Kjell Sandved/PhotoResearchers, Inc.

Figure 1. Nature's geometries include wavy edges that sometimes assume the complex shapes called fractals, where a pattern repeats on different scales. One family of such patterns includes the complex wavy structures that are found along the edges of thin living tissues. The origin of this complexity turns out to be one of the more tractable problems in biological pattern formation. By applying simple growth laws and principles from physics and geometry and testing their ideas with flexible synthetic membranes, the authors have found

Leaves, Flowers and Garbage Bags: Making Waves

Rippled fractal patterns on thin plastic sheets and biological membranes offer elegant examples of the spontaneous breaking of symmetry

Eran Sharon, Michael Marder and Harry L. Swinney

The emergence of patterns is one of the world's most durable mysteries. Some patterns—clouds, snowflakes—form in space. Others—the ebb and flow of tides, seasonal wet and dry spells—are patterns that form in time. Natural patterns are mysterious because they are complex, organized and interconnected, even though the laws of physics on which they rest—Newton's classical laws of motion—are simple.

The living world presents the ultimate examples of pattern formation. The patterns in biological systems are the most stunningly complex of any we encounter. Consider: In order to form a complex organism from an initial featureless collection of identical cells, a system must undergo myriad transitions that break its spatial symmetries and trigger the differentiation of cells at selected sites. How are these sites selected? How complex and controlled must a growth process be to direct that particular things happen in sequence and at the right sites?

It is difficult to imagine how the impersonal interactions of atoms can lead to the growth of a plant or an animal from inanimate matter. Yet in fact this is what happens with the birth and development of every living thing. Some of the simplest features of biological shapes can be explained by basic physical laws. We will describe here an elegant example: the edges of flowers and leaves, where complex rippled shapes give the impressions of ruffles and frills. We suspected that very simple growth processes might provide the mechanism that shapes thin membranes and sheets into complex shapes in space, and indeed we have found that they do. By themselves, these processes do not break

Yva Momatiuk and John Eastcott/PhotoResearchers, Inc.

that a leaf or flower—just like a torn sheet of plastic—can use an enhanced, uniform growth at its margins to generate such complex patterns. Examples of wavy edges in nature include, from left to right, some lichens (shown, *Sticta limbata*), orchids (shown, *Schomborgkia beysiana*), sea slugs (represented by *Glossodoris hikuerensis*) and ornamental cabbage. (Lichen photograph courtesy of Stephen Sharnoff; sea slug photograph courtesy of Jeff Jeffords.)

any symmetry. Instead, the complex patterns emerge from the elastic and geometric properties of the thin membranes of which the flowers and leaves are constructed.

Spontaneous Symmetry Breaking

One of the main concepts used to explain how complex patterns can be teased from simple laws is *spontaneous symmetry breaking*. Symmetry breaking is significant in almost every field of physics, but it is especially important in searching for the origin of patterns.

To define spontaneous symmetry breaking, we first must define symmetry. A two-dimensional object is symmetrical if you can pick it up, move it or rotate it and place it in a new location, and then find that the resulting pattern is a perfect overlay of the pattern that was present before you began. An example appears in Figure 2.

The most symmetrical pattern of all is one that is featureless and uniform—a void. Empty space is symmetrical in this way, and the equations of physics are too. The equations are indifferent to where objects are located in space. Objects can be anywhere or nowhere, and the laws of physics will apply to them.

Spontaneous symmetry breaking happens whenever equations that are featureless and uniform have solutions that are not. More generally, spontaneous symmetry breaking describes any case where the solutions of equations have less symmetry than the equations themselves.

Here is an example. Imagine that you have picked up a thin plastic ruler. Ignoring the marks and labels on the ruler, you can think of it as uniform and featureless in the horizontal direction. Now grab the ruler at its two ends and gently press inward. The stresses within the ruler are distributed uniformly within it, and it is still uniform and featureless in the horizontal direction. However, as you compress the ruler more and more, it will eventually give way and buckle.

This buckling is a spontaneous breaking of symmetry. At all interior points away from your fingers, the ruler used to be flat and patternless. Under compression, a solitary half of a horizontal wave suddenly emerges; the symmetry in the direction perpendicular to the ruler's original plane has been broken.

Because buckling will be very important for understanding the shapes we will discuss later, we should describe it in a bit more detail. As you press the ruler inward from its two ends by a given amount, it must decide between deforming in two different ways. It can deform simply by compressing in the horizontal direction—squeezing, like a spring—without breaking any symmetry *(see Figure 3)*. In this configuration the energy of the ruler is proportional to its thickness, which we'll denote as t.

