The Life of Pi

History and Computation

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Prepared for

AUSTRALIAN COLLOQUIA

June 21-July 17, 2003

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Revised: June 1, 2003

The Life of Pi

"My name is

Piscine Molitor Patel

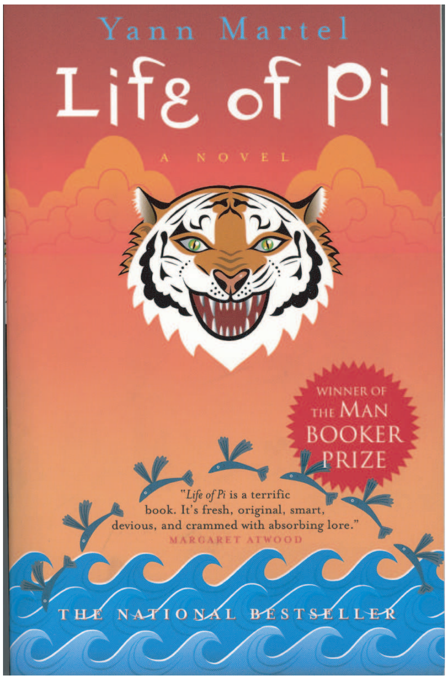
known to all as Pi Patel

For good measure I added

$$\pi = 3.14^*$$

and I then drew a large circle which I sliced in two with a diameter, to evoke that basic lesson of geometry."

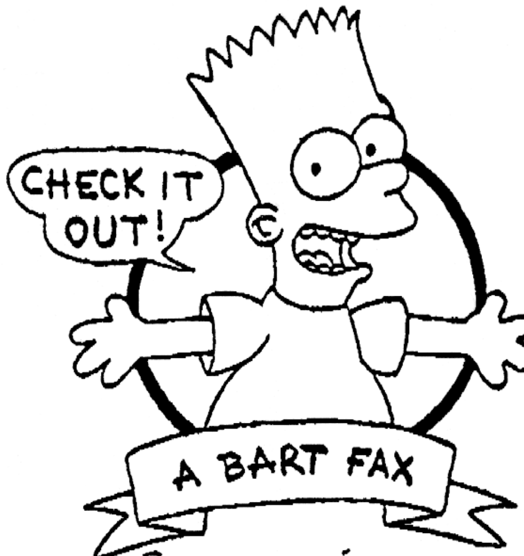
*The Notation of π was introduced by Euler in 1737.



Abstract. The desire, and originally the need, to calculate ever more accurate values of π, the ratio of the circumference of a circle to its diameter, has challenged mathematicians for many centuries and, especially recently, π has provided fascinating examples of computational mathematics. It is also part of *the popular imagination*.*

*The "MacTutor" website, at the University of St. Andrews — my home town in Scotland — `http://www-gap.dcs.st-and.ac.uk/~history` is rather a good history source.

The Simpsons



TO: DAVID BAILEY
FROM: JACQUELINE ATKINS
DATE: 10/9/92
NUMBER OF PAGES: 1

FAX (310) 203-3852
PHONE (310) 203-3959

A Professor at UCLA told me that you might be able to give me the answer to: What is the 40,000th digit of Pi?

We would like to use the answer in our show. Can you help?

Why π is not $\frac{22}{7}$

Even *Maple or Mathematica* 'knows' this since

$$(1) \qquad 0 < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\,dx = \frac{22}{7} - \pi,$$

though it would be prudent to ask 'why' it can perform the integral and 'whether' to trust it?

Assume we trust it. Then the integrand is strictly positive on $(0,1)$, and the answer in (1) is an area and so strictly positive, despite millennia of claims that π is $22/7$.

Of course $22/7$ is one of the early continued fraction approximations to π. The first 4 are

$$3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}.$$

In this case, the indefinite integral provides immediate reassurance. We obtain

$$\int_0^t \frac{x^4(1-x)^4}{1+x^2}\,dx \;=\; \frac{1}{7}t^7 - \frac{2}{3}t^6 + t^5 - \frac{4}{3}t^3 + 4\,t \;-\; 4\arctan(t)\,, \tag{2}$$

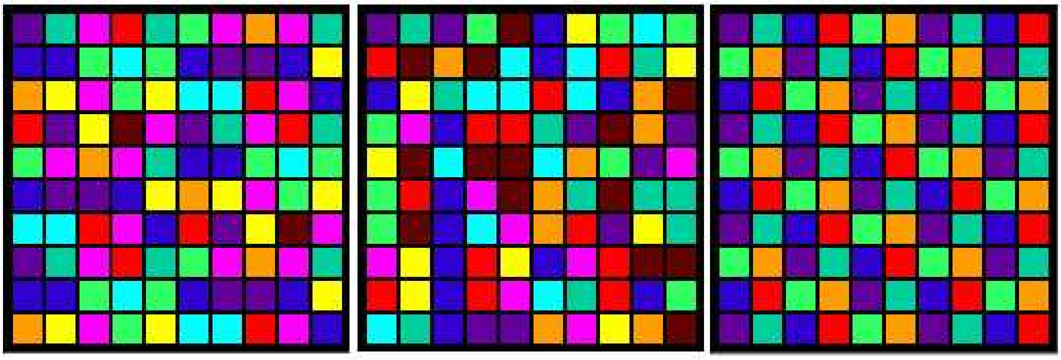
as differentiation easily confirms, and the fundamental theorem of calculus proves (1).

One can take this idea a bit further. Note that

$$\int_0^1 x^4(1-x)^4\,dx \;=\; \frac{1}{630}, \tag{3}$$

and we observe that

$$\begin{aligned} \frac{1}{2}\int_0^1 x^4(1-x)^4\,dx \;&<\; \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\,dx \\ &<\; \int_0^1 x^4(1-x)^4\,dx. \end{aligned} \tag{4}$$



Archimedes: 223/71 < π < 22/7

Combine this with (1) and (3) to derive: $223/71 < 22/7 - 1/630 < \pi < 22/7 - 1/1260 < 22/7$ and so re-obtain Archimedes famous computation

$$3\frac{10}{71} < \pi < 3\frac{10}{70}. \tag{5}$$

The Figure shows the estimate graphically.

- The derivation above seem first to have been written down in *Eureka*, the Cambridge student journal in 1971. The integral in (1) was shown by Kurt Mahler to his students in the mid sixties.

The Childhood of Pi

About 2000 BCE, the Babylonians used the approximation $3\frac{1}{8} = 3.125$. At this same time or earlier, according to an ancient papyrus, Egyptians assumed a circle with diameter nine has the same area as a square of side eight, which implies $\pi = \frac{256}{81} = 3.1604\ldots$.

Some have argued that the ancient Hebrews used $\pi = 3$:

> "Also, he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about." (I Kings 7:23; see also 2 Chron. 4:2)

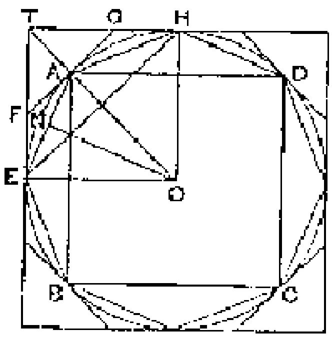
Pi(es)

Archimedes (ca. 250 BCE) was the first to show that the ‘two Pi’s‘ are the same:

Area$= \pi_1\, r^2$ and **Perimeter** $= 2\,\pi_2\, r$.

'ea of any circle is equal to a right-angled triangle in of the sides about the right angle is equal to the radius, 'ier to the circumference, of the circle.

BCD be the given circle, *K* the triangle described.



The first rigorous mathematical calculation of π was also due to Archimedes, who used a brilliant scheme based on **doubling inscribed and circumscribed polygons** ($6 \mapsto 12 \mapsto 24 \mapsto 48 \mapsto 96$) to obtain the bounds $3\frac{10}{71} < \pi < 3\frac{1}{7}$.

Archimedes' scheme constitutes the first true algorithm for π, in that it is capable of producing an arbitrarily accurate value for π.

