L^3\ln\epsilon}{32}\left(R^{\mu\nu}\,R_{\mu\nu} \;-\; \frac{R^2}{3}\right)\right) \tag{5.2.18}$$

in the minimal subtraction renormalization scheme. The action $\mathcal{S}$ we started with is diffeomorphism invariant, but the introduction of the counterterms spoils this symmetry. More precisely, the Penrose-Brown-Henneaux transformation of $\mathcal{S} - \mathcal{S}_{\rm ct}$ is given by

$$\delta(\mathcal{S} - \mathcal{S}_{\rm ct}) \quad = \quad -2\,\epsilon\,\frac{\partial}{\partial\epsilon}\,\mathcal{S}_{\rm ct} \quad = \quad \frac{L^3}{64\pi G}\,\left(R^{\mu\nu}\,R_{\mu\nu} \; - \; \frac{R^2}{3}\right)\,. \tag{5.2.19}$$

The Weyl variation at the boundary gives rise to the energy momentum trace

$$g^{\mu\nu}\,\langle T_{\mu\nu}\rangle \quad = \quad \frac{L^3}{8\pi G}\,\left(\frac{R^{\mu\nu}\,R_{\mu\nu}}{8} \; - \; \frac{R^2}{24}\right) \quad = \quad \frac{N^2}{32\pi^2}\,\left(R^{\mu\nu}\,R_{\mu\nu} \; - \; \frac{R^2}{3}\right) \tag{5.2.20}$$

with $d = 5$ Newton constant $G = \frac{G_{10}}{\mathrm{vol}(S^5)} = \frac{\pi L^3}{2N^2}$. Again, the gravity result conincides with the $N \to \infty$ limit of the field theory pendant (5.2.8).

Chapter 6

Generalizations of AdS/CFT

6.1 Holographic renormalization group flows

In the context of the AdS/CFT correspondence, the term *holography* represents the fact that the number of bulk degrees of freedom equals the number of boundary degrees of freedom. So far, we have checked this phenomenon for $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$ and symmetry group $SO(4,2) \times SO(6) \subset SU(2,2|4)$ and its gravity dual. However, this correspondence is far away from reality because of the following points:

- correspondence valid in the $N \to \infty$ limit
- conformal symmetry and supersymmetry
- fields in the adjoint representation of the gauge group

On the other hand, with QCD, we have the well-established theory of strong interactions which differs from $\mathcal{N} = 4$ SYM in several points:

- gauge group $SU(3)$, i.e. small finite N
- no supersymmetry
- confinement incompatible with conformal symmetry
- quarks in the fundamental representation of the gauge group

Adapting the gauge group $SU(N) \to SU(3)$ in the AdS/CFT correspondence will probably not be achieved in the near future. But looking at large N QCD and performing a $1/N$ expansion is still a useful approximation.

As to the other features in the list: We may break conformal symmetry and supersymmetry in a controlled way, and we may add quark degrees of freedom. To find a modest beginning, let us first of all break some of supersymmetry ($\mathcal{N}=4$ to $\mathcal{N}=1$) as well as conformal invariance. On the field theory side, this is done by adding relevant operators to the Lagrangian:

$$\mathcal{L} \;=\; \mathcal{L}_{\mathcal{N}=4} \;+\; \frac{m_{ij}}{2}\,\mathrm{Tr}\big\{X^i\,X^j\big\} \;+\; \frac{M_{ab}}{2}\,\mathrm{Tr}\big\{\psi^a\,\psi^b\big\} \;+\; b_{ijk}\,\mathrm{Tr}\big\{X^i\,X^j\,X^k\big\} \tag{6.1.1} \qquad \boxed{\texttt{6,1}}$$

This modification triggers renormalization group flows. In general QFTs, renormalization group equations express the invariance of physics under difference choices of the renormalization scale μ. Let Γ denote some vertex function (depending on couplings g_i), then the RG equation assumes the form

$$0 \;=\; \mu\,\frac{\mathrm{d}\Gamma[\mu,g_i]}{\mathrm{d}\mu} \;=\; \mu\,\frac{\partial\Gamma[\mu,g_i]}{\partial\mu} \;+\; \beta_i(g)\,\partial_i\Gamma[\mu,g_i]\;, \qquad \beta_i(g) \;=\; \mu\,\frac{\mathrm{d}g_i(\mu)}{\mathrm{d}\mu}\;. \tag{6.1.2} \qquad \boxed{\texttt{6,2}}$$

6.1.1 Renormalization group flow in supergravity

To find an AdS analogue of field theory RG equations we now look for a toy model of a supergravity RG flow. The idea is to obtain equations like (6.1.2) as equations of motion in extra dimensions. For that purpose, consider five dimensional gravity with a single scalar field:

$$\mathcal{S} \;=\; \frac{1}{4\pi\,G}\int \mathrm{d}^5x\;\sqrt{|g|}\;\left(-\frac{R}{4} \;+\; \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi \;+\; V(\varphi)\right) \tag{6.1.3} \qquad \boxed{\texttt{6,3}}$$

For simplicity set $4\pi G = 1$ and split spacetime as $\mathrm{d}^5x = \mathrm{d}^4x\,\mathrm{d}r$, then φ is dimensionless. Moreover, choose the potential $V(\varphi)$ such that is has one or more critical points with $V'(\varphi)=0$. The equations of motion for φ and $g_{\mu\nu}$ read

$$\frac{1}{\sqrt{|g|}}\,\partial_\mu\left(\sqrt{|g|}\,g^{\mu\nu}\,\partial_\nu\varphi\right) \;-\; V'(\varphi) \;=\; 0 \tag{6.1.4} \qquad \boxed{\texttt{6,4}}$$

as well as

$$R_{\mu\nu} \;-\; \frac{R}{2}\,g_{\mu\nu} \;=\; 2\,\partial_\mu\varphi\,\partial_\nu\varphi \;-\; g_{\mu\nu}\,\partial_\lambda\varphi\,\partial^\lambda\varphi \;+\; 2\,V(\varphi) \;=:\; 2\,T_{\mu\nu} \tag{6.1.5} \qquad \boxed{\texttt{6,5}}$$

At the critical points φ_i, there is a trivial solution of the scalar equation of motion $\varphi(r)=\varphi_i$. Here, the Einstein equation reduces to $R_{\mu\nu}-\frac{R}{2}g_{\mu\nu} = -2g_{\mu\nu}V(\varphi_i)$. This is identical to the Einstein equation of AdS space $G_{\mu\nu}-\Lambda g_{\mu\nu}=0$ if we identify $\Lambda_i = 2V(\varphi_i) = -\frac{d(d-1)}{L_i^2}$. In other words, constant scalar fields with AdS_{d+1} geometry of scale L_i are critical solutions.

A more general ansatz for solving the equations of motion to (6.1.3) involves a metric with so-called *warp factor* $A(r)$:

$$\mathrm{d}s^2 \;=\; e^{2A(r)}\,\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \;+\; \mathrm{d}r^2 \tag{6.1.6} \qquad \boxed{\texttt{6,6}}$$

This is known as the *domain wall ansatz* for the metric, it reduces to another form of the AdS metric if we make the linear choice $A(r) = r/L$. The idea is to identify the radial coordinate with the RG scale $r = \frac{1}{\mu}$.
The components of the Riemann tensor due to (6.1.6) read

$$\begin{aligned} R^{ij}{}_{kl} &= -\,A'(r)^2 \left(\delta^i_k\,\delta^j_l \;-\; \delta^i_l\,\delta^j_k\right) \\ R^{i5}{}_{j5} &= -\left(A''(r) \;+\; A'(r)^2\right)\delta^i_j \\ R^{ij}{}_{k5} &= 0 \end{aligned} \tag{6.1.7}$$

6,7

(where $i, j, k, l \in \{0, 1, 2, 3\}$ and $r \equiv 5$) and the resulting Ricci tensor is given by

$$\begin{aligned} R_{ij} &= -\,e^{2A(r)}\left(A''(r) \;+\; d\,A'(r)^2\right)\delta_{ij} \\ R_{55} &= -\,d\left(A''(r) \;+\; A'(r)\right)^2 \\ R_{i5} &= 0\,. \end{aligned} \tag{6.1.8}$$

6,8

This gives rise to Einstein equations

$$\begin{aligned} G^i{}_j &= (d-1)\,\delta^i_j\left(A'' \;+\; \frac{d}{2}\,(A')^2\right) \;=\; 2\,T^i{}_j \\ G^5{}_5 &= \frac{d\,(d-1)}{2}\,(A')^2 \;=\; 2\,T^5{}_5\,. \end{aligned} \tag{6.1.9}$$

6,8a

By carefully considering the difference $G^i{}_i - G^5{}_5$ (without sum over i), one can extract a bound on the second derivative of the warp factor from (6.1.9):

$$A'' \;=\; \frac{2}{d-1}\left(T^i{}_i \;-\; T^5{}_5\right) \;=\; -\frac{2}{d-1}\left(\varphi'\right)^2 \quad\Rightarrow\quad A'' \;<\; 0 \tag{6.1.10}$$

6,10

This is consistent with the *weak energy condition* saying that every Poincaré invariant matter distribution satisfies $T^i{}_i - T^5{}_5 < 0$.
The $d = 4$ equations of motion in terms of A read

$$\begin{aligned} \varphi'' \;+\; 4\,A'\,\varphi' &= \frac{\mathrm{d}V(\varphi)}{\mathrm{d}\varphi} \\ (\varphi')^2 \;-\; 2\,V(\varphi) &= 6\,(A')^2\,. \end{aligned} \tag{6.1.11}$$

6,11

Note that $A(r) = \frac{r}{L_i}$ at critical points.
Our goal is to find a general solution of (6.1.11) which interpolates between two critical points. In $\mathrm{AdS}_5 \times S^5$ language, we are looking for a domain wall solution interpolating between AdS of radius L_1 for $r \to +\infty$ and AdS of radius L_2 for $r \to -\infty$. This is dual to an RG flow between two fixed points.

