Theoretical analysis of Link Analysis Ranking

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Link Analysis Ranking

- Link Analysis Ranking (LAR) algorithm:
  - Given a (directed) graph G, determine the importance of the nodes in the graph using the information of the edges (links) between the nodes.
- Inuition:
  - A link from node p to node q denotes endorsement. Node p considers node q an authority on a subject
  - mine the graph of recommendations, assign an authority value to every page
- Applications:
  - Assess the importance of Web pages using link information.
  - Recommendation systems

Why theoretical analysis of Link Analysis Ranking?

- Plethora of LAR algorithms: we need a formal way to compare and analyze them
- Need to define properties that are useful
  - stability of the algorithm
- Axiomatic characterization of LAR algorithms
  - extension of social choice theory to recommendation systems

Link Analysis Ranking algorithm

- A LAR algorithm is a function that maps a graph to a real vector

$$A: G_n \rightarrow \mathbf{R}^n$$

- G_n : class of graphs of size n
- LAR vector w: the output $A(G)$ of an algorithm A on a graph G
  - w_i : the authority weight of node i

Popular LAR algorithms

- InDegree algorithm
  - w_i = in-degree(i)
- PageRank algorithm [BP98]
  - perform a random walk on G with random resets (with probability 1-a)
  - w = stationary distribution of the random walk
- HITS algorithm [K98]
  - compute the left (hub) and right (authority) singular vectors of the adjacency matrix W
  - w = right singular vector

Properties of Interest

- Stability
  - small changes in the graph should cause small changes in the output of the algorithm
- Similarity
  - the output of two algorithms are close

Under what conditions (for which classes of graphs) is an algorithm stable, or are two algorithms similar?

- Axiomatic characterizations

Distance between LAR vectors

- Geometric distance: how close are the numerical weights of vectors w_1, w_2?

$$d_2(w_1, w_2) = \sqrt{\sum \left| w_1[i] - w_2[i] \right|^2}$$

- Assumption: Weights are normalized under norm L_2
  - normalization makes a difference

Distance between LAR vectors

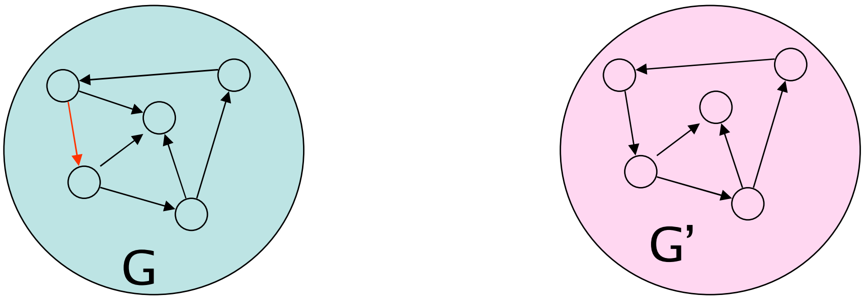
- Rank distance: how close are the ordinal rankings induced by the vectors w_1, w_2?
  - Kendal's τ distance

$$d_r(w_1, w_2) = \frac{\text{pairs ranked in a different order}}{\text{total number of distinct pairs}}$$

Stability: graph distance

- Definition: Link distance between graphs G=(P,E) and G'=(P,E')

$$d_{\ell}(G,G') = |E \cup E'| - |E \cap E'|$$



$$d_{\ell}(G,G') = 2$$

Stability

- $C_k(G)$: set of graphs G' such that $d_{\ell}(G,G') \leq k$

- Definition: Algorithm A is stable if for any fixed k

$$\max_{G \in G_n} \max_{G' \in C_k(G)} d_2(A(G), A(G')) = o(1)$$

- Definition: Algorithm A is rank stable if for any fixed k

$$\max_{G} \max_{G' \in C_k(G)} d_r(A(G), A(G')) = o(1)$$

Stability: Results

- InDegree is (rank) stable on G_n [BRRT05]
- HITS, PageRank, are (rank) unstable on G_n

