The Inner Product

(Many slides adapted from Octavia Camps and Amitabh Varshney)

Much of material in Appendix A

Goals

- Remember the inner product
- See that it represents distance in a specific direction.
- Use this to represent lines and planes.
- Use this to represent half-spaces.

Vectors

- Ordered set of numbers: (1,2,3,4)
- Example: (*x*,*y*,*z*) coordinates of pt in space.

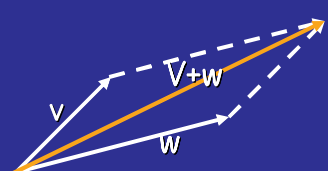
$$v = (x_1, x_2, \dots, x_n)$$

$$\|v\| = \sqrt{\sum_{i=1}^{n} x_i^2}$$

$$\text{If } \|v\| = 1, v \text{ is a unit vector}$$

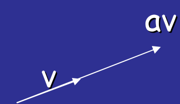
Vector Addition

$$\mathbf{v} + \mathbf{w} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

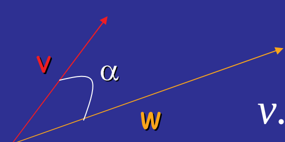


Scalar Product

$$a\mathbf{v} = a(x_1, x_2) = (ax_1, ax_2)$$



Inner (dot) Product



$$v.w = (x_1, x_2).(y_1, y_2) = x_1 y_1 + x_2.y_2$$

The inner product is a SCALAR!

$$v.w = (x_1, x_2).(y_1, y_2) = \| v \| \cdot \| w \| \cos\alpha$$

$$v.w = 0 \Leftrightarrow v \perp w$$

First, we note that if we scale a vector, we scale its inner product. That is, <sv,w> = s<v,w>. This follows pretty directly from the definition.

This means that the statement <v,w> = ||v|| ||w|| cos(alpha) is true if and only if it is the case that when v and w are unit vectors, <v,w> = cos(alpha), because:

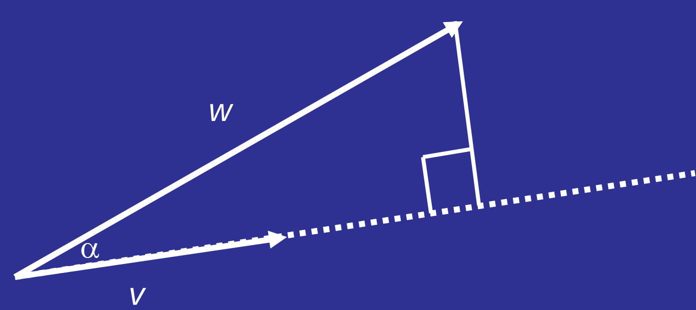
<v,w> = <(v/||v||),(w/||w||)> ||v|| ||w||. So from now on, we can assume that w, v are unit vector.

Then, as an example, we can consider the case where w = (1,0). It follows from the definition of cosine that <v,w> = cos(alpha). We can also see that taking <v,(1,0)> and <v,(0,1)> produces the (x,y) coordinates of v. That is, if (1,0) and (0,1) are an orthonormal basis, taking inner products with them gives the coordinates of a point relative to that basis. This is why the inner product is so useful. We just have to show that this is true for any orthonormal basis, not just (1,0) and (0,1).

How do we prove these properties of the inner product? Let's start with the fact that orthogonal vectors have 0 inner product. Suppose one vector is (x,y), and WLOG x,y>0. Then, if we rotate that by 90 degrees counterclockwise, we'll get (y, -x). Rotating the vector is just like rotating the coordinate system in the opposite direction. And (x,y)*(y,-x) = xy – yx = 0.

Next, note that if w1 + w2 = w, then v*w = v*(w1+w2) = v*w1 + v*w2. For any w, we can write it as the sum of w1+w2, where w1 is perpendicular to v, and w2 is in the same direction as v. So v*w1 = 0. v*w2 = ||w2||, since v*w2/||w2|| = 1. Then, if we just draw a picture, we can see that cos alpha = ||w2|| = v*w2 = v*w.

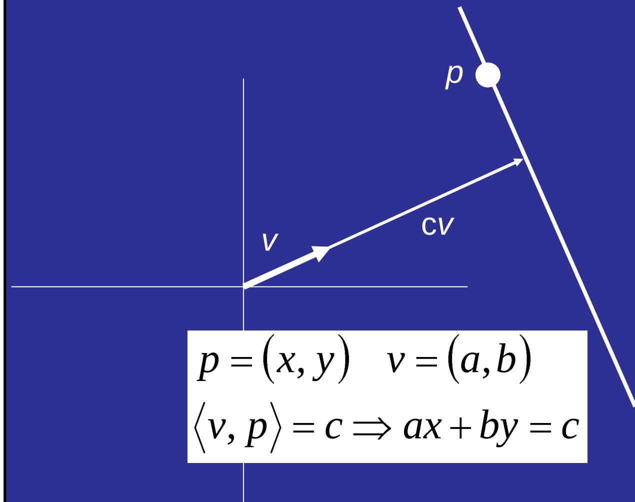
Inner product and direction



This tells us that if v is a unit vector (and w isn't) that

<v,w> = ||w|| cos(α). This is the *projection* of *w* onto *v.* It means that to get to *w*, we go a distance of <v,w> in the direction v, and then some distance in a direction orthogonal to v.

