On the Universality of the Riemann zeta-function

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This course gives an introduction to the value distribution theory of the Riemann zeta-function $\zeta(s)$ with a special emphasis on its remarkable universality property. Our main goal is to derive Voronin's celebrated *universality theorem* which states, roughly speaking, that any(!) non-vanishing analytic function on a sufficiently small disc can be uniformly approximated by $\zeta(s)$. For that aim we go along Voronin's original proof; the modern approach via limit theorems for weak convergent probability measures is beyond the scope of this course. We state some consequences as there is the classical result due to H. Bohr that the set of values of $\zeta(s)$ taken on vertical lines in the right half of the critical strip lies dense in the complex plane, and an answer to Hilbert's question whether the zeta-function satisfies an algebraic differential equation. Finally, we shall discuss some open questions in this field (the problem of effectivization and Bagchi's reformulation of Riemann's hypothesis) and the recent progress done towards their solution.

This course bases mainly on the nicely written monography [25] on the Riemann zeta-function. However, we also have to use some results from other fields different than analytic number theory, namely approximation theory of numbers and of functions, the theory of linear operators in Hilbert spaces and complex analysis, for which we refer to [14] and [55], respectively. Unfortunately, we cannot give a detailed proof of all facts which are required to prove Voronin's theorem (as for example the seperation theorem from functional analysis). However,the motivated reader can fill by the given references all these gaps.

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1 Introduction

In this introduction we give a brief historical overview on the phenomenon of *universality*; more details can be found in [21].

The first in the mathematical literature appearing *universal* object was discovered by Fekete in 1914/15 (see [42]); he proved that *there exists a real power series* $\sum_{n=1}^{\infty} a_n x^n$ *with the property that for any continuous function on the interval* $[-1,1]$ *with* $f(0)=0$ *there is a sequence of positive integers* (m_k) *such that the partial sums* $\sum_{1\le n\le m_k} a_n x^n$ *converge to* $f(x)$ *uniformly on* $[-1,1]$. We shall prove the following variant due to Luh [38]:

Theorem 1.1 *There exists a real power series*

$$\sum_{n=0}^{\infty} a_n x^n$$

with the property that for any interval $[a,b]$ *with* $0 \notin [a,b]$ *and any continuous function* $f(x)$ *on* $[a,b]$ *there exists a sequence of positive integers* (m_k) *such that*

$$\sum_{n=0}^{m_k} a_n x^n \quad \longrightarrow \quad f(x) \qquad \text{uniformly on } [a,b]$$

as $k \to \infty$.

The proof relies mainly on *Weierstrass' approximation theorem*, which states that *each continuous function* $f(x)$ *on a closed interval is the limit of a uniformly convergent sequence of polynomials*; for a proof see [14].

Proof. Denote by (Q_n) the sequence of polynomials with rational coefficients; this is obviously a countable set. We construct a sequence of polynomials (P_n) as follows: let $P_0 = Q_0$, and assume that for $n \in \mathbb{N}$ the polynomials $P_0, \ldots, P_{n-1}$ are known. Let d_n be the degree of P_{n-1}. Further, let $\varphi_n(x)$ be a continuous function on $[-n,n]$ such that

$$\varphi_n(x) = \left(Q_n(x) - \sum_{\nu=0}^{n-1} P_\nu(x)\right) x^{-d_n-1} \qquad \text{for} \quad x \in \mathcal{I}_n := \left[-n, -\frac{1}{n}\right] \cup \left[\frac{1}{n}, n\right].$$

Then, by Weierstrass' approximation theorem, there exists a polynomial F_n with

$$\max_{x\in[-n,n]} |F_n(x) - \varphi_n(x)| < n^{-d_n-2}.$$

Setting $P_n(x) = F_n(x) x^{d_n+1}$, the sequence (P_n) is constructed, and P_n satisfies

$$\max_{x\in\mathcal{I}_n} \left| \sum_{k=0}^{n} P_k(x) - Q_n(x) \right| = \max_{x\in\mathcal{I}_n} |(F_n(x) - \varphi_n(x)) x^{d_n+1}| < \frac{1}{n}.$$

Obviously, distinct P_n have no powers in common. Thus we can rearrange formally the polynomial series into a power series:

$$\sum_{n=0}^{\infty} a_n x^n := \sum_{n=0}^{\infty} P_n(x)$$

(note that this would be impossible if infinitely many P_n would have a term x^n in common).

Again by Weiertstrass' approximation theorem, to any continuous function $f(x)$ on $[a,b]$ there exists a sequence of positive integers (n_k), tending to infinity, such that

$$\max_{x\in[-1,1]} |f(x) - Q_{n_k}(x)| < \frac{1}{k}.$$

For sufficiently large k the interval $[a,b]$ is contained in $\mathcal{I}_{n_k}$. In view of the above estimates we obtain

$$\begin{aligned}\max_{x\in[a,b]} \left| f(x) - \sum_{n=0}^{d_{n_k}+1} a_n x^n \right| &= \max_{x\in[a,b]} \left| f(x) - \sum_{n=0}^{n_k} P_n(x) \right| \\ \leq \max_{x\in[a,b]} |f(x) - Q_{n_k}(x)| &+ \max_{x\in[a,b]} \left| Q_{n_k}(x) - \sum_{n=0}^{n_k} P_n(x) \right| < \frac{1}{k} + \frac{1}{n_k},\end{aligned}$$

which tends to zero as $k \to \infty$. Thus, putting $m_k = d_{n_k} + 1$, the assertion of the theorem follows. •

In the years after Fekete's discovery many of such *universal* approximations were found. Here we have to mention Birkhoff [6] who proved *the existence of an entire function* $f(z)$ *with the property that to any given entire function* $g(z)$ *there exists a sequence* (a_n) *such that*

$$f(z + a_n) \to g(z) \qquad \textit{locally uniformly in } \mathbb{C}.$$

It was Marcinkiewicz [39] in 1935 who was the first to use the notion *universality* when he proved *the existence of a continuous function whose difference quotients can approximate any measurable function in the sense of convergence almost everywhere.*

In all these examples there are two characteristic aspects of universality, namely the existence of a single object which

- is *maximal* divergent, and
- (via a countable process) allows to approximate a *maximal* class of objects.

This observation led to understand universality as a phenomenon which occurs quite naturally in limiting processes which diverge or behave irregularly in some cases. Meanwhile it turned out that the phenomenon of universality is anything but a rare event in analysis! See [21] for an interesting survey on more or less all known types of universalities and an approach to unify them all.

However, for a long time no *explicit* example of a universal object was found. Surprisingly, Voronin discovered in 1975 that a very famous function has a remarkable *universal* property.

Let $s = \sigma + it$ with $i^2 = -1$ and $\sigma, t \in \mathbb{R}$ be a complex variable (this mixture of latin and greek letters is tradition in number theory). Then the **Riemann zeta-function** is given by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{for} \quad \sigma > 1. \tag{1.1}$$

Since $|n^s| = n^\sigma$, it follows easily (by the integral test) that the series converges absolutely in the half-plane $\sigma > 1$. Series of the type $\sum_n \frac{a(n)}{n^s}$ are called **Dirichlet series**; see [54], Section IX for details. However, the true analytic character of the Riemann zeta-function becomes only visible by continuing $\zeta(s)$ to the left of $\sigma = 1$. Later we will prove the identity

$$\zeta(s) = \frac{s}{s-1} + s\int_1^\infty \frac{[x]-x}{x^{s+1}}\,\mathrm{d}x \qquad \text{for} \quad \sigma > 0; \tag{1.2}$$

here $[x] := \max\{z \in \mathbb{Z} : z \leq x\}$ is the integer part function. Since the appearing integral converges, we have found an analytic continuation for $\zeta(s)$ to the half-plane $\sigma > 0$ except for a simple pole at $s = 1$ with residue 1. As we shall see later on, the value distribution of the Riemann zeta-function in the so-called **critical strip** $0 \leq \sigma \leq 1$ is of special interest out of different points of view.

Now, since we have an expression for $\zeta(s)$ in the critical strip, we are in the position to formulate Voronin's result [57].

Theorem 1.2 *Let $0 < r < \frac{1}{4}$ and let $f(s)$ be a non-vanishing continuous function on the disc $|s| \leq r$, which is analytic in the interior. Then, for any $\varepsilon > 0$ there exists a $\tau > 0$ such that*

$$\max_{|s|\leq r} \left| \zeta\left(\frac{3}{4} + s + i\tau\right) - f(s) \right| < \varepsilon.$$

Thus, $\zeta(s)$ approximates any such function with any precision somewhere in $\frac{1}{2} < \sigma < 1$. Interpreting the absolute value of an analytic funtion as an analytic landscape over the complex plane, we see that any analytic landscape can be found in the analytic landscape of $\zeta(s)$ (up to an arbitrarily small error). Therefore, in German we say:

Wer die Zetafunktion kennt, kennt die Welt!

However, we do not know the zeta-function good enough; we even have not proved the analytic continuation (1.2).

2 The Riemann zeta-function

In this section we shall prove first properties of $\zeta(s)$ which also give a first impression on the leading role of the zeta-function in multiplicative number theory.

In the introduction we have seen that the series in (1.1) converges absolutely for $\sigma > 1$. Since, for $\sigma \geq \sigma_0 > 1$,

$$\begin{aligned}\left|\sum_{n=1}^{\infty} \frac{1}{n^s}\right| &\leq \sum_{n=1}^{\infty} \frac{1}{n^{\sigma_0}} \leq 1 + \sum_{n=2}^{\infty} \int_{n-1}^{n} \frac{\mathrm{d}u}{u^{\sigma_0}} \\ &= 1 + \int_1^{\infty} u^{-\sigma_0}\,\mathrm{d}u = 1 + \frac{1}{\sigma_0 - 1},\end{aligned} \tag{2.1}$$

the series in question converges uniformly in any half-plane $\sigma > \sigma_0$ with $\sigma_0 > 1$. A well-known theorem of Weierstrass states that *the limit of a uniformly convergent sequence of holomorphic functions is holomorphic*; see [54], §2.8. Thus

Theorem 2.1 *$\zeta(s)$ is analytic for $\sigma > 1$ and satisfies in this half-plane the identity*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{2.2}$$

Here and in the sequel p denotes always a prime number. The product in (2.2) is taken over all primes; it is called **Euler product** since it was discovered by Euler in 1737. As we shall see in the proof below it can be regarded as the analytic version of the *unique prime factorization of the integers.*

Proof. It remains to show the identity between the series and the product in (2.2). In view of the geometric series expansion and the unique prime factorization of the integers,

$$\prod_{p\leq x} \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_{p\leq x} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots\right) = \sum_{\substack{n \\ p|n \Rightarrow p\leq x}} \frac{1}{n^s};$$

as usual, we write $d|n$ if the integer d divides the integer n, and $d\not| n$ otherwise. Since

$$\left|\sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{\substack{n \\ p|n\Rightarrow p\leq x}} \frac{1}{n^s}\right| \leq \sum_{n>x} \frac{1}{n^{\sigma}} \leq \int_x^{\infty} u^{-\sigma}\,\mathrm{d}u = \frac{x^{1-\sigma}}{\sigma - 1}$$

tends to zero as $x \to \infty$, we get identity (2.2) by sending $x \to \infty$. The theorem is proved. •

We shall see later on that the Euler product (2.2) is the key to prove Voronin's universality theorem (in spite of the fact that the product does not converge in the region of universality). However, the representation (2.2) gives also a first glance on the close connection between $\zeta(s)$ and the distribution of prime numbers. A first immediate consequence is Euler's proof of the infinitude of the prime numbers: assuming that there are only finitely many primes, the product in (2.2) is finite, and therefore convergent throughout the whole complex plane, contradicting the fact that the $\zeta(s)$ defining Dirichlet series reduces to the divergent harmonic series as $s \to 1+$. Hence, *there are infinitely many prime numbers.* This fact is well known since Euclid's elementary proof, but the analytic access gives much deeper knowledge on the distribution of prime numbers.

It was the young Gauss who conjectured in 1792 for the number of primes under a given magnitide:

$$\pi(x) := \sharp\{p \leq x\} \sim \mathrm{Li}(x), \tag{2.3}$$

where the **logarithmic integral** is defined by

$$\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}u}{\log u};$$

here and in the sequel we write $f(x) = O(g(x))$, resp. $f(x) \ll g(x)$, with a positive function $g(x)$ if

$$\limsup_{x \to \infty} \frac{|f(x)|}{g(x)}$$

is bounded, and we write $f(x) \sim g(x)$ if the latter quantity is actually a limit and equals one. By partial integration it is easily seen that

$$\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}u}{\log u} = \sum_{k=1}^{n} \frac{x(k-1)!}{(\log x)^k} + O\left(\frac{x}{(\log x)^{n+1}}\right).$$

Thus, Gauss' conjecture states that, in first approximation, the number of primes $\leq x$ is asymptotically $\frac{x}{\log x}$. Čebyshev proved in 1852 by elementary methods that for sufficiently large x

$$0.921\ldots \leq \pi(x) \cdot \frac{\log x}{x} \leq 1.055\ldots,$$

and additionally that if

$$\lim_{x \to \infty} \pi(x) \cdot \frac{\log x}{x}$$

exists, then the limit has to be equal to one, which supports (2.3).

However, as Riemann showed Euler's analytic access, i.e. the link between the zeta-function and the primes by (2.2), encodes much more arithmetic information than Čhebyshev's elemenatry approach. In his only but outstanding paper [49] on number theory Riemann noticed the importance of studying $\zeta(s)$ as a function of a complex variable (Euler dealt only with real s). Riemann proved:

- *the function*

$$\zeta(s) - \frac{1}{s-1}$$

is entire; consequently, $\zeta(s)$ has an anlytic continuation throughout the whole complex plane except for $s = 1$ where $\zeta(s)$ has a simple pole with residue 1.

- *$\zeta(s)$ satisfies the functional equation*

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Note that the Gamma-function, defined by

$$\Gamma(z) = \int_0^\infty u^{z-1}\exp(-u)\,\mathrm{d}u \qquad \text{for} \quad \mathrm{Re}\, z > 0,$$

plays an important role in the theory of the zeta-function; see [54], §1.86 and §4.41, for a collection of its most important properties.