6.1.2 Leigh Strassler flow

A very nice example is the *Leigh Strassler flow* which can be best described in superfield language. It establishes a field operator map in particular for the relevant operators added to $\mathcal{L}_{\mathcal{N}=4}$. We can describe the $\mathcal{N}=4$ theory in $\mathcal{N}=1$ notation by reorganizing the $\mathcal{N}=4$ supermultiplet (one vector A_μ, four fermions λ_a and six real scalars X^i) as follows

- one $\mathcal{N}=1$ vector multiplet $(A_\mu, \lambda_{a=4})$
- three $\mathcal{N}=1$ chiral multiplets $(\lambda_a, \varphi_a = X^{2a-1} + iX^{2a})$ with $a = 1, 2, 3$

Superspace is an 8 dimensional space spanned by four standard spacetime coordinates x^μ, $\mu = 0, 1, 2, 3$ and two Weyl spinorial Grassmann variables θ_α, $\bar\theta^{\dot\beta}$ (where $\alpha, \dot\beta = 1, 2$). Scalars and three fermions of $\mathcal{N}=4$ SYM are aligned into $\mathcal{N}=1$ *chiral superfields* with theta expansion

$$\Phi^i \;=\; \varphi^i \,+\, \theta^\alpha \lambda^i_\alpha \,+\, \text{higher order in } \theta\,. \qquad (6.1.12) \quad \boxed{\texttt{6,12}}$$

Chiral superfields are annihilated by half the supercharges $\bar Q_{\dot\alpha}\Phi = 0$. Clearly, the expansion terminates after a $\theta\theta\,\bar\theta\bar\theta$ term.

In addition, a vector superfield V captures the $\mathcal{N}=4$ vector A_μ and the fourth fermion. One can bring the theta expansion into the form

$$V \;=\; \theta\,\sigma^\mu\,\bar\theta\,A_\mu \,+\, (\theta\,\theta)\,\bar\theta_{\dot\alpha}\,\bar\lambda^{\dot\alpha}_4 \,+\, (\bar\theta\,\bar\theta)\,\theta^\alpha\,\lambda_{\alpha,4} \qquad (6.1.13) \quad \boxed{\texttt{6,12a}}$$

Let us write down the Lagrangian of the $\mathcal{N}=4$ theory in $\mathcal{N}=1$ superspace language:

$$\begin{aligned} \mathcal{L}_{\mathcal{N}=4} \;&=\; \sum_{i=1}^{3} \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\;\mathrm{Tr}\big\{e^{-gV}\,\bar\Phi^i\,e^{gV}\,\Phi^i\big\} \\ &\quad+\, \left(g\int \mathrm{d}^2\theta\;\mathrm{Tr}\big\{\Phi^3\,\big[\Phi^1\,,\,\Phi^2\big]\big\} \,+\, \frac{1}{g^2}\int \mathrm{d}^2\theta\;\mathrm{Tr}\big\{W^\alpha\,W_\alpha\big\} \,+\, h.c.\right) \end{aligned} \qquad (6.1.14) \quad \boxed{\texttt{6,13}}$$

In this formulation, $\mathcal{L}_{\mathcal{N}=4}$ exhibits manifest $\mathcal{N}=1$ supersymmetry with R symmetry group $SU(3)\times U(1)$. The first $\mathrm{d}^2\theta$ integrand $\sim g\Phi^3\,\big[\Phi^1\,,\,\Phi^2\big]$ is referred to as a *superpotential.* The gauge kinetic term involves superfields W_α which contain the non-abelian field strength $F_{\mu\nu}$ derived from A_μ.

The general relevant deformation which preserves $\mathcal{N}=1$ supersymmetry is obtained by adding an additional superpotential contribution

$$U \;=\; h\,\mathrm{Tr}\big\{\Phi^3\,\big[\Phi^1\,,\,\Phi^2\big]\big\} \,+\, \frac{M_{AB}}{2}\,\mathrm{Tr}\big\{\Phi^A\,\Phi^B\big\}\,. \qquad (6.1.15) \quad \boxed{\texttt{6,14}}$$

What is left to show is the fact that this theory flows to a non-trivial fixed point in the IR.

The Leigh Strassler flow is triggered by the deformation

$$U_{\rm LS} \;:=\; h\,\mathrm{Tr}\big\{\Phi^3\,\big[\Phi^1\,,\,\Phi^2\big]\big\} \;+\; \frac{m}{2}\,\mathrm{Tr}\big\{(\Phi^3)^2\big\}\;. \qquad (6.1.16) \quad \boxed{\texttt{6,15}}$$

Because of the scaling dimensions $[h]=0$ and $[m]=1$, the former term is referred to as marginal, the mass term as relevant.

The deformation (6.1.16) leads to a reduced R symmetry $SU(2)\times U(1)$, the former acting on the $\Phi^{1,2}$ fields. The $U(1)$ charges of the chiral superfields $\Phi^{1,2,3}$ are $(1/2,1/2,-1)$.

A necessary condition for an IR fixed point is $\beta^i=0$, and luckily the beta function for the gauge coupling $\beta(g)$ is well-known for $\mathcal{N}=1$ theories to all orders in perturbation theory. It is given by the NSVZ beta function (named after Novikov, Shifman, Vainstein and Zakharov)

$$\beta(g) \;=\; -\,\frac{g^3\,T(R_A)}{8\pi^2}\,\frac{3\,T(G)\;-\;\sum_A T(R_A)\,(1\,-\,2\,\gamma_A)}{1\;-\;g^2\,T(G)/(8\pi^2)} \qquad (6.1.17) \quad \boxed{\texttt{6,16}}$$

Here, γ_A denotes the anomalous dimension of the superfield Φ^A and the Dynkin index $T(R_A)$ of the representation R_A is defined by the normalizaton of the two-trace,

$$\mathrm{Tr}_A\big\{T^a\,T^b\big\} \;=\; T(R_A)\,\delta^{ab}\;. \qquad (6.1.18) \quad \boxed{\texttt{6,17}}$$

Here we are dealing with $G=SU(N)$ and all the fields transform in the adjoint representation. Therefore, $T(R_A)=T(G)=N$ and

$$\beta(g) \;\sim\; 2N\,(\gamma_1+\gamma_2+\gamma_3)\;. \qquad (6.1.19) \quad \boxed{\texttt{6,18}}$$

The β functions for matter fields are simple due to non-renormalization theorems in SUSY theories. Hence, the running of the parameters h,m in (6.1.16) is governed by

$$\beta_h \;=\; \gamma_1+\gamma_2+\gamma_3\;, \qquad \beta_m \;=\; 1\;-\;2\,\gamma_3\;. \qquad (6.1.20) \quad \boxed{\texttt{6,19}}$$

The condition $\beta(g)=\beta_h=\beta_m$ has a unique $SU(2)$ invariant solution

$$\gamma_1 \;=\; \gamma_2 \;=\; -\,\frac{\gamma_3}{2} \;=\; -\,\frac{1}{4}\;. \qquad (6.1.21) \quad \boxed{\texttt{6,20}}$$

The IR fixed point theory with given values of the anomalous dimensions has $\mathcal{N}=1$ superconformal symmetry under $SU(2,2|1)$. The engineering dimensions of the superfields are given by $\Delta_A=1+\gamma_A$. According to the field operator map, conformal primary operators $\mathcal{O}$ are constructed from gauge invariant combinations of Φ^1, Φ^2 and the field strength superfield W_α (which contains $F_{\mu\nu}$).

$\mathcal{O}$	$\mathrm{Tr}\{\Phi^i\,\Phi^j\}$	$\mathrm{Tr}\{W_\alpha\,\Phi^i\}$	$\mathrm{Tr}\{W^\alpha\,W_\alpha\}$	$\mathrm{Tr}\{\bar\Phi^\dagger_i\,(T^a)^i{}_j\,\Phi^j\}$	$\mathrm{Tr}\{W_\alpha\,\bar W_{\dot\beta}\}$
Δ	3/2	9/4	3	2	3

The $\Phi^{i=1,2}$ form an $SU(2)$ doublet, and T^a denote the associated $SU(2)$ generators.

