1. For the questions below, just provide the requested answer—no explanation or justification required. As always, answers should be left in raw, unevaluated form, involving binomials, factorials, powers, etc. (such as $\binom{10}{3}2^3$). Do **not** use notations $C(n,k)$ or $P(n,k)$ in your answers.

   All of the questions have a simple answer and require virtually no hand computations.

   Write legibly and circle or $\boxed{\text{box}}$ your answer.

   (a) How many elements does the **power set** of $\{1,2,\dots,10\}$ have? $\boxed{2^{10}}$

   (b) What is the number of **reflexive** relations on the set $A = \{1,2,\dots,10\}$? $\boxed{2^{10\cdot 9} = 2^{90}}$

   (c) What is the number of **bijections** from the set $\{1,2,\dots,10\}$ to the set $\{1,2,\dots,10\}$? $\boxed{10!}$

   (d) What is the number of **one-to-one** functions from the set $\{1,2,3\}$ to the set $\{1,2,\dots,10\}$? $\boxed{10\cdot 9\cdot 8 = 10!/7!}$

   (e) What is the constant term (i.e., the coefficient of x^0) in the expansion of $(2x^3 - \frac{1}{x})^{100}$. $\boxed{-\binom{100}{25}2^{25}}$

   (f) Evaluate the sum $x^{100} + x^{99}(1-x)^1 + x^{98}(1-x)^2 + x^{97}(1-x)^3 + \cdots + x^1(1-x)^{99} + (1-x)^{100}$ for $0 < x < 1$. $\boxed{\frac{x^{101}-(1-x)^{101}}{x-(1-x)}}$

2. Suppose you roll an ordinary die 10 times. Determine the following probabilities. As always, leave your answers in "raw" form, in terms of fractions, factorials, or binomial coefficients.

   (a) The probability that **at least** 2 sixes come up.

   Solution: By the complement trick and the binomial distribution, this is

$$\boxed{1 - \binom{10}{0}\left(\frac{5}{6}\right)^{10} - \binom{10}{1}\left(\frac{1}{6}\right)^{1}\left(\frac{5}{6}\right)^{9}}$$

   (b) The probability that **exactly** 4 sixes come up.

   Solution: This is a standard binomial probability (with $n = 10$, $k = 4$, and $p = 1/6$):

$$\boxed{\binom{10}{4}\left(\frac{1}{6}\right)^{4}\left(\frac{5}{6}\right)^{6}}$$

   (c) The probability that the 4th six occurs at the last (i.e., 10-th) roll:

   Solution: The given event is equivalent to getting (i) a six in trial 10 (which as prob. 1/6) and (ii) exactly 3 sixes in trials 1–9 (which has probability $\binom{9}{3}(1/6)^3(5/6)^6$. Multiplying these two probabilities gives the answer:

$$\boxed{\binom{9}{3}\left(\frac{1}{6}\right)^{4}\left(\frac{5}{6}\right)^{6}}$$

   (d) The probability that a six comes up in rolls 1, 4, and 8, but in none of the other rolls.

   Solution: This corresponds to getting the *specific* success/failure sequence SFFSFFFSFF, which has probability $\boxed{(1/6)^3(5/6)^7}$.

   (e) The probability that exactly two distinct numbers show up in the ten rolls.

   Solution: This is similar to a wordcounting example worked out in class. First pick the two numbers ($\binom{6}{2}$ ways), then multiply by the number of 10-letter "words" consisting of exactly these two numbers ($2^{10} - 2$): $\boxed{\frac{1}{6^{10}}\binom{6}{2}(2^{10}-2)}$

(f) The probability that exactly three 2's, exactly three 4's and exactly four 6's show up in the ten rolls.

Solution: The numerator in this probability is the number of 10-letter "words" consisting of three 2's, three 4's, and four 6's. There are $10!/(3!3!4!)$ of these, and dividing by 6^{10} gives the probability $\boxed{\dfrac{10!}{6^{10}3!3!4!}}$

3. The questions below are independent of each other. Provide the requested answer, *along with a brief explanation/justification* (e.g., by citing an appropriate theorem).

(a) Does there exist a graph with 9 vertices, with degree sequence $1, 1, 2, 2, 3, 3, 4, 4, 5$? Justify your answer!

Solution: $\boxed{\text{NO}}$ By the Handshake Theorem, the number of odd-degree vertices must be even. The given graph has an odd number of vertices with an odd degree.

(b) If a planar graph has 12 vertices, *all of the same degree* d, and divides the plane into 20 regions, what is the degree d of the vertices? Justify your answer.

Solution: By Euler's formula, the graph has $e = r + v - 2 = 20 + 12 - 2 = 30$ edges. By the Handshake Theorem, $2e = vd = 12d$, so $d = 2e/12 = 2 \cdot 30/12 = \boxed{5}$.

(c) Given a positive integer n, consider the graph Q_n whose vertices are bit strings of length n, and such that two bitstrings are connected by an edge if and only if they differ in exactly one bit. Answer the following questions and give a brief justification/explanation (one sentence is enough).