In view of the Euler product (2.2) it is easily seen that $\zeta(s)$ has no zeros in the half-plane $\sigma > 1$. Using the functional equation, it turns out that $\zeta(s)$ vanishes in $\sigma < 0$ exactly at the so-called **trivial zeros**

$$\zeta(-2n) = 0 \qquad \text{for} \qquad n \in \mathbb{N};$$

this is caused by the simple poles of the Gamma-function at the non-positive integers. All other zeros of $\zeta(s)$ are said to be **nontrivial**, and it comes out that they are all non-real, and that there location is in fact a *nontrivial* task. We denote the nontrivial zeros by $\varrho = \beta + i\gamma$. Obviously, they have to lie in the critical strip $0 \leq \sigma \leq 1$ The functional equation, in addition with the reflection principle

$$\zeta(\overline{s}) = \overline{\zeta(s)}, \tag{2.4}$$

show some symmetries of $\zeta(s)$. In particular, the nontrivial zeros of $\zeta(s)$ have to be distributed symmetrically with respect to the real axis and the so-called **critical line** $\sigma = \frac{1}{2}$. It was Riemann's ingenious contribution to number theory to point out how the distribtuion of these nontrivial zeros is linked to the distribution of prime numbers. Riemann conjectured:

- *the number $N(T)$ of nontrivial zeros $\varrho = \beta + i\gamma$ with $0 \leq \gamma \leq T$ (counted according multiplicities) satisfies*

$$N(T) \sim \frac{T}{2\pi}\log\frac{T}{2\pi e};$$

this was proved in 1895/1905 by von Mangoldt, who found more precisely

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T). \tag{2.5}$$

- *all nontrivial zeros lie on the critical line $\sigma = \frac{1}{2}$, or equivalently,*

$$\zeta(s) \neq 0 \qquad \textit{for} \quad \sigma > \frac{1}{2}; \tag{2.6}$$

this is the famous, yet unproved **Riemann hypothesis**. Riemann worked with $\zeta(\frac{1}{2}+it)$ and wrote *"...und es ist sehr wahrscheinlich, dass alle Wurzeln reell sind. Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen..."*. Note that Riemann also calculated the *first* zeros: for example, the first is $\varrho = \frac{1}{2}+i\cdot 14.134\ldots$. Further, Riemann conjectured

- *there exist some constants* A, B *such that*

$$\frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \exp(A+Bs)\prod_{\varrho}\left(1-\frac{s}{\varrho}\right)\exp\left(\frac{s}{\varrho}\right);$$

- *the* **explicit formula***: for any* $x \geq 2$

$$\begin{aligned} \pi(x)+\sum_{n=2}^{\infty}\frac{\pi(x^{1/n})}{n} &= \mathrm{Li}(x) - \sum_{\substack{\varrho=\beta+i\gamma \\ \gamma>0}}\left(\mathrm{Li}(x^{\varrho})+\mathrm{Li}(x^{1-\varrho})\right) \\ &\quad + \int_x^{\infty}\frac{du}{u(u^2-1)\log u} - \log 2; \end{aligned} \tag{2.7}$$

this was proved in 1895 by von Mangoldt whereas the last but one conjecture was proved by Hadamard. The explicit formula follows from both product representations of $\zeta(s)$, the Euler product on one side and the Hadamard product on the other side.

Riemann's ideas led to the first proof of Gauss' conjecture (2.3), the celebrated prime number theorem, by Hadamard and de La Vallée Poussin (independendly) in 1896.

3 The approximate functional equation

In this section we shall derive not only an analytic continuation for $\zeta(s)$ to the half-plane $\sigma > 0$ but also a quite good approximation which will be very useful later on. Unfortunately, the proof is rather technical; we follow [55], §IV.

What happens on the abscissa of convergence $\sigma = 1$? At $s = 1$ the zeta-function defining series reduces to the harmonic series. To obtain an analytic continuation for $\zeta(s)$ we have to seperate this singularity. For that purpose we need **Abel's partial summation**:

Lemma 3.1 *Let* $\lambda_1 < \lambda_2 < \ldots$ *be a divergent sequence of real numbers, define for* $\alpha_n \in \mathbb{C}$ *the function* $A(u) = \sum_{\lambda_n \leq u}\alpha_n$, *and let* $F : [\lambda_1, \infty) \to \mathbb{C}$ *be a continuous differentiable function. Then*

$$\sum_{\lambda_n \leq x}\alpha_n F(\lambda_n) = A(x)F(x) - \int_{\lambda_1}^{x} A(u)F'(u)\,du.$$

For those who are familiar with the Riemann-Stieltjes integral there is nearly nothing to show. Anyway,

Proof. We have

$$A(x)F(x) - \sum_{\lambda_n \le x} \alpha_n F(\lambda_n) = \sum_{\lambda_n \le x} \alpha_n (F(x) - F(\lambda_n)) = \sum_{\lambda_n \le x} \int_{\lambda_n}^{x} \alpha_n F'(u)\, \mathrm{d}u.$$

Since $\lambda_1 \le \lambda_n \le u \le x$, interchanging integration and summation yields the assertion. •

Now we apply partial summation to finite pieces of the series (1.1). Let $N < M$ be positive integers and $\sigma > 1$. Then, application of Lemma 3.1 with $F(u) = u^{-s}, \alpha_n = 1$ and $\lambda_n = n$ yields

$$\begin{aligned}
\sum_{N<n\le M} \frac{1}{n^s} &= M^{1-s} - N^{1-s} + s\int_N^M \frac{[u]}{u^{s+1}}\, \mathrm{d}u \\
&= M^{1-s} - N^{1-s} + s\int_N^M \frac{[u]-u}{u^{s+1}}\, \mathrm{d}u + s\int_N^M u^{-s}\, \mathrm{d}u \\
&= \frac{1}{s-1}(N^{1-s} - M^{1-s}) + s\int_N^M \frac{[u]-u}{u^{s+1}}\, \mathrm{d}u.
\end{aligned}$$

Sending $M \to \infty$ we obtain

Theorem 3.2 *For $\sigma > 0$,*

$$\zeta(s) = \sum_{n \le N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} + s\int_N^\infty \frac{[u]-u}{u^{s+1}}\, du.$$

Thus, $\zeta(s)$ has an analytic continuation to the half-plane $\sigma > 0$ except for one simple pole at $s = 1$ with residue 1.

Putting $N = 1$ in the formula of Theorem 3.2, we obtain the analytic continuation (1.2) for $\zeta(s)$ from the introduction. But this integral representation gives also a very useful approximation of $\zeta(s)$ in the critical strip. By the periodicity of the function $[u] - u$ we expect that the contribution of the integral is small. In order to give a rigorous proof of this idea we have to do some preliminary observations.

Let $f(u)$ be any function with continuous derivative on the interval $[a, b]$. Using the lemma on partial summation with $\alpha_n = 1$ if $n \in (a, b]$, and $\alpha_n = 0$ otherwise, we get

$$\sum_{a<n\le b} f(n) = ([b] - [a])f(b) - \int_a^b ([u] - [a])f'(u)\, \mathrm{d}u = [b]f(b) - [a]f(a) - \int_a^b [u]f'(u)\, \mathrm{d}u.$$

Obviously,

$$-\int_a^b [u]f'(u)\, \mathrm{d}u = \int_a^b \left(u - [u] - \frac{1}{2}\right) f'(u)\, \mathrm{d}u - \int_a^b \left(u - \frac{1}{2}\right) f'(u)\, \mathrm{d}u.$$

Applying partial integration to the last integral on the right hand side, we deduce

Lemma 3.3 (Euler's summation formula) *Assume that* $f : [a,b] \to \mathbb{R}$ *has a continuous derivative. Then*

$$\begin{aligned}\sum_{a<n\le b} f(n) &= \int_a^b f(u)\,du + \int_a^b \left(u-[u]-\frac{1}{2}\right) f'(u)\,du \\ &\quad + \left(a-[a]-\frac{1}{2}\right) f(a) - \left(b-[b]-\frac{1}{2}\right) f(b).\end{aligned}$$

An easy application is the well-known asymptotic formula

$$\sum_{n\le x} \frac{1}{n} = \log n + \gamma + O\left(\frac{1}{x}\right) \qquad \text{as} \quad x\to\infty, \tag{3.1}$$

where $\gamma = 0.577\ldots$ is the **Euler-Mascheroni constant**. This formula describes very precisely the rate of divergence of the harmonic series. Our application is more difficult. First, we replace in Euler's summation formula the function $u-[u]-\frac{1}{2}$ by its Fourier series expansion.

Lemma 3.4 *For* $u \notin \mathbb{Z}$,

$$\left| u - \frac{1}{2} - \sum_{\substack{|m|\le M\\ m\neq 0}} \frac{\exp(-2\pi i m u)}{2\pi i m}\right| \le \frac{1}{2\pi M(u-[u])},$$

and, for $u\in\mathbb{R}$,

$$\sum_{m\neq 0}^{\infty} \frac{\exp(-2\pi i m u)}{2\pi i m} = \begin{cases} u-[u]-\frac{1}{2} & \textit{if} \quad u\notin\mathbb{Z},\\ 0 & \textit{if} \quad u\in\mathbb{Z},\end{cases}$$

where the terms with $\pm m$ *have to be added together; the partial sums are uniformly bounded in* α *and* M.

Proof. By symmetry and periodicity it suffices to consider only the case $0<u\le\frac{1}{2}$. Since

$$\int_u^{\frac{1}{2}} \exp(-2\pi i m x)\,\mathrm{d}x = \frac{(-1)^{m+1}+\exp(-2\pi i m u)}{2\pi i m} \qquad \text{for} \quad 0\neq m\in\mathbb{Z},$$

we obtain

$$\begin{aligned}\sum_{\substack{|m|\le M\\ m\neq 0}} \frac{\exp(-2\pi i m u)}{2\pi i m} - u + \frac{1}{2} &= \int_u^{\frac{1}{2}} \sum_{|m|\le M} \exp(2\pi i m x)\,\mathrm{d}x \\ &= \int_u^{\frac{1}{2}} \frac{\sin((2M+1)\pi x)}{\sin(\pi x)}\,\mathrm{d}x.\end{aligned} \tag{3.2}$$

By the mean-value theorem there exists a $\xi\in(u,\frac{1}{2})$ such that the latter integral equals

$$\int_u^{\xi} \frac{\sin((2M+1)\pi x)}{\sin(\pi u)}\,\mathrm{d}x.$$

This implies immediately both formulas of the lemma. It remains to show that the partial sums of the Fourier series are uniformly bounded in u and M. Substituting $y = (2M+1)\pi x$ in (3.2), we get

$$\begin{aligned}\int_u^{\frac{1}{2}} \frac{\sin((2M+1)\pi x)}{\sin(\pi x)}\,\mathrm{d}x &= \int_u^{\frac{1}{2}} \frac{\sin((2M+1)\pi x)}{\pi x}\,\mathrm{d}x \\ &\quad + \int_u^{\frac{1}{2}} \sin((2M+1)\pi x)\left(\frac{1}{\sin(\pi x)} - \frac{1}{\pi x}\right)\mathrm{d}x \\ &\ll \int_0^{\infty} \frac{\sin(y)}{y}\,\mathrm{d}y + \int_0^{\frac{1}{2}} \left|\frac{1}{\sin(\pi x)} - \frac{1}{\pi x}\right|\mathrm{d}x\end{aligned}$$

with an implicit constant not depending on α and M; obviously both integrals exist, which gives the uniform boundedness. •

Further, we will make use of the following estimate of exponential integrals.

Lemma 3.5 *Assume that $F : [a,b] \to \mathbb{R}$ has a continuous non-vanishing derivative and that $G : [a,b] \to \mathbb{R}$ is continuous. If $\frac{G}{F'}$ is monotonic on $[a,b]$, then*

$$\left|\int_a^b G(u)\exp(iF(u))\,du\right| \le 4\left|\frac{G}{F'}(a)\right| + 4\left|\frac{G}{F'}(b)\right|.$$

Proof. First, we assume that $F'(u) > 0$ for $a \le u \le b$. Since $(F^{-1}(v))' = F'(F^{-1}(v))^{-1}$, substituting $u = F^{-1}(v)$ leads to

$$\int_a^b G(u)\exp(iF(u))\,\mathrm{d}u = \int_{F(a)}^{F(b)} \frac{G(F^{-1}(v))}{F'(F^{-1}(v))}\exp(iv)\,\mathrm{d}v.$$

Application of the mean-value theorem gives, in case of a monotonically increasing $\frac{G}{F'}$,

$$\mathrm{Re}\left\{\int_{F(a)}^{F(b)} \frac{G(F^{-1}(v))}{F'(F^{-1}(v))}\exp(iv)\,\mathrm{d}v\right\} = \frac{G}{F'}(F(a))\int_{F(a)}^{\xi} \cos v\,\mathrm{d}v + \frac{G}{F'}(F(b))\int_{\xi}^{F(b} \cos v\,\mathrm{d}v$$

with some $\xi \in (a,b)$. This gives the desired estimate. The same idea applies to the imaginary part. Furthermore, the case $F'(u) < 0$ can be treated analogously. The lemma is proved. •

Now we are in the position to prove **van der Corput's summation formula**.

Theorem 3.6 *To any given $\eta > 0$ there exists a positive constant $C = C(\eta)$ only depending on η with the following property: assume that $f : [a,b] \to \mathbb{R}$ is a function with continuous derivative, $g : [a,b] \to [0,\infty)$ is differentiable function, and that f', g and $|g'|$ are all monotically decreasing. Then*

$$\sum_{a<n\le b} g(n)\exp(2\pi i f(n)) = \sum_{f'(a)-\eta<m<f'(b)+\eta} \int_a^b g(u)\exp(2\pi i(f(u)-mu))\,du + \mathcal{E},$$

where

$$|\mathcal{E}| \le C(\eta)\left(|g'(a) + g(a)\log(|f'(a)| + |f'(b)| + 2)\right).$$

Van der Corput's summation formula looks rather technical but the idea is simple as we will shortly explain. The integral

$$\int_a^b g(u)\exp(2\pi i(f(u)-mu))\,\mathrm{d}u$$

is (up to a constant factor) the Fourier transform of $g(u)\exp(2\pi i f(u))$ at $u = m$ (therefore, we may interpret Theorem 3.6 as an approximate version of Poisson's summation formula).