When buckling sets in, the ruler deforms mainly by bending. In this type of deformation, the ruler breaks the orthogonal symmetry. It uses the third

Eran Sharon is a member of the nonlinear dynamics group at the Racah Institute of Physics at The Hebrew University of Jerusalem, where he received his Ph.D. in physics. His research interests include the spontaneous formation of structured fluid flows as well as the origins of mechanical instabilities and their role in the growth of plants. This article was developed from a talk Sharon delivered at the Kavli Institute of Theoretical Physics at the University of California, Santa Barbara. Michael Marder and Harry L. Swinney are professors of physics in the Center for Nonlinear Dynamics at the University of Texas at Austin, where Sharon was formerly a research associate. Marder is also co-director of UTeach, one of the nation's largest programs preparing secondary, math, science and computer science teachers. His research focuses on the mechanics of solids, particularly the fracture of brittle materials. He is author of Condensed Matter Physics *(Wiley). Swinney is director of UT Austin's Center for Nonlinear Dynamics. He studies how ordered and chaotic spatial patterns arise and evolve in fluid flows, chemical systems and granular media. He is a member of the National Academy of Sciences. Address for Sharon: The Racah Institute of Physics, The Hebrew University of Jerusalem, Givat-Ram, Jerusalem 91904, Israel. Internet: erans@vms.huji.ac.il*

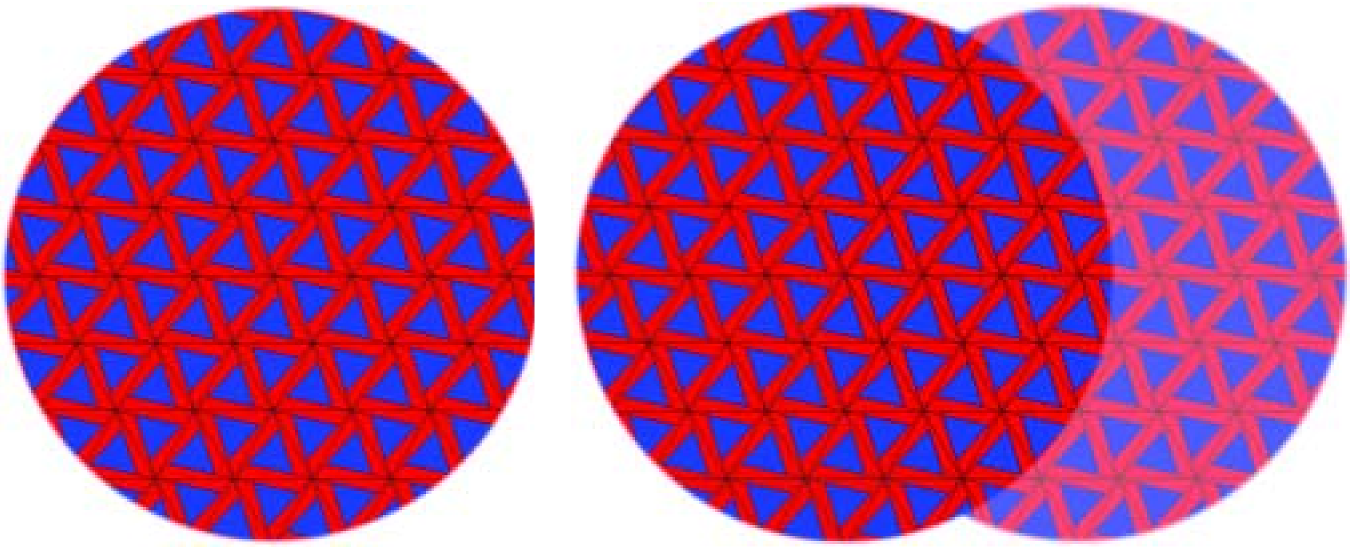


Figure 2. Physical laws are symmetrical; they are indifferent to where objects are located in space. In order for a physical process to produce a complex pattern, symmetry breaking must take place. Symmetry is illustrated by the example above: The object at left is symmetric under rotations by 120 degrees. When it is rotated through 120 degrees and laid down on itself, the original object and rotated object match perfectly *(right)*. Symmetry is spontaneously broken whenever equations that are featureless and uniform produce solutions that are not symmetrical.

dimension to fulfill the displacement on its boundaries, while preserving its length, *L*, along its center plane. In this case its energy is proportional to t^3. If we consider rulers of smaller and smaller thickness, we note that "t^3" decreases much faster than does "t." Physical objects follow the path of lowest energy; thus, at a small enough