As discovered in the 19th century, this scheme can be stated as a simple recursion, as follows. Set $a_0 := 2\sqrt{3}$ and $b_0 := 3$. Then define

$$\begin{aligned} a_{n+1} &= \frac{2a_n b_n}{a_n + b_n} && (H) \\ b_{n+1} &= \sqrt{a_{n+1} b_n} && (G) \end{aligned} \tag{6}$$

This converges to π, with the error decreasing by a factor of four with each iteration.

Variations of Archimedes' geometrical scheme were the basis for all high-accuracy calculations of π for the next 1800 years — well beyond its 'best before' date.

For example, in fifth century CE China, Tsu Chung-Chih used a variation of this method to get π correct to seven digits.

A millennium later, Al-Kashi in Samarkand "**who could calculate as eagles can fly**" computed 2π in **sexagecimal**:

$$2\pi = 6 + \frac{16}{60^1} + \frac{59}{60^2} + \frac{28}{60^3} + \frac{01}{60^4} + \frac{34}{60^5} + \frac{51}{60^6} + \frac{46}{60^7} + \frac{14}{60^8} + \frac{50}{60^9},$$

good to 16 decimal places (using $3 \cdot 2^{28}$-gons).

Precalculus π Calculations

Name	Year	Digits
Babylonians	2000? BCE	1
Egyptians	2000? BCE	1
Hebrews (1 Kings 7:23)	550? BCE	1
Archimedes	250? BCE	3
Ptolemy	150	3
Liu Hui	263	5
Tsu Ch'ung Chi	480?	7
Al-Kashi	1429	14
Romanus	1593	15
Van Ceulen (**Ludolph's number***)	1615	35

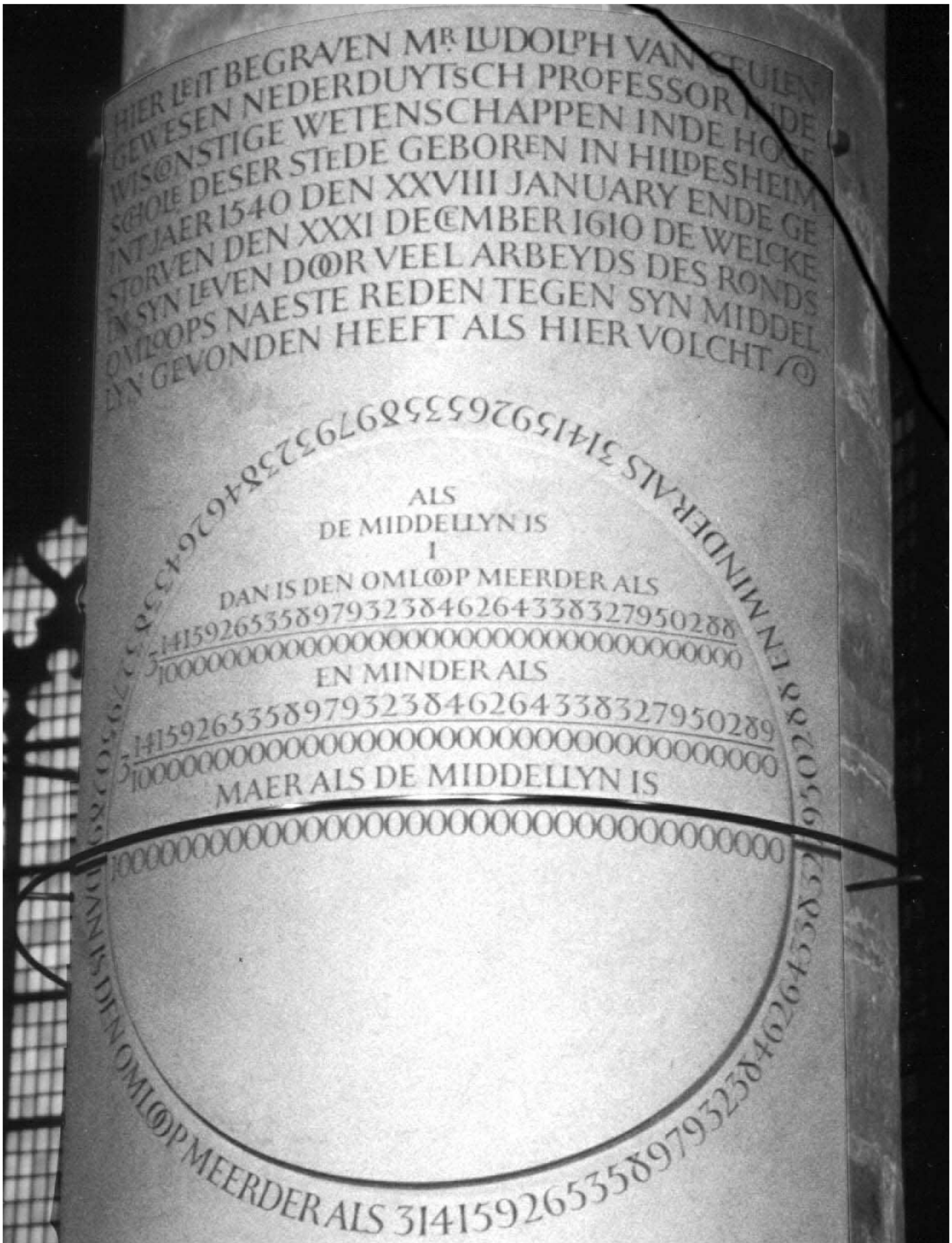
- * Using 2^{62}-gons—to 39 places with 35 correct—published posthumously.

- Little progress was made in Europe during the 'dark ages', but a significant advance arose in India (450 CE): *modern positional, zero-based decimal arithmetic* — the "Indo-Arabic" system. This greatly enhanced arithmetic in general, and computing π in particular.

Ludolph's Rebuilt Tombstone in Leiden

Ludolph van Ceulen (1540-1610)

- Tombstone reconsecrated July 5, 2000.


HIER LEIT BEGRAVEN MR LUDOLPH VAN
GEWESEN NEDERDUYTSCH PROFESSOR
WISCONSTIGE WETENSCHAPPEN INDE HO
SCHOLE DESER STEDE GEBOREN IN HILDESHEIM
INT JAER 1540 DEN XXVIII JANUARY ENDE GE
STORVEN DEN XXXI DECEMBER 1610 DE WELCKE
IN SYN LEVEN DOOR VEEL ARBEYDS DES RONDS
OMLOOPS NAESTE REDEN TEGEN SYN MIDDEL
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The Indo-Arabic system came to Europe around 1000 CE. Resistance ranged from accountants who didn't want their livelihood upset to clerics who saw the system as 'diabolical,' since they incorrectly assumed its origin was Islamic. European commerce resisted until the 18th century, and even in scientific circles usage was limited into the 17th century.

The prior difficulty of doing arithmetic* is indicated by college placement advice given a wealthy German merchant in the 16th century:

> "If you only want him to be able to cope with addition and subtraction, then any French or German university will do. But if you are intent on your son going on to multiplication and division — assuming that he has sufficient gifts — then you will have to send him to Italy." (George Ifrah, p. 577)

*Claude Shannon had 'Throback 1' built to compute in Roman, at Bell Labs in 1953.

Pi's Adolescence

The dawn of modern mathematics appears in *Viéte's product* (1579)

$$\frac{\sqrt{2}}{2}\frac{\sqrt{2+\sqrt{2}}}{2}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots = \frac{2}{\pi}$$

considered to be the first truly infinite formula; and in the *first continued fraction* for $2/\pi$ given by Lord Brouncker (1620-1684):

$$\frac{2}{\pi} = \cfrac{1}{1+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\cdots}}}}$$

based on *John Wallis's 'interpolated' product*

$$\prod_{k=1}^{\infty}\frac{4k^2-1}{4k^2} = \frac{2}{\pi}, \tag{7}$$

which lead to the discovery of the Gamma function and much more.

(7) may be derived from Euler's product formula for π, (8) with $x = 1/2$, or by repeatedly integrating $\int_0^{\pi/2} \sin^{2n}(t)\, dt$ by parts.