Proof of Theorem 3.6. Using Euler's summation formula with $F(u) = g(u)\exp(2\pi i f(u))$ and the Fourier series expansion of Lemma 3.4, we get

$$\begin{aligned}\sum_{a<n\leq b} g(n)\exp(2\pi i f(n)) &= \int_a^b g(u)\exp(2\pi i f(u))\,\mathrm{d}u + O(g(a)) \\ &\quad + \int_a^b \sum_{m\neq 0} \frac{\exp(-2\pi i m u)}{2\pi i m}\frac{\mathrm{d}}{\mathrm{d}u}\left(g(u)\exp(2\pi i f(u))\right)\,\mathrm{d}u.\end{aligned}$$

Since the series on the right hand side converges uniformly on each compact subset, which is free of integers, and since its partial sums are uniformly bounded, we may interchange summation and integration. This yields

$$\begin{aligned}\sum_{a<n\leq b} g(n)\exp(2\pi i f(n)) &= \int_a^b g(u)\exp(2\pi i f(u))\,\mathrm{d}u \\ &\quad + \sum_{m\neq 0}\frac{1}{m}\left(\mathcal{I}_1(m)+\frac{1}{2\pi i}\mathcal{I}_2(m)\right) + O(g(a),\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\mathcal{I}_1(m) &:= \int_a^b f'(u)g(u)\exp(2\pi i(f(u)-mu))\,\mathrm{d}u, \\ \mathcal{I}_2(m) &:= \int_a^b g'(u)\exp(2\pi i(f(u)-mu))\,\mathrm{d}u.\end{aligned}$$

Partial integration gives

$$\begin{aligned}\mathcal{I}_1(m) &= \left[\frac{\exp(2\pi i(f(u)-mu))g(u)}{2\pi i}\right]_{u=a}^b \\ &\quad - \int_a^b \frac{\exp(2\pi i f(u))}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}u}g(u)\exp(-2\pi i m u)\,\mathrm{d}u, \\ &= O(g(a)) - \frac{1}{2\pi i}\mathcal{I}_2(m) + m\int_a^b g(u)\exp(2\pi i(f(u)-mu))\,\mathrm{d}u.\end{aligned}$$

Thus,

$$\sum_{\substack{f'(a)-\eta<m<f'(b)+\eta \\ m\neq 0}} \frac{1}{m}\left(\mathcal{I}_1(m)+\frac{1}{2\pi i}\mathcal{I}_2(m)\right)$$

$$= \sum_{\substack{f'(a)-\eta<m<f'(b)+\eta \\ m\neq 0}} \int_a^b g(u)\exp(2\pi i(f(u)-hu))\,\mathrm{d}u$$
$$+O\left(\sum_{\substack{f'(a)-\eta<m<f'(b)+\eta \\ m\neq 0}} \frac{g(a)}{|m|} \right).$$

Now assume that $m > f'(a)+\eta$ and $f'(b)>0$. Then $f'(u)>0$ for $a \le u \le b$. Using Lemma 3.5 with $F(u) = 2\pi(f(u)-mu)$ and $G = gf'$, we find

$$\mathcal{I}_1(m) \ll \left| \frac{g(a)f'(a)}{f'(a)-m} \right|.$$

Hence,

$$\sum_{\substack{m>f'(a)+\eta \\ m\neq 0}} \left| \frac{\mathcal{I}_1(m)}{m} \right| \ll g(a) \sum_{0<m\le 2|f'(a)|} \frac{1}{m} + g(a) \sum_{m>|f'(a)|} \frac{|f'(a)|}{m^2}.$$

The contribution arising from $m < f'(b)-\eta$ can be treated similarly. This gives

$$\sum_{\substack{m\notin[f'(b)-\eta, f'(a)+\eta] \\ m\neq 0}} \left| \frac{\mathcal{I}_1(m)}{m} \right| \ll g(a)\log(|f'(a)|+|f'(b)|+2).$$

Now assume $m > f'(a)+\eta$ and $m \neq 0$. Then, by the mean-value theorem,

$$\operatorname{Re} \mathcal{I}_2(m) = -\int_a^b |g'(u)| \cos 2\pi(f(u)-mu)\,\mathrm{d}u = g'(a)\int_a^\xi \cos 2\pi(f(u)-mu)\,\mathrm{d}u$$

with some $\xi \in (a,b)$. Partial integration yields

$$\begin{aligned} \int_a^\xi \cos 2\pi(f(u)-mu)\,\mathrm{d}u &= \left[-\operatorname{Re} \frac{\exp(2\pi i(f(u)-mu)}{2\pi i m} \right]_{u=a}^{\xi} \\ &\quad +\operatorname{Re} \frac{1}{m}\int_a^\xi f'(u)\exp(2\pi i(f(u)-mu))\,\mathrm{d}u \\ &\ll \frac{1}{|m|}\left(1+\frac{|f'(a)|}{|f'(a)-m|}\right). \end{aligned}$$

Therefore,

$$\sum_{m>f'(a)+\eta} \left| \frac{\operatorname{Re} \mathcal{I}_2(m)}{m} \right| \ll g'(a).$$

With slight modifiactions this method applies also to the cases $\operatorname{Im} \mathcal{I}_2(m)$ and $m \le f'(b)-\eta$. Further, if $0 \notin [f'(b)-\eta, f'(a)+\eta]$, then Lemma 3.5 gives

$$\int_a^b g(u)\exp(2\pi i f(u))\,\mathrm{d}u \ll g(a).$$

In view of (3.3) the theorem follows from the above estimates under the condition $f'(b) > 0$. If this condition is not fulfilled, then argue with $f(u) - ku$, where $k := 1 - [f'(b)]$, instead of $f(u)$. •

Now we apply van der Corput's summation formula to the zeta-function. Let $\sigma > 0$. By Theorem 3.2 we have

$$\zeta(s) = \sum_{n\leq x} \frac{1}{n^s} + \sum_{x<n\leq N} \frac{\exp(-it\log n)}{n^\sigma} + \frac{N^{1-s}}{s-1} + s\int_N^\infty \frac{[u]-u}{u^{s+1}}\,\mathrm{d}u.$$

Setting $g(u) = u^{-\sigma}$ and $f(u) = -\frac{t}{2\pi}\log u$, we get $f'(u) = -\frac{t}{2\pi u}$. Assume that $|t| \leq 4x$, then $|f'(u)| \leq \frac{7}{8}$. With the choice $\varepsilon = \frac{1}{10}$ the interval $(f'(b)-\eta, f'(a)+\eta)$ contains only the integer $m = 0$. Thus Theorem 3.6 yields

$$\sum_{x<n\leq N} \frac{\exp(-it\log n)}{n^\sigma} = \int_x^N u^{-s}\,\mathrm{d}u + O(x^{-\sigma}) = \frac{N^{1-s}-x^{1-s}}{1-s} + O(x^{-\sigma}).$$

In addition with

$$s\int_N^\infty \frac{[u]-u}{u^{s+1}}\,\mathrm{d}u \ll |s|N^{-\sigma}$$

we deduce

Theorem 3.7 *We have, uniformly for $\sigma \geq \sigma_0 > 0, |t| \leq 4x$,*

$$\zeta(s) = \sum_{n\leq x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} + O\left(x^{-\sigma}\right).$$

This so-valled **approximate functional equation** was found by Hardy and Littlewood [22] in 1921 (the name comes from the appearing quantities s and $1-s$ as in the functional equation) but was also known by Riemann (see Siegel's paper [50] on Riemann's unpublished papers on $\zeta(s)$). Meanwhile there are better approximate functional equations known, that means an approximation by shorter sums with a smaller error term. However, to indicate the power of this simple approximation we note

$$\zeta\left(\frac{1}{2}+it\right) \ll t^{\frac{1}{2}} \qquad \text{as} \quad t\to\infty \tag{3.4}$$

for any $\varepsilon > 0$. Also here much better estimates known. For example, using the functional equation and the Phragmén-Lindelöf principle (that is a kind of maximum principle for unbounded domains), one can obtain $\frac{1}{4}+\varepsilon$ instead of $\frac{1}{2}$ for any positive ε; Huxley [23] holds the record with the exponent $\frac{89}{570}+\varepsilon$. The yet unproved **Lindelöf hypothesis** states

$$\zeta\left(\frac{1}{2}+it\right) \ll t^{\varepsilon} \qquad \text{as} \quad t\to\infty; \tag{3.5}$$

note that the truth of the Riemann hypothesis would imply this estimate but not vice versa.

4 Density theorems

By use of the approximate functional equation, we shall first derive a mean-square formula for $\zeta(s)$ in the half-plane $\sigma > \frac{1}{2}$. Such mean-square formulae are an important tool in the theory of the Riemann zeta-function; in particular, they give information on the number of zeros as we shall see below. We follow [55], §VII and §IX.

Theorem 4.1 *For* $\sigma > \frac{1}{2}$,

$$\int_1^T |\zeta(\sigma+it)|^2 \, dt = \zeta(2\sigma)T + O(T^{2-2\sigma}\log T).$$

Proof. By the approximate functional equation,

$$\zeta(\sigma+it) = \sum_{n<t} \frac{1}{n^{\sigma+it}} + O(t^{-\sigma}).$$

Thus, by the reflection principle (2.4),

$$\int_1^T \left| \sum_{n<t} \frac{1}{n^{\sigma+it}} \right|^2 \mathrm{d}t = \int_1^T \sum_{m,n<t} \frac{1}{n^{\sigma+it}m^{\sigma-it}} \, \mathrm{d}t = \sum_{m,n<T} \frac{1}{(mn)^\sigma} \int_\tau^T \left(\frac{m}{n}\right)^{it} \mathrm{d}t$$

with $\tau := \max\{m,n\}$. The diagonal terms $m = n$ give the contribution

$$\sum_{n<T} \frac{T-n}{n^{2\sigma}} = T\left(\zeta(2\sigma) - \sum_{n\geq T} \frac{1}{n^{2\sigma}}\right) - \sum_{n<T} \frac{1}{n^{2\sigma-1}} = \zeta(2\sigma)T + O(T^{2-2\sigma})$$

by the trick (2.1). The non-diagonal terms $m \neq n$ contribute

$$\sum_{\substack{m,n<T\\ m\neq n}} \frac{1}{(mn)^\sigma} \frac{\left(\frac{m}{n}\right)^{iT} - \left(\frac{m}{n}\right)^{i\tau}}{i\log\frac{n}{m}} \ll \sum_{0<m<n<T} \frac{1}{(mn)^\sigma \log\frac{n}{m}}.$$

If $m < \frac{n}{2}$ then $\log\frac{n}{m} > \log 2 > 0$, and hence

$$\sum_{n<T} \sum_{m<\frac{n}{2}} \frac{1}{(mn)^\sigma \log\frac{n}{m}} \ll \left(\sum_{n<T} \frac{1}{n^\sigma}\right)^2 \ll T^{2-2\sigma}.$$

If $m \geq \frac{n}{2}$ we write $n = m + r$ with $1 \leq r \leq \frac{n}{2}$. By the Taylor series expansion of the logarithm,

$$\log\frac{n}{m} = -\log\left(1 - \frac{r}{n}\right) > \frac{r}{n}.$$

This gives, in view of (3.1),

$$\sum_{n<T} \sum_{r\leq\frac{n}{2}} \frac{1}{(mn)^\sigma \log\frac{n}{m}} \ll \sum_{n<T} n^{1-2\sigma} \sum_{r\leq\frac{n}{2}} \frac{1}{r} \ll T^{2-2\sigma}\log T.$$

Collecting together, the assertion of the theorem follows. •

Obviously, with regard to the simple pole of the zeta-function, the mean-square formula above cannot hold on the critical line: $\zeta(2\sigma)$ is unbounded as $\sigma \to \frac{1}{2}+$.

We can derive from the above theorem some information on the zero distribution of $\zeta(s)$. This observation dates back to H. Bohr and Landau [11], resp. Littlewood [37]. For that purpose we have to use some facts from classical function theory.

The *theorem of residues* states: *let $\mathcal{C}$ be a closed path in the complex plane without double points, and let $f(s)$ be meromorphic with poles at $s_1, \ldots, s_m$ inside $\mathcal{C}$, then*

$$\frac{1}{2\pi i}\int_{\mathcal{C}} f(s)\,\mathrm{d}s = \sum_{j=1}^{m} \mathrm{Res}_{s=s_j} f(s), \tag{4.1}$$

where

$$\mathrm{Res}_{s=s_j} f(s) := \frac{1}{2\pi i}\oint_{|s-s_j|=\varepsilon} f(s)\,\mathrm{d}s$$

is the residue of $f(s)$ at $s = s_j$ (which coincides with the coefficient a_{-1} in the Laurent expansion of $f(s)$ at s_j, i.e. $f(s) = \sum_{m=-\infty}^{\infty} a_m(s-s_j)^m$). The special case of a logarithmic derivative gives the possibility to count zeros and poles. *If $f(s)$ and $\mathcal{C}$ satisfies the conditions above, and if $N(0)$ denotes the number of zeros, $N(\infty)$ the number of poles of $f(s)$ (according multiplicities) inside $\mathcal{C}$, then*

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f'}{f}(s)\,\mathrm{d}s = N(0) - N(\infty). \tag{4.2}$$

In particular, we obtain the so-called *argument principle*, which states that *if $f(s)$ is analytic, then the change in the argument of* $\arg f(s)$ *as s varies on $\mathcal{C}$ (in positive direction) equals the number of zeros of $f(s)$ inside $\mathcal{C}$.*

All these statements are well-known facts in complex analysis; see [54], Section III. Further, we need Littlewood's lemma, i.e. the following integrated version of (4.2):

Lemma 4.2 *Let $f(s)$ be regular in and upon the boundary of the rectangle $\mathcal{R}$ with vertices $b, b+iT, a+iT, a$, and not zero on $\sigma = b$. Denote by $n(\sigma, T)$ the number of zeros $\varrho = \beta + i\gamma$ of $f(s)$ inside the rectangle with $\beta > \sigma$ including those with $\gamma = T$ but not $\gamma = 0$. Then*

$$\int_{\mathcal{R}} \log f(s)\, ds = -2\pi i \int_b^a n(\sigma, T)\, d\sigma.$$

Since the *complex* logarithm is a multi-valued function, we have to be careful! Obviously, $f(s)$ is non-vanishing in the neighbourhood of $\sigma = b$, and thus we define $\log f(s)$ here starting with any one value of the logarithm, and for other points s of the rectangle by analytic continuation along the polygon with corners $b + it, s = \sigma + it$, provided that the path does not cross a zero or pole of $f(s)$; if it does, put $\log f(s) = \lim_{\varepsilon \to 0+} \log f(\sigma + it + i\varepsilon)$.