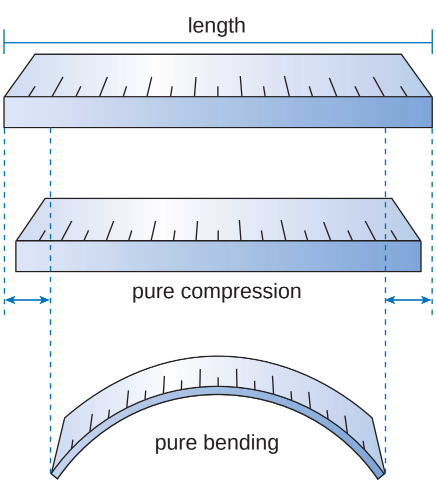


Figure 3. Buckling is an example of symmetry breaking. When a ruler is pressed inward from the ends, it first absorbs the displacement by compressing in its plane. Then it gives way, breaks the up-down symmetry and buckles, while hardly changing its length. Under in-plane compression, the elastic energy of the ruler is proportional to its thickness. The bending energy is different: It is proportional to the cube of the thickness. Very thin objects such as the pages in this magazine therefore must buckle under compression, rather than significantly alter their length.

thickness, the buckled state becomes energetically favorable. Indeed, when considering very thin objects such as the sheets of paper bound together to make this magazine, it becomes clear that under compression the sheet "must" buckle, while hardly changing its length.

Permanent Buckling

A bent plastic ruler is not much like a leaf or a flower. As soon as you let go of it, the ruler snaps back to its original flat shape. But it is easy to carry out another simple household experiment where spontaneous symmetry breaking leads to a permanent and much richer pattern.

Take a garbage bag or other thin sheet of plastic. Cut out a square 15 centimeters (six inches) on a side. Cut a slit into one side, about one centimeter long. Now grab the plastic on either side of the slit and pull it apart, slowly ripping the plastic into two pieces. You should see the torn edge begin to curl up into a pattern of waves upon waves. It is tempting to think that the complex wavy pattern results from small variations in the pull your hands exert in ripping apart the plastic. However, this is not the case.

Figure 5 shows a close-up of a piece of plastic while it is being ripped in a carefully controlled laboratory setting. Our ruler was made of a rigid material, but the garbage bag is a flexible membrane. The plastic stretches permanently in the vicinity of the point where it tears. But if you keep the tip of the crack at the center of your field of view, you will see that the amount of deformation is constant as the tear moves through the plastic. Along the direction of the tear's progress, the plastic deforms in a completely symmetrical way. The rippled pattern that emerges is a new example of spontaneous symmetry breaking. The symmetry is broken in the perpendicular direction, as in the ruler's buckling, but there is a new, additional symmetry breaking in the direction of the propagation of the crack.

Let's look at the pattern more closely. Figure 6 shows photographs of the edge of a piece of plastic. This particular piece of plastic was 0.12 millimeter (8 one-thousandths of an inch) thick, and because it was so thin, it was highly susceptible to buckling. The top image shows a region 30 millimeters (a little over an inch) across. Now take

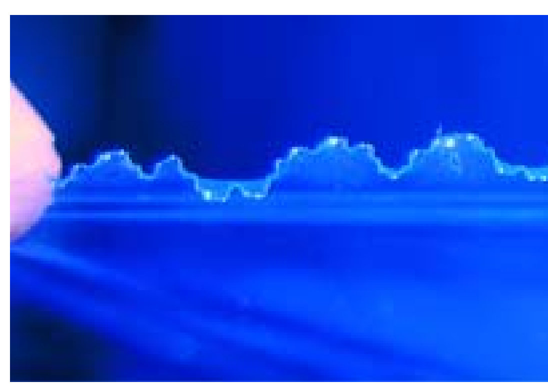


Figure 4. Buckling cascades can easily be produced with household garbage bags. Here a