A. How many vertices does this graph have?

Solution: $\boxed{2^n}$ (number of binary strings of length n)

B. How many edges does this graph have?

Solution: $\boxed{n \cdot 2^{n-1}}$. A vertex is connected to exactly n other vertices, namely those in which exactly one of the n bits of this of the bitstring representing this vertex is flipped. Hence the degree of the vertices is n. Since there are 2^n vertices, by the Handshake Theorem, the number of edges is $e = (1/2)2^n n = n2^{n-1}$.

C. For which values of n does this graph have an Euler circuit?

Solution: An Euler circuit exists if and only if all degrees are even. Since in the graph Q_n all degrees are n, an Euler circuit exists if and only if $\boxed{n \text{ is even}}$

(d) Complete the following definition by filling in the blanks with the appropriate **graph-theoretic** term (consisting of exactly two words).

A **Gray code** is a ... ... in the graph Q_n.

Solution: A Gray code is a **Hamilton circuit** in Q_n

For which values of n does there exist a Gray code? (Just state the answer. No justification required for this part.)

Solution: $\boxed{\text{For all } n.}$

(e) The Königsberg bridge problem (see picture) can be modeled by a certain graph. Draw this graph (in the usual way, with dots as vertices and lines/curves as edges), and find its adjacency matrix.

Solution: The regions A,B,C,D in the graph correspond to the vertices, and the bridges to the edges. The adjacency matrix, with rows and columns labeled A,B,C,D, is $\boxed{\begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 2 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}}$.

4. Evaluate the following sums and products (where n denotes a (general) positive integer). The answer should be a simple function of n (e.g., a polynomial, factorial, binomial coefficients, or a power, or some simple combination of these).

   (a) $\sum_{i=1}^{n}(n-i)$ $\boxed{\frac{n(n-1)}{2}}$

   (b) $\prod_{i=1}^{n}\frac{n+i}{i}$ $\boxed{\frac{(2n)!}{n!^2}=\binom{2n}{n}}$

   (c) $\prod_{i=1}^{n} n^{n-i}$ $\boxed{n^{n(n-1)/2}}$

   (d) $1-3+3^2-3^3+\cdots+(-1)^n 3^n$ $\boxed{\frac{1-(-3)^{n+1}}{1-(-3)}=\frac{1}{4}(1+(-1)^{n+1}3^{n+1}) \text{ (geom. series formula)}}$

   (e) $1+\frac{1}{2n}+\frac{1}{4n^2}+\frac{1}{8n^3}+\frac{1}{16n^4}+\cdots$ (Note that this is an infinite sum.) $\boxed{\frac{1}{1-(1/2n)}=\frac{2n}{2n-1}}$

5. The following problems are independent of each other.

   (a) Solve the recurrence $a_n = 2n^2 a_{n-1}$ with initial condition $a_0 = 1$.

   Solution: By iteration,

   $$a_n = 2n^2 a_{n-1} = 2^2 n^2 (n-1)^2 a_{n-2} = \cdots = 2^n n^2 (n-1)^2 \dots 2^2 \cdot 1 a_0 = \boxed{2^n (n!)^2}.$$

   (b) Find the generating function of the sequence defined by $a_0 = 1$ and $a_n = 3a_{n-1} + 2$ for $n \ge 1$. (No need to solve the recurrence. Just find the generating function.)

   Solution: (Cf. 7.4:33) $G(x) = \frac{1}{1-3x} + \frac{2x}{(1-x)(1-3x)}$.

6. The following questions are independent of each other.

   (a) Consider the relation R on the set of positive integers defined by $(x, y) \in R$ if x and y have **no** common prime factor. Determine whether this relation is (a) reflexive, (b) symmetric, (c) transitive. If the property holds, just say so; if it does not hold, give a *specific* counterexample.

   Solution: **Reflexive:** NO. Counterexample: $x = 2$. 2 has a common prime factor 2 with itself, so $(2, 2) \notin R$.
   Symmetric: YES. (Obvious.)
   Transitive: NO. Let $x = 2$, $y = 1$, and $z = 2$. Then x and y have no common prime factor, so $(x, y) \in R$, y and z have no common prime factor, so $(y, z) \in R$, but x and z have common prime factor 2, so $(x, y) \notin R$.

   (b) Write down a formula for the number of positive integers less than or equal to 1000 that are divisible by **none of** the numbers 5, 7, and 11? The answer should be in "raw" form and involve expressions like $\lfloor \frac{213}{2\cdot 13} \rfloor$.

   Solution: By inclusion/exclusion, the number sought is

   $$1000 - \lfloor\frac{1000}{5}\rfloor - \lfloor\frac{1000}{7}\rfloor - \lfloor\frac{1000}{11}\rfloor + \lfloor\frac{1000}{5\cdot 7}\rfloor + \lfloor\frac{1000}{5\cdot 11}\rfloor + \lfloor\frac{1000}{7\cdot 11}\rfloor - \lfloor\frac{1000}{5\cdot 7\cdot 11}\rfloor$$

7. For the following questions, either give a specific example with the requested properties, or explain (in one or two sentences) why no such example exists.

   (a) A function from $\mathbb{Z}_+$ to $\mathbb{Z}$ that is a bijection.