Proof. Without loss of generality we may assume that the lines $t=0$ and $t=T$ are free off zeros and poles of $f(s)$. Obviously,

$$\begin{aligned}\int_{\mathcal{R}} \log f(s)\,\mathrm{d}s &= \int_a^b \log f(\sigma)\,\mathrm{d}\sigma - \int_a^b \log f(\sigma+iT)\,\mathrm{d}\sigma \\ &\quad + \int_0^T (\log f(b+it) + \log f(a+it))\, i\,\mathrm{d}t. \end{aligned} \tag{4.3}$$

The last integral equals

$$\int_0^T i \int_a^b \frac{f'}{f}(\sigma+it)\,\mathrm{d}\sigma\,\mathrm{d}t = \int_a^b \int_\sigma^{\sigma+iT} \frac{f'}{f}(s)\,\mathrm{d}s\,\mathrm{d}\sigma.$$

By (4.2),

$$\int_\sigma^{\sigma+iT} \frac{f'}{f}(s)\,\mathrm{d}s = \left\{\int_\sigma^b + \int_b^{b+iT} - \int_{\sigma+iT}^{b+iT}\right\} \frac{f'}{f}(s)\,\mathrm{d}s - 2\pi i n(\sigma,T).$$

Substituting this in formula (4.3) proves the lemma. •

Note that Littlewood's lemma can be used, in addition with *Stirling's formula* and some facts about entire functions, to prove the Riemann-von Mangoldt formula (2.5) (see [5]).

We finish our short excursion to function theory and continue with our investigations on the zeros of the Riemann zeta-function. Let $N(\sigma,T)$ denote the number of zeros $\varrho=\beta+i\gamma$ of $\zeta(s)$ with $\beta>\sigma, 0<\gamma\leq T$ (counting multiplicities). Then, application of Littlewood's lemma with fixed $b=\sigma_0>\frac{1}{2}$ yields

$$\begin{aligned}2\pi\int_{\sigma_0}^1 N(\sigma,T)\,\mathrm{d}\sigma &= \int_0^T \log|\zeta(\sigma_0+it)|\,\mathrm{d}t - \int_0^T \log|\zeta(2+it)|\,\mathrm{d}t \\ &\quad + \int_2^{\sigma_0} \arg\zeta(\sigma+iT)\,\mathrm{d}\sigma - \int_2^{\sigma_0} \arg\zeta(\sigma))\,\mathrm{d}\sigma. \end{aligned} \tag{4.4}$$

The main contribution comes from the first integral on the right hand side. The last integral does not depend on T and so it is bounded. Since $\zeta(s)$ has an Euler product representation (2.2), the logarithm has a Dirichlet series representation:

$$\log\zeta(s) = -\sum_p \log\left(1-\frac{1}{p^s}\right) = \sum_{p,k}\frac{1}{kp^{ks}} \qquad \text{for} \quad \sigma>1, \tag{4.5}$$

where k runs through the positive integers; here we choose that branch of the logarithm which is real on the positive real axis. Hence we obtain

$$\int_0^T \log|\zeta(2+it)|\,\mathrm{d}t = \mathrm{Re}\,\left\{\sum_{p,k}\frac{1}{kp^{2k}}\int_0^T \exp(-itk\log p)\,\mathrm{d}t\right\} \ll \sum_{n=2}^\infty \frac{1}{n^2} \ll 1.$$

It remains to estimate $\arg\zeta(\sigma+iT)$. We may assume that T is not the ordinate of zero. Since $\arg\zeta(2)=0$ and

$$\arg\zeta(s) = \arctan\left(\frac{\mathrm{Im}\,\zeta(s)}{\mathrm{Re}\,\zeta(s)}\right),$$

where

$$\operatorname{Re}\,\zeta(2+it)=\sum_{n=1}^{\infty}\frac{\cos(it\log n)}{n^2}\geq 1-\sum_{n=2}^{\infty}\frac{1}{n^2}>1-\int_1^{\infty}\frac{\mathrm{d}u}{u^2}=0,$$

we have by the argument principle

$$|\arg\zeta(2+iT)|\leq\frac{\pi}{2}.$$

Now assume that $\operatorname{Re}\,\zeta(\sigma+iT)$ vanishes q times as $\frac{1}{2}\leq\sigma\leq 2$. Devide the interval $[\frac{1}{2}+iT,2+iT]$ into $q+1$ parts, throughout each of which $\operatorname{Re}\,\zeta(s)$ is of constant sign. Hence, again by the argument principle, in each part the variation of $\arg\zeta(s)$ does not exceed π. This gives

$$|\arg\zeta(s)|\leq\left(q+\frac{3}{2}\right)\pi\qquad\text{for}\quad\sigma\geq\frac{1}{2}.$$

Further, q is the number of zeros of the function

$$g(z)=\frac{1}{2}\left(\zeta(z+iT)+\zeta(z-iT)\right)$$

for $\operatorname{Im}\,z=0$ and $\frac{1}{2}\leq\operatorname{Re}\,z\leq 2$. Thus, $q\leq n(\frac{3}{2})$, where $n(r)$ is the number of zeros of $\zeta(s)$ for $|z-2|\leq r$. Obviously,

$$\int_0^2\frac{n(r)}{r}\,\mathrm{d}r\geq\int_{\frac{3}{2}}^2\frac{n(r)}{r}\,\mathrm{d}r\geq n\left(\frac{3}{2}\right)\int_{\frac{3}{2}}^2\frac{\mathrm{d}r}{r}=n\left(\frac{3}{2}\right)\log\frac{4}{3}.$$

Jensen's formula states that *if* $f(s)$ *is an analytic function for* $|s|\leq R$ *with zeros* $s_1,\ldots,s_m$ *(according their multiplicities) and* $f(0)\neq 0$, *then*

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f(r\exp(i\theta))|\,\mathrm{d}\theta=\log\frac{r^m|f(0)|}{|s_1\cdot\ldots\cdot s_m|}$$

for $r<R$ (in a sense, this is nothing else than Poisson's integral formula; see [54], §3.61). This gives here

$$\int_0^2\frac{n(r)}{r}\,\mathrm{d}r=\frac{1}{2\pi}\int_0^{2\pi}\log|\zeta(2+r\exp(i\theta))|\,\mathrm{d}\theta-\log|\zeta(2)|.$$

In view of (3.4) we obtain

$$q\leq n\left(\frac{3}{2}\right)\leq\frac{1}{\log\frac{4}{3}}\int_0^2\frac{n(r)}{r}\,\mathrm{d}r\ll\log T.$$

This yields

$$\arg\zeta(\sigma+iT)\ll\log T\qquad\text{uniformly for}\quad\sigma\geq\frac{1}{2},$$

and, consequently, the same bound holds by integration with respect to $\frac{1}{2} \leq \sigma \leq 2$. The restriction that T has not to be an imaginary part of a zero of $\zeta(s)$ can be removed from considerations of continuity. Therefore, we may replace (4.4) by

$$\int_{\sigma_0}^{1} N(\sigma, T)\, \mathrm{d}\sigma = \frac{1}{2\pi} \int_0^T \log |\zeta(\sigma_0 + it)|\, \mathrm{d}t + O(\log T). \tag{4.6}$$

Now we need a further analytic fact due to Jensen: *Jensen's inequality* states that *for any continuous function* $f(u)$ *on* $[a, b]$,

$$\frac{1}{b-a} \int_a^b \log f(u)\, \mathrm{d}u \leq \log \left(\frac{1}{b-a} \int_a^b f(u)\, \mathrm{d}u \right)$$

(for instance, this can be deduced from the *arithmetic-geometric mean inequality*, or see [54], §9.623).

Hence, we obtain for any fixed $\sigma_0 > \frac{1}{2}$

$$\int_0^T \log |\zeta(\sigma + it)|\, \mathrm{d}t \leq \frac{T}{2} \log \left(\frac{1}{T} \int_0^T |\zeta(\sigma + it)|^2\, \mathrm{d}t \right) \ll T$$

by applying Theorem 4.1. Thus,

$$\int_{\sigma_0}^{1} N(\sigma, T)\, \mathrm{d}\sigma \ll T.$$

Let $\sigma_1 = \frac{1}{2} + \frac{1}{2}(\sigma_0 - \frac{1}{2})$, then we get

$$N(\sigma_0, T) \leq \frac{1}{\sigma_0 - \sigma_1} \int_{\sigma_1}^{\sigma_0} N(\sigma, T)\, \mathrm{d}\sigma \leq \frac{2}{\sigma_0 - \frac{1}{2}} \int_{\sigma_1}^{1} N(\sigma, T) \ll T.$$

With view to (4.6) we have proved

Theorem 4.3 *For any fixed* $\sigma > \frac{1}{2}$,

$$N(\sigma, T) \ll T.$$

The theorem above is a first *density theorem.* In view of the Riemann-von Mangoldt formula (2.5) we see that, supporting Riemann's hypothesis, all but an infinitesimal proportion of the zeros of $\zeta(s)$ lie in the strip $\frac{1}{2} - \varepsilon < \sigma < \frac{1}{2} + \varepsilon$, however small ε may be!

However, for later applications we need a stronger result.

Theorem 4.4 *For any fixed* σ *in* $\frac{1}{2} < \sigma < 1$,

$$N(\sigma, T) \ll T^{4\sigma(1-\sigma)} (\log T)^{10}.$$

Proof. For $2 \le V \le T$ let $N_1(\sigma, V)$ count the zeros $\varrho = \beta + i\gamma$ of $\zeta(s)$ with $\beta \ge \sigma$ and $\frac{1}{2}V < \gamma \le V$. Taking $x = V$ in Theorem 3.7

$$\zeta(s) = \sum_{k \le V} \frac{1}{k^s} + \frac{V^{1-s}}{s-1} + O\left(V^{-\sigma}\right)$$

for $\frac{1}{2}V < t \le V$ and $\frac{1}{2} \le \sigma \le 1$. Multiplying this with the Dirichlet polynomial

$$M_X(s) := \sum_{m \le X} \frac{\mu(m)}{m^s},$$

where $X = V^{2\sigma-1}$, gives

$$\zeta(s) M_X(s) = P(s) + R(s),$$

where

$$P(s) := \sum_{m \le X} \frac{\mu(m)}{m^s} \sum_{k \le V} \frac{1}{k^s} = \sum_{n \le XV} \frac{a(n)}{n^s}$$

with

$$a(n) := \sum_{\substack{m|n \\ m \le X, n \le mV}} \mu(m) = \begin{cases} 1 & \text{if} \quad m = 1, \\ 0 & \text{if} \quad 1 < n \le X, \end{cases} \tag{4.7}$$

and

$$R(s) \ll |M_X(s)| V^{-\sigma}.$$

Note that $M_X(s)$, as the truncated Dirichlet series of the reciprocal of $\zeta(s)$, *mollifies* $\frac{1}{\zeta(s)}$. We shall use $P(s)$ as a *zero-detector*. Let $s = \varrho$ be a zero of the zeta-function with $\frac{1}{2}V < \gamma \le V$. Then,

$$\begin{aligned} 1 &\le \left| \sum_{X < n \le XV} \frac{a(n)}{n^\varrho} \right| + O(|M_X(\varrho)| V^{-\beta}), \\ 1 &\ll \left| \sum_{X < n \le XV} \frac{a(n)}{n^\varrho} \right|^2 + O(|M_X(\varrho)|^2 V^{-2\beta}). \end{aligned}$$

Then, summing up both sides of the latter inequality over all such N zeros leads to

$$N_1(V) \ll \sum_{\substack{\sigma \le \beta \le 1 \\ \frac{1}{2}V < \gamma \le V}} \left(\left| \sum_{X < n \le XV} \frac{a(n)}{n^\varrho} \right|^2 + |M_X(\varrho)|^2 V^{-2\sigma} \right). \tag{4.8}$$

Now we divide the interval $[\frac{1}{2}V, V]$ into subintervals of length 1 of the form $[2m+n-1, 2m+n]$, where $n = 1, 2$ and $\frac{1}{4}V - 1 \le m \le \frac{1}{2}V$. Then, we may write

$$\begin{aligned} \sum_{\substack{\sigma \le \beta \le 1 \\ \frac{1}{2}V < \gamma \le V}} &\le \sum_{\frac{1}{4}V - 1 \le m \le \frac{1}{2}V} \sum_{n=1}^{2} \sum_{2m+n-1 < \gamma \le 2m+n} \\ &\le 2 \max_{1 \le n \le 2} \sum_{\frac{1}{4}V - 1 \le m \le \frac{1}{2}V} \sum_{2m+n-1 < \gamma \le 2m+n} . \end{aligned}$$

In view of the Riemann-von Mangoldt formula (2.5) there are only $\ll \log V$ many zeros with $2m+n-1 < \gamma \le 2m+n$. Now let $\sum_\varrho'$ denote the largest of the related sums according to $2m+n-1<\gamma\le 2m+n$. Then

$$\sum_{\substack{\sigma\le\beta\le 1\\ \frac{1}{2}V<\gamma\le V}} \ll \log V \sum_\varrho{}',$$

resp. in (4.8)

$$N_1(V) \ll \log V \sum_\varrho{}' \left(\left| \sum_{X<n\le XV} \frac{a(n)}{n^\varrho}\right|^2 + \left|\sum_{m\le X}\frac{\mu(m)}{m^\varrho}\right|^2 V^{-2\sigma}\right). \tag{4.9}$$

First of all we shall give a bound for

$$S(Y) := \sum_\varrho{}' \left|\sum_{Y<n\le U} \frac{b(n)}{n^\varrho}\right|^2,$$

where $U\le 2Y$ and $V\ge Y\ge 1$ and

$$b(n) \ll \sum_{d|n} 1 =: \sigma_0(n). \tag{4.10}$$

By partial summation, for fixed $\varrho=\beta+i\gamma$,

$$\sum_{Y<n\le U}\frac{b(n)}{n^\varrho} = \int_Y^U C(u)\,\mathrm{d}u^{-\beta} \qquad \text{with} \quad C(u) := \sum_{Y<n\le u}\frac{b(n)}{n^{i\gamma}}.$$

Applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
\left|\sum_{Y<n\le U}\frac{b(n)}{n^\varrho}\right| &\ll Y^{-\beta-1}\int_Y^U |C(u)|\,\mathrm{d}u + Y^{-\beta}|C(U)|,\\
\left|\sum_{Y<n\le U}\frac{b(n)}{n^\varrho}\right|^2 &\ll Y^{-2\beta-1}\int_Y^U |C(u)|^2\,\mathrm{d}u + Y^{-2\beta}|C(U)|^2.
\end{aligned}$$

This leads to

$$S(Y) \ll Y^{-2\sigma}\sum_\varrho{}' \left|\sum_{Y<n\le W}\frac{b(n)}{n^{i\gamma}}\right|^2,$$

where $W\le U$. Since the distance of the imaginary parts of counted zeros $\varrho_r=\beta_r+i\gamma_r$ is ≥ 1, we can find

$$\begin{aligned}
\left|\sum_{Y<n\le W} b(n)n^{i\gamma_{r+1}}\right|^2 \le{} & \int_{\gamma_r}^{\gamma_{r+1}} \left|\sum_{Y<n\le W} b(n)n^{it}\right|^2 \mathrm{d}t\\
&+2\int_{\gamma_r}^{\gamma_{r+1}} \left|\sum_{Y<n\le W} b(n)n^{it}\cdot \sum_{Y<m\le W} b(m)\log m\cdot m^{it}\right| \mathrm{d}t.
\end{aligned}$$

Summation over r and application of Cauchy-Schwarz yields

$$S(Y) \ll Y^{-2\sigma}(I_1 + \sqrt{I_1 I_2}),$$

where

$$I_1 := \int_{\frac{1}{2}V}^{V} \left| \sum_{Y<n\leq W} b(n) n^{it} \right|^2 \,\mathrm{d}t\ , \quad I_2 := \int_{\frac{1}{2}V}^{V} \left| \sum_{Y<n\leq W} b(n) \log n \cdot n^{it} \right|^2 \,\mathrm{d}t.$$

Taking (4.7) into account, $|a(n)|$ satisfies condition (4.10) on $b(n)$. By elementary estimates one can show that

$$\sum_{n\leq x} \sigma_0^k(n) \ll x(\log x)^k,$$

where the implicit constant depends only on k; a proof can be found in [25]. This yields

$$\begin{aligned} I_1 &\ll (V+Y)\log V \sum_{Y<n\leq 2Y} \sigma_0^2(n) \ll (VY+Y^2)(\log V)^5, \\ I_2 &\ll (VY+Y^2)(\log V)^7. \end{aligned}$$

Now dividing the first sum on the right hand side of (4.9) into $\ll \log V$ sums, application of the latter estimates yields

$$\log V {\sum_{\varrho}}' \left| \sum_{X<n\leq VX} \frac{a(n)}{n^{\varrho}} \right|^2 \ll (VX^{1-2\sigma} + (VX)^{2-2\sigma})(\log V)^9.$$

Similarly, we get for the second term

$$V^{-2\sigma}(\log T)^2 {\sum_{\varrho}}' \left| \sum_{m\leq X} \frac{\mu(m)}{m^{\varrho}} \right|^2 \ll V^{-2\sigma}(V + X^{2-2\sigma})(\log V)^9.$$

Substituting this in (4.9) with regard to $X = V^{2\sigma-1}$, we obtain

$$N_1(V) \ll V^{4\sigma(1-\sigma)}(\log V)^9.$$

Using this with $V = T^{1-n}$ and summing up over all $n \in \mathbb{N}$, proves the theorem. •

Theorem 4.4 is due to Bohr and Landau [12]. There are stronger estimates known. For instance, the strongest unconditional estimate which holds throughout the right half of the critical strip is

$$N(\sigma, T) \ll T^{2.4(1-\sigma)}(\log T)^{18.2}$$

due to Gritsenko [20]. The **density hypothesis** states that

$$N(\sigma, T) \ll T^{(2+\varepsilon)(1-\sigma)}$$

for all $\varepsilon > 0$; one can show that the Lindelöf hypothesis (3.5) implies the density hypothesis.

5 A zero-free region

In order to prove Gauss' conjecture we need further knowledge on the zero distribution of $\zeta(s)$. We shall establish a zero-free region for $\zeta(s)$ which covers the abscissa of absolute convergence $\sigma = 1$. In this delicate problem we follow (with slight modifications) the ideas of de La Vallée Poussin; see [55], Section 3.

In the sequel we only argue for $s = \sigma + it$ from the upper half-plane; with regard to (2.4) all estimates below can be reflected with respect to the real axis.

Lemma 5.1 *We have, for* $t \geq 8, 1 - \frac{1}{2}(\log t)^{-1} \leq \sigma \leq 2$,

$$\zeta(s) \ll \log t \qquad \textit{and} \qquad \zeta'(s) \ll (\log t)^2.$$

Proof. Let $1 - (\log t)^{-1} \leq \sigma \leq 3$. If $n \leq t$, then

$$|n^s| = n^\sigma \geq n^{1-(\log t)^{-1}} = \exp\left(\left(1 - \frac{1}{\log t}\right) \log n\right) \gg n.$$

Thus, the approximate functional equation (Theorem 3.7) in addition with (3.1) implies

$$\zeta(s) \ll \sum_{n \leq t} \frac{1}{n} + t^{-1} \ll \log t.$$

The estimate for $\zeta'(s)$ follows immediately from *Cauchy's formula* (which is actually a consequence of the theorem of residues (4.1))

$$\zeta'(s) = \frac{1}{2\pi i} \oint \frac{\zeta(z)}{(z-s)^2} \, \mathrm{d}z,$$

and standard estimates of integrals, or alternatively, by differentiation of the formula of Theorem 3.2. •

In view of the Euler product (2.2) we have for $\sigma > 1$

$$|\zeta(\sigma + it)| = \exp(\mathrm{Re}\ \log \zeta(s)) = \exp\left(\sum_{p,k} \frac{\cos(kt \log p)}{kp^{k\sigma}}\right).$$

Since

$$17 + 24 \cos \alpha + 8 \cos(2\alpha) = (3 + 4 \cos \alpha)^2 \geq 0,$$

it follows that

$$\zeta(\sigma)^{17} |\zeta(\sigma + it)|^{24} |\zeta(\sigma + 2it)|^8 \geq 1. \tag{5.1}$$

This inequality is the main idea for our following observations. By the approximate functional equation, Theorem 3.7, we have for small $\sigma > 1$

$$\zeta(\sigma) \ll \frac{1}{\sigma - 1}.$$

Assuming that $\zeta(1+it)$ has a zero for $t = t_0 \neq 0$, it would follow that

$$|\zeta(\sigma + it_0)| \ll \sigma - 1,$$

leading to

$$\lim_{\sigma \to 1+} \zeta(\sigma)^{17} |\zeta(\sigma + it_0)|^{24} = 0,$$

which contradicts (5.1). Thus

$$\zeta(1 + it) \neq 0.$$

It can be shown that the non-vanishing of $\zeta(1+it)$ is equivalent to Gauss' conjecture (2.3), i.e. the prime number theorem without error term; see [55], §3.7. But we are interested in a prime number theorem with error term. A simple refinement of the argument allows a lower estimate for the modulus of $\zeta(1+it)$: for $t \geq 1$ and $1 < \sigma < 2$, we deduce from (5.1) and Lemma 5.1

$$\frac{1}{|\zeta(\sigma + it)|} \leq \zeta(\sigma)^{\frac{17}{24}} |\zeta(\sigma + 2it)|^{\frac{1}{3}} \ll (\sigma - 1)^{-\frac{17}{24}} (\log t)^{\frac{1}{3}}.$$

Furthermore, with Lemma 5.1,

$$\zeta(1+it) - \zeta(\sigma + it) = -\int_1^\sigma \zeta'(u + it) \, du \ll |\sigma - 1| (\log t)^2. \tag{5.2}$$

Hence

$$\begin{aligned} |\zeta(1+it)| &\geq |\zeta(\sigma + it)| - c_1(\sigma - 1)(\log t)^2 \\ &\geq c_2(\sigma - 1)^{\frac{17}{24}} (\log t)^{-\frac{1}{3}} - c_1(\sigma - 1)(\log t)^2, \end{aligned}$$

where c_1, c_2 are certain positive constants. Chosing a constant $B > 0$ such that $A := c_2 B^{\frac{17}{24}} - c_1 B > 0$ and putting $\sigma = 1 + B(\log t)^{-8}$, we obtain

$$|\zeta(1+it)| \geq \frac{A}{(\log t)^6}. \tag{5.3}$$

This gives an estimate on the left of the line $\sigma = 1$.

Lemma 5.2 *There exists a positive constant δ such that*

$$\zeta(s) \neq 0 \qquad \textit{for} \qquad \sigma \geq 1 - \delta \min\{1, (\log t)^{-8}\}.$$

Proof. In view of Lemma 5.1 the estimate (5.2) holds for $1 - \delta(\log t)^{-8} \leq \sigma \leq 1$. Using (5.3), it follows that

$$|\zeta(\sigma + it)| \geq \frac{A - c_1\delta}{(\log t)^6},$$

where the right hand side is positive for sufficiently small δ. This yields the zero-free region of Lemma 5.2. •

The largest known zero-free region for the zeta-function was found by Vinogradov [56] and Korobov [26]. Using Vinogradov's ingenious method for exponential sums, they proved

$$\zeta(s) \neq 0 \quad \text{in} \quad \sigma \geq 1 - \frac{c}{(\log |t|)^{\frac{1}{3}} (\log \log |t|)^{\frac{2}{3}}} \tag{5.4}$$

for some positive constant c and sufficiently large $|t|$; see [25], §IV.3.

6 The prime number theorem

The aim of this section is to prove Gauss' conjecture (2.3), the celebrated prime number theorem.

Out of technical reasons we work with the logarithmic derivative of $\zeta(s)$ (instead of $\log\zeta(s)$ as Riemann did). Logarithmic differentiation of the Euler product (2.2), resp. differentiation of (4.5), gives for $\sigma > 1$

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}, \qquad \text{where} \quad \Lambda(n) := \begin{cases} \log p & \text{if } n = p^k, \\ 0 & \text{otherwise,} \end{cases}$$

is the **von Mangoldt Λ-function**. Since $\zeta(s)$ does not vanish in the half-plane $\sigma > 1$, the logarithmic derivative is analytic for $\sigma > 1$. As we shall see below all information on $\pi(x)$ is encoded in

$$\psi(x) := \sum_{n\leq x}\Lambda(n) = \sum_{p\leq x}\log p + O\left(x^{\frac{1}{2}}\right). \tag{6.1}$$

The idea of proof is ingenious but simple. Partial summation gives

$$-\frac{\zeta'}{\zeta}(s) = s\int_1^{\infty}\psi(x)\frac{\mathrm{d}x}{x^{s+1}}.$$

If we can now transform this into a formula in which $\psi(x)$ is isolated, we may hope to find an asymptotic formula for $\psi(x)$ by contour integration methods. The first step of this program can be done by a type of Fourier transformation.

Lemma 6.1 *Let c and y be positive and real. Then*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s}\,ds = \begin{cases} 0 & \textit{if} \quad 0 < y < 1, \\ \frac{1}{2} & \textit{if} \quad y = 1, \\ 1 & \textit{if} \quad y > 1. \end{cases}$$

Proof. If $y = 1$, then the integral in question equals

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}t}{c+it} = \frac{1}{\pi}\lim_{T\to\infty}\int_0^T\frac{c}{c^2+t^2}\,\mathrm{d}t = \frac{1}{\pi}\lim_{T\to\infty}\arctan\frac{T}{c} = \frac{1}{2},$$

by well-known properties of the arctan-function. Now assume that $0 < y < 1$ and $r > c$. Since the integrand is analytic in $\sigma > 0$, *Cauchy's theorem* (resp. the theorem of residues (4.1)) implies, for $T > 0$,

$$\int_{c-iT}^{c+iT}\frac{y^s}{s}\,\mathrm{d}s = \left\{\int_{c-iT}^{r-iT} + \int_{r-iT}^{r+iT} + \int_{r+iT}^{c+iT}\right\}\frac{y^s}{s}\,\mathrm{d}s.$$

It is easily seen that

$$\begin{aligned}\int_{r\pm iT}^{c\pm iT}\frac{y^s}{s}\,\mathrm{d}s &\ll \frac{1}{T}\int_r^c y^{\sigma}\,\mathrm{d}\sigma \ll \frac{y^c}{T|\log y|}, \\ \int_{r-iT}^{r+iT}\frac{y^s}{s}\,\mathrm{d}s &\ll \frac{y^r}{r} + y^r\int_1^T\frac{\mathrm{d}t}{t} \ll y^r\left(\frac{1}{r}+\log T\right).\end{aligned}$$

Sending now r and then T to infinity, the first case follows. Finally, if $y > 1$, then we bound the corresponding integrals over the rectangular contour with corners $c \pm iT, -r \pm iT$, analogously. Now the pole of the integrand at $s = 0$ with residue

$$\operatorname{Res}_{s=0} \frac{y^s}{s} = \lim_{s \to 0} \frac{y^s}{s} \cdot s = 1$$

gives the values $2\pi i$ for the integral in this case. •

We apply this to the logarithmic derivative of the zeta-function and obtain for $x \notin \mathbb{Z}$ and $c > 1$

$$\int_{c-i\infty}^{c+i\infty} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \frac{x^s}{s} \, \mathrm{d}s = \sum_{n=1}^{\infty} \Lambda(n) \int_{c-i\infty}^{c+i\infty} \left(\frac{x}{n}\right)^s \frac{\mathrm{d}s}{s};$$

here interchanging integration and summation is allowed by the absolute convergence of the series. In view of Lemma 6.1 it follows that

$$\sum_{n \le x} \Lambda(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \frac{x^s}{s} \, \mathrm{d}s,$$

resp.

$$\psi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(-\frac{\zeta'}{\zeta}(s)\right) \frac{x^s}{s} \, \mathrm{d}s;$$

this is **Perron's formula**. Since

$$\int_{c\pm iT}^{c\pm i\infty} \frac{y^s}{s} \, \mathrm{d}s = \frac{y^s}{s \log y}\bigg|_{s=c\pm iT}^{c\pm i\infty} + \frac{1}{\log y} \int_{c\pm iT}^{c\pm i\infty} \frac{y^s}{s^2} \, \mathrm{d}s \ll \frac{y^c}{T|\log y|}$$

for $0 < y \neq 1$ and $T > 0$, it follows that

$$\int_{c\pm iT}^{c\pm i\infty} \left(\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s}\right) \frac{x^s}{s} \, \mathrm{d}s \ll \frac{x^c}{T} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^c \left|\log \frac{x}{n}\right|} \ll \frac{x^c}{T} \left|\frac{\zeta'}{\zeta}(c)\right| + \frac{x(\log x)^2}{T} + \log x.$$

This yields

$$\psi(x) = -\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta'}{\zeta}(s) \frac{x^s}{s} \, \mathrm{d}s + O\left(\frac{x^c}{T} \left|\frac{\zeta'}{\zeta}(c)\right| + \frac{x(\log x)^2}{T} + \log x\right), \tag{6.2}$$

which holds for arbitrary x. To find an asymptotic formula for the integral above we move the path of integration to the left, excluding $s = 0$. By the theorem of residues we expect contributions to the main term from the poles of the integrand, i.e.