Perturbations of PageRank

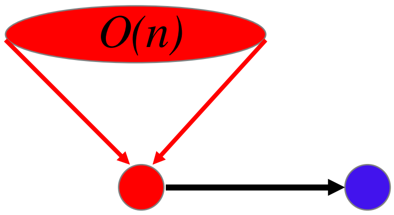
- Perturbations to unimportant nodes have small effect on the PageRank values [NZJ01][BGS03]

$$d_1(A(G), A(G')) \leq \frac{2\alpha}{1-2\alpha} \sum_{i \in P} A(G)[i]$$

- Lee and Borodin 2003: PageRank is stable
  - HITS remains unstable

Instability of PageRank

- PageRank is unstable



- PageRank is rank unstable [Lempel Moran 2003]
- Open question: Can we derive conditions for the stability of PageRank in the general case?

Singular Value Decomposition

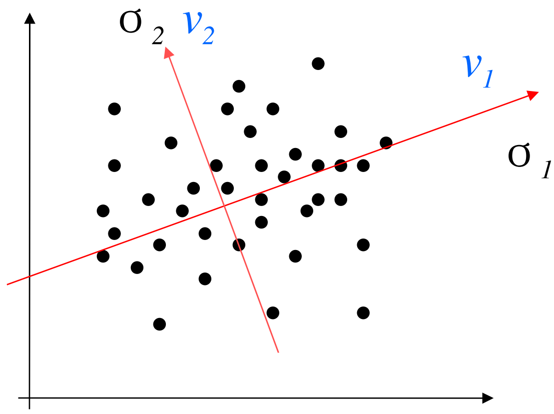
$$A = U \quad \Sigma \quad V^T = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix} \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_r \end{bmatrix}$$

$[n\times r]\ [r\times r]\ [r\times n]$

- **r** : rank of matrix A
- $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r$: singular values (square roots of eig-vals AA^T, A^TA)
- $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_r$: left singular vectors (eig-vectors of AA^T)
- $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_r$: right singular vectors (eig-vectors of A^TA)
- $A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$