One may divine (8) as Euler did by *considering* $\sin(\pi x)$ *as an 'infinite' polynomial* and obtaining a product in terms of the roots $0, \{1/n^2\}$. It is thus plausible that

$$(8) \quad \zeta(2) = \frac{\sin(\pi\, x)}{x} \;=\; c \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right).$$

Euler argued that, like a polynomial, c was the value at zero, and the coefficient of x^2 in the Taylor series the sum of the roots:

$$\sum_n \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This also leads to the evaluation of $\zeta(2n)$ as a rational multiple of π^{2n}: $\zeta(4) = \pi^4/90, \zeta(6) = \pi^6/945, \zeta(8) = \pi^8/9450, \ldots$ (in terms of Bernoulli numbers).

• In 1976 Apéry showed $\zeta(3)$ irrational; and we now know *one of* $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is.

Pi's Adult Life with Calculus

> "I am ashamed to tell you to how many figures I carried these computations, having no other business at the time."
> (Issac Newton, 1666)

In the 17th century, Newton and Leibniz discovered calculus, and this powerful tool was quickly exploited to find new formulas for π. One early calculus-based formula comes from the integral: $\tan^{-1} x$

$$= \int_0^x \frac{dt}{1+t^2} = \int_0^x (1 - t^2 + t^4 - t^6 + \cdots)\, dt$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

Substituting $x = 1$ *formally* gives the well-known **Gregory–Leibniz formula** (1671–74)

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

Calculus π Calculations

Name	Year	Correct Digits
Sharp (and Halley)	1699	71
Machin	1706	100
Strassnitzky and Dase	1844	200
Rutherford	1853	440
Shanks	1874	(707) 527
Ferguson (**Calculator**)	1947	808
Reitwiesner et al. (**ENIAC**)	1949	2,037
Genuys	1958	10,000
Shanks and Wrench	1961	100,265
Guilloud and Bouyer	1973	1,001,250

- Done naively, this is useless — so slow that hundreds of terms are needed to compute two digits. [Sharp used $\tan^{-1}(1/\sqrt{3})$.]

However, Euler's (1738) trigonometric identity

$$\tan^{-1}(1) = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) \tag{9}$$

produces the geometrically convergent

$$\begin{aligned} \frac{\pi}{4} = & \frac{1}{2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} + \cdots \\ & + \frac{1}{3} - \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} - \frac{1}{7\cdot 3^7} + \cdots \end{aligned} \tag{10}$$

An even faster formula, found earlier by John Machin, lies similarly in the identity

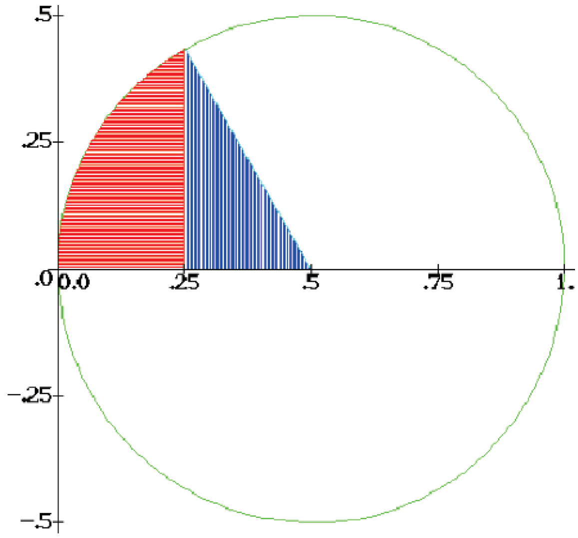
$$(11)\quad \frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right).$$

- This was used in numerous computations of π (starting in 1706) and culminating with Shanks' computation of π to 707 decimal digits accuracy in 1873 (although it was *found in 1945 to be wrong* after the 527-th decimal place, by Ferguson).

Newton discovered a different (disguised arcsin) formula. He considering the area A of the left-most red region shown in the next Figure. Now, A is the integral

$$(12)\qquad A = \int_0^{1/4} \sqrt{x - x^2}\, dx.$$

Newton's arcsin



Also, A is the area of the circular sector, $\pi/24$, less the area of the triangle, $\sqrt{3}/32$. Newton used his **binomial theorem** in (12):

$$\begin{aligned}
A &= \int_0^{\frac{1}{4}} x^{1/2}(1-x)^{1/2}\,dx \\
&= \int_0^{\frac{1}{4}} x^{1/2}\left(1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{1}6 - \frac{5x^4}{128} - \cdots\right) dx \\
&= \int_0^{\frac{1}{4}} \left(x^{1/2} - \frac{x^{3/2}}{2} - \frac{x^{5/2}}{8} - \frac{x^{7/2}}{16} - \frac{5x^{9/2}}{128} \cdots\right) dx
\end{aligned}$$

Integrate term-by-term and combining the above:

$$\pi = \frac{3\sqrt{3}}{4} + 24\left(\frac{1}{3\cdot 8} - \frac{1}{5\cdot 32} - \frac{1}{7\cdot 128} - \frac{1}{9\cdot 512} \cdots\right).$$

Newton used this formula to compute 15 digits of π. As noted, he later 'apologized' for "having no other business at the time."*

- The Viennese *computer* Johan Zacharias Dase demonstrated his computational skill by multiplying

  $$79532853 \times 93758479 = 7456879327810587$$

  in 54 seconds; two 20-digit numbers in six minutes; two 40-digit numbers in 40 minutes; two 100-digit numbers in $8\frac{3}{4}$ hours.

- In 1844, after being shown

  $$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right)$$

  he calculated π to 200 places *in his head* in two months.

*The great fire of London that ended the plague year took place in September 1666. A standard chronology says "Newton never tried to compute π."

- Dase later calculated a seven-digit logarithm table, and extended a table of integer factorizations to 10,000,000. Gauss requested that Dase be permitted to assist him, but Dase died shortly afterwards.

One motivation for computations of π was very much in the spirit of modern experimental mathematics: to see if the decimal expansion of π repeats, which would mean that π is the ratio of two integers (i.e., rational), or to recognize π as an algebraic constant.

The question of the *rationality of* π was settled in the late 1700s, when Lambert and Legendre proved (using continued fractions) that the constant is irrational.

The question of whether π is algebraic was settled in 1882, when Lindemann proved that π *is transcendental*.

- Lindemann's proof also settled, once and for all, the ancient Greek question of whether the circle could be squared with ruler and compass.

  It cannot, because numbers that are the lengths of lines that can be constructed using ruler and compasses (often called *constructible numbers*) are necessarily algebraic, and squaring the circle is equivalent to constructing the value π.

- Aristophanes knew this and derided 'circle-squarers' ($\tau\varepsilon\tau\rho\alpha\gamma\omega\sigma\iota\varepsilon\iota\nu$) in his play "The Birds" of 414 BCE.

The Irrationality of π

Niven's 1947 proof that π is irrational. Let $\pi = a/b$, the quotient of positive integers. We define the polynomials

$$f(x) = \frac{x^n(a-bx)^n}{n!}$$

$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x)$$

the positive integer being specified later. Since $n!f(x)$ has integral coefficients and terms in x of degree not less than n, $f(x)$ and its derivatives $f^{(j)}(x)$ have integral values for $x = 0$; also for $x = \pi = a/b$, since $f(x) = f(a/b - x)$. By elementary calculus we have

$$\begin{aligned} & \frac{d}{dx}\{F'(x)\sin x - F(x)\cos x\} \\ = \; & F''(x)\sin x + F(x)\sin x = f(x)\sin x \end{aligned}$$

and

$$\begin{aligned} \int_0^\pi f(x)\sin x dx &= [F'(x)\sin x - F(x)\cos x]_0^\pi \\ &= F(\pi) + F(0). \qquad (13) \end{aligned}$$

Now $F(\pi) + F(0)$ is an *integer*, since $f^{(j)}(0)$ and $f^{(j)}(\pi)$ are integers. But for $0 < x < \pi$,

$$0 < f(x)\sin x < \frac{\pi^n a^n}{n!},$$

so that the integral in (13) is *positive but arbitrarily small* for n sufficiently large. Thus (13) is false, and so is our assumption that π is rational. **QED**

- This is an excellent intimation of more sophisticated irrationality and transcendence proofs.