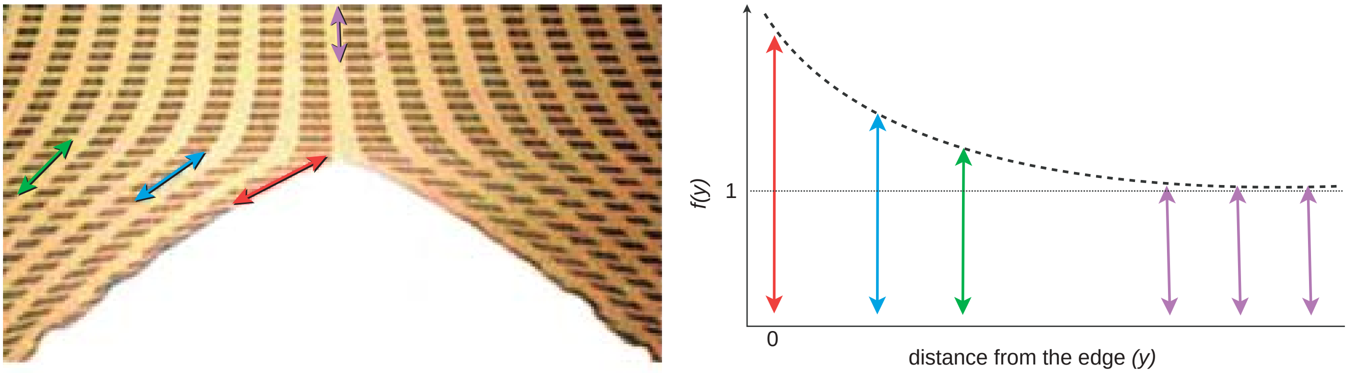


Figure 5. Buckling similar to that seen in biological membranes is observed at the edge of a plastic sheet as it is torn (*Figure 4*). To understand this buckling, the authors printed a grid of dots on a plastic sheet and measured the distances between the dots after tearing (*above left*, crack propagating upward). The distance between dots in the direction in which the crack was propagating changed in an irreversible way as the sheet was tearing. This elongation, demonstrated by the lengths of the colored arrows, was found to depend only on the distance from the edge and to increase at a steeper and steeper rate approaching the edge. In the graph at right, the function that describes the elongation expresses the new metric of the sheet. This metric requires a new geometry; it cannot be accommodated within the sheet's Euclidean (flat) geometry but only on a surface with a negative Gaussian curvature, on which every point is a saddle-like point. (Photograph courtesy of the authors.)

approximately one-third of the image, the part enclosed in a box on the left-hand side, and magnify it by a factor of 3.2. The result is printed just below the original. Remarkably, the magnified image looks nearly identical to the original edge. But the process does not end there. The magnified image can be magnified again, and the result again, and again, and again, and again, each time producing essentially the same pattern. This property of the pattern—the fact that it looks the same after successive magnifications—means that the edge of the piece of plastic can be called a *fractal*.

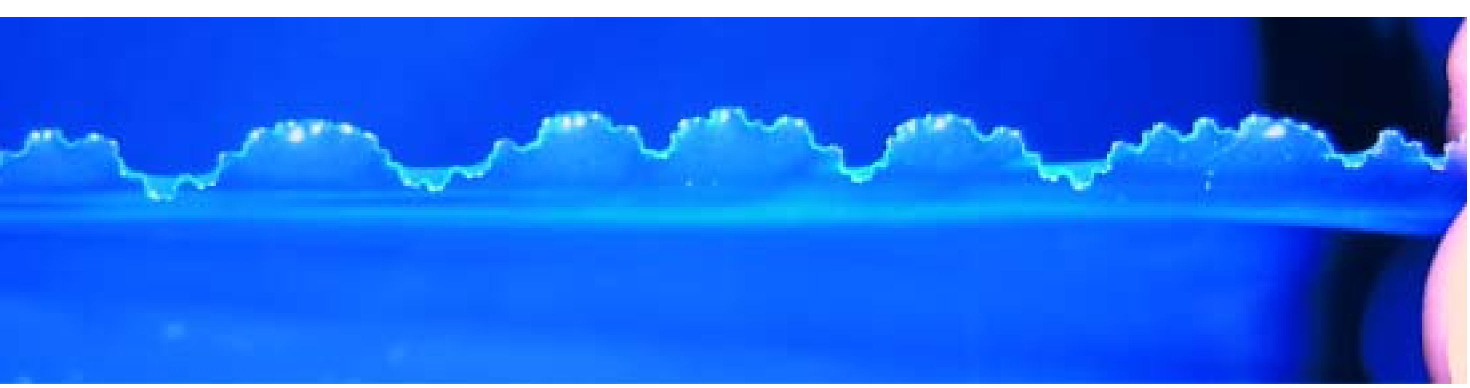
The Role of Metrics

The deformation of the plastic in our experiment was uniform and symmetrical, yet it led to something with an extra dimension of complexity beyond normal buckling—a fractal pattern. What is the key feature of the deformation that leads to the pattern? The answer is the *metric*.