6.1.3 Holographic flows in supergravity

The next goal is to construct a gravity dual to the Leigh Strassler flow. For that purpose, let us start with some general remarks about holographic flows in supergravity. Consider a metric of domain wall type

$$\mathrm{d}s^2 \ = \ e^{2A(r)}\,\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \ + \mathrm{d}r^2 \tag{6.1.22}$$

6,21

with boundary at $r \rightarrow \infty$. Recall that it reduces to an AdS spacetime if $A(r) = \frac{r}{L}$, i.e. these cases are dual to conformal field theories. An RG flow between two CFTs can be mapped to an $A(r)$ solution which interpolates between two linear regimes of different slope.
As a simplest case, we put a single scalar field into the spacetime (6.1.22), the equations of motion for this system have been given in (6.1.11). They can be simplified by introducing an auxiliary function $W(\varphi)$,

$$V(\varphi) \ = \ \frac{1}{2}\left(\frac{\mathrm{d}W}{\mathrm{d}r}\right)^2 \ - \ \frac{4}{3}\,W^2 \ , \tag{6.1.23}$$

6,22

namely they become a first order *gradient flow* (after using $A' = -\frac{2}{3}W$)

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} \ = \ \frac{\mathrm{d}W}{\mathrm{d}\varphi} \ . \tag{6.1.24}$$

6,23

We look for solutions of the equations of motions which interpolate between two conformal fixed points on the field theory side:

$$\lim_{r\rightarrow\infty} A(r) \ = \ \frac{r}{L_1} \ , \qquad \lim_{r\rightarrow -\infty} A(r) \ = \ \frac{r}{L_2} \tag{6.1.25}$$

6,24

The two AdS radii L_i lead to different central charges for the CFTs, say $L_1 \leftrightarrow c_{\mathrm{UV}}$ and $L_2 \leftrightarrow c_{\mathrm{IR}}$. Recall that in $d = 4$ SYM, the central charge is the prefactor relating $T^\mu{}_\mu$ with $R^{\mu\nu}R_{\mu\nu} - \frac{R^2}{3}$. In quadratic approximation, the gravity potential $V(\varphi)$ in the neighbourhood of the UV critical point φ_i is given by

$$V(\varphi) \ = \ V(\varphi_i) \ + \ \frac{m_i^2}{2\,L_i^2}\,h^2 \ + \ \mathcal{O}(h^3) \ . \tag{6.1.26}$$

6,25

in terms of $h := \varphi - \varphi_i$ with masses $m_i^2 = L_i^2 V''(\varphi_i)$. The value at φ_i itself is the cosmological constant,

$$V(\varphi_i) \ = \ -\frac{d\,(d-1)}{4\,L_i^2} \ . \tag{6.1.27}$$

6,26

The fluctuation $h(r,\vec{x})$ can be viewed as the gravity dual to some operator $\mathcal{O}_\Delta(\vec{x})$ where $m_i^2 = \Delta(\Delta - 4)$.
The marginal operator $\frac{1}{g^2}\mathrm{Tr}\{F_{\mu\nu}F^{\mu\nu}\}$ for instance has dimension $\Delta = 4$, so it must couple to some massless field. A careful analysis of the $d = 10$ DBI action identifies the dilaton with that scalar coupling to $\mathrm{Tr}\{F^2\}$. The dilaton being constant then leads to the vanishing of the β function.

The asymptotic behaviour of $h(r,\vec{x})$ at the boundary of $(d+1)$ dimensional AdS space is governed by

$$h(r,\vec{x}) \ \stackrel{r\to\infty}{\longrightarrow}\ e^{(\Delta-d)r}\left(\varphi_i \,+\, \bar{h}(\vec{x})\right) \,, \qquad (6.1.28) \quad \boxed{\texttt{6,27}}$$

cf. the earlier interpretation $\varphi(r) \sim \varphi_0 e^{(\Delta-d)r} + \langle \mathcal{O} \rangle\, e^{-\Delta r}$.

As a generalization of the AdS/CFT generating functional, consider

$$\left\langle \exp\left(-\,\mathcal{S}_{\rm CFT} + \int \mathrm{d}^d x\ \mathcal{O}_\Delta(\vec{x})\left(\varphi_i \,+\, \bar{h}(\vec{x})\right)\right)\right\rangle \ =\ e^{-\mathcal{S}_{\rm grav}[\bar{h}]}\ . \qquad (6.1.29) \quad \boxed{\texttt{6,28}}$$

The new piece $\Delta\mathcal{S} := \int \mathrm{d}^d x\, \mathcal{O}_\Delta(\varphi_i + \bar{h})$ belongs to a non-conformal action $\mathcal{S} = \mathcal{S}_{\rm CFT} + \Delta\mathcal{S}$, i.e. $\Delta\mathcal{S}$ is an operator deformation of the CFT.

Correlation functions on the field theory side can be obtained from the gravitational action

$$\langle \mathcal{O}_\Delta(\vec{x}_1)\ldots\mathcal{O}_\Delta(\vec{x}_n)\rangle \ =\ \frac{(-1)^{n-1}\,\delta^n}{\delta\bar{h}(\vec{x}_1)\ldots\delta\bar{h}(\vec{x}_n)}\ \mathcal{S}_{\rm grav}[\varphi_i + \bar{h}]\,\Big|_{\bar{h}=0}\ . \qquad (6.1.30) \quad \boxed{\texttt{6,29}}$$

Negative mass squares occur if the critical point is a local maximum. If $0 > m^2 > -\frac{d^2}{4}$, then $d > \Delta > \frac{D}{2}$ and $\mathcal{O}_\Delta$ is a relevant deformation driving the field theory away from the fixed point.

If V has a local minimum at the fixed point then $m^2 > 0$ and $\mathcal{O}_\Delta$ has conformal dimension $\Delta > d$. This is an example for an irrelevant operator which drives the flow into an IR fixed point. We look at *interpolating flows*, i.e. at solutions of the combined equations of motion (of Einstein- and scalar type) for which the scalar field $\varphi(r)$ corresponds to a maximum of $V(\varphi)$ in the UV ($r \to \infty$) and to a minimum in the IR ($r \to -\infty$).

Let us revisit the expansion around the critical point, this time in the form

$$\varphi(r) \ =\ \varphi_i \,+\, h(r)\,, \qquad A' \ =\ \frac{1}{L_i} \,+\, a'(r) \qquad (6.1.31) \quad \boxed{\texttt{6,30}}$$

where $a' = \mathcal{O}(h^2)$. The linearized equation of motion for the scalar fluctuation reads

$$h'' \,+\, \frac{d}{L_i}\,h' \,-\, \frac{m_i^2}{L_i^2}\,h \ =\ 0 \qquad (6.1.32) \quad \boxed{\texttt{6,31}}$$

with general solution

$$h(r) \ =\ B\,e^{(\Delta_i-d)r/L_i} \,+\, C\,e^{-\Delta_i r/L_i}\,, \qquad m_i^2 \ =\ \Delta_i\,(\Delta_i \,-\, 4)\ . \qquad (6.1.33) \quad \boxed{\texttt{6,32}}$$

In the limit $r \to \pm\infty$ we have

$$\begin{aligned} \varphi(r\to+\infty) &= \varphi_1 \,+\, B_1\,e^{(\Delta_1-d)r/L_1} \,+\, C_1\,e^{-\Delta_1 r/L_1} \\ \varphi(r\to-\infty) &= \varphi_2 \,+\, B_2\,e^{(\Delta_2-d)r/L_2} \,+\, C_2\,e^{-\Delta_2 r/L_2}\ . \end{aligned} \qquad (6.1.34) \quad \boxed{\texttt{6,33}}$$

One gets the impression that the domain wall flow sees the IR fixed point only in the deep interior $r \to -\infty$. To establish the field operator map for CFT at the IR fixed point, we have to extend the IR geometry to a complete AdS space with radius $L = L_{\rm IR}$. Note that the bound $A'' < 0$ implies $L_{\rm UV} > L_{\rm IR}$.

6.2 The holographic c theorem

TO BE FILLED BY JOHANNA

6.3 Last lecture of 2009

Domain wall flow $\mathrm{d}s^2 = e^{2A(r)}\eta_{ij}\mathrm{d}x^i\mathrm{d}x^j + \mathrm{d}r^2$. Case $A(r) = \frac{r}{L}$ corresponds to AdS.

Other possibility: Confinement

Plots: - A(r) with pole at finite negative value of r, and linear for $r \to \infty$ - Coupling in QCD

Characterise confinement by calculating Wilson loop (area law)

What we also want to do: So far: N=4, all fields in adjoint rep of gauge group $\phi \to e^{-i\Lambda}\phi e^{i\Lambda}$ Need for QCD: Quarks are in fundamental rep. $\phi \to e^{i\Lambda}\phi$

Also on gravity side, need more degrees of freedom. One possibility: Embedding of probe D7 branes → flavour (global symmetry)

In the Maldacena limit, the gauge coupling on the D7 brane goes to zero → global symmetry. In the probe limit (N D3-branes, $N \to \infty$, 1 or 2 D7-branes) we may ignore the backreaction of the D7-branes on the D3-branes. Separation of D3, D7 in 8,9 directions corresponds to quark mass.

New duality: 1) Standard AdS/CFT 2) N=2 SUSY hypermultiplet in fundamental rep of gauge group ⇔ fluctuations of D7 in AdS5xS5 as described by the Dirac-Born-Infeld action for the D7.

SUSY embeddings

$D3$	x	x	x	x	0	0	0	0	0	0	
$D3$	x	x	0	0	x	x	0	0	0	0	$AdS_3 \times S_1$
$D5$	x	x	x	0	x	x	x	0	0	0	$AdS_4 \times S_2$
$D7$	x	x	x	x	x	x	x	x	0	0	$AdS_5 \times S_3$

Black Hole in AdS_5 ¡-¿ Field theory at finite temperature Hawking Temperature ¡-¿ temperature in QFT

6.4 Applications 1: Field Theories at finite temperature

Switching on a temperature breaks all of the supersymmetry (still we keep the degrees of freedom of $\mathcal{N} = 4$, with possibly different masses, and the $N \to \infty$ planar limit!). Simplest case: $t \to -i\tau$, $e^{iHt} \to e^{-\beta H}$, $\beta = \frac{1}{k_B T}$, field theory in thermal equilibrium.