- With the development of computer technology in the 1950s, π was computed to thousands and then millions of digits. These computations were facilitated by the discovery of advanced algorithms for the underlying high-precision arithmetic operations.

- For example, in 1965 it was found that the newly-discovered *fast Fourier transform* (FFT) could be used to perform high-precision multiplications much more rapidly than conventional schemes.

  Such methods (e.g., for $\div, \sqrt{x}$) dramatically lowered the time required for computing π and other constants to high precision.

- In spite of these advances, until the 1970s all computer evaluations of π still employed classical formulas, usually of Machin-type.

Ballantine's (1939) Series for π

Another formula of Euler for arccot is

$$x \sum_{n=0}^{\infty} \frac{(n!)^2\, 4^n}{(2\,n+1)!\,\left(x^2+1\right)^{n+1}} \;=\; \arctan\left(\frac{1}{x}\right)$$

This allows one to rewrite the formula, used by Guilloud and Boyer in 1973 to compute a million digits of Pi, viz, $\pi/4 =$

$12 \arctan(1/18) + 8 \arctan(1/57) - 5 \arctan(1/239)$

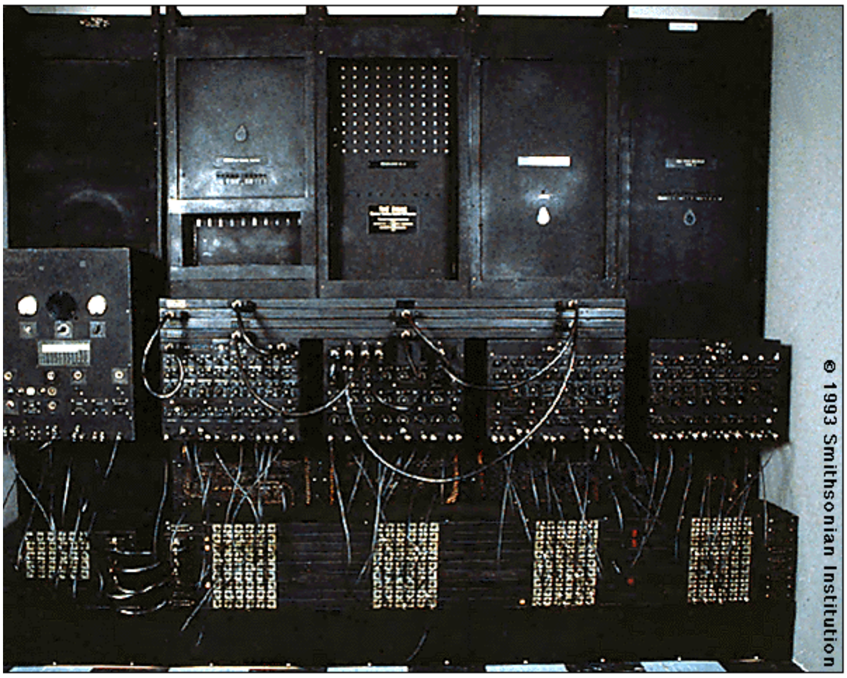
in the efficient form

$$\begin{aligned}\pi &= 864 \sum_{n=0}^{\infty} \frac{(n!)^2\, 4^n}{(2\,n+1)!\, \mathbf{325}^{n+1}} \\ &+ 1824 \sum_{n=0}^{\infty} \frac{(n!)^2\, 4^n}{(2\,n+1)!\, \mathbf{3250}^{n+1}} \\ &- 20 \arctan\left(\frac{1}{239}\right),\end{aligned}$$

where the terms of the second series are just decimal shifts of the first.

ENIAC: Integrator and Calculator

SIZE/WEIGHT: ENIAC had 18,000 vacuum tubes, 6,000 switches, 10,000 capacitors, 70,000 resistors, 1,500 relays, was 10 feet tall, occupied 1,800 square feet and weighed 30 tons.



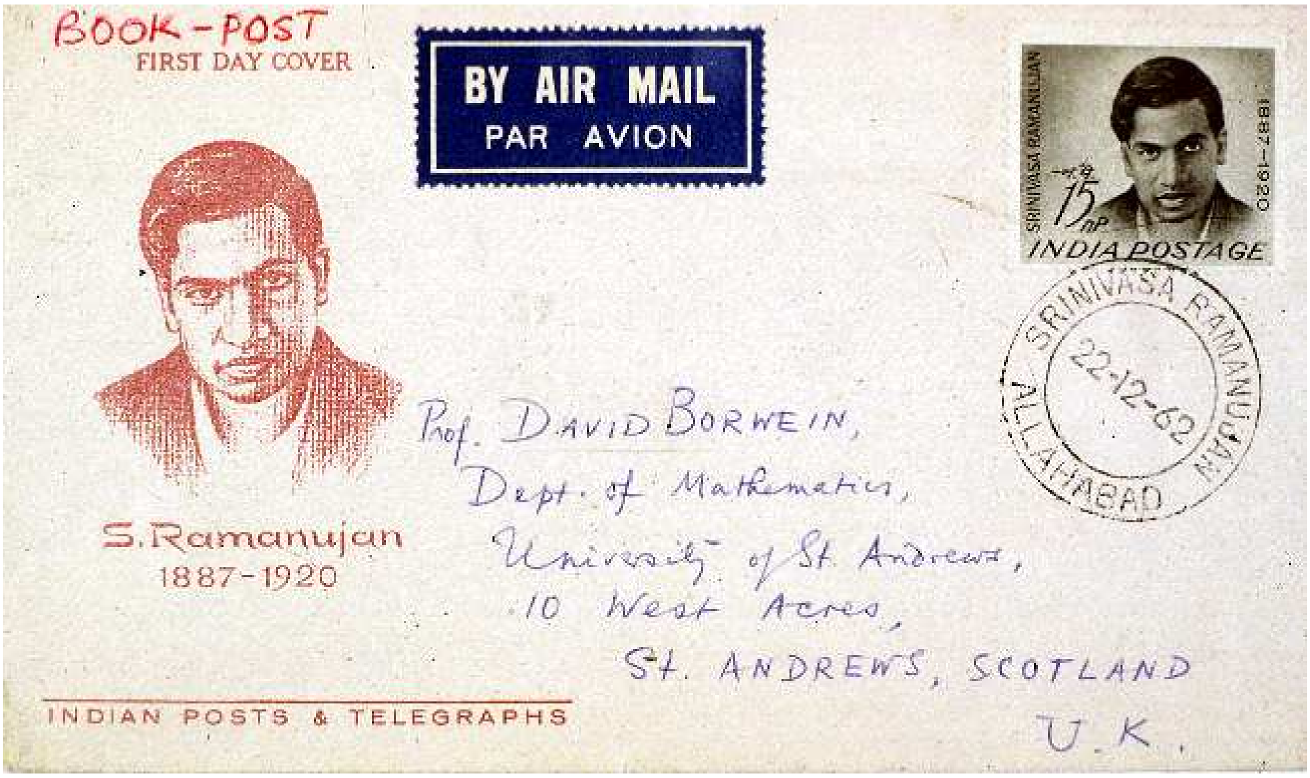
SPEED/MEMORY: A 1.5GHz Pentium does 3 million adds/sec. ENIAC did 5,000 — 1,000 times faster than any earlier machine. The first stored-memory computer, ENIAC could store 200 digits.

ARCHITECTURE: Data flowed from one accumulator to the next, and after each accumulator finished a calculation, it communicated its results to the next in line.

The accumulators were connected to each other manually.

- The 1949 computation of π to 2,037 places took 70 hours.

Pi in the Digital Age



Ramanujan's Seventy-Fifth Birthday Stamp.

Truly new infinite series formulas were discovered by Ramanujan around 1910, but were not well known (nor fully proven) until quite recently when his writings were widely published.

One of these series is the remarkable:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!\,(1103 + 26390k)}{(k!)^4 396^{4k}}$$

(14)

Each term of this series produces an additional *eight* correct digits in the result. When Gosper used this formula to compute 17 million digits of (the continued fraction for) π in 1985, **this concluded the first proof of (14)**!

