CS 70 Spring 2005 — Discrete Mathematics for CS — Clancy/Wagner — HW 5

Due Thursday, February 24th

Coverage: This assignment involves topics from the February 15 and 17 lectures.

Administrative reminders: We will accept only unformatted text files or PDF files for homework submission. Include your name, login name, section number, and partner list in your submission. Give the command `submit hw5` to submit your answers to this assignment.

Homework exercises:

1. **(8 pts.) College admissions**
   Consider the problem of admitting n students to k colleges. The jth college can admit up to n_j students. Each student has an order of preference for the colleges, and each college has an order of preference for the students. A *stable admission* is an assignment of the students to the colleges where there is no "rogue couple", that is, no situation where there is a student A who is not admitted to college a, but who prefers a to his/her college and is preferred by a to at least one of a's students.

   Describe how to use the Traditional Matching Algorithm as a subroutine to produce a stable admission. (That is, you shouldn't change the TMA code.) Also justify why your procedure produces a stable admission.

2. **(8 pts.) Longest time to stable marriage**
   We know that there can be no more than n^2 stages of the TMA algorithm, because at least one girl is deleted from at least one list at each stage. Construct an instance with n boys and n girls so that *at least* $c \cdot n^2$ stages are required for some constant $c > 0$, and justify your runtime bound.

   We are looking for a *general pattern* here, one that results in $c \cdot n^2$ stages for all n. c might be less than 1, say 1/10 or 1/100.

3. **(16 pts.) You can't please everyone**
   A person x is said to *prefer* a matching M to a matching M' if x strictly prefers her/his partner in M to her/his partner in M'. Given two stable matchings M and M', a person may prefer one to the other or be indifferent if she/he is matched with the same person in both.

   Suppose now that M and M' are stable matchings, and suppose that m and w are partners in M but not in M'. Prove that one of m and w prefers M to M', and the other prefers M' to M.

4. **(16 pts.) Revenge of the girls**
   Consider the following variation on the stable marriage problem: In Hawkinsville this year, we have n eligible boys and girls. Everyone knows the preferences of all the boys, but the girls have managed to keep their preferences totally secret. Tomorrow night, the girls will throw a secret meeting (no boys allowed!), privately share their (true) preferences, and collude to come up with a set of fake

preferences that they will announce to the village elders. As a twist, each girl may declare some boys to be unacceptable (she'd rather be single than marry any of them). Next week, the village elders will run the traditional marriage algorithm on the boys' (true) preferences along with the girls' announced (and totally fake) preferences. The output of the traditional marriage algorithm will be used to marry off all the eligible youngsters in Hawkinsville.

Show that the girls can conspire to choose fake preferences so that when the wedding bells ring next week, we'll end up with a girl-optimal pairing. (By "optimal," we mean optimal with respect to everyone's true preferences.)

5. (12 pts.) Cake-cutting

Consider this protocol for three-party cake cutting:

1. Alice cuts the cake into three equal chunks (equal by her measure).
2. Bob cuts each of these three chunks in half (so that both halves of each chunk are equal by his measure).
3. Among these 6 pieces, Carol chooses the two best pieces (by her measure).
4. Among the 4 remaining pieces, Alice chooses the best two (by her measure).
5. Bob gets the last 2 pieces.

Justify each of your answers below briefly.

(a) Is this protocol fair for Alice? for Bob? for Carol?

(b) Is this protocol envy-free for Alice? for Bob? for Carol?

Recall that a protocol is envy-free for X if it has the following property: if X follows the protocol, then X gets at least as much (by her measure) as anyone else gets, no matter how the other parties behave.