2nd step: consider black hole in Minkowski signature AdS space $\to$ dynamical processes (transport, relaxation) near-equilibrium.

Quark-Gluon plasma: The to date most successful application of generalized AdS/CFT is to descriptions of the QGP. This is a state of (QCD) matter at finite temperature and/or density.

Phase diagram of QCD: See plot

Perturbative QCD is not suited for describing the strongly coupled quark-gluon-plasma. Lattice gauge theory is difficult at finite temperature and density, and not suited for describing dynamical processes such as scattering.

AdS/CFT at finite temperature is well-suited for describing strongly coupled $\mathcal{N} = 4$ theory in the planar limit, in particular dynamical processes. In some important examples, comparison to QCD is possible.

Most famous example: (Shear viscosity / entropy density) ratio.

$$\eta \;=\; \frac{1}{\omega} \int \mathrm{d}^4 p \; e^{i\omega t} \; \langle T_{xy}(\vec{p})\, T_{xy}(-\vec{p}) \rangle \tag{6.4.1}$$

(Kubo formula)

AdS/CFT: $\frac{\eta}{s} = \frac{1}{4\pi}$ very small value, almost ideal fluid!

Perturbative QCD result: factor 10 larger

AdS/CFT result is in agreement with measurements at RHIC (Large experimental error, however).

Review of Black Holes:

1) Flat space

Schwarzschild Metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + (1 - \frac{2GM}{r})^{-1} dr^2 + r^2 d\Omega^2 \tag{6.4.2}$$

$r = 0$ is a singularity (true curvature singularity), $R^{\mu\nu\sigma\rho} R_{\mu\nu\sigma\rho} = \frac{48 G^2 M^2}{r^6}$.

$r = 2GM$ Schwarzschild radius

At the Schwarzschild radius, the curvature is finite. It corresponds to the event horizon of the black hole. For an external observer, the light cone closes up as $r \to 2GM$.

Other coordinates (Eddington-Finkelstein): Light cones tilt over

Hawking Temperature: Perform a Bogolubov transformation between states in coordinate systems at and far away from the black hole.

2) Calculate number density $\to$ thermal spectrum. $n_\Omega = \frac{1}{exp(\frac{E}{\kappa T_H})-1}$, $T_H = \frac{kappa}{2\pi} = \frac{1}{8\pi GM}$. κ surface gravity: acceleration needed to keep an object at the horizon (Schwarzschild radius).

6.5 Gauge gravity duality at finite temperature and density

At finite density and temperature: Let φ^* be a saddle point of some Euclidean action $\mathcal{S}_{\mathrm{E}}[\varphi]$, then we can approximate the generating functional semiclassically as

$$\mathcal{Z} \;=\; \int \mathcal{D}\varphi \; e^{-\mathcal{S}_{\mathrm{E}}[\varphi]} \;\approx\; e^{-\mathcal{S}_{\mathrm{E}}[\varphi^*]} \; . \tag{6.5.1}$$ 8,1

According to the weak form of the AdS/CFT correspondence, the partition function of the classical bulk theory with asymptotically AdS boundary conditions is equivalent to the partition function of the large N QFT. The metric g then takes the role of the φ field above:

$$\mathcal{Z}_{\mathrm{grav}} \;=\; e^{-\mathcal{S}_{\mathrm{E}}[g^*]} \tag{6.5.2}$$ 8,2

The gravitational action contains a *Gibbons Hawking boundary term* required for finiteness,

$$\mathcal{S}_{\mathrm{E}}[g] \;=\; -\frac{1}{2\,\kappa^2}\int \mathrm{d}^{d+1}x\,\sqrt{g}\left(\mathcal{R} \,+\, \frac{d\,(d-1)}{L^2}\right) \,+\, \frac{1}{2\,\kappa^2}\int_{r\to 0} \mathrm{d}^d x\,\sqrt{g}\left(-\,2\,\mathcal{K} \,+\, \frac{2\,(d-1)}{L^2}\right) . \tag{6.5.3}$$ 8,3

Here, $\mathcal{K}$ denotes the trace of the extrinsic curvature,

$$\mathcal{K} \;=\; \gamma^{\mu\nu}\,\nabla_\mu\, n_\nu \; , \tag{6.5.4}$$ 8,4

where $\gamma^{\mu\nu}$ is the induced metric on the boundary at $r \to 0$ and n^μ an outward pointing unit normal vector on the boundary.

A saddle point, i.e. a solution to the equations of motion, is obtained by analytic continuation of the AdS Schwarzschild metric

$$\mathrm{d}s^2 \;=\; \frac{L^2}{r^2}\left(f(r)\,\mathrm{d}\tau^2 \,+\, \frac{\mathrm{d}r^2}{f(r)} \,+\, \mathrm{d}x^i\,\mathrm{d}x^i\right) , \qquad f(r) \;=\; 1 \,-\, \frac{r^4}{r_{\mathrm{H}}^4} \; . \tag{6.5.5}$$ 8,5

The periodicity requirement for regularity fixes the temperature to

$$T \;=\; \frac{d}{4\pi\, r_{\mathrm{H}}} \; . \tag{6.5.6}$$ 8,6

We obtain further thermodynamic quantities by evaluating the partition function at the saddle point $e^{-\mathcal{S}_{\mathrm{E}}[g^*]}$. The action as given in (6.5.3), evaluated at the Euclidean Schwarzschild metric, is found to be

$$\mathcal{S}_{\mathrm{E}} \;=\; -\,\frac{L^{d-1}}{2\,\kappa^2\, r_{\mathrm{H}}^d}\,\frac{V_{d-1}}{T} \;=\; -\,\frac{(4\pi)^d\, L^{d-1}\, V_{d-1}\, T^{d-1}}{2\,\kappa^2\, d^d} \tag{6.5.7}$$ 8,7

where V_{d-1} is the spatial volume of the associated QFT.

In order to be in the classical gravity regime, we need that the spacetime is weakly curved in Planck units, i.e. that $\frac{L^{d-1}}{\kappa^2} \ll 1$. The dual field theory analogue of $\frac{L^{d-1}}{\kappa^2} \ll 1$ is $N \to \infty$, recall that $L^4 = 4\pi g_{\mathrm{s}} N \alpha'^2$.

From the action given by (6.5.7) we obtain the free energy and entropy as

$$F \;=\; -T\ln\mathcal{Z} \;=\; T\,\mathcal{S}_{\mathrm{E}}[g^*] \;=\; -\,\frac{(4\pi)^d\,L^{d-1}\,V_{d-1}\,T^d}{2\,\kappa^2\,d^d} \tag{6.5.8}$$

$$S \;=\; -\,\frac{\partial F}{\partial T} \;=\; \frac{(4\pi)^d\,L^{d-1}\,V_{d-1}\,T^{d-1}}{2\,\kappa^2\,d^{d-1}}\;. \tag{6.5.9}$$

The expression for the entropy is equal to the area of the event horizon divided by $4G_{\mathrm{N}} = \frac{\kappa^2}{2\pi}$. This area entropy relation is universally expected to be true for event horizons.

6.5.1 Finite density

Consider gravitational theories which are dual to a QFT with an additional global $U(1)$ symmetry. What is the gravity dual of this symmetry? Generically, in gauge gravity duality, the correspondence is

$$\left.\begin{array}{c}\text{global symmetry of field}\\ \text{theory in } d \text{ dimensions}\end{array}\right\} \;\Leftrightarrow\; \left\{\begin{array}{c}\text{local symmetry of gravity}\\ \text{in } d+1 \text{ dimensions}\end{array}\right. \tag{6.5.10}$$

The current J_μ of global $U(1)$ symmetry in field theory is dual to a gauge field A_M in the $d+1$ dimensional gravity theory, more precisely to its pull back A_μ to the boundary.
To give another example of the global-local dictionary: A global $SO(d-1)$ rotation symmetry in the spatial directions of a QFT becomes part of the diffeomorphisms of general relativity. Gauge symmetries include subgroups of "large" gauge transformations which act non-trivially as global symmetries on the boundary of spacetime. In the AdS/CFT correspondence, this is precisely the global symmetry at the boundary.
To describe the physics of the global $U(1)$ symmetry, we therefore have to add a Maxwell field to the bulk spacetime. This leads to Einstein Maxwell theory (which for instance arises naturally from the graviphoton in supersymmetric theories). In Minkowski signature, its action reads

$$\mathcal{S}[g,A] \;=\; -\,\frac{1}{2\,\kappa^2}\int \mathrm{d}^{d+1}x\,\sqrt{g}\,\left(\mathcal{R} \,+\, \frac{d\,(d-1)}{L^2}\right) \,-\, \frac{1}{4\,g^2}\int \mathrm{d}^{d+1}x\,\sqrt{g}\,F_{\mu\nu}\,F^{\mu\nu}\;. \tag{6.5.11}$$ 8,8