At about the same time, David and Gregory Chudnovsky found the following variation of Ramanujan's formula:

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k\,(6k)!(13591409 + 545140134k)}{(3k)!\,(k!)^3\,640320^{3k+3/2}}$$

Each term of this series produces an additional 14 correct digits.

The Chudnovskys implemented this formula using a clever scheme that enabled them to utilize the results of an initial level of precision to extend the calculation to even higher precision.

They used this in several large calculations of π, culminating with a **then record computation** to over four billion decimal digits in 1994.

- Relatedly, the Ramanujan-type series

  $$\frac{1}{\pi} = \sum_{n=0}^{\infty} \left(\frac{\binom{2n}{n}}{16^n} \right)^3 \frac{42\,n+5}{16}. \tag{15}$$

  allows one to compute the billionth binary digit of $1/\pi$, or the like, *without computing the first half* of the series.

- While the Ramanujan and Chudnovsky series are considerably more efficient than classical formulas, they share the property that the number of terms needed increases linearly with the number of digits desired.

  That is, **if you want to compute π to twice as many digits, you have to evaluate twice as many terms** of the series.

- In 1976, Eugene Salamin and Richard Brent independently discovered a *reduced complexity* algorithm for π.

  It is based on the **arithmetic-geometric mean iteration** (AGM) and some other ideas due to Gauss and Legendre around 1800 (although Gauss never directly saw the connection to computing π).

The **Salamin–Brent algorithm** is:

Set $a_0 = 1, b_0 = 1/\sqrt{2}$ and $s_0 = 1/2$. Calculate

$$\mathbf{a_k} = \frac{\mathbf{a_{k-1}} + \mathbf{b_{k-1}}}{\mathbf{2}}, \qquad \mathbf{b_k} \;=\; \sqrt{a_{k-1}b_{k-1}}$$

$$c_k = a_k^2 - b_k^2, \qquad s_k \;=\; s_{k-1} - 2^k c_k$$

$$\text{(16)} \qquad \text{and compute} \quad p_k \;=\; \frac{2a_k^2}{s_k}.$$

Then p_k converges *quadratically* to π.

- Each iteration *doubles* the correct digits —**successive iterations produce 1, 4, 9, 20, 42, 85, 173, 347 and 697 digits** of π, and takes $\log N$ operations for N digits.

- Twenty-five iterations computes π to over 45 million decimal digit accuracy. However, each of these iterations must be performed to the precision of the final result.

In 1985, my brother Peter and I discovered other algorithms of this type.

A1: set $a_0 = 1/3$ and $s_0 = (\sqrt{3}-1)/2$. Iterate

$$
\begin{aligned}
r_{k+1} &= \frac{3}{1+2(1-s_k^3)^{1/3}} \\
s_{k+1} &= \frac{r_{k+1}-1}{2} \\
\text{and} \\
a_{k+1} &= r_{k+1}^2 a_k - 3^k (r_{k+1}^2 - 1).
\end{aligned}
$$

Then $1/a_k$ **converges cubically** to π — each iteration triples the number of correct digits.

A2: set $a_0 = 6-4\sqrt{2}$ and $y_0 = \sqrt{2}-1$. Iterate

$$
\begin{aligned}
y_{k+1} &= \frac{1-(1-y_k^4)^{1/4}}{1+(1-y_k^4)^{1/4}} \qquad \text{and} \\
a_{k+1} &= a_k(1+y_{k+1})^4 \\
&\quad - 2^{2k+3} y_{k+1}(1+y_{k+1}+y_{k+1}^2).
\end{aligned}
$$

Then $1/a_k$ **converges quartically** to π.

With $a_0 = 6 - 4\sqrt{2}$, $y_0 = \sqrt{2} - 1$ **and**

$$y_1 = \frac{1 - \sqrt[4]{1 - {y_0}^4}}{1 + \sqrt[4]{1 - {y_0}^4}}, a_1 = a_0 (1 + y_1)^4 - 2^3 y_1 \left(1 + y_1 + {y_1}^2\right)$$

$$y_2 = \frac{1 - \sqrt[4]{1 - {y_1}^4}}{1 + \sqrt[4]{1 - {y_1}^4}}, a_2 = a_1 (1 + y_2)^4 - 2^5 y_2 \left(1 + y_2 + {y_2}^2\right)$$

$$y_3 = \frac{1 - \sqrt[4]{1 - {y_2}^4}}{1 + \sqrt[4]{1 - {y_2}^4}}, a_3 = a_2 (1 + y_3)^4 - 2^7 y_3 \left(1 + y_3 + {y_3}^2\right)$$

$$y_4 = \frac{1 - \sqrt[4]{1 - {y_3}^4}}{1 + \sqrt[4]{1 - {y_3}^4}}, a_4 = a_3 (1 + y_4)^4 - 2^9 y_4 \left(1 + y_4 + {y_4}^2\right)$$

$$y_5 = \frac{1 - \sqrt[4]{1 - {y_4}^4}}{1 + \sqrt[4]{1 - {y_4}^4}}, a_5 = a_4 (1 + y_5)^4 - 2^{11} y_5 \left(1 + y_5 + {y_5}^2\right)$$

$$y_6 = \frac{1 - \sqrt[4]{1 - {y_5}^4}}{1 + \sqrt[4]{1 - {y_5}^4}}, a_6 = a_5 (1 + y_6)^4 - 2^{13} y_6 \left(1 + y_6 + {y_6}^2\right)$$

$$y_7 = \frac{1 - \sqrt[4]{1 - {y_6}^4}}{1 + \sqrt[4]{1 - {y_6}^4}}, a_7 = a_6 (1 + y_7)^4 - 2^{15} y_7 \left(1 + y_7 + {y_7}^2\right)$$

$$y_8 = \frac{1 - \sqrt[4]{1 - {y_7}^4}}{1 + \sqrt[4]{1 - {y_7}^4}}, a_8 = a_7 (1 + y_8)^4 - 2^{17} y_8 \left(1 + y_8 + {y_8}^2\right)$$

$$y_9 = \frac{1 - \sqrt[4]{1 - {y_8}^4}}{1 + \sqrt[4]{1 - {y_8}^4}}, a_9 = a_8 (1 + y_9)^4 - 2^{19} y_9 \left(1 + y_9 + {y_9}^2\right)$$

$$y_{10} = \frac{1 - \sqrt[4]{1 - {y_9}^4}}{1 + \sqrt[4]{1 - {y_9}^4}}, a_{10} = a_9 (1 + y_{10})^4 - 2^{21} y_{10} \left(1 + y_{10} + {y_{10}}^2\right)$$

$$y_{11} = \frac{1 - \sqrt[4]{1 - {y_{10}}^4}}{1 + \sqrt[4]{1 - {y_{10}}^4}}, a_{11} = a_{10} (1 + y_{11})^4 - 2^{23} y_{11} \left(1 + y_{11} + {y_{11}}^2\right)$$

$$y_{12} = \frac{1 - \sqrt[4]{1 - {y_{11}}^4}}{1 + \sqrt[4]{1 - {y_{11}}^4}}, a_{12} = a_{11} (1 + y_{12})^4 - 2^{25} y_{12} \left(1 + y_{12} + {y_{12}}^2\right)$$

$$y_{13} = \frac{1 - \sqrt[4]{1 - {y_{12}}^4}}{1 + \sqrt[4]{1 - {y_{12}}^4}}, a_{13} = a_{12} (1 + y_{13})^4 - 2^{27} y_{13} \left(1 + y_{13} + {y_{13}}^2\right)$$

$$y_{14} = \frac{1 - \sqrt[4]{1 - {y_{13}}^4}}{1 + \sqrt[4]{1 - {y_{13}}^4}}, a_{14} = a_{13} (1 + y_{14})^4 - 2^{29} y_{14} \left(1 + y_{14} + {y_{14}}^2\right)$$

$$y_{15} = \frac{1 - \sqrt[4]{1 - {y_{14}}^4}}{1 + \sqrt[4]{1 - {y_{14}}^4}}, a_{15} = a_{14} (1 + y_{15})^4 - 2^{31} y_{15} \left(1 + y_{15} + {y_{15}}^2\right)$$