- the nontrivial zeros of $\zeta(s)$,
- the pole of $\zeta(s)$ at $s = 1$.

However, for our purpose it is sufficient to exclude the zeros of the zeta-function. In view of the zero-free region of Lemma 5.2 we put $c = 1+\lambda$ with $\lambda = \delta(\log t)^{-8}$, where δ is given by Lemma 5.2, and integrate over the boundary of the rectangle $\mathcal{R}$ given by the corners $1 \pm \lambda \pm iT$. By this choice $\zeta(s)$ does not vanish in and on the boundary of $\mathcal{R}$. The theorem of residues (4.1) implies

$$\begin{aligned}\int_{c-iT}^{c+iT} \left(-\frac{\zeta'}{\zeta}(s)\right) \frac{x^s}{s}\,\mathrm{d}s &= \left\{\int_{1+\lambda-iT}^{1-\lambda-iT} + \int_{1-\lambda-iT}^{1-\lambda+iT} + \int_{1-\lambda+iT}^{1+\lambda-iT}\right\} \left(-\frac{\zeta'}{\zeta}(s)\right) \frac{x^s}{s}\,\mathrm{d}s \\ &\quad +2\pi i \mathrm{Res}_{\,s=1} \left(-\frac{\zeta'}{\zeta}(s)\right) \frac{x^s}{s}.\end{aligned}$$

For the logarithmic derivative of $\zeta(s)$ we have

$$-\frac{\zeta'}{\zeta}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = \frac{1}{s-1} + O(1)$$

as $s \to 1$. Thus, we obtain for the residue

$$\mathrm{Res}_{\,s=1} \left(-\frac{\zeta'}{\zeta}(s)\right) \frac{x^s}{s} = \lim_{s\to 1}(s-1)\cdot\left(\frac{1}{s-1} + O(1)\right)\frac{x^s}{s} = x.$$

It remains to bound the integrals. For the horizontal integrals we find with regard to Lemma 5.2

$$\int_{1-\lambda\pm iT}^{1+\lambda\pm iT} \left(-\frac{\zeta'}{\zeta}(s)\right) \frac{x^s}{s}\,\mathrm{d}s \ll \frac{x^{1+\lambda}}{T}.$$

Further, for the vertical integral,

$$\int_{1-\lambda-iT}^{1+\lambda+iT} \left(-\frac{\zeta'}{\zeta}(s)\right) \frac{x^s}{s}\,\mathrm{d}s \ll x^{1-\lambda}(\log T)^9.$$

Collecting together, we dedcue from (6.2)

$$\psi(x) = x + O\left(\frac{x^{1+\lambda}}{T\lambda} + x^{1-\lambda}(\log T)^9 + \frac{x(\log x)^2}{T} + \log x\right).$$

Choosing $T = \exp(\delta^{\frac{1}{10}}(\log x)^{\frac{1}{9}})$, we arrive at

$$\psi(x) = x + O\left(x\exp(-c(\log x)^{\frac{1}{9}})\right).$$

Setting

$$\theta(x) := \sum_{p\le x}\log p,$$

it follows from (6.1) that also

$$\theta(x) = x + O\left(x\exp(-c(\log x)^{\frac{1}{9}})\right).$$

Applying now partial summation, we find

$$\pi(x) = \sum_{p\le x} \log p \cdot \frac{1}{\log p} \quad = \quad \frac{\theta(x)}{\log x} - \int_2^x \theta(u)\frac{\mathrm{d}}{\mathrm{d}u}\frac{1}{\log u}\,\mathrm{d}u$$
$$= \quad \frac{x}{\log x} - \int_2^x u\frac{\mathrm{d}}{\mathrm{d}u}\frac{1}{\log u}\,\mathrm{d}u + O\left(x\exp\left(-c(\log x)^{\frac{1}{9}}\right)\right).$$

Partial integration leads to **prime number theorem**:

Theorem 6.2 *There exists a positive constant c such that for $x \ge 2$*

$$\pi(x) = Li(x) + O\left(x\exp\left(-c(\log x)^{\frac{1}{9}}\right)\right).$$

Thus, the simple pole of the zeta-function is not only the key in Euler's proof of the infinitude of primes (Section 2) but gives also the main term of the asymptotic formula in the prime number theorem.

We see that the prime numbers, which - on a first look - seem to be randomly distributed among the positive integers, satisfy a strong distribution law! For example, the prime number theorem implies that, if p_n is the n-th prime number, then

$$p_n \sim n\log n.$$

In view of the largest known zero-free region (5.4) one can obtain

$$\pi(x) = \mathrm{Li}(x) + O\left(x\exp\left(-c\frac{(\log x)^{\frac{3}{5}}}{(\log\log x)^{\frac{1}{5}}}\right)\right).$$

With a little bit more effort and a little bit more facts from the theory of functions it is possible to prove the analogue of Riemann's explicit formula (2.7). Integrating over the full complex plane one can show for $x \ne p^k$ the exact(!) **explicit formula**

$$\psi(x) = x - \sum_{\varrho}\frac{x^{\varrho}}{\varrho} - \frac{1}{2}\log\left(1-\frac{1}{x^2}\right) - \log(2\pi),$$

resp. its truncated version

$$\psi(x) = x - \sum_{|\gamma|\le T}\frac{x^{\varrho}}{\varrho} + O\left(\frac{x}{T}(\log(xT))^2\right). \tag{6.3}$$

This shows a deep relation between the error term in the prime number theorem and the distribution of the nontrivial zeros of the zeta-function.

Theorem 6.3 *For fixed $\theta \in [\frac{1}{2}, 1)$,*

$$\psi(x) - x \ll x^{\theta+\varepsilon} \qquad \Longleftrightarrow \qquad \zeta(s) \ne 0 \quad \textit{for} \quad \sigma > \theta.$$

With regard to known zeros of $\zeta(s)$ on the critical line it turns out that an error term with $\theta < \frac{1}{2}$ is impossible; see [24] for more details.

Unfortunately, for the proof of one implication of Theorem 6.3 we have to use some facts we have not proved; the interested reader may find all missing details in [25].

Proof. By partial summation we obtain for $\sigma > 1$

$$-\frac{\zeta'}{\zeta}(s) = \frac{s}{s-1} + s\int_1^\infty \frac{\psi(u)-u}{u^{s+1}}\,\mathrm{d}u.$$

If $\psi(x) - x \ll x^{\theta+\varepsilon}$, then the integral above converges for $\sigma > \theta$, giving an analytic continuation for

$$\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1}$$

to the half-plane $\sigma > \theta$, and, in particular, $\zeta(s)$ does not vanish there.

Conversely, if all nontrivial zeros $\varrho = \beta + i\gamma$ satisfy $\beta \leq \theta$, then it follows from (6.3) that

$$\psi(x) - x \ll x^\theta \sum_{|\gamma|\leq T} \frac{1}{|\gamma|} + \frac{x}{T}(\log(xT))^2. \tag{6.4}$$

In view of the Riemann-von Mangoldt-Formel (2.5) we have

$$N(T+1) - N(T) \ll \log T,$$

and therefore

$$\sum_{|\gamma|\leq T} \frac{1}{|\gamma|} \ll \sum_{m=1}^{[T]+1} \frac{\log m}{m} \ll (\log T)^2.$$

Substituting this in (6.4) leads to

$$\psi(x) - x \ll x^\theta (\log T)^2 + \frac{x}{T}(\log(xT))^2.$$

Now the choice $T = x^{1-\theta}$ finishes the proof of this implication. •

Theorem 6.3 shows that the Riemann's hypothesis (2.6) is true if and only if

$$\psi(x) = x + O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

and since the latter estimate is best possible (there are zeros on the critical line), Riemann's hypothesis states that the prime numbers are as uniformly distributed as possible!

In order to prove Voronin's universality theorem we have to do some preliminaries.

7 Diophantine approximation

In the theory of diophantine approximations one investigates how *good* an irrational number can be approximated by rational numbers; this has a plenty of applications in various fields of mathematics and natural sciences. We follow [25], A.8.

For abbreviation we denote vectors of $\mathbb{R}^N$ by $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$, we write $\tau\mathbf{x} = (\tau x_1, \ldots, \tau x_N)$ for $\tau \in \mathbb{R}$ and $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \ldots + x_N y_N$. Further, for $\mathbf{x} \in \mathbb{R}^N$ and $\gamma \subset \mathbb{R}^N$ we write $x \in \gamma \bmod 1$ if there exists $\mathbf{y} \in \mathbb{Z}^N$ such that $\mathbf{x} - \mathbf{y} \in \gamma$. Moreover, we have to introduce the notion of the Jordan volume of a region $\gamma \subset \mathbb{R}^N$. Therefore, we consider the sets of parallelepipeds γ_1 and γ_2 with sides parallel to the axes and of volume Γ_1 and Γ_2 with $\gamma_2 \subset \gamma \subset \gamma_2$; if there are γ_1 and γ_2 such that $\limsup_{\gamma_1} \Gamma_1$ coincides with $\liminf_{\gamma_2} \Gamma_2$, then γ has the **Jordan volume**

$$\Gamma = \limsup_{\gamma_1} \Gamma_1 = \limsup_{\gamma_2} \Gamma_2.$$

The Jordan sense of volume is more restrictive than the one of Lebesgue, but if the Jordan volume exists it is also defined in the sense of Lebesgue and equal to it.

Weyl [59] proved

Theorem 7.1 *Let $a_1, \ldots, a_N \in \mathbb{R}$ be linearly independent over the field of rational numbers, and let γ be a subregion of the N-dimensional unit cube with Jordan volume Γ. Let $\mathbf{a} = (a_1, \ldots, a_N)$, then*

$$\lim_{T\to\infty} \frac{1}{T} meas\{\tau \in (0,T) \,:\, \tau\mathbf{a} \in \gamma \bmod 1\} = \Gamma.$$

Proof. From the definition of the Jordan measure it follows that for any $\varepsilon > 0$ there exist two finite sets of open parallelepipeds $\{\prod_j^-\}$ and $\{\prod_j^+\}$ inside the unit cube such that

$$\overline{\bigcup_j \prod^-} \subset \operatorname{int}\gamma \subset \bigcup_j \prod^+ \qquad \text{and} \qquad \text{meas}\left(\bigcup_j \prod^+ \setminus \bigcup_j \prod^-\right) < \varepsilon; \tag{7.1}$$

here, as usual, $\overline{M}$ denotes the closure of the set M, and $\operatorname{int} M$ its interior. Denote by $\chi^{\pm}$ the characteristic function of $\bigcup \prod_j^{\pm}$, i.e.

$$\chi^{\pm}(\mathbf{x}) = \begin{cases} 1 & \text{if} \quad \mathbf{x} \in \bigcup \prod_j^{\pm}, \\ 0 & \text{if} \quad \mathbf{x} \notin \bigcup \prod_j^{\pm}. \end{cases}$$

Further, let χ be the characteristic function of $\gamma \bmod 1$. Consequently,

$$0 \le \chi^-(\mathbf{x}) \le \chi(\mathbf{x}) \le \chi^+(\mathbf{x}) \le 1,$$

and

$$\int_{[0,1]^N} (\chi^+(\mathbf{x}) - \chi^-(\mathbf{x}))\, \mathrm{d}\mathbf{x} < \varepsilon,$$

where the integral is N-dimensional with $\mathrm{d}\mathbf{x} = \mathrm{d}x_1 \cdots \mathrm{d}x_N$. Define

$$\Phi(x) = \begin{cases} 0 & \text{if } \ |x| \geq \frac{1}{2}, \\ c\exp\left(-\left(\frac{1}{x+\frac{1}{2}} + \frac{1}{x-\frac{1}{2}}\right)\right) & \text{if } \ |x| < \frac{1}{2}, \end{cases}$$

where c is given by

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \Phi(x)\,\mathrm{d}x = 1.$$

Consequently, $\Phi(x)$ is an infintely differentiable function, and hence the functions, defined by

$$\tilde{\chi}^{\pm}(\mathbf{x}) = \delta^{-N} \int_{[0,1]^N} \chi^{\pm}(\mathbf{y}) \Phi\left(\frac{x_1 - y_1}{\delta}\right) \cdots \Phi\left(\frac{x_N - y_N}{\delta}\right) \mathrm{d}\mathbf{y}.$$

for $0 < \delta < 1$, are infinitely differentiable functions, too. From (7.1) it follows that for sufficiently small δ we have

$$0 \leq \tilde{\chi}^{-}(\mathbf{x}) \leq \chi(\mathbf{x}) \leq \tilde{\chi}^{+}(\mathbf{x}) \leq 1,$$

and

$$0 \leq \int_{[0,1]^N} (\tilde{\chi}^{+}(\mathbf{x}) - \tilde{\chi}^{-}(\mathbf{x}))\,\mathrm{d}\mathbf{x} < 2\varepsilon. \tag{7.2}$$

We conclude

$$\int_0^T \tilde{\chi}^{-}(\tau\mathbf{a})\,\mathrm{d}\tau \leq \mathrm{meas}\,\{(\tau \in (0,T)\,:\,\tau\mathbf{a} \in \gamma \bmod 1\} \leq \int_0^T \tilde{\chi}^{+}(\tau\mathbf{a})\,\mathrm{d}\tau \tag{7.3}$$

and

$$0 \leq \int_0^T \tilde{\chi}^{+}(\tau\mathbf{a})\,\mathrm{d}\tau - \int_0^T \tilde{\chi}^{-}(\tau\mathbf{a})\,\mathrm{d}\tau \leq 2\varepsilon T.$$