If the Einstein Maxwell action is derived from a supersymmetric theory, then the couplings κ and g are related. Moreover, the supercurrent of the four dimensional supergravity theory contains both the R symmetry current R_μ and the energy momentum tensor in its θ expansion:

$$J_\mu \;=\; R_\mu \,+\theta^\alpha\,Q_{\alpha\mu} \,+\, \bar\theta_{\dot\alpha}\,\bar Q^{\dot\alpha}_\mu \,+\, \theta^\alpha\,\sigma^\nu_{\alpha\dot\beta}\,\bar\theta^{\dot\beta}\,T_{\mu\nu} \tag{6.5.12}$$ 8,8a

6.5.2 Chemical potential in quantum field theory

Consider a QFT containing a scalar, fermion and a gauge field with Lagrangian

$$\mathcal{L} \;=\; (D_\mu\varphi)^*\,D^\mu\varphi \,+\, i\,\bar\psi\not{D}\psi \,+\, \frac{1}{g^2}\,F_{\mu\nu}\,F^{\mu\nu}\;. \tag{6.5.13}$$ 8,9

The $U(1)$ gauge field A_μ enters the covariant derivative via $D_\mu = \partial_\mu + iA_\mu$. Let us give its time component a non-vanishing VEV of the form $\langle A_0 \rangle = \mu$ such that

$$A_0 \;=\; \langle A_0 \rangle \;+\; \delta A_0 \; . \tag{6.5.14}$$
8,10

then a potential is generated of the form

$$V \;=\; -\,\mu^2\,\varphi^* \;-\; \mu\,\psi^\dagger\,\psi \; . \tag{6.5.15}$$
8,11

This is an upside down mass term for the scalar causing instability, and the extra term for the fermion can be interpreted as a density operator $\mathcal{N}_\psi = \psi^\dagger\psi$. The coefficient of the overall potential $-\mu\mathcal{N}$ is interpreted as the *chemical potential.*

The corresponding thermodynamical potential of the grand canonical ensemble is the *Gibbs free energy*

$$\Omega \;=\; E \;-\; T\,S \;-\; \mu\,\mathcal{N} \; . \tag{6.5.16}$$
8,12

A similar structure is present in the gravity dual. For this, we have to find a solution to the equation of motion of Einstein Maxwell theory with $A = A_t(r)\mathrm{d}t$. The background Maxwell potential of the field theory is read off from the boundary values of the bulk Maxwell field $A_\mu(r) = A_\mu^{(0)} + \ldots$ as $r \to 0$. The Einstein equations of motion involve the energy momentum tensor of the field strength $F_{\mu\nu}$

$$R_{\mu\nu} \;-\; \frac{\mathcal{R}}{2}\,g_{\mu\nu} \;-\; \frac{d\,(d-1)}{2\,L^2}\,g_{\mu\nu} \;=\; \frac{\kappa^2}{g^2}\,\left(F_{\mu\lambda}\,F_\nu{}^\lambda \;-\; \frac{1}{4}\,g_{\mu\nu}\,F_{\lambda\rho}\,F^{\lambda\rho}\right) \tag{6.5.17}$$
8,13

whereas the Maxwell equations remain in their standard form $\nabla_\mu F^{\mu\nu} = 0$.

A particular solution of the Maxwell field of the form $A = A_t(r)\mathrm{d}t$ is the *Reissner Nordström AdS black hole*

$$\begin{aligned} \mathrm{d}s^2 \;&=\; \frac{L^2}{r^2}\,\left(-\,f(r)\,\mathrm{d}t^2 \;+\; \frac{\mathrm{d}r^2}{f(r)} \;+\; \mathrm{d}x^i\,\mathrm{d}x^i\right) \\ f(r) \;&=\; 1 \;-\; \left(1 \;+\; \frac{r_{\mathrm{H}}^2\,\mu^2}{\gamma^2}\right)\left(\frac{r}{r_{\mathrm{H}}}\right)^d \;+\; \frac{r_{\mathrm{H}}^2\,\mu^2}{\gamma^2}\,\left(\frac{r}{r_{\mathrm{H}}}\right)^{2(d-1)} \\ \gamma \;&=\; \frac{(d-1)\,L^2\,g^2}{(d-2)\,\kappa^2} \end{aligned} \tag{6.5.18}$$
8,14

This satisfies the boundary condition that $A_t(r)$ has to vanish at the horizon since ∂_t is not well-defined as a Killing vector there. Moreover we have

$$A_t(r) \;=\; \mu\,\left[1 \;-\; \left(\frac{r}{r_{\mathrm{H}}}\right)^{d-2}\right] \; . \tag{6.5.19}$$
8,15

This identifies the μ parameter in the solution (6.5.18) with the chemical potential. The temperature is again fixed by analytic continuation to the Euclidean regime and is given by

$$T = \frac{1}{4\pi\, r_{\mathrm{H}}} \left(d - \frac{(d-2)\, r_{\mathrm{H}}^2\, \mu^2}{\gamma^2} \right) . \qquad (6.5.20)$$ 8,16

In the grand canonical ensemble, by evaluating the Euclidean action on the solution, we find the following Gibbs free energy

$$\Omega = -T \ln \mathcal{Z} = -\frac{L^{d-1}}{2\,\kappa^2\, r_{\mathrm{H}}^d} \left(1 + \frac{r_{\mathrm{H}}^2\, \mu^2}{\gamma^2} \right) V_{d-1} = \mathcal{F}\left(\frac{T}{\mu} \right) V_{d-1}\, T^d . \qquad (6.5.21)$$ 8,17

From this, we may obtain the charge density (wlog in $d = 3$ dimensions)

$$\rho = -\frac{1}{V_2} \frac{\partial \Omega}{\partial \mu} = \frac{2\, L^2\, \mu}{\kappa^2\, r_{\mathrm{H}}\, \gamma^2} . \qquad (6.5.22)$$ 8,18

6.6 Dissipative dynamics close to equilibrium

So far, we have considered time independent homogeneous backgrounds. A natural next step is to include small space- and timedependent perturbations about equilibrium. The idea of *linear response theory* will be implemented in the context of the AdS/CFT correspondence in the following. This is particularly useful for describing experimentally relevant processes such as transport coefficients and spectroscopy.

The basic object in the linear response theory is the so-call retarded Green function relating linear sources to corresponding expectation values. The theory allows to relate two point correlation functions to transport coefficients.

6.6.1 Retarded Green functions in QFT

Consider the response of a system to the presence of weak external fields φ_i (with possible Lorentz indices suppressed) coupled to a set of operators $\mathcal{O}^i(x)$. The Hamiltonian is the modified by a term of the form

$$\delta \mathcal{H} = -\int \mathrm{d}^d x\ \varphi_i(t, \vec{x})\, \mathcal{O}^i(t, \vec{x}) . \qquad (6.6.1)$$ 8,19

From time dependent perturbation theory, we know that these external fields will produce a change in the expectation values of the operators,

$$\delta \langle \mathcal{O}^i \rangle = \int \mathrm{d}^{d+1} y\ G_{\mathrm{R}}^{ij}(x, y)\, \varphi_j(y) + \mathcal{O}(\varphi^2) \qquad (6.6.2)$$ 8,20a

$$G_{\mathrm{R}}^{ij}(x, y) = i\, \Theta(t_x - t_y) \left\langle \left[\mathcal{O}^i(x)\, ,\, \mathcal{O}^j(y) \right] \right\rangle , \qquad (6.6.3)$$ 8,20

where $G_{\mathrm{R}}^{ij}(x,y)$ is the *retarded Green function.* It is nonvanishing only in the forward light cone and therefore provides a causality structure. In Fourier space we have

$$\delta\langle\mathcal{O}^i(k)\rangle \;=\; G_{\mathrm{R}}^{ij}(k)\,\varphi_j(k) \;+\; \mathcal{O}(\varphi^2)\;, \qquad G_{\mathrm{R}}^{ij}(k) \;=\; \int \mathrm{d}^{d+1}x\; e^{-ik\cdot x}\, G_{\mathrm{R}}^{ij}(x,0)\;. \tag{6.6.4}$$ 8,21

To explain the relation between Green functions and transport coefficients, it is convenient to start with the instructive example of *Ohm's law.* It relates an electric source field E_j representing a linear perturbation to the (electric) response current J^i via conductivity matrix σ:

$$J^i(\omega) \;=\; \sigma^{ij}(\omega)\, E_j(\omega) \tag{6.6.5}$$ 8,22

To make contact with the general notation, let us identify φ_i with the external vector potential A_μ and the operator $\mathcal{O}^i$ with the conserved current J^μ. Choosing temporal gauge with $A_t = 0$, the electric field becomes $E_i = -\partial_t A_i$. In Fourier space, using $A_i \sim e^{-i\omega t}$, we have $E_i = i\omega A_i$. Comparing Ohm's law with (6.6.4) we see that the conductivity and the current-current Green's function as defined by (6.6.3) are proportional:

$$\sigma^{ij}(\omega) \;=\; \frac{G_{\mathrm{R}}^{ij}(\omega)}{i\omega} \tag{6.6.6}$$ 8,23

6.6.2 The gravity side of Green functions

The AdS/CFT correspondence conjectures that the current-current correlation can be computed from a higher dimensional gravity theory by varying its action with respect to boundary values $A_\mu^{(0)}$ of a source field, i.e.

$$\langle J_\mu(x) J_\nu(y)\rangle \;\sim\; \frac{\delta^2}{\delta A_\mu^{(0)}(x)\delta A_\nu^{(0)}(y)}\; e^{i\mathcal{S}_{\mathrm{grav}}}\;. \tag{6.6.7}$$ 8,24

In general on the gravity side, taking operator mixing into account we have

$$\delta\langle\mathcal{O}_A(\omega,k)\rangle \;=\; G_{\mathrm{R}}^{\mathcal{O}_A\mathcal{O}_B}(\omega,k)\,\delta\varphi_B^{(0)}(\omega,k)\;. \tag{6.6.8}$$ 8,25

The source is now the boundary value $\varphi^{(0)}$ of a field in curved space. Consider fluctuations of the bulk fields of the form $\varphi_A(r) \to \varphi_A(r) + \delta\varphi_A(r)e^{-i\omega t + ik\cdot x}$ where φ_A solves the equations of motion in gravity dual space.