$$y_{16} = \frac{1 - \sqrt[4]{1 - {y_{15}}^4}}{1 + \sqrt[4]{1 - {y_{15}}^4}}, a_{16} = a_{15} (1 + y_{16})^4 - 2^{33} y_{16} \left(1 + y_{16} + {y_{16}}^2\right)$$

$$y_{17} = \frac{1 - \sqrt[4]{1 - {y_{16}}^4}}{1 + \sqrt[4]{1 - {y_{16}}^4}}, a_{17} = a_{16} (1 + y_{17})^4 - 2^{35} y_{17} \left(1 + y_{17} + {y_{17}}^2\right)$$

$$y_{18} = \frac{1 - \sqrt[4]{1 - {y_{17}}^4}}{1 + \sqrt[4]{1 - {y_{17}}^4}}, a_{18} = a_{17} (1 + y_{18})^4 - 2^{37} y_{18} \left(1 + y_{18} + {y_{18}}^2\right)$$

$$y_{19} = \frac{1 - \sqrt[4]{1 - {y_{18}}^4}}{1 + \sqrt[4]{1 - {y_{18}}^4}}, a_{19} = a_{18} (1 + y_{19})^4 - 2^{39} y_{19} \left(1 + y_{19} + {y_{19}}^2\right)$$

$$y_{20} = \frac{1 - \sqrt[4]{1 - {y_{19}}^4}}{1 + \sqrt[4]{1 - {y_{19}}^4}}, \mathbf{a_{20}} = a_{19} (1 + y_{20})^4 - 2^{41} y_{20} \left(1 + y_{20} + {y_{20}}^2\right).$$

Then the transcendental number

$$Pi$$

and the algebraic number

$$\frac{1}{a_{20}}$$

actually agree through more than 1.5 trillion decimal places !!!

Star Trek

Kirk asks:

> " **Aren't there some mathematical problems that simply can't be solved**?"

And Spock 'fries the brains' of a rogue computer by telling it:

> " **Compute to the last digit the value of Pi**."

- This algorithm, together with the Salamin–Brent scheme, has been employed by Yasumasa Kanada in Tokyo in various computations of π over the past 15 years or so, including 200 billion decimal digits in 1999.

- Shanks in 1963 was confident that a billion digit computation was *forever impossible*.

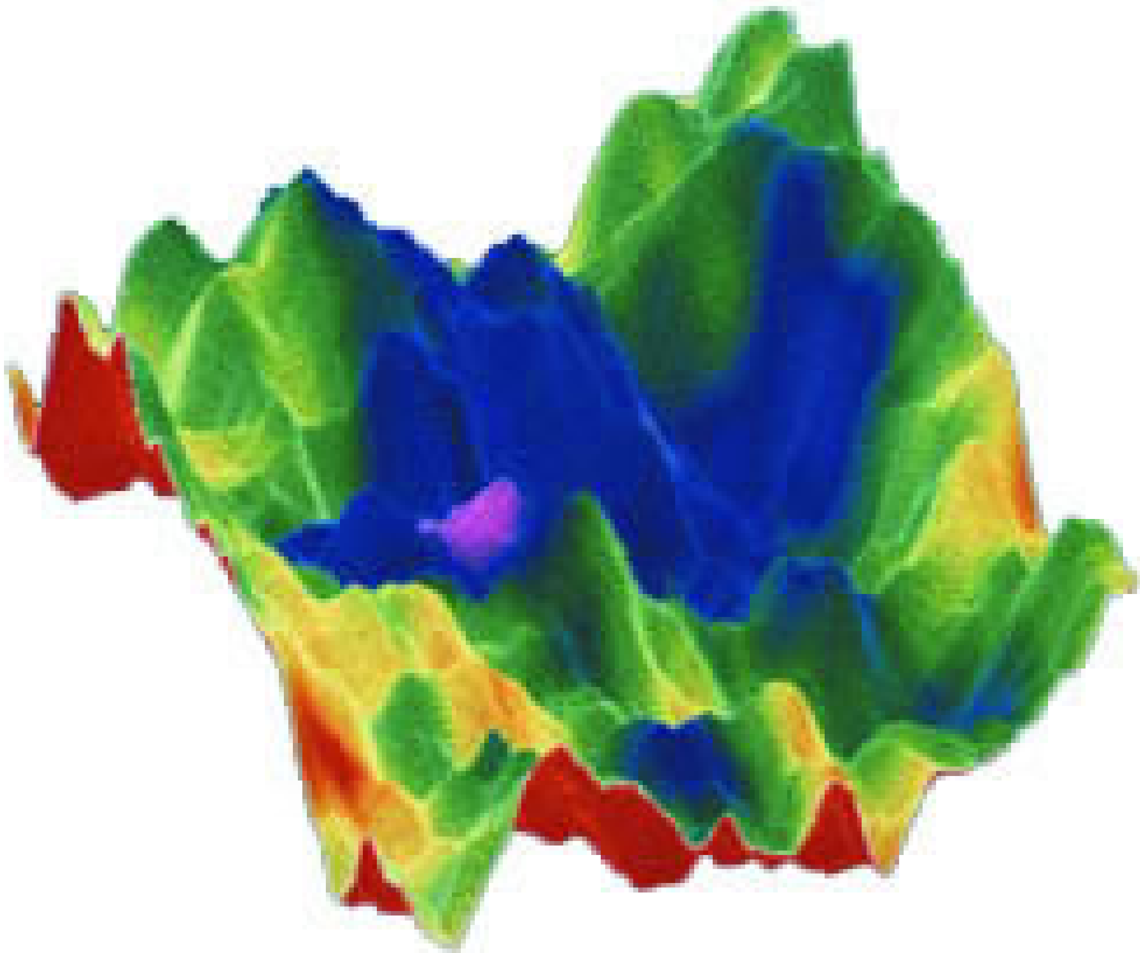
- In 1997 the first occurrence of the sequence **0123456789** was found (late) in the decimal expansion of π starting at the

  $$\mathbf{17,387,594,880}\text{-th digit}$$

  after the decimal point.

  In consequence the status of several famous *intuitionistic examples* due to Brouwer and Heyting has changed.

The First Million Digits of π



- Pi as a random walk.

Modern π Calculations

Name	Year	Correct Digits
Miyoshi and Kanada	1981	2,000,036
Kanada-Yoshino-Tamura	1982	16,777,206
Gosper	1985	17,526,200
Bailey	Jan. 1986	29,360,111
Kanada and Tamura	Sep. 1986	33,554,414
Kanada and Tamura	Oct. 1986	67,108,839
Kanada et. al	Jan. 1987	134,217,700
Kanada and Tamura	Jan. 1988	201,326,551
Chudnovskys	May 1989	480,000,000
Kanada and Tamura	Jul. 1989	536,870,898
Kanada and Tamura	Nov. 1989	1,073,741,799
Chudnovskys	Aug. 1991	2,260,000,000
Chudnovskys	May 1994	4,044,000,000
Kanada and Takahashi	Oct. 1995	6,442,450,938
Kanada and Takahashi	Jul. 1997	51,539,600,000
Kanada and Takahashi	Sep. 1999	206,158,430,000
Kanada-Ushiro-Kuroda	Dec. 2002	1,241,100,000,000

Back to the Future

In December 2002, Kanada computed π to over **1.24 trillion decimal digits**. His team first computed π in hexadecimal (base 16) to 1,030,700,000,000 places, using the following two arctangent relations:

$$\pi = 48\tan^{-1}\frac{1}{49} + 128\tan^{-1}\frac{1}{57} - 20\tan^{-1}\frac{1}{239} + 48\tan^{-1}\frac{1}{110443}$$

$$\pi = 176\tan^{-1}\frac{1}{57} + 28\tan^{-1}\frac{1}{239} - 48\tan^{-1}\frac{1}{682} + 96\tan^{-1}\frac{1}{12943}$$

due to Takano (1982) and Störmer (1896). Kanada verified the results of these two computations agreed, and then converted the hex digit sequence to decimal.
The resulting decimal expansion was checked by converting it back to hex.*

*These conversions are themselves non-trivial, requiring

- This scheme is quite different from earlier ones. One reason is that the Salamin-Brent and Borwein quartic algorithms, whused in the past, require full-scale multiply, divide and square root operations, which in turn require large-scale FFT operations.