When the metric on a sheet changes, this means that distances on the sheet change. The way this happens is illustrated in Figure 5. The violet arrow shows the distance between two points on the plastic before the crack arrives. Beyond the point where the sheet rips, the distance between these same two points increases, because of stretching, to the value indicated by the red arrow. This increase in length is permanent and remains after the pulling process stops and the sheet is laid down. We say that the metric of the sheet in the tearing direction has increased.

Notice also in the picture that if you keep an eye on the material at the locations of the blue and green arrows, you see that the stretching farther from the tear is less. Apparently, the increase in the metric is not uniform. To quantify this we plot $f(y)$, a function showing how the metric increases as a function of y, the distance from the edge. Far from the edge, where no irreversible deformation occurred, the metric was not changed, so $f = 1$. Within the zone of irreversible deformation, though, $f(y)$ increases at an accelerating pace approaching the edge. The tearing process has provided the sheet with a new metric. This metric reflects the fact that the sheet's edge is now "too long." Like the thin ruler, it has buckled out of the plane—but here the compression comes from the expansion of a flexible membrane, the material's ability to change its metric. We suggest that the observed fractal cascade of waves upon waves upon waves is the configuration that minimizes the energy of the elongated sheet.

The metric describes distances on a surface, but it does much more. In one of the most fundamental theorems of differential geometry—the so-called Theorema Egregium—Carl Friedrich Gauss showed that the metric of a surface defines its shape in space. The shape of the function $f(y)$ defines the *Gaussian curvature* of the surface, which determines whether the local topography at y will

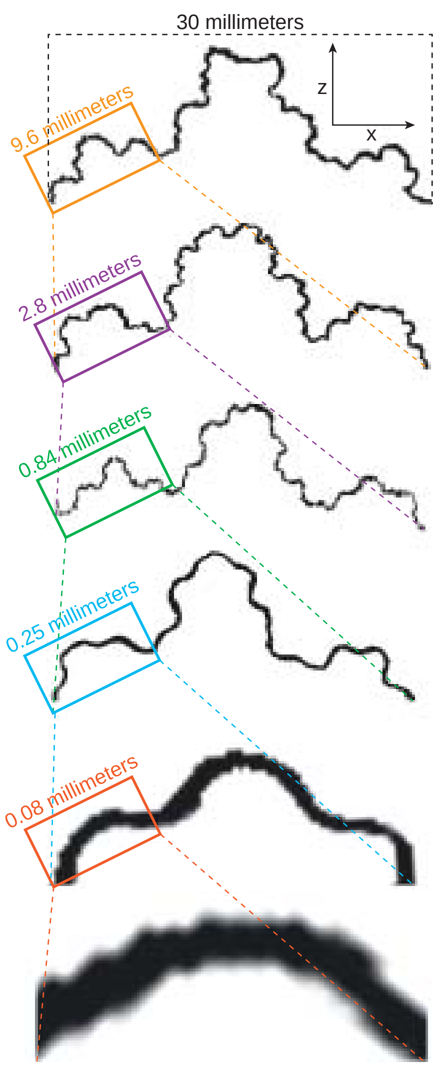


thin plastic bag has been torn. Its edges are far from featureless; they exhibit a rich pattern of waves within waves within waves. What is the principle that forces the sheet to select such a complex pattern—waves at various scales—as its energy minimum? (Photograph by Eran Sharon.)

be flat (zero Gaussian curvature), curved like a top of a hill (positive Gaussian curvature) or will have a saddle-like shape (negative Gaussian curvature). We find that whenever f gets steeper toward the edge, the sheet *cannot* be flat! It must have a saddle-like shape at every point within the entire deformed region close to the edge. We now understand why these sheets buckle spontaneously and permanently: Their new metric is not flat, and so their shape must include curves. The distances between the dots on the surface after tearing cannot be met if the sheet is flat. Then, to avoid the expensive compression energy, the sheet happily pays cheap bending energy, as it buckles out of the plane, while trying to generate saddle points everywhere.

But where does the complexity expressed in the fractal patterns come from? Why would a sheet with such a featureless metric adopt such a complex shape? Can't it find a simpler one? Is it the best it can do? Apparently it is!

We live in ordinary Euclidean space, described by three linear dimensions. This geometry places severe limitations on the possible shapes that can live within it. In Euclidean space, it is very difficult and in fact impossible to find a simple surface, connected to the flat part of the sheet, that has saddle-like points everywhere. If our sheets were placed in another space—say, one with four dimensions—they might have adopted a featureless shape. But in our ordinary world they are "compressed" by space itself; they are forced to break the symmetry again and again and to form complex structure. Actually, without the ability to do experiments with sheets, which minimize energy while using their floppiness, it would be very difficult to guess that surfaces with such simple metrics must be so complicated.