The equation of motion for $\delta\varphi_A$ is obtained by substituting the perturbed solution into the equations of motion and by then linearizing. Take boundary conditions at the AdS boundary $r = 0$

$$\delta\varphi_A(r) \;=\; r^{d-\Delta}\,\delta\varphi_A^{(0)} \;+\; \ldots \qquad : \; r\to 0\;. \tag{6.6.9}$$ 8,26

Since we are interested in the field theory's behaviour at finite temperature, we assume the gravity dual to possess an AdS black hole background. The second boundary condition supplementing (6.6.9) has to be imposed at the black hole's horizon. As one can read off from the conformal diagram below, there are two possibilities leading to retarded and advanced Green functions:

conf diagram

Fluctuations of the time slice ending on the future event horizon are associated with the retarded Green function. The future horizon at $r = r_{\mathrm{H}}$ (where $g_{tt} = 0$) is a null surface beyond which events cannot causally propagate to the asymptotically AdS region (the boundary region) any more. On a future horizon, regularity requires that modes are ingoing (they can propagate into the black hole but cannot escape from it). For the horizon at $r = r_{\mathrm{H}}$ with nonzero temperature, this implies

$$\delta\varphi_A(r) \quad = \quad (r - r_{\mathrm{H}})\, e^{-4\pi i\omega/T} \Big[const \; + \; ... \Big] \qquad : r \to r_{\mathrm{H}} \; . \tag{6.6.10}$$ 8,27

In all of the subsequent, we impose the given boundary conditions (6.6.9) and (6.6.10).
Given a mode $\delta\varphi_A$ satisfying the required boundary conditions and linearized equations of motion, we obtain from (6.6.8) that

$$G_{\mathrm{R}}^{\mathcal{O}_A\mathcal{O}_B}(\omega, k) \quad = \quad \frac{\delta\langle\mathcal{O}_A\rangle}{\delta\varphi_B^{(0)}}\Big|_{\delta\varphi=0} \quad = \quad \lim_{r\to 0} \frac{\delta\Pi_A}{\delta\varphi_B^{(0)}}\Big|_{\delta\varphi=0} \tag{6.6.11}$$ 8,28

where Π_A is obtained in the following way:

$$\langle\mathcal{O}_A\rangle \quad = \quad -i\,\frac{\delta\mathcal{Z}_{\mathrm{bulk}}[\varphi^{(0)}]}{\delta\varphi_A^{(0)}} \overset{N\to\infty}{=} -i\,\frac{\delta\mathcal{S}_{\mathrm{grav}}[\varphi^{(0)}]}{\delta\varphi_A^{(0)}} \tag{6.6.12}$$ 8,29

Taking regularization at the boundary into account, we have

$$\frac{\delta\mathcal{S}_{\mathrm{grav}}[\varphi^{(0)}]}{\delta\varphi_A^{(0)}} \quad = \quad \lim_{r\to 0}\left[-\frac{\delta\mathcal{S}_{\mathrm{grav}}[\varphi^{(0)}]}{\delta(\partial_r\varphi_A^{(0)})} + \frac{\delta\mathcal{S}_{\mathrm{bdy}}[\varphi^{(0)}]}{\delta\varphi_A^{(0)}} \right] \quad = \quad \lim_{r\to 0} \Pi_A[\varphi^{(0)}] \; . \tag{6.6.13}$$ 8,30

The boundary version of the action contains appropriate counterterms necessary to make $\mathcal{S}_{\mathrm{grav}}$ finite when evaluated at the boundary. The underlying procedure is known as *holographic renormalization.* For a scalar field, the boundary term is

$$\mathcal{S}_{\mathrm{bdy}} \quad = \quad -\int_{r\to 0} \mathrm{d}^d x\, \sqrt{\gamma}\, \left(\varphi\, n^\mu\, \nabla_\mu\varphi \; + \; \frac{\Delta}{2\,L}\, \varphi^2 \right) \; . \tag{6.6.14}$$ 8,31

Inserting the near boundary value of the scalar field ...

HERE THE LAST LINES FROM LAST LECTURE ARE MISSING

6.6.3 Example: Holographic computation of Ohm's law

Consider the gravity dual for a strongly interacting 2+1 dimensional field theory, $\mathcal{N} = 8,\ d = 3,\ SU(N)$ SYM theory in 2+1 dimensions. The coupling $g^2_{\rm YM}$ has dimensions of mass in $d = 3$, so it has to appear in the dimensionless ration $\frac{g^2_{\rm YM}}{E}$ with E denoting the energy scale considered. We expect the field theory to be strongly coupled at low energies and to have an IR fixed point. This field theory is conjectured to be dual to M theory on $\mathrm{AdS}_4 \times S^7$. A certain sector of the dual gravity theory is described by the four dimensional effective action

$$\mathcal{S} \;=\; \frac{1}{2\,\kappa^2}\int \mathrm{d}^4x\,\sqrt{|g|}\,\big(\mathcal{R}\;-\;2\,\Lambda\big)\;-\;\frac{1}{4\,g^2}\int \mathrm{d}^4x\,\sqrt{|g|}\,F_{AB}\,F^{AB} \tag{6.6.15}$$ 8,32

with negative cosmological constant $\Lambda = -3/L^2$.
Take A_μ to be the dual field to the current J_μ of a $U(1)$ subgroup of the global $SO(8)$ R symmetry. The classical gravitational description is valid at large N where $\frac{1}{\kappa^2} \sim N^{3/2}$. The couplings κ and g in the Einstein Maxwell action (6.6.15) are related by supersymmetry $\kappa^2 = 2g^2L^2$.
An important solution to the equations of motion for the action $\mathcal{S}$ is the dyonic black hole (which is only possible in $d = 4$ on the gravity side), i.e. a black hole with both electric and magnetic charge (q, h):

$$\begin{aligned}
\frac{\mathrm{d}s^2}{L^2} &= \frac{1}{r^2}\left(-\,f(r)\,\mathrm{d}t^2\;+\;\mathrm{d}x^2\;+\;\mathrm{d}y^2\right)\;+\;\frac{1}{r^2}\,\frac{\mathrm{d}r^2}{f(r)}\\
A &= \frac{h\,x}{r_{\rm H}}\,\mathrm{d}y\;-\;q\left(1\;-\;\frac{r}{r_{\rm H}}\right)\mathrm{d}t\\
f(r) &= 1\;+\;\alpha\,(h^2\;+\;q^2)\,\frac{r^4}{r_{\rm H}^4}\;-\;\big(1\;+\;\alpha\,(h^2\;+\;q^2)\big)\,\frac{r^3}{r_{\rm H}^3}
\end{aligned} \tag{6.6.16}$$ 8,33

We have defined $\alpha := \frac{\kappa^2 r_{\rm H}^2}{2g^2L^2}$. This solution is dual to the 2+1 dimensional field theory at finite T, B and electric charge density n:

$$T \;=\; \frac{3\;-\;(h^2+q^2)\,\alpha}{4\pi\,r_{\rm H}}\,, \qquad B \;=\; \frac{h}{r_{\rm H}}\,, \qquad n \;=\; \langle J^t\rangle \tag{6.6.17}$$ 8,34

Now let us compute the current. Near the boundary, we may expand $A_\mu = a_\mu + rb_\mu + \ldots$. On shell, the Maxwell part of the action reduces to a boundary term of the form

$$\delta\mathcal{S}_{\rm max} \;=\; \frac{1}{g^2}\int \mathrm{d}^3x\;\eta^{\mu\nu}\,\delta A_\mu\,\partial_r A_\nu\Big|_{r=0} \;=\; \frac{1}{g^2}\int \mathrm{d}^3x\;\eta^{\mu\nu}\,\delta a_\mu\,b_\nu\;. \tag{6.6.18}$$ 8,35

We use the radial gauge $A_r = 0$. Then, the current takes the expectation value

$$\langle J^\mu\rangle \;=\; \frac{\delta\mathcal{S}_{\rm max}}{\delta a_\mu} \;=\; \frac{b^\mu}{g^2}\;. \tag{6.6.19}$$ 8,36

For the dyonic black hole, $b_t = \frac{q}{r_{\mathrm{H}}}$ and $a_t = -q =: \mu$ such that

$$\langle J^t \rangle \;=\; \langle n \rangle \;=\; -\frac{q}{g^2\, r_{\mathrm{H}}} \;=\; \frac{\mu}{g^2\, r_{\mathrm{H}}} \; . \tag{6.6.20}$$ 8,37

On the way towards Ohm's law, we define $J_\pm = J_x \pm i J_y$ and $E_\pm = E_x \pm i E_y$. The previous calculation gave

$$\langle J_\pm \rangle \;=\; \frac{1}{g^2} \lim_{r\to 0} \partial_r A_\pm \; . \tag{6.6.21}$$ 8,38

We can think of $\pm i\partial_r A_\pm$ as a bulk magnetic field $\mathcal{B}_\pm$ and of $E_\pm$ as the boundary limit of a bulk electric field $\mathcal{E}_\pm$. From the gravity point of view, Ohm's law may be written as

$$\sigma_\pm \;=\; \frac{\pm i \,\langle J_\pm \rangle}{E_\pm} \;=\; \lim_{r\to 0} \frac{\mathcal{B}_\pm(r)}{g^2\, \mathcal{E}_\pm(r)} \; . \tag{6.6.22}$$ 8,39

In the Maxwell part of the action (6.6.15), either F_{AB} or its dual $\tilde{F}_{AB} = \frac{1}{2}\varepsilon_{ABCD}F^{CD}$ could be the fundamental field strength. The action is classically invariant under switching the electric and magnetic field. Note that this electric magnetic duality is a special feature of four dimensional spacetime.