   Solution: Example: the mapping $1 \to 0, 2 \to 1, 3 \to -1, 4 \to 2, 5 \to -2$, etc.

   (b) A function from $\mathbb{R}$ to $\mathbb{Z}$ that is onto.

Solution: Example: The floor function $f(x) = \lfloor x \rfloor$.

(c) An infinite set of real numbers in the interval $[0, 1]$ that is countable.

Solution: Example: The rational numbers in this interval..

8. The following questions are independent of each other.

(a) A partition of a set A is a collection of nonempty subsets A_i of A satisfying certain properties. State these properties.

Solution: The properties defining a partition of S are:
(i) **Pairwise disjoint:** $A_i \cap A_j = \emptyset$ for all $i \neq j$.
(ii) **Union is A:** $A_1 \cup A_2 \cup A_3 \cup \cdots = A$.

(b) The equivalence class of 213 in the congruence relation modulo 7 is a certain set. Write out this set *explicitly* (by listing its elements), using proper set-theoretic notation. You can use ellipses (...), provided the pattern is completely clear.

Solution: $\{3, 10, 17, 24, \dots, -4, -11, -18, \dots\}$

(c) Complete the following definition:

A function $f(x)$ is said to be of order $O(g(x))$ with witnesses C and k if ...

Solution: ... $\boxed{|f(x)| \leq C|g(x)| \text{ for all } x \geq k}$

9. A test has 5 questions, each with possible scores of 0, 1, 2, or 3 points. The scores are recorded on a score sheet that shows the number of points the student obtained on each question—just like the table on the cover page of this exam. Two score sheets are considered equal if and only if, on *each* of the questions, both sheets show the same score. (Thus, for example, two students receiving scores of 2,1,1,3,0 and 1,2,1,3,0, respectively, on Questions 1–5 would have different score sheets since the first got 2 points on Question 1 and 1 point on Question 2 and the second got 1 point on Question 1 and 2 points on Question 2.

The following questions depend on these assumptions, but are otherwise independent of each other.

(a) What is the probability that, in a class of 40, there are at least two students with the same score sheets?

Solution: A score sheet can be represented by a tuple $(s_1, s_2, s_3, s_4, s_5)$, with $s_i \in \{0, 1, 2, 3\}$. There are $4^5 = 2^{10} = 1024$ such tuples, so there are 1024 possible score sheets.
The given problem is a birthday type problem with the 1024 possible score sheets corresponding to the 365 possible birthdays. The probability that all score sheets are distinct is $(1024 \cdot 1023 \dots 995)/1024^{40}$, and the probability that at least two score sheets are identical is 1 minus the above, i.e.,

$$\boxed{1 - \frac{1024 \cdot 1023 \dots 985}{1024^{40}}}$$

Remark: The numerical value of the above quantity is $0.5377\dots$). Thus, in a class of 40 there is already a greater than 50-50 chance that two score sheets are identical. That only 40 students should be needed for this is surprising, but consistent with the birthday paradox.

(b) What is the minimal number of students in a class needed in order to **guarantee** that there are two students with identical score sheets? Clearly explain/justify your reasoning, citing appropriate theorems if necessary.

Solution: This is pigeonhole problem. Letting the pigeonholes be the score sheets, and the pigeons the students, the pigeonhole principle guarantees that if the number of students is greater than the number of score sheets, then there will be at least two students with the same score sheet. By the first part, there are 1024 different score sheets, so the number of students needed is one more, namely $\boxed{1025}$. (That this is indeed the *minimal* such number is clear, since if there are no more students than score sheets, each can obviously have a different score sheet.

10. Let $F_n = \sum_{i=1}^n f_i$, where $f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8, f_7 = 13, \dots$ are the Fibonacci numbers. Use induction to prove that $F_n = f_{n+2} - 1$ for all positive integers n. Your write-up of the induction proof must be mathematically rigorous and complete, be presented in logical order with all necessary steps and explanations where needed (e.g., say things like "by the definition of ... ", "by the induction hypothesis", or "by algebra" at the appropriate places). A list of disconnected formulas does not constitute a proof.

Solution: Let $(*)$ denote the formula $F_n = f_{n+2} - 1$ we seek to prove..
Base step: For $n = 1$, we have $F_1 = 1 = 2 - 1 = f_3 - 1$, so $(*)$ holds in this case.
Inductive step: Assume $(*)$ holds for $n = k$, where k is a positive integer. Then

$$\begin{aligned}
F_{k+1} &= \sum_{i=1}^{k+1} f_i \quad \text{(by def. of } F_{k+1}\text{)}\\
&= \sum_{i=1}^{k} f_i + f_{k+1} \quad \text{(algebra)}\\
&= F_k + f_{k+1} \quad \text{(by def. of } F_k\text{)}\\
&= f_{k+2} - 1 + f_{k+1} \quad \text{(by ind. hyp.)}\\
&= f_{k+3} - 1 \quad \text{(by the recurrence for } f_n\text{)}.
\end{aligned}$$

This proves that $(*)$ holds for $n = k + 1$, so the induction is complete.
Conclusion: By the principle of induction, $(*)$ holds for all positive integers n.