Feature 6. Photographs of a torn plastic sheet's deformed edge show the complexity that characterizes its buckling cascade. Here, the first image has a width of 30 millimeters. In the second image, a 9.6-millimeter section is enlarged; this and successive images, magnified by a constant ratio of 3.2, show that the same pattern is repeated at multiple scales, an example of a pattern that is fractal. Fractal patterns are typically produced by dynamical processes that are nonlinear. (Images courtesy of the authors.)

Time to Leave

In nature, the edge of a plant leaf can be either smooth or wavy. If you look at the edge of a wavy leaf, you might notice that there are visual similarities between the leaf edge and the edges of torn plastic sheets. This suggested to our group that the reasons for the buckling of leaf edges might be similar. We carried out experiments in order to discern whether the shape similarities are coincidental—or whether, by changing the metric of a leaf near its edge, we might create buckling patterns as we did in plastic.

Fortunately biologists know enough about plant growth that we could enlist the plant's own chemistry in demonstrating how physical and mathematical laws might apply to leaf-edge buckling. Rather than creating new metrics in plants by tearing them, we created new metrics in plants using the growth of cells. Eggplant leaves are normally flat and smooth. We found that we could create wavy-edged leaves by applying the growth-regulating plant hormone auxin (indoleacetic acid). We applied auxin along a thin strip to the edges of the leaves to increase the rate of tissue growth, expecting the leaves to grow more along their edges where the auxin had been placed. We waited to see: What shape would the leaf adopt?

Indeed, after few days of treatment with auxin, waviness appeared along the edges of the leaves. The waviness had nothing to do with the leaves' vein structures. The amplitudes of the waves increased continuously over the course of the experiment. We were able to show that a symmetrical pattern of growth, one that is nearly uniform at the edge of the leaf, can lead to short-wavelength buckling.

Of course, in our experiment the growth rate was not the same across the entire leaf. Gauss's rules tell us that whenever there is a difference in growth rates between a leaf's edge and center, buckling should be expected.

In fact, given the opportunities for buckling, it is much more puzzling that leaves could be flat than that they might display rippled edges. The expression of genes in plant growth appears to serve as a powerful regulator of the proliferation and then expansion of cells during leaf growth.

Utpal Nath (now at the University of Bombay) and his colleagues at the John Innes Centre in Norwich, U.K., demonstrated that the distribution of growth regions is genetically regulated across leaves. When this regulation mechanism is interrupted, leaves that would otherwise be flat grow and form curved surfaces. So genetic coding does affect leaf shape by controlling growth rates along the edges of the leaf but does not necessarily provide a map of sites where symmetry should be broken by the growth law.

A second demonstration of the symmetry breaking underlying leaf-edge buckling explores the intrinsic geometry of naturally wavy leaves. Can their shapes result from growth that is invariant along the edge? Take a leaf and carefully cut thin strips parallel to its edge. To see their geometry, flatten these strips between two glass plates. When this procedure is applied to flat leaves we see no surprises: The leaf is built of arcs whose diameter increases as you move outward. However, this is not so in the case of wavy leaves: We find that the diameter of the arcs gets smaller and smaller as the edge is approached. When waviness is more severe, we can find arcs of very small diameter near the edge.

Figure 7. Can a leaf that is normally flat be induced to become wavy? Here the growth hormone auxin is applied to the edge of a normally flat leaf from an eggplant, causing enhanced growth only near its margins. This growth imposes a negative Gaussian curvature on the leaf, similar to that in the torn plastic sheets in Figures 5 and 6. After 10 days of such a treatment, waves have developed; at 12 and 14 days the waves have grown bigger, and waves within waves become discernible. (Photographs courtesy of the authors.)

The observed curvature of the arcs when they are flattened is called the *geodesic curvature* along these lines—another property controlled by their metric. An important observation is that the geodesic curvature along the edges of the wavy leaf in Figure 8 is nearly constant. We do not see any big variations in this curvature that are correlated either with the vein structure or with the waviness of the leaf. The tissue along the edge grew nearly uniformly, the growth law was uniform, and the leaf grew as a simple leaf. Like the plastic sheets, it should have been a simple featureless leaf, but because of the geometrical limitations of space, it was forced to break the symmetry and to adopt a wavy shape.