Both integrands above are infinitely differentiable functions which are 1-periodic in each variable. Thus, we have the Fourier expansion

$$\tilde{\chi}^{\pm}(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^N} c_{\mathbf{n}}^{\pm} \exp(2\pi i \mathbf{n} \cdot \mathbf{x}),$$

where

$$c_{\mathbf{n}}^{\pm} = \int_{[0,1]^N} \tilde{\chi}^{\pm}(\mathbf{x}) \exp(-2\pi i \mathbf{n} \cdot \mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Note that $c_{\mathbf{0}}^{\pm}$ is the volume of $\bigcup \prod_j^{\pm}$. Integration by parts gives

$$c_{\mathbf{n}}^{\pm} \ll \prod_{j=1}^{N} (|n_j| + 1)^{-k} \qquad \text{for} \quad k = 1, 2, \ldots,$$

where the implicit constant depends only on k. This shows that the Fourier series converges absolutely, and hence, for every $\varepsilon > 0$ there exists a finite set $\mathcal{M} \subset \mathbb{Z}^N$ such that

$$\tilde{\chi}^{\pm}(\mathbf{x}) = \sum_{\mathbf{n} \in \mathcal{M}} c_{\mathbf{n}}^{\pm} \exp(2\pi i \mathbf{n} \cdot \mathbf{x}) + R(\mathbf{x}) \qquad \text{with} \quad |R(\mathbf{x})| < \varepsilon.$$

This yields

$$\frac{1}{T}\int_0^T \tilde{\chi}^{\pm}(\tau\mathbf{a})\,\mathrm{d}\tau = \frac{1}{T}\int_0^T \sum_{\mathbf{n}\in\mathcal{M}} c_{\mathbf{n}}^{\pm}\exp(2\pi i\tau\mathbf{n}\cdot\mathbf{x}))\,\mathrm{d}\tau + \theta\varepsilon$$

with $|\theta| < 1$. Consequently,

$$\frac{1}{T}\int_0^T \tilde{\chi}^{\pm}(\tau\mathbf{a})\,\mathrm{d}\tau = c_{\mathbf{0}}^{\pm} + \sum_{\mathbf{0}\neq\mathbf{n}\in\mathcal{M}} \frac{c_{\mathbf{n}}^{\pm}}{T}\int_0^T \exp(2\pi i\tau\mathbf{n}\cdot\mathbf{a})\,\mathrm{d}\tau + \theta\varepsilon.$$

Since the a_n are linearly independent over $\mathbb{Q}$, we have $\mathbf{n}\cdot\mathbf{a}\neq 0$ for $\mathbf{n}\neq 0$. It follows for such $\mathbf{n}$ that

$$\int_0^T \exp(2\pi i\tau\mathbf{n}\cdot\mathbf{a}))\,\mathrm{d}\tau \ll 1.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \tilde{\chi}^{\pm}(\tau\mathbf{a})\,\mathrm{d}\tau = c_{\mathbf{0}}^{\pm}.$$

This gives with regard to (7.3)

$$\begin{aligned} c_{\mathbf{0}}^{-} - \varepsilon &\leq \liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\,\{(\tau\in(0,T)\ :\ \tau\mathbf{a}\in\gamma \bmod 1\} \\ &\leq \limsup_{T\to\infty}\frac{1}{T}\mathrm{meas}\,\{(\tau\in(0,T)\ :\ \tau\mathbf{a}\in\gamma \bmod 1\} \leq c_{\mathbf{0}}^{+} + \varepsilon \end{aligned}$$

for any positive ε. From (7.2) it follows that $0 \leq c_{\mathbf{0}}^{+} - c_{\mathbf{0}}^{-} \leq 2\varepsilon$. Now sending $\varepsilon\to 0$, the theorem is proved. •

As an immediate consequence of Theorem 7.1 we get the classical inhomogeneous **Kronecker approximation theorem**:

Corollary 7.2 *Let $\alpha_1,\ldots,\alpha_N\in\mathbb{R}$ be linearly independent over the field of rationals, let $\beta_1,\ldots,\beta_N$ be arbitrary real numbers, and let q be a positive number. Then there exists a number $\tau > 0$ and integers $x_1,\ldots,x_N$ such that*

$$|\tau\alpha_n - \beta_n - x_n| < \frac{1}{q} \qquad \textit{for} \quad 1\leq n\leq N.$$

We give an application to the value distribution of the zeta-function. As we have seen above $\zeta(s)$ is in the half-plane $\sigma > 1$ given by an absolute convergent Dirichlet series. However, the value distribution of the zeta-function in that region is anything but boring. Answering a question of Hilbert, H. Bohr and Landau [10], [13] showed that $|\zeta(s)|$ takes arbitrarily large and arbitrarily small values in $\sigma > 1$ - in spite of the absence of zeros!

Theorem 7.3 *For any $\varepsilon > 0$ and any real θ there exists an infinite sequence of $s = \sigma + it$ with $\sigma \to 1+$ and $t \to \infty$ such that*

$$Re\,\{\exp(i\theta)\log\zeta(\sigma+it)\} \geq (1-\varepsilon)\log\zeta(\sigma) + O(1).$$

In particular,

$$\liminf_{\sigma>1,t\geq 1} |\zeta(s)| = 0 \qquad \textit{and} \qquad \limsup_{\sigma>1,t\geq 1} |\zeta(s)| = \infty.$$

Our proof differs slightly from the original one.

Proof. By (4.5) one easily finds

$$\log\zeta(s) = \sum_p \frac{1}{p^s} + O(1) \qquad \text{for} \quad \sigma > 1. \tag{7.4}$$

Thus, we have for any $t \geq 1$ and $x \geq 2$

$$\text{Re}\,\{\exp(i\theta)\log\zeta(\sigma+it)\} \;\geq\; \sum_{p\leq x} \frac{\cos(t\log p - \theta)}{p^\sigma} - \sum_{p>x} \frac{1}{p^\sigma} + O(1). \tag{7.5}$$

Here we use diophantine approximation. By the unique (!) prime factorization of the integers the logarithms of the prime numbers are linearly independent. Denote by p_n the n-th prime, then Kronecker's approximation theorem implies that for any given integers q, N the existence of some real number $\tau > 0$ and integers $x_1, \ldots, x_N$ with

$$\left|\tau\frac{\log p_n}{2\pi} - \frac{\theta}{2\pi} - x_n\right| < \frac{1}{q} \quad \text{for} \quad n = 1, \ldots, N.$$

Obviously, we get with $q \to \infty$ infinitely many τ with the above property. Setting $N = \pi(x)$, we obtain

$$\cos(\tau\log p - \theta) \geq \cos\left(\frac{2\pi}{q}\right) \quad \text{for all} \quad p \leq x,$$

provided that $q \geq 4$. Therefore, we deduce from (7.5)

$$\text{Re}\,\{\exp(i\theta)\log\zeta(\sigma+i\tau)\} \geq \cos\left(\frac{2\pi}{q}\right)\sum_{p\leq x}\frac{1}{p^\sigma} - \sum_{p>x}\frac{1}{p^\sigma} + O(1),$$

resp.

$$\text{Re}\,\{\exp(i\theta)\log\zeta(\sigma+i\tau)\} \geq \cos\left(\frac{2\pi}{q}\right)\log\zeta(\sigma) - 2\sum_{p>x}\frac{1}{p^\sigma} + O(1) \tag{7.6}$$

in view of (7.4). Sending $q, x \to \infty$ we obtain the estimate of the theorem. In view of the simple pole of $\zeta(s)$ at $s = 1$ we get with $\theta = 0$, resp. $\theta = \pi$, by sending $\sigma \to 1+$ the further assertions follow. •

It is even possible to give quantitative estimates for the rate of divergence; see [55], §8.6, and [51].

We conclude with the notion of uniform distribution modulo 1. Let $\gamma(\tau)$ be a continuous function with domain of definition $[0,\infty)$ and range $\mathbb{R}^N$. Then the curve $\gamma(\tau)$ is said to be **uniformly distributed** $\bmod 1$ in $\mathbb{R}^N$ if for every parallelepiped $\prod = [\alpha_1,\beta_1]\times\ldots\times[\alpha_N,\beta_N]$ with $0\le\alpha_j<\beta_j\le 1$ for $1\le j\le N$

$$\lim_{T\to\infty}\frac{1}{T}\text{meas}\,\{(\tau\in(0,T)\,:\,\gamma(\tau)\in\prod \bmod 1\}=\prod_{j=1}^{N}(\beta_j-\alpha_1).$$

In a sense, a curve is uniformly distributed mod 1 if the *right* proportion of values lies in a given subset of the unit cube.

Since in questions about uniform distribution mod 1 one is interested in the fractional part only, we define for a curve $\gamma(\tau)$ in $\mathbb{R}^N$

$$\{\gamma(\tau)\}=(\gamma_1(\tau)-[\gamma_1(\tau)],\ldots,\gamma_N(\tau)-[\gamma_N(\tau)]);$$

recall that $[x]$ is the integral part of $x\in\mathbb{R}$.

Theorem 7.4 *Suppose that the curve $\gamma(\tau)$ is uniformly distributed* mod 1 *in $\mathbb{R}^N$. Let D be a closed and Jordan measurable subregion of the unit cube in $\mathbb{R}^N$ and let Ω be a family of complex-valued continuous functions defined on D. If Ω is uniformly bounded and equicontinuous, then*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(\{\gamma(\tau)\})\chi_D^{\gamma}(\tau)\,d\tau=\int_D f(\mathbf{x})\,d\mathbf{x}$$

uniformly with respect to $f\in\Omega$, where $\chi_D^{\gamma}(\tau)$ is equal to 1 if $\gamma(\tau)\in D$ mod 1, *and zero otherwise.*

Proof. By the definition of the Riemann-integral as a limit of Riemann sums, we have for any Riemann integrable function F on the unit cube in $\mathbb{R}^N$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T F(\{\gamma(\tau)\})\,\mathrm{d}\tau=\int_{[0,1]^N}F(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{7.7}$$

By the assumptions on Ω, for every $\varepsilon>0$ there exist $f_1,\ldots,f_n\in\Omega$ such that for every $f\in\Omega$ there is an f_j with $1\le j\le n$, and

$$\sup_{\mathbf{x}\in D}|f(\mathbf{x})-f_j(\mathbf{x})|<\varepsilon.$$

By (7.7) there exists T_0 such that for any $T>T_0$ and for each function $f_1,\ldots,f_n$ one has

$$\left|\int_D f_j(\mathbf{x})\,\mathrm{d}\mathbf{x}-\frac{1}{T}\int_0^T f_j(\{\gamma(\tau)\})\chi_D^{\gamma}(\tau)\,\mathrm{d}\tau\right|<\varepsilon.$$

Now, for any $f\in\Omega$,

$$\begin{aligned}&\left|\int_D f(\mathbf{x})\,\mathrm{d}\mathbf{x}-\frac{1}{T}\int_0^T f(\{\gamma(\tau)\})\chi_D^\gamma(\tau)\,\mathrm{d}\tau\right|\\ &\le\ \left|\int_D f_j(\mathbf{x})\,\mathrm{d}\mathbf{x}-\frac{1}{T}\int_0^T f_j(\{\gamma(\tau)\})\chi_D^\gamma(\tau)\,\mathrm{d}\tau\right|+\left|\int_D (f(\mathbf{x})-f_j(\mathbf{x}))\,\mathrm{d}\mathbf{x}\right|\\ &\quad+\frac{1}{T}\left|\int_0^T (f(\{\gamma(\tau)\})-f_j(\{\gamma(\tau)\}))\chi_D^\gamma(\tau)\,\mathrm{d}\tau\right|\end{aligned}$$

By the estimates above, this is bounded by 3ε. Since $\varepsilon>0$ is arbitrary, the assertion of the theorem follows. •

In the next section we shall meet the heart of Voronin's universality theorem.

8 Conditionally convergent series

A series $\sum_n a_n$ of real numbers a_n is called **conditionally convergent** if $\sum_n |a_n|$ is divergent but $\sum_n a_n$ is convergent for an appropiate rearrangement of the terms a_n. Riemann proved that *any conditionally convergent series can be rearranged such that its sum converges to an arbitrary preassigned real number*; see [2]; hence, every convergent series, which does not converge absolutely, is conditionally convergent. For instance, to any given $c\in\mathbb{R}$ there exists a permutation π of $\mathbb{N}$ (i.e. a one-to-one mapping on $\mathbb{N}$) such that

$$\sum_{n=1}^\infty \frac{(-1)}{\pi(n)}=c;$$

here and in what follows do not confuse the permutation π with the prime counting function. Thus, all conditionally convergent series are in a certain sense *universal* with respect to $\mathbb{R}$!

It is the aim of this section to extended Riemann's rearrangement theorem to Hilbert spaces; recall that a complete normed linear space with inner product is called **Hilbert space**. We shall give an example which will be of interest later on. Let R be a positive real number, then the **Hardy space** $\mathcal{H}_2^R$ is the set of functions $f(s)$ which are analytic for $|s|<R$ and for which

$$\|f\|:=\lim_{r\to R-}\int\int_{|s|<r}|f(s)|\,\mathrm{d}\sigma\,\mathrm{d}t<\infty.$$

We define on $\mathcal{H}_2^R$ an inner product by

$$\langle f,g\rangle\ =\mathrm{Re}\int\int_{|s|\le R} f(s)\overline{g(s)}\,\mathrm{d}\sigma\,\mathrm{d}t. \tag{8.1}$$

This makes $\mathcal{H}_2^R$ into a real Hilbert space.

Pechersky [45] proved

Theorem 8.1 *Suppose that a series $\sum_n u_n$ of vectors in a real Hilbert space $\mathcal{H}$ satisfies the condition*

$$\sum_{n=1}^{\infty} \|u_n\|^2 < \infty,$$

and for any $e \in \mathcal{H}$ with $\|e\| = 1$ the series $\sum_n \langle u_n, e\rangle$ converges conditionally. Then for any $v \in \mathcal{H}$ there is a permutation π of $\mathbb{N}$ such that

$$\sum_{n=1}^{\infty} u_{\pi(n)} = v$$

in the norm of $\mathcal{H}$.

It is obvious how the notion of *conditionally convergent* series has to be extended to real Hilbert spaces.