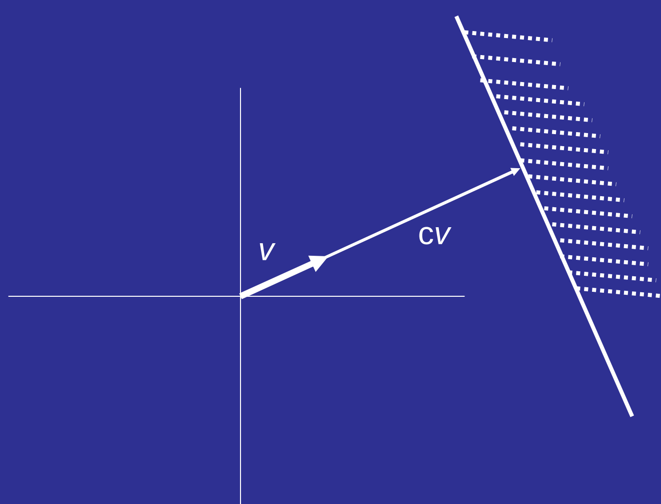
2D Lines



$$p = (x, y) \quad v = (a, b)$$
$$\langle v, p \rangle = c \Rightarrow ax + by = c$$

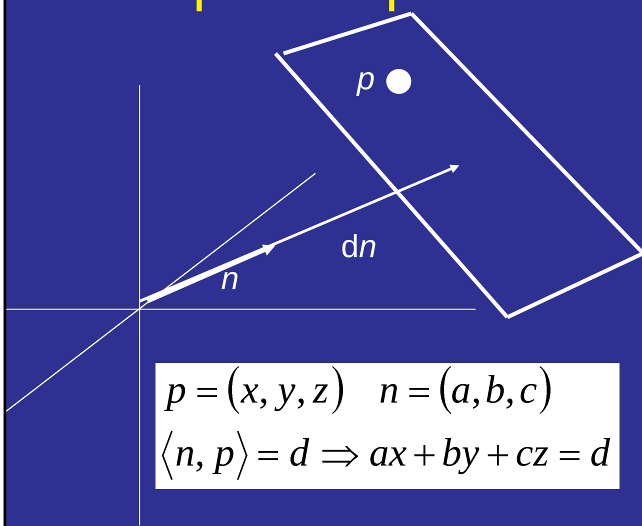
Consider any line. Suppose *v=(a,b)* is a unit vector in the direction orthogonal to it. Then we can describe any point, *p=(x,y)*, on the line by saying we go a fixed distance c in the direction *v,* and then some distance orthogonal to *v.* So, <*v,p*> = c and the equation for a line is: *ax+by=c*

2D Half-spaces



A line divides the plane in two halves. If we go less than c in the direction *v,* we are in one half-space. More than c, we cross the line and enter the other half space. So a half-space is defined by: *ax+by < c*

Implicit Equation of a Plane



$$p = (x, y, z) \quad n = (a, b, c)$$
$$\langle n, p \rangle = d \Rightarrow ax + by + cz = d$$

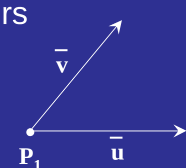
Likewise, we reach any point in a plane by going a distance d in a direction *n=(a,b,c)* that is perpendicular to it, and then moving within the plane. *n* is orthogonal to any vector in the plane.

Normal of a Plane

Plane: sum of a point and two vectors

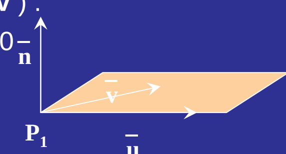
$\mathbf{P} = \mathbf{P}_1 + \alpha\, \bar{\mathbf{u}} + \beta\, \bar{\mathbf{v}}$

$\mathbf{P} - \mathbf{P}_1 = \alpha\, \bar{\mathbf{u}} + \beta\, \bar{\mathbf{v}}$



If $\bar{\mathbf{n}}$ is orthogonal to $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ ($\mathbf{n} = \bar{\mathbf{u}} \times \bar{\mathbf{v}}$) :

$\bar{\mathbf{n}}^{T} \bullet (\mathbf{P} - \mathbf{P}_1) = \alpha\, \bar{\mathbf{n}}^{T} \bullet \bar{\mathbf{u}} + \beta\, \bar{\mathbf{n}}^{T} \bullet \bar{\mathbf{v}} = 0$



The Cross-Product

- (a,b,c)x(d,e,f) = (bf-ce, cd – af, ae-bd)
- Verify <(a,b,c)x(d,e,f), (a,b,c)> =

(abf-ace+bcd-baf+cae-cbd) = 0.

- Similar for <(a,b,c)x(d,e,f), (d,e,f)>
- Direction obeys right-hand rule.
- Length v x w = ||v|| ||w|| sin(θ)

3D Half-spaces

- Similar to 2D with lines.
- Plane divides space into two parts.
- In one part, we go less than d in direction *n,* in other part we go more than d.
- $ax + by + cd < d$, $ax + by + cz > d$

3D Lines

- There are two direction orthogonal to line.
- Move some amount in each direction to get to line, then any amount in 3^{rd} direction orthogonal to both of these.
- *a1x + b1y + c1z = d1 & a2x + b2y + c2z = d2* (Two equations with three unknowns).
- Equivalently, a line is the intersection of two planes.
- Or: start at some point, *p=(x0,y0,z0),* on the line, and move in the tangent direction *(a,b,c)* by some distance t:

(x,y,z) = (x0, y0, z0) + t(a,b,c) (Three equations with four unknowns.

Points

Using these facts, we can represent points. Note:

(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)

x = (x,y,z).(1,0,0)   y = (x,y,z).(0,1,0)

z = (x,y,z).(0,0,1)