Wrapping Up

Flowers, like leaves, form complex buckled shapes. Geometrically, the main difference between the two is that leaves form essentially from long, free-standing strips, whereas flowers have more complex geometries; the central tube of a daffodil, for example, closes on itself like a cylinder. What happens to such a cylinder or tube when we apply to it a metric that increases toward its edge? Just as the leaf grows from the center, we can think about "growing" such cylinders starting from a ring of cells and adding rings on top of one another. If the rings all have the same number of cells, they will have the same diameter and will form a cylinder. However, as the number of cells that form a ring grows exponentially upward, the metric of the cylinder increases also, leading to an increasing diameter of the cylinder in its upper part and to a trumpet-like shape.

As the metric of the flower increases, the edge of the flower splays outward more and more. Eventually, it splays out so much that the edge of the flower is perpendicular to the direction of the stem along which it is growing. It forms a circle with a radius we'll call R. That marks the end of this phase of flower growth. If cells continue to attach to the end of the flower, causing the metric to grow at an ever-steeper rate as the flower grows sideways, the perimeter of the edge will have to be longer than $2\pi R$. This is known to be impossible in our Euclidean space without breaking the axial symmetry. The edge of the flower must buckle.

In Figure 9a we show the result of an experimental study using thin tubes made of polyacrylamide gel. This gel changes its volume depending on its environment. It swells in water, but shrinks in acetone. We used this property to change the metric of the tube. First, we dipped the tube in acetone, causing it to shrink uniformly. Next we dipped one end of the tube in water, allowing the water to diffuse into the tube. As a result, the tube swelled, its local diameter dependent on the local water-to-acetone ratio. This ratio was high near the edge that was dipped into water and decreased away from the water, leading to a variation in the metric and to a trumpet-like shape.

In Figure 9b we show the result of a computer simulation of the same effect. The computer was instructed to create a rubber-like material and to make it expand on the left-hand side just like the experimental gel. The model results match the experimental observations well. When the transition between water and acetone happens over a short distance rather than gradually, the metric of the cylinder changes quickly (along *y*), and it is impossible for the cylinder to respond with a symmetrical trumpet-like shape. The edges of the cylinder buckle *(Figure 9c.)*

In 9d we display another computer model, where now the metric has been made to vary rapidly along the axis of the cylinder. The simulated tube displays a rippled wavy edge that resem-

Figure 8. Comparison of narrow strips cut from the edges of flat and wavy leaves *(top)* and flattened between glass plates *(bottom)* reveals the difference in the leaves' intrinsic geometries. The strips cut from the flat leaf *(left)* show the expected pattern of arcs, with the radius of the arcs increasing from the center. In contrast, strips from the edges of the wavy leaf *(right)* have arcs that are smaller than the inner strips. A geometry with a decreasing radius of curvature as the edge of the leaf is approached cannot exist within a plane; it requires a negative Gaussian curvature. the constant curvature along the length of each strip indicates a *uniform* growth at the edge of the leaf. The two leaves were collected from the same bay leaf bush. (Photographs courtesy of the authors.)