  These require huge amounts of memory, and massive all-to-all communication between nodes of a large parallel system.

- For this latest computation, even the very large system available did not have sufficient memory and network bandwidth to perform these operations at reasonable efficiency levels, at least not for trillion-digit computations.

massive computation.

- This used a 1 Tbyte main memory system, as the previous computation, yet got six times as many digits. Hex and decimal evaluations included, it ran 600 hours on a 64-node Hitachi, the main segment at nearly 1 Tflop/sec.

Yasumasa Kanada

手にしているのはπの値が入ったカートリッジテープ

Why Pi?

- What is the motivation behind modern computations of π, given that questions such as the irrationality and transcendence of π were settled more than 100 years ago?

- One motivation is the raw challenge of harnessing the stupendous power of modern computer systems. Programming such calculations are definitely not trivial, especially on large, distributed memory computer systems.

- There have been substantial practical spin-offs. For example, some new techniques for performing the fast Fourier transform (FFT), heavily used in modern science and engineering computing, had their roots in attempts to accelerate computations of π.

- Beyond practical considerations is the abiding interest in the fundamental question of the *normality (digit randomness)* of π.

  Kanada, for example, has performed detailed statistical analysis of his results to see if there are any statistical abnormalities that suggest π is not normal.

- Indeed the first computer computation of π and e on ENIAC was so motivated by John von Neumann.

- The digits of π have been studied more than any other single constant, in part because of the widespread fascination with π.

  Both Kanada's counts are entirely reasonable.

Decimal Digit	Occurrences
0	99999485134
1	99999945664
2	100000480057
3	99999787805
4	100000357857
5	99999671008
6	99999807503
7	99999818723
8	100000791469
9	99999854780
Total	**1000000000000**

- According to Kanada, the 10 decimal digits ending in position one trillion are 6680122702, while the 10 hexadecimal digits ending in position one trillion are `3F89341CD5`.

Hex Digit	Occurrences
0	62499881108
1	62500212206
2	62499924780
3	62500188844
4	62499807368
5	62500007205
6	62499925426
7	62499878794
8	<u>62500</u>216752
9	62500120671
A	62500266095
B	62499955595
C	62500188610
D	62499613666
E	62499875079
F	62499937801
Total	**1000000000000**

- In retrospect, I wonder why in antiquity π was not *measured* to an accuracy in excess of 22/7?

  Perhaps it reflects not an inability to do so but a very different mind set to a modern (Baconian) experimental one.

- In the same vein, one reason that Gauss and Ramanujan did not further develop the ideas in their identities for π is that an iterative algorithm, as opposed to explicit results, was not as satisfactory for them (especially Ramanujan).

  Ramanujan much preferred formulae like

$$\frac{3}{\sqrt{163}}\log(640320) \approx \pi$$

  correct to *15 decimal places* and

$$\frac{3}{\sqrt{67}}\log(5280) \approx \pi$$

  correct to *9 decimal places*.

Discovering the Cubic & Quartic Iterations

The genesis of the π algorithms and related material is an illustrative example of experimental mathematics. For positive integer N, the function

$$\alpha(\mathbf{N}) = \frac{\mathbf{E}'(\mathbf{k_N})}{\mathbf{K}(\mathbf{k_N})} - \frac{\pi}{4\,\mathbf{K}^2(\mathbf{K_n})}$$

arose, where k_N is the N-th *singular value* and K and K and E' are complete *elliptic integrals*.

For present purposes it suffices that $\alpha(N)$ is <u>very easy to compute</u>.

For example, the first few non–composite values are (to 20 digit accuracy):

$$\begin{aligned}
\alpha(1) &\approx 0.49999999999999999999 \\
\alpha(2) &\approx 0.41421356237309504880 \\
\alpha(3) &\approx 0.36602540378443864678 \\
\alpha(5) &\approx \underline{0.33188261099247156221} \\
\alpha(7) &\approx 0.32287565553229529536
\end{aligned}$$

- It is obvious that $\alpha(1) = 1/2$ and easy to spot that $\alpha(2) = \sqrt{2} - 1$, from which it was quickly observed that $\alpha(3) = (\sqrt{3} - 1)/2$ and that $\alpha(7) = (\sqrt{7} - 2)/2$, but $\alpha(5)$ *did not appear* to be a quadratic.

- Twenty years ago such identification was not easy and it was only when it occurred to us that quadratic fields congruent to ± 1 mod 4 behave differently that we stumbled upon (experimentally) the identity

$$\alpha(5) = \frac{\sqrt{5} - \sqrt{2\sqrt{5} - 2}}{2}. \tag{17}$$

- Nowadays this is almost trivial: a "Minpoly calculation" immediately returns

$$29 - 80x - 24x^2 + 16x^4 = 0$$

and this has the surd above as its smallest positive root.

- At this point we could have used known results only to prove the value of $\alpha(1), \alpha(2)$ and $\alpha(3)$. Those for $\alpha(5)$ and $\alpha(7)$ remained conjectural.

There was however an empirical family of algorithms for π: let $\alpha_0 = \alpha(N)$ and $k_0 := k'_N$ (where $k' = \sqrt{1-k^2}$) and iterate

$$
\begin{aligned}
k_{n+1} &= \frac{1-k'_n}{1+k'_n} \\
&\text{and} \\
\alpha_{n+1} &= (1+k_{n+1})^2 \alpha_n - \sqrt{N}\, 2^{n+1} k_{n+1}.
\end{aligned}
$$

Then

$$
\lim_{n\to\infty} \alpha_n^{-1} = \pi. \tag{18}
$$

Again, (18) was provable for $N = 1, 2, 3$ and only conjectured for $N = 5, 7$.

- In each case the algorithm *appeared* to converge quadratically to π. On closer inspection while the provable cases were correct to $5,000$ digits, the empirical versions of (18) agreed with π to roughly 100 places only.

- Now in many ways to have discovered a "natural" number that agreed with π to that level — and no more — would have been more interesting than the alternative. That seemed unlikely but recoding and rerunning the iterations kept producing identical results.

- Twenty years ago very high precision calculation was less accessible, and the code was being run remotely over a phone-line in a Berkeley Unix integer package.

After about six weeks, it transpired that the package's **the square root algorithm was badly flawed**, but **only if run with an odd precision of more than sixty digits**!

- And for idiosyncratic reasons that had only been the case in the two unproven cases.

- Needless to say, tracing the bug was a salutory and somewhat chastening experience.

Borweins and Plouffe (MSNBC, 1987)

Computing Individual Digits of π

An outsider might be forgiven for thinking that essentially everything of interest with regards to π has been discovered.

This sentiment is suggested in the closing chapters of Beckmann's 1971 book *A History of* π.

- Ironically, the Salamin–Brent quadratically convergent iteration was discovered only five years later, and the higher-order convergent algorithms followed in the 1980s.

- In 1990, Rabinowitz and Wagon discovered a 'spigot'' algorithm for π. This permits successive digits of π (in any desired base) to be computed by a relatively simple recursive algorithm based on the *all previously* generated digits.

But even insiders are sometimes surprised by a new discovery.

Prior to 1996, most folks thought if you want to determine the d-th digit of π, you had to generate the (order of) the entire first d digits.

This is not true, at least for hex (base 16) or binary (base 2) digits of π. In 1996, P, Borwein, Plouffe, and Bailey found an algorithm for computing individual hex digits of π. It:

(1) produces a modest-length string hex or binary digits of π, beginning at an arbitrary position, using no prior bits;

(2) is implementable on any modern computer;

(3) requires no multiple precision software;

(4) requires very little memory; and

(5) has a computational cost growing only slightly faster than the digit position.

For example, the millionth hexadecimal digit (four millionth binary digit) of π can be found in under a minute on a present computer.

The new algorithm is not fundamentally faster than best known schemes for computing all digits of π up to some position, but its elegance and simplicity are of considerable interest.