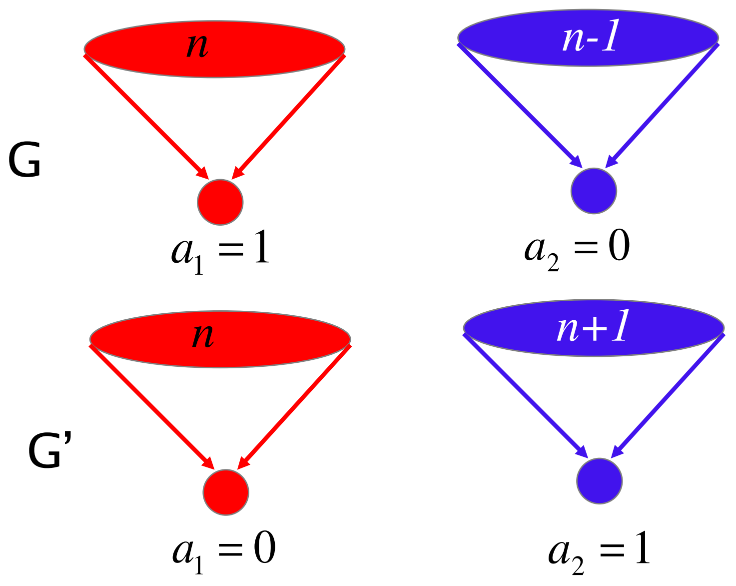
Singular Value Decomposition

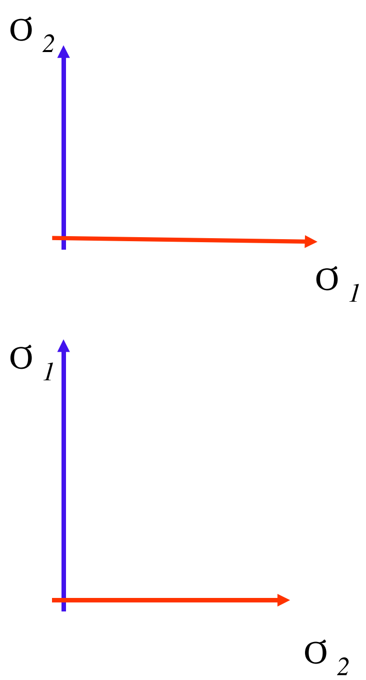
- Linear trend **v** in matrix A:
  - the tendency of the row vectors of A to align with vector **v**
  - strength of the linear trend: A**v**
- SVD discovers the linear trends in the data
- $\mathbf{u}_i\mathbf{v}_i^T$: the i-th strongest linear trend
- σ_i : the strength of the i-th strongest linear trend



- HITS ranks according to the strongest linear trend $\mathbf{v}_i$ in the authority space

Instability of HITS





Eigengap $\sigma_1 - \sigma_2 = 1$

Stability of HITS

- Theorem: HITS is stable if

$$\sigma_1(W) - \sigma_2(W) = \omega(1)$$

  - The two strongest linear trends are well separated

- [Ng, Zheng, Jordan 2001]: HITS is stable if

$$\sigma_1^2 - \sigma_2^2 = \omega\left(\sqrt{d_{out}}\right)$$

Similarity

- Definition: Two algorithms A_1, A_2 are similar if

$$\max_{G \in G_n} d_2\left(A_1(G), A_2(G)\right) = o(1)$$

- Definition: Two algorithms A_1, A_2 are rank similar if

$$\max_{G \in G_n} d_r\left(A_1(G), A_2(G)\right) = o(1)$$

- Definition: Two algorithms A_1, A_2 are rank equivalent if

$$\max_{G \in G_n} d_r\left(A_1(G), A_2(G)\right) = 0$$

Similarity: Results

- No pairwise combination of InDegree, HITS, PageRank algorithms is similar, or rank similar on the class of all possible graphs G_n

- Can we get better results if we restrict ourselves to smaller classes of graphs?
  - We focus on simialrity of HITS and InDeggree algorithms [DLT05]

Product Graphs

- Latent authority and hub vectors a, h
  - h_i = probability of node i being a good hub
  - a_j = probability of node j being a good authority

- Generate a link $i \rightarrow j$ with probability $h_i a_j$

$$W[i,j] = \begin{cases} 1 & \text{with probability } h_i a_j \\ 0 & \text{with probability } 1 - h_i a_j \end{cases}$$

  - Azar, Fiat, Karlin, McSherry, Saia 2001, Michail, Papadimitriou 2002,Chung, Lu, Vu 2002

- The class of product graphs G_n^p
  - (a.k.a. “graphs with given expected degree sequences”)

Product Graphs

$$W = M + R$$

- M: rank-1 matrix ha^T

$$M = \vec{h}\vec{a}^T = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} = \begin{bmatrix} h_1a_1 & h_1a_2 & \cdots & h_1a_n \\ h_2a_1 & h_2a_2 & \cdots & h_2a_n \\ \vdots & \vdots & \ddots & \vdots \\ h_na_1 & h_na_2 & \cdots & h_na_n \end{bmatrix}$$

- R: rounding matrix

$$R[i,j] = \begin{cases} 1 - h_ia_j & \text{with probability } h_ia_j \\ -h_ia_j & \text{with probability } 1 - h_ia_j \end{cases}$$

Product Graphs

- Idea[AFK+01]: View the product graph W=M+R as a pertubation of the rank-1 matrix M by the matrix R
- HITS and InDegree are identical on rank-1 matrix M
- How do the outputs change after perturbing M by R ?