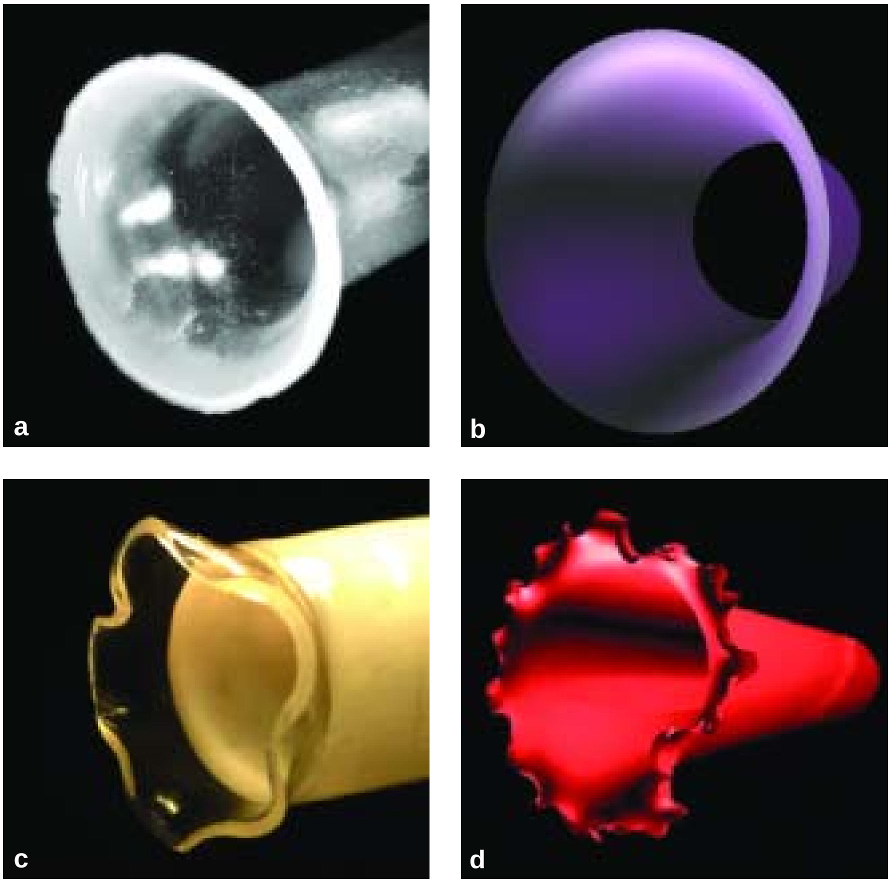


Figure 9. Tubes can deform along their edges just as sheets can. The authors modeled the growth of a cylindrical flower part by applying to a cylinder a metric that increases toward its edge. First they obtained thin tubes made of polyacrylamide gel, which swells in water but shrinks in acetone. By dipping a tube into acetone, then dipping the edge into water, they created a trumpet shape *(a)*. A computer simulation shows the shape *(b)*. By making the acetone-to-water transition take place over a short distance, causing the metric of the cylinder to increase steeply, they forced the gel tube to break the circular symmetry, to buckle and to produce a wavy edge *(c)*. This process, simulated in the computer, produced a trumpet shape with a complex wavy edge, like that of a daffodil *(d)*. (Images courtesy of the authors.)

Figure 10. Daffodil trumpets' edges display the same buckling behavior as the polyacrylamide tubes in Figure 9. This suggests that the entire complex, three-dimensional shape of the daffodil crown could result from a constant and uniform growth law of its tissue. Thus geometry and elasticity can produce a complex shape without need for complex genetic instructions. (Photograph by Eran Sharon.)

bles the daffodil. Thus the entire beautiful, complex three-dimensional shape of the crown of the daffodil could result from a constant and uniform growth law of its cells, which themselves do not break any symmetry. It is purely as a result of geometry and elasticity that wrinkles, of a selected wavelength, appear along the edge.

The central conclusion to draw from the idea of spontaneous symmetry breaking is that one hardly needs complex equations or complex conditions to produce complex shapes. We have shown how the buckled shapes of leaves and flowers can result from very simple deformations of sheets and cylinders. Uniform deformations can produce fractals.

Biological systems do not necessarily produce complex structures in simple ways. Genetic coding is also capable of producing complex structures, such as eyes and hands, through complex, detailed specification of where individual parts are to be located. However, in the general program relating biological pattern formation to physical law, it is comforting to have some cases where the patterns can be understood in elementary terms. Physics and biology meet at the rippled edges of leaves and flowers to provide one of these rare tractable problems.

Bibliography

Ball, P. 1998. *The Self-Made Tapestry: Pattern Formation in Nature.* Cambridge, U.K.: Cambridge University Press.

Cho, A. 2004. Life's patterns: No need to spell it out. *Science* 303:782–783.

Coen, E. 2000. *The Art of Genes: How Organisms Make Themselves.* Oxford, U.K.: Oxford University Press.

Mandelbrot, B. B. 1982. *The Fractal Geometry of Nature* (2nd edition). San Francisco: W. H. Freeman and Co.

Marder, M., E. Sharon, S. Smith and B. Roman. 2003. Theory of edges of leaves. *Europhysics Letters* 62:498–504.

Nath, U., B. C. W. Crawford, R. Carpenter and E. Coen. 2003. Genetic control of surface curvature. *Science* 929:1404–1407.

O'Neill, B. 1992. *Elementary Differential Geometry* (2nd edition). New York: Academic Press.

Sharon, E., B. Roman, M. Marder, G.-S. Shin and H. L. Swinney. 2002. Buckling cascades in thin sheets. *Nature* 419:579.

Stewart, I. 1992. F*earful Symmetry: Is God a Geometer?* New York: Blackwell Publishers.

Thompson, D. W. 1942. *On Growth and Form* (2nd edition). Cambridge, U.K.: Cambridge University Press.

For relevant Web links, consult this issue of *American Scientist Online:*

http://www.americanscientist.org/IssueTOC/issue/601