For the dyonic black hole, the duality transformation is

$$\mathcal{B}_\pm \;\mapsto\; -\mathcal{E}_\pm \; , \qquad \mathcal{E}_\pm \;\mapsto\; \mathcal{B}_\pm \; , \qquad (q,h) \;\mapsto\; (h,-q) \; . \tag{6.6.23}$$ 8,40

For the numerical computation of $\sigma_\pm$, see Hartnoll, Herzog. The result satisfies the constraints from electric magnetic duality

$$\sigma_\pm(q,h) \;=\; \lim_{r\to 0} \frac{\mathcal{B}_\pm(q,h)}{g^2\, \mathcal{E}_\pm(q,h)} \;=\; -\lim_{r\to 0} \frac{\mathcal{E}_\pm(h,-q)}{g^2\, \mathcal{B}_\pm(h,-q)} \;=\; -\frac{1}{g^4\, \sigma_\pm(h,-q)} \; . \tag{6.6.24}$$ 8,41

6.7 Hydrodynamics and shear viscosity in AdS/CFT

6.7.1 Relativistic hydrodynamics

According to the work of Son and Starinets, we will work with a metric of signature $\eta^{\mu\nu} = (-,+,+,+)$. In hydrodynamics, one considers a system in equilibrium subject to small perturbations. This is a perfect framework for an effective theory describing dynamics at large distances and time scales. It describes dissipation in thermal media.

In the simplest case, the hydrodynamic equations are just the laws of energy momentum conservation $\partial_\mu T^{\mu\nu} = 0$. The number of independent components of $T^{\mu\nu}$ is reduced by the assumption of local thermal equilibrium: If perturbations have long wavelengths or small frequencies, then the state of the system considered at a given time is determined by the temperature as a function of the coordinates, $T(x)$, and the local fluid four velocities $u^\mu(x)$. Because $u_\mu u^\mu = -1$, only three components are actually independent.

The number of variables is thus four (e.g. u_1, u_2, u_3, T), equal to the number of independent equations $\partial_\mu T^{\mu\nu} = 0$. To express $T^{\mu\nu}(x)$ in terms of $T(x), u^\mu(x)$, it is convenient to expand in powers of spatial derivatives. To zeroth order, we have an ideal fluid (without dissipation) for which

$$T^{\mu\nu} \ = \ (\varepsilon \, + \, P)\, u^\mu \, u^\nu \ + \ P\, g^{\mu\nu} \ + \ \mathcal{O}(\partial) \ , \tag{6.7.1}$$ 8,42

ε and P denote the energy density and the pressure. From thermodynamic laws $\mathrm{d}\varepsilon = T\mathrm{d}S$, $\mathrm{d}P = s\mathrm{d}T$ and $\varepsilon + P = Ts$ (with entropy density s), one can deduce conservation of the entropy current

$$\partial_\mu \, (s\, u^\mu) \ = \ 0 \ . \tag{6.7.2}$$ 8,43

To describe dissipation or entropy production, we have to proceed to the next order in the derivative expansion,

$$T^{\mu\nu} \ = \ (\varepsilon \, + \, P)\, u^\mu \, u^\nu \ + \ P\, g^{\mu\nu} \ - \ \sigma^{\mu\nu} \ + \ \mathcal{O}(\partial^2) \ . \tag{6.7.3}$$ 8,44

For simplification, we go to a local rest frame in which $u^i(x) = 0$. In this frame, $\sigma^{00} = \sigma^{0i} = 0$ equivalent to $T^{00} = \varepsilon$ and $T^{0i} = 0$. The only nonzero entries of the dissipative energy momentum contribution are

$$\sigma_{ij} \ = \ \eta \, \left(\partial_i u_j \ + \ \partial_j u_i \ - \ \frac{2}{3} \, \delta_{ij} \, \partial_k u^k \right) \ + \ \zeta \, \delta_{ij} \, \partial_k u^k \ , \tag{6.7.4}$$ 8,45

parametrized by shear viscosity η and bulk viscosity ζ. In a general frame, this leads to

$$\sigma^{\mu\nu} \ = \ P^{\mu\alpha} \, P^{\nu\beta} \left[\eta \, \left(\partial_\alpha u_\beta \ + \ \partial_\beta u_\alpha \ - \ \frac{2}{3} \, \delta_{\alpha\beta} \, \partial_\lambda u^\lambda \right) \ + \ \zeta \, \delta_{\alpha\beta} \, \partial_\lambda u^\lambda \right] \tag{6.7.5}$$ 8,46

where $P^{\mu\nu} := g^{\mu\nu} + u^\mu u^\nu$ is the projector onto directions perpendicular to u^μ.
Charged fluids have an additional conserved $U(1)$ current $\partial_\mu J^\mu = 0$ given by

$$j^\mu \ = \ \rho \, u^\mu \ - \ D \, P^{\mu\nu} \, \partial_\nu \alpha \ . \qquad WHAT \ IS \ \alpha \ ??? \tag{6.7.6}$$ 8,47

In the fluid's rest frame, we rediscover Fick's law of diffusion $\vec{j} = -D \vec{\nabla} \rho$.

6.7.2 Kubo formula from linear response theory

Let us now concentrate on the particular case when metric perturbatiions are time dependent but homogeneous in space, i.e.

$$g_{ij}(t, \vec{x}) \ = \ \delta_{ij} \ + \ h_{ij}(t) \ , \qquad h_{ii} = 0 \tag{6.7.7}$$ 8,48

$$g_{00}(t, \vec{x}) \ = \ -1 \ , \qquad g_{0i}(t, \vec{x}) \ = \ 0 \ . \tag{6.7.8}$$ 8,49

The velocity vector hence depends on time only, $u^i = u^i(t)$. Consider the case where the fluid remains at rest at all times, $u^\mu = (1,0,0,0)$.
In curved spacetime, equation (6.7.5) for the $\mathcal{O}(\partial)$ contributions to $T^{\mu\nu}$ generalizes to

$$\sigma^{\mu\nu} = P^{\mu\alpha}\, P^{\nu\beta} \left[\eta \left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha \right) + \left(\zeta - \frac{2\,\eta}{3} \right) g_{\alpha\beta}\, \nabla_\lambda u^\lambda \right] . \qquad (6.7.9)$$ 8,50

In the situation considered above, this simplifies to

$$\sigma_{xy} = 2\,\eta\,\Gamma^0{}_{xy} = \eta\,\partial_0 h_{xy} \; . \qquad (6.7.10)$$ 8,51

By comparison with linear response theory, we find the zero spatial momentum, low frequency limit of the retarded Green function of T_{xy}:

$$G^{\mathrm{R}}_{xy,xy}(\omega,\vec{0}) = \int \mathrm{d}t\, \mathrm{d}^3x\; e^{i\omega t}\, \Theta(t) \,\langle \left[T_{xy}(t,\vec{x})\, , \, T_{xy}(0,\vec{0}) \right] \rangle = -\, i\eta\, \omega + \mathcal{O}(\omega^2) \qquad (6.7.11)$$ 8,52

The associated Kubo formula is

$$\eta = -\lim_{\omega\to 0} \frac{1}{\omega}\, \mathrm{Im} \left\{ G^{\mathrm{R}}_{xy,xy}(\omega,\vec{0}) \right\} . \qquad (6.7.12)$$ 8,53

Let us now explain the notion of hydrodynamic modes, determined by the poles of the retarded Green function. They also give for instance the poles in the spectral function. Poles of correlators are obtained from solutions of the linearized hydrodynamic equations, i.e. from plane wave solutions $e^{-i\omega t + i\vec{k}\cdot\vec{x}}$. Dissipation is described by complex ω with negative imaginary part.

figure spectral function and ω poles

Charge diffusion is governed by the following dispersion relation,

$$\left(\partial_t - D\,\vec{\nabla}^2 \right) \rho = 0 \quad \Rightarrow \quad \omega = iD\,\vec{k}^2 \; , \qquad (6.7.13)$$ 8,54

which determines a pole in the current-current Green function $\langle J_\mu(p) J_\nu(-p) \rangle$.