HITS and InDegree on Product Graphs

- Theorem: HITS and InDegree are similar with high probability on the class of product graphs, G_n^p subject to some assumptions

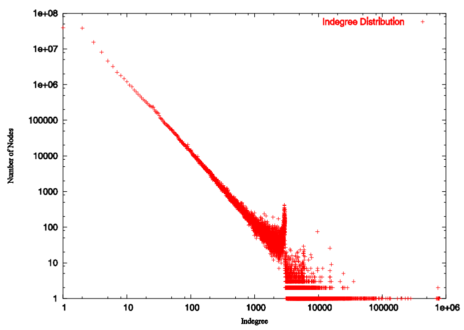
Assumption 1: $\sigma_1(M) = \|h\|_2 \|a\|_2 = \omega(\sqrt{n})$

Assumption 2: Let $H = \sum hi$ then $H\|a\|_2 = \omega(\sqrt{n}\log n)$

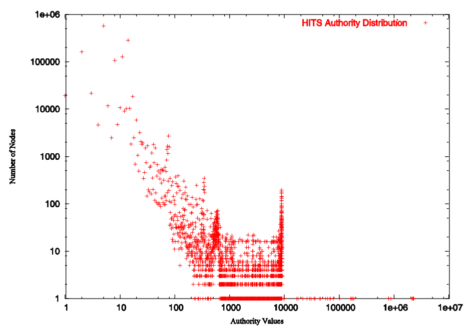
- Assumptions 1 and 2 are general enough to include graphs with (expected) degrees that follow power law distribution with $\alpha > 3$

Experiments with real web graphs

- Dataset: The Stanford WebBase project
- Correlation coefficient between authority and in-degree vector: 0.93



in-degree distribution



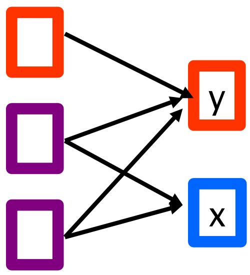
HITS authority values distribution

- Correlation coefficient between hub and out-degree vectors: 0.05

Monotonicity

- Monotonicity: Algorithm A is strictly monotone if for any nodes x and y

$$B_N(x) \subset B_N(y) \Leftrightarrow A(G)[x] < A(G)[y]$$



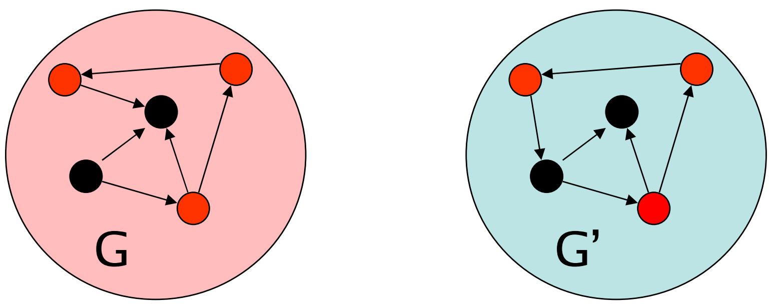
$$w_x < w_y$$

Locality

- Locality: An algorithm A is strictly rank local if, for every pair of graphs G=(P,E) and G'=(P,E'), and for every pair of nodes x and y, if $B_G(x)=B_{G'}(x)$ and $B_G(y)=B_{G'}(y)$ then

$$A(G)[x] < A(G)[y] \Leftrightarrow A(G')[x] < A(G')[y]$$

  - the relative order of the nodes remains the same if their back links are not affected



- The InDegree algorithm is strictly rank local

Label Independence

- Label Independence: An algorithm is label independent if a permutation of the labels of the nodes yields the same permutation of the weights
  - the weights assigned by the algorithm do not depend on the labels of the nodes

Axiomatic characterization of the InDegree algorithm

- Theorem: Any algorithm that is strictly rank local, strictly monotone and label independent is rank equivalent to the InDegree algorithm

- All three properties are needed

Other work

- An axiomatic characterization of PageRank algorithm
  - "Ranking Systems: The PageRank axioms" Alon Altman, Moshe Tenneholtz, ACM Conference on Electronic Commerce, 2005

Open questions

- What is the necessary condition for the stability of the HITS algorithm?
  - can the results of [NZJ01] be proven for 0/1 matrices?
- Can we say anything about other LAR algorithms on product graphs?
  - e.g. PageRank
- Can we prove anything when we consider rank distance?
- Can we define other properties?
  - e.g., is spam sensitivity different from stability?

Thank you!