It is based on the following new formula for π:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right) \tag{19}$$

which was *discovered numerically* in the form using *integer relation methods* for several months in CECM:

$$\pi = 4F(1, \frac{1}{4}; \frac{5}{4}, -\frac{1}{4}) + 2\tan^{-1}(\frac{1}{2}) - \log 5$$

where $F([1, 1/4]; 5/4, -1/4) = 0.955933837\ldots$ is a hypergeometric function.

Proof. For $0 < k < 8$,

$$\int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1-x^8}\,dx = \int_0^{1/\sqrt{2}} \sum_{i=0}^{\infty} x^{k-1+8i}\,dx$$
$$= \frac{1}{2^{k/2}} \sum_{i=0}^{\infty} \frac{1}{16^i(8i+k)}$$

Thus one can write

$$\sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right)$$
$$= \int_0^{1/\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1-x^8}\,dx,$$

which on substituting $y := \sqrt{2}x$ becomes

$$\int_0^1 \frac{16\,y - 16}{y^4 - 2\,y^3 + 4\,y - 4}\,dy$$
$$= \int_0^1 \frac{4y}{y^2-2}\,dy$$
$$- \int_0^1 \frac{4y-8}{y^2-2y+2}\,dy = \pi.$$

QED

In 1997, Fabrice Bellard of INRIA computed 152 binary digits of π starting at the trillionth position.

The computation took 12 days on 20 workstations working in parallel over the Internet.

Bellard's scheme is actually based on the following variant of (19):

$$\pi = 4\sum_{k=0}^{\infty}\frac{(-1)^k}{4^k(2k+1)} - \frac{1}{64}\sum_{k=0}^{\infty}\frac{(-1)^k}{1024^k}\left(\frac{32}{4k+1}+\frac{8}{4k+2}+\frac{1}{4k+3}\right)$$

This formula permits individual hex or binary digits of π to be calculated roughly 43% faster than (19).

In 1998 Colin Percival, a 17-year-old student at Simon Fraser University, utilized 25 machines to calculate first the five trillionth hexadecimal digit, and then the ten trillionth hex digit.

In September, 2000, he found the quadrillionth binary digit is **0**, a computation that required **250 CPU-years, using 1734 machines in 56 countries**.

The table below gives computational results.

Position	Hex Digits Beginning At This Position
10^6	26C65E52CB4593
10^7	17AF5863EFED8D
10^8	ECB840E21926EC
10^9	85895585A0428B
10^{10}	921C73C6838FB2
10^{11}	9C381872D27596
1.25×10^{12}	07E45733CC790B
2.5×10^{14}	E6216B069CB6C1

BBP Formulas

Constants α of the form

$$\alpha = \sum_{k=0}^{\infty} \frac{p(k)}{q(k)2^k}, \tag{20}$$

where $p(k)$ and $q(k)$ are integer polynomials, are said to be in the class of *binary BBP numbers*.

I illustrate for $\log 2$ why this permits one to calculate isolated digits in the binary expansion:

$$\log 2 = \sum_{k=0}^{\infty} \frac{1}{k2^k}. \tag{21}$$

We wish to compute a few binary digits beginning at position $d+1$.

This is equivalent to calculating $\{2^d \log 2\}$, where $\{\cdot\}$ denotes fractional part.

We can write

$$\begin{aligned}\{2^d \log 2\} &= \left\{\left\{\sum_{k=0}^{d} \frac{2^{d-k}}{k}\right\} + \left\{\sum_{k=d+1}^{\infty} \frac{2^{d-k}}{k}\right\}\right\} \\ (22) \qquad &= \left\{\left\{\sum_{k=0}^{d} \frac{\mathbf{2^{d-k}}\ \mathrm{mod}\ \mathbf{k}}{k}\right\} + \left\{\sum_{k=d+1}^{\infty} \frac{\mathbf{2^{d-k}}}{\mathbf{k}}\right\}\right\}.\end{aligned}$$

The key observation is: the numerator of the first sum in (22), $2^{d-k} \bmod k$, can be calculated rapidly by the *binary algorithm for exponentiation*, performed modulo k.

That is, exponentiation is economically performed by a factorization based on the binary expansion of the exponent. For example,

$$3^{17} = ((((3^2)^2)^2)^2) \cdot 3$$

uses only five multiplications, not the usual 16.

- It is important to reduce each product module k — 3^{17} mod 10 is done $3^2 = 9; 9^2 = 1; 1^2 = 1; 1^2 = 1; 1 \times 3 = 3$.

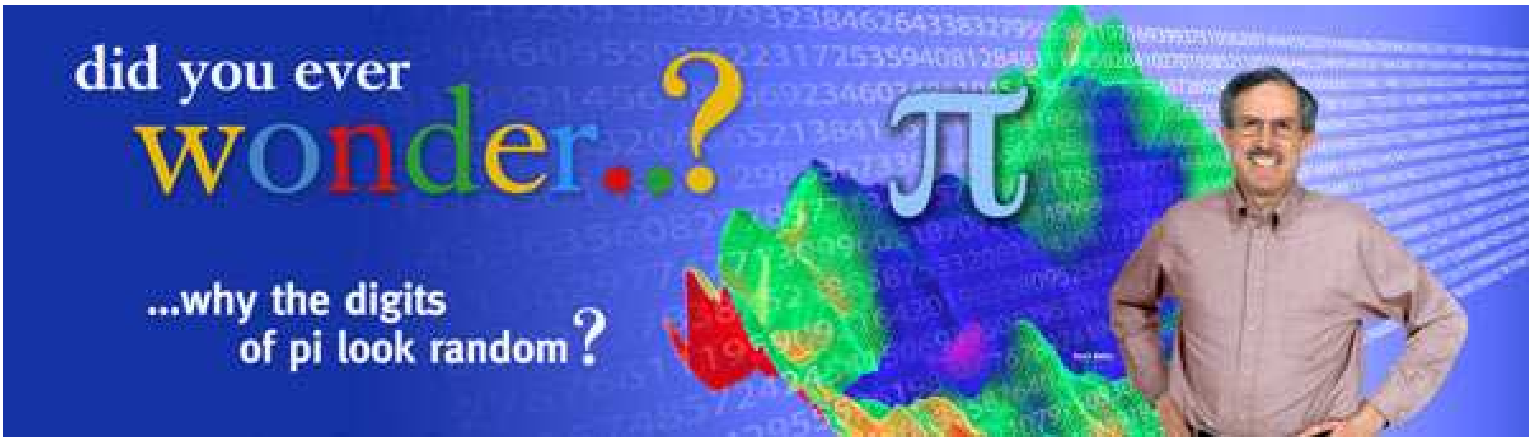
One question that arose in the wake of this discovery is whether there is a formula of this type and an associated computational scheme to compute individual *decimal* digits of π.

Searches conducted by numerous researchers have been unfruitful.

- Recently D. Borwein (my father) W. Gallway and I have shown that there are no BBP formulas of the *Machin-type* of (19) unless the base is a power of two.

- Bailey and Crandall have shown exciting connections between the existence of a b-ary BBP formula for α and its *normality* base b.

David Bailey



- A *ternary BBP formula*

$$\pi^2 = \frac{2}{27}\sum_{k=0}^{\infty}\frac{1}{3^{9k}}\left\{\frac{243}{(12k+1)^2}-\frac{405}{(12k+2)^2}\right.$$
$$-\frac{81}{(12k+4)^2}-\frac{27}{(12k+5)^2}-\frac{72}{(12k+6)^2}$$
$$-\frac{9}{(12k+7)^2}-\frac{9}{(12k+8)^2}-\frac{5}{(12k+10)^2}$$
$$\left.+\frac{1}{(12k+11)^2}\right\}$$

... Life of Pi.

- At the end, Piscine (Pi) Molitor writes

> I am a person who believes in form, in harmony of order. Where we can, we must give things a meaningful shape. For example—I wonder—could you tell my jumbled story in exactly one hundred chapters, not one more, not one less? I'll tell you, that's one thing I hate about my nickname, the way that number runs on forever. It's important in life to conclude things properly. Only then can you let go.

We may not share the sentiment, but we should celebrate that Pi knows Pi to be irrational.

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