6.7.3 Shear modes and sound modes

Shear modes correspond to fluctuations of pairs of components T^{0a} and T^{3a} where $a = 1,2$.

$$\begin{aligned} T^{3a} &= -\,\eta\,\partial_3 u^a = -\,\frac{\eta}{\varepsilon + P}\,\partial_3 T^{0a} \\ \partial_0 T^{0a} &= -\,\frac{\eta}{\varepsilon+P}\,\partial_3^2 T^{aa} = 0 \end{aligned} \qquad (6.7.14)$$ 8,55

For plane waves $h \sim e^{-i\omega t + i\vec{k}\cdot\vec{x}}$, we find $\omega = \frac{-i\eta\vec{k}^2}{\varepsilon+P}$.

Sound waves, on the other hand, are longitudinal fluctuations of T^{00}, T^{03}, T^{33} with speed $c_S = \sqrt{\frac{\mathrm{d}P}{\mathrm{d}\varepsilon}}$ and frequency

$$\omega \;=\; c_S\, k \;-\; \frac{i}{2}\left(\frac{4\,\eta}{3} \;+\; \zeta\right)\frac{\vec{k}^2}{\varepsilon + P}\;. \tag{6.7.15}$$ 8,56

In weakly coupled theories, the viscosity is governed by the mean free path $\ell_{\mathrm{mfp}} \sim (n\sigma v)^{-1}$ where n denotes the density, σ the cross section for interactions and v a typical velocity. In $\lambda\varphi^4$ theory at finite temperature, one can derive perturbatively that $n \sim T^3$ and $\sigma \sim \left(\frac{\lambda}{T}\right)^2$. The viscosity is then obtained by multiplying with the energy density ε (for which a Stefan Boltzmann law $\varepsilon \sim T^4$ is assumed):

$$\eta \;\sim\; \varepsilon\,\ell_{\mathrm{mfp}} \;\sim\; \frac{T^3}{\lambda^2} \tag{6.7.16}$$ 8,57

The entropy density scales in the same way with temperature, $s \sim T^3$, so the quotient

$$\frac{\eta}{s} \;\sim\; \frac{1}{\lambda^2} \tag{6.7.17}$$ 8,58

depends on λ only and becomes large at weak coupling $\lambda \ll 1$.

6.7.4 AdS/CFT calculation of the shear viscosity

In order to compute $\langle T_{xy} T_{xy}\rangle$ in the field theory, we have to examine the propagation of the dual graviton h_{xy} in AdS spacetime. For this purpose, let us start from the Einstein Hilbert action in five dimensions. Consider a scalar metric fluctuation h_{xy}, denote it by φ in the following. The quadratic part of the Einstein Hilbert action in φ is given by

$$\mathcal{S}_{\mathrm{quad}}[\varphi] \;=\; \frac{N^2}{8\pi^2\,L^3}\int \mathrm{d}^4x\,\mathrm{d}r\;\sqrt{-g}\,\left(-\frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\varphi\,\partial_\nu\varphi\right) \tag{6.7.18}$$ 8,59

and gives rise to the linearized equation of motion $\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) = 0$. Perform a Fourier transformation of the boundary coordinates, then a boundary condition of type $\varphi(p, r = 0) = \varphi_0(p)$ can be imposed. It is convenient to factorize

$$\varphi(p,r) \;=\; f_p(r)\,\varphi_0(p) \tag{6.7.19}$$ 8,60

where f_p is called zero mode function and satisfies

$$\left(\frac{f_p'}{r^3}\right)' \;-\; \frac{p^2}{r^3}\,f_p \;=\; 0\,, \qquad f_p(0) \;=\; 1. \tag{6.7.20}$$ 8,61

An exact solution in terms of a Bessel function K_2 exists,

$$f_p(r) \;=\; \frac{(pr)^2}{2}\,K_2(pr) \;=\; 1 \;-\; \frac{(pr)^2}{4} \;-\; \frac{(pr)^4}{16}\,\ln(pr) \;+\; \mathcal{O}\big((pr)^4\big)\;. \tag{6.7.21}$$ 8,62

The other solution $(pr)^2 I_2(pr)$ is ruled out since it blows up for $r \to \infty$. The on shell action for f_p reads

$$\begin{aligned} \mathcal{S}_{\text{quad}}[\varphi] &= \frac{N^2}{16\pi^2} \int \mathrm{d}^4x \, \frac{1}{r^3} \, \varphi(x,r) \, \varphi'(x,r) \Big|_{r=0} \\ &= \int \frac{\mathrm{d}^4p}{(2\pi)^4} \, \varphi_0(-p) \, \mathcal{F}(p,r) \, \varphi_0(p) \Big|_{r=0} \qquad (6.7.22) \quad \boxed{\texttt{8,59}} \\ \mathcal{F}(p,r) &= \frac{N^2}{16\pi^2 r^3} \, f_{-p}(r) \, \partial_r f_p(r) \; . \qquad (6.7.23) \quad \boxed{\texttt{8,60}} \end{aligned}$$

It then follows that

$$\langle T_{xy}(p) \, T_{xy}(-p) \rangle \; = \; -2 \lim_{r \to 0} \mathcal{F}(p,r) \; = \; \frac{N^2}{64\pi^2} \, p^4 \, \ln p^2 \; . \qquad (6.7.24) \quad \boxed{\texttt{8,61}}$$

At nonzero temperature, the metric

$$\mathrm{d}s^2 \; = \; \frac{r^2}{L^2} \left(- f \, \mathrm{d}t^2 \; + \; \mathrm{d}\vec{x}^2 \right) \; + \; \frac{L^2}{r^2 \, f} \, \mathrm{d}r^2 \; + \; L^2 \, \mathrm{d}\Omega_5^2 \qquad (6.7.25) \quad \boxed{\texttt{8,62}}$$

gives rise to the Hawking temperature $T_{\mathrm{H}} = \frac{r_{\mathrm{H}}}{\pi L^2}$. The entropy is given by the Bekenstein Hawking formula $S = \frac{A}{4\pi}$ with A the area of the black hole horizon. For the density, one finds

$$s \; = \; \frac{S}{V} \; = \; \frac{\pi^2}{2} \, N^2 \, T^3 \; . \qquad (6.7.26) \quad \boxed{\texttt{8,62a}}$$

This is 3/4 of the entropy density in $\mathcal{N} = 4$ SYM theory at vanishing t'Hooft coupling. Define a new coordinate $u = \frac{r_{\mathrm{H}}^2}{r^2}$, then the boundary is situated at $u = 0$, the horizon at $u = 1$. In terms of u, the metric is

$$\mathrm{d}s^2 \; = \; \frac{(\pi T L)^2}{u^2} \left(- f(u) \, \mathrm{d}t^2 \; + \; \mathrm{d}\vec{x}^2 \right) \; + \; \frac{L^2}{4 \, u^2 \, f(u)} \, \mathrm{d}u^2 \; + \; L^2 \, \mathrm{d}\Omega_5^2 \; . \qquad (6.7.27) \quad \boxed{\texttt{8,62b}}$$

In real time AdS/CFT, consider again the factorization (6.7.23), then the φ equation of motion implies (HIER IST ETWAS UNGUENSTIG, DASS f DOPPELT VERGEBEN IST...)

$$f_p'' \; - \; \frac{1+u^2}{u \, f(u)} \, f_p' \; + \; \frac{W^2}{u \, f^2(u)} \, f_p \; - \; \frac{q^2}{u \, f(u)} \, f_p \; = \; 0 \qquad (6.7.28) \quad \boxed{\texttt{8,63}}$$

with shorthands $W = \frac{\omega}{2\pi T}$ and $q = \frac{k}{2\pi T}$. Near $u = 0$, the two solutions behave as $f_1 \sim 1$ and $f_2 \sim u^2$. In Minkowski space, there are two finite solutions near the horizon $f_p \sim (1-u)^{-iW/2}$ and $f_p^* \sim (1-u)^{iW/2}$. Any linear combination is possible, the solution is not unique! This is a problem in defining the Green function.

A thorough analysis of the real time formalism in AdS/CFT leads to the result that the retarded Green function is related to $\mathcal{F}$ by the same formula that was found at zero temperature:

$$G^{\mathrm{R}}(p) \; = \; -2 \lim_{u \to 0} \mathcal{F}(p,u) \qquad (6.7.29) \quad \boxed{\texttt{8,64}}$$

This has no contribution from the horizon, but to obtain f_p, we need infalling boundary conditions of course. The correlator $\langle T_{xy}(p) T_{xy}(-p) \rangle$ comes from $h_{xy} = \varphi$ with

$$\varphi_p'' \; - \; \frac{1+u^2}{u\,f}\,\varphi_p' \; + \; \frac{W^2 \; - \; q^2\,f}{u\,f^2}\,\varphi_p \;\; = \;\; 0 \tag{6.7.30}$$ 8,65

giving rise to the incoming wave solution $f_p(r) \sim (1-u^2)^{-iW/2} + \mathcal{O}(W^2, q^2)$. Using (6.7.29), the resulting Green function is

$$G^{\mathrm{R}}(\omega, \vec{k}) \;\; = \;\; -\frac{\pi^2\,N^2\,T^4}{4}\,i\omega \; . \tag{6.7.31}$$ 8,66

This is the famous Kuba formula for the viscosity

$$\eta \;\; = \;\; \frac{\pi}{8}\,N^2\,T^3 \; , \qquad \frac{\eta}{s} \;\; = \;\; \frac{1}{4\pi} \; . \tag{6.7.32}$$ 8,67

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