The proof is a bit more complicated than the one for Riemann's rearrangement theorem but the idea behind is still the same. We start with

Lemma 8.2 *Under the assumptions of Theorem 8.1, for any $v \in \mathcal{H}$ and any $\varepsilon > 0$ there exist a positive integer N and numbers $\varepsilon_1, \ldots, \varepsilon_N$, equal to 0 or 1, such that*

$$\Big\| s - \sum_{n=1}^{N} \varepsilon_n u_n \Big\| < \varepsilon.$$

Proof. We choose an integer m so that

$$\sum_{n=m}^{\infty} \|u_n\|^2 < \frac{\varepsilon^2}{9}.$$

Denote by P_m the set of all linear combinations

$$\sum_{n=m}^{N} \lambda_n u_n \qquad \text{with} \quad \lambda_n \in [0,1] \quad \text{and} \quad N = m, m+1, m+2, \ldots;$$

obviously, P_m is convex. Let $\overline{P_m}$ be the closure of P_m with respect to the norm of $\mathcal{H}$; consequently $\overline{P_m}$ is a closed convex set. Now we shall show that $\overline{P_m}$ coincides with $\mathcal{H}$.

The *seperation theorem* for linear operators states that *if X is a normed linear space and D is a convex subset of X which is closed in the norm of X, then for any $s \in X \setminus D$ there exist $\varepsilon > 0$ and a linear functional F on X such that*

$$F(x) \le F(s) - \varepsilon \qquad \textit{for all} \quad x \in D.$$

The proof follows from the well-known *theorem of Hahn-Banach*, which relates linear functionals to convex sets; see [14], §V.2.7. A simple consequence is that for any proper convex subset D of real Hilbert space $\mathcal{H}$, which is closed in the norm of $\mathcal{H}$, there exists a vector $e \in \mathcal{H}$ with $\|e\| = 1$ such that

$$\sup_{x \in D} \langle x, e\rangle < \infty.$$

We return to our problem: suppose that $\overline{P_m} \neq \mathcal{H}$, then, by the above reasoning, there exists $e \in \mathcal{H}$ with $\|e\| = 1$ such that $\sup_{x \in \overline{P_m}} \langle x, e \rangle < \infty$. Since, by assumption, the series $\sum_{n \geq m}^{\infty} \langle u_n, e \rangle$ converges conditionally with some arrangement of the terms, the series of the positive terms of the series only is divergent. Thus, for any C there exist an N and a sequence $\varepsilon_m, \ldots, \varepsilon_N$, equal to 0 or 1, such that

$$\sum_{n=m}^{N} \varepsilon_n \langle u_n, e \rangle \ > C.$$

Since $\sum_m^N \varepsilon_n u_n \in \overline{P_m}$, it follows that $\sup_{x \in \overline{P_m}} \langle x, e \rangle = \infty$, giving the contradiction.

So we have shown $\overline{P_m} = \mathcal{H}$. Consequently, there exist $N \geq m$ and $\lambda_m, \ldots, \lambda_N \in [0,1]$ such that

$$\Big\| v - \sum_{n=m}^{N} \lambda_n u_n \Big\| < \frac{\varepsilon}{3}.$$

By induction we can construct $\varepsilon_m, \ldots, \varepsilon_N$, equal to 0 or 1, such that for any M with $m \leq M \leq N$ the inequality

$$\Big\| \sum_{n=m}^{M} \lambda_n u_n - \sum_{n=m}^{M} \varepsilon_n u_n \Big\| \leq \sum_{n=m}^{M} \|u_n\|^2$$

holds. Therefore, we may set $\varepsilon_m = 1$ and suppose that $\varepsilon_m, \ldots, \varepsilon_M$ have been chosen so that the last inequality is fulfilled. With ε_{M+1}, equal to 0 or 1, satisfying

$$(\lambda_{M+1} - \varepsilon_{M+1}) \Big\langle \sum_{n=m}^{M} (\lambda_n - \varepsilon_n) u_n, u_n \Big\rangle \leq 0,$$

we get

$$\Big\| \sum_{n=m}^{M+1} \lambda_n u_n - \sum_{n=m}^{M+1} \varepsilon_n u_n \Big\|^2 \leq \Big\| \sum_{n=m}^{M} (\lambda_n - \varepsilon_n) u_n \Big\|^2 + \|u_{M+1}\|^2 \leq \sum_{n=m}^{M+1} \|u_n\|^2.$$

In particular,

$$\Big\| \sum_{n=m}^{N} \lambda_n u_n - \sum_{n=m}^{N} \varepsilon_n u_n \Big\|^2 \leq \sum_{n=m}^{N} \|u_n\|^2 < \frac{\varepsilon^2}{9},$$

which proves the lemma. •

The next step is

Lemma 8.3 *Under the assumptions of Theorem 8.1, there exists a permutation $\{n_k\}$ of $\mathbb{N}$ such that some subsequence of the partial sums of the series $\sum_k u_{n_k}$ converges to v in the norm of $\mathcal{H}$.*

Proof. Let $n_1 = 1$. Applying Lemma 8.2 to the series $\sum_{n \geq 2} u_n$, yields the existence of a finite set $T_1 \subset \{2, 3, \ldots\}$ such that

$$\Big\| v - u_1 - \sum_{n \in T_1} u_n \Big\| < \frac{1}{2}.$$

Now write the indices in T_1 in an arbitrary order. If $2 \notin T_1$, then write 2. Denote by T_2 the set of all indices we have so far, and define $N_1 = \max\{n \in T_2\}$. Applying (8.2) to the series $\sum_{n=N_1+1}^{\infty} u_n$, shows that there exists a finite set $T_3 \subset \{N_1+1, N_1+2, \ldots\}$ such that

$$\Big\|v - \sum_{n \in T_2} u_n - \sum_{n \in T_3} u_n\Big\| < \frac{1}{4}.$$

Continuing this process, the assertion of the lemma follows. •.

Further, we have to prove

Lemma 8.4 *Let $v_1, \ldots, v_N$ be arbitrary elements in a real Hilbert space $\mathcal{H}$. Then there exists a permutation π of the set $\{1, \ldots, N\}$ such that*

$$\max_{1 \le m \le N} \Big\| \sum_{n=1}^{N} v_{\pi(n)} \Big\| \le \Big(\sum_{n=1}^{N} \|v_n\|^2 \Big)^{\frac{1}{2}} + 2 \Big\| \sum_{n=1}^{N} v_n \Big\|.$$

Proof. First, suppose that

$$\sum_{n=1}^{N} v_n = 0.$$

Then we shall construct by induction a permutation $\{n_1, \ldots, n_N\}$ of $\{1, \ldots, N\}$ such that

$$\max_{1 \le m \le N} \Big\| \sum_{k=1}^{m} v_{n_k} \Big\| \le \Big(\sum_{n=1}^{N} \|v_n\|^2 \Big)^{\frac{1}{2}}. \tag{8.2}$$

Therefore, set $n_1 = 1$ and suppose that $n_1, \ldots, n_j$ with $1 \le j \le N-1$ have been chosen, satisfying

$$\max_{1 \le m \le j} \Big\| \sum_{k=1}^{m} v_{n_k} \Big\|^2 \le \sum_{n=1}^{j} \|v_n\|^2.$$

Then we choose n_{j+1} from the remaining numbers such that

$$\langle \sum_{k=1}^{j} v_{n_k}, v_{n_{j+1}} \rangle \le 0;$$

obviously, such an n_{j+1} exists since otherwise

$$\sum_{j \ne n_k} \Big\langle \sum_{k=1}^{j} v_{n_k}, v_j \Big\rangle = \Big\langle \sum_{k=1}^{j} v_{n_k}, -\sum_{k=1}^{j} v_{n_k} \Big\rangle > 0.$$

Hence,

$$\Big\| \sum_{k=1}^{j+1} v_{n_k} \Big\|^2 \le \sum_{k=1}^{j+1} \|v_{n_k}\|^2.$$

This yields the existence of a permutation π such that (8.2) holds under the assumption $\sum_{n=1}^{N} v_n = 0.$

For arbitrary $v_1,\ldots,v_N$ define

$$v_{N+1}=-\sum_{n=1}^N v_n,$$

and apply the already proved fact; obviously this leads to the additional term

$$\|v_{N+1}\|^2=\Big\|\sum_{n=1}^N v_n\Big\|^2,$$

multiplied by 2, in (8.2). The lemma is shown. •

Now we are in the position for the

Proof of Theorem 8.1. Without loss of generality we may assume, by Lemma 8.3, that some subsequence of the partial sums of the series $\sum_k u_k$ converges to v in the norm of $\mathcal{H}$; we define

$$U_n=\sum_{k=1}^n u_k,$$

and suppose that the sequence of the U_{n_j} converges to v. For each $j\in\mathbb{N}$ there is a permutation π of the set of vectors $\{U_{n_j+1},\ldots,U_{n_{j+1}}\}$ in such a way that the value of

$$m_j:=\max_{1\leq m\leq n_{j+1}-n_j}\Big\|\sum_{n=n_j+1}^{n_j+m} u_{\pi(n)}\Big\|$$

is minimal. By Lemma 8.4 it follows that

$$m_j\leq\left(\sum_{n=n_j+1}^{\infty}\|u_n\|^2\right)^{\frac{1}{2}}+2\|U_{n_{j+1}}-U_{n_j}\|,$$

which obviously tends to zero as $j\to\infty$. Hence, the corresponding series converges to v in the norm of $\mathcal{H}$. Theorem 8.1 is proved. •

In the following section we shall return to the zeta-function and start with the proof of Voronin's universality theorem.

9 Finite Euler products

As we have seen in the beginning the Euler product (2.2) does not represent the zeta-function inside the critical strip. However, as Bohr [7] discovered in his investigations on the value distribution of $\zeta(s)$, an appropriate truncated Euler product (2.2) converges almost everywhere inside the critical strip to the zeta-function; see (10.5) below. This important observation can be used for our approximation problem: if we are able to approximate a given function by a finite Euler product, then we finally have *only* to switch from the finite Euler product to the zeta-function itself!

Let Ω denote the set of all sequences of real numbers indexed by the primes, that are all infinite vectors of the form $\omega := (\omega_2, \omega_3, \ldots)$ with $\omega_p \in \mathbb{R}$. Then we define for any finite subset M of the set of all primes, any $\omega \in \Omega$ and all complex s

$$\zeta_M(s,\omega) = \prod_{p\in M} \left(1 - \frac{\exp(-2\pi i \omega_p)}{p^s}\right)^{-1}.$$

Obviously, $\zeta_M(s,\omega)$ is an analytic function in s without zeros in the half-plane $\sigma > 0$. Consequently, its logarithm exists and equals

$$\log \zeta_M(s,\omega) = -\sum_{p\in M} \log\left(1 - \frac{\exp(-2\pi i \omega_p)}{p^s}\right)$$

(out of technical reasons we prefer to consider series than products); here as for $\log\zeta(s)$ we may take the principal branch of the logarithm.

The first step in the proof of Voronin's theorem is to show

Theorem 9.1 *Let $0 < r < \frac{1}{4}$ and suppose that $g(s)$ is continuous on $|s| \leq r$ and analytic in the interior. Further, let $\omega_0 = \left(\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \ldots\right)$. Then for any $\varepsilon > 0$ and any $y > 0$ there exists a finite set M of prime numbers, containing at least all primes $p \leq y$, such that*

$$\max_{|s|\leq r} \left|\log \zeta_M\left(s + \frac{3}{4}, \omega_0\right) - g(s)\right| < \varepsilon.$$

Unfortunately, the proof makes use of some classical results from analysis, which proofs are beyond the scope of this course.

Proof. Since $g(s)$ is continuous for $|s| \leq r$, there exists $\kappa > 1$ such that $\kappa^2 r < \frac{1}{4}$ and

$$\max_{|s|\leq r} \left|g\left(\frac{s}{\kappa^2}\right) - g(s)\right| < \frac{\varepsilon}{2}. \tag{9.1}$$

The function $g\left(\frac{s}{\kappa^2}\right)$ is bounded on the disc $|s| \leq \kappa r =: R$, and thus belongs to the Hardy space $\mathcal{H}_2^R$.

Denote by p_k the k-th prime number. We consider the series

$$\sum_{k=1}^{\infty} u_k(s) \qquad \text{with} \quad u_k(s) := \log\left(1 - \frac{\exp(-2\pi i \omega_{p_k})}{p_k^{s+\frac{3}{4}}}\right)^{-1}.$$

First, we shall prove that for every $v \in \mathcal{H}_2^R$ there exists a rearrangement of the series $\sum u_k(s)$ for which

$$\sum_{k=1}^{\infty} u_{j_k}(s) = v(s).$$

In view of the Taylor expansion of the logarithm the series $\sum_k u_k(s)$ differs from

$$\sum_{k=1}^{\infty} \eta_k(s) = \sum_{k=1}^{\infty} \exp\left(-\frac{2\pi i k}{4}\right) p_k^{-s-\frac{3}{4}}$$

by an absolute convergent series. Therefore, it is sufficient to verify the conditions of the rearrangement theorem 8.1 for the series $\sum_k \eta_k(s)$. Since $R < \frac{1}{4}$,

$$\sum_{k=1}^{\infty} \|\eta_k\|^2 \ll \sum_p \frac{1}{p^{\frac{3}{2}}} < \infty.$$

Further, we have to check that for any $\varphi \in \mathcal{H}_2^R$ with $\|\varphi\|^2 = 1$ the series

$$\sum_{k=1}^{\infty} \langle \eta_k, \varphi \rangle \tag{9.2}$$

is conditionally convergent for some rearrangement of its terms. With view to the Cauchy-Schwarz inequality,

$$\sum_{k=1}^{\infty} \langle \eta_k, \varphi \rangle \leq \Big\| \sum_{k=1}^{\infty} \eta_k \Big|^{\frac{1}{2}} \cdot \|\varphi\big|^{\frac{1}{2}} = \Big\| \sum_{k=1}^{\infty} \eta_k \Big|^{\frac{1}{2}} < \infty,$$

it suffices to show that there exist two subseries of (9.2), where one is diverging to $+\infty$ and the other one to $-\infty$.

By (8.1),

$$\begin{aligned} \langle \eta_k, \varphi \rangle &= \mathrm{Re} \int\!\!\int_{|s|\leq R} \exp\left(-\frac{2\pi i k}{4}\right) p_k^{-s-\frac{3}{4}} \overline{\varphi(s)}\, \mathrm{d}\sigma\, \mathrm{d}t \\ &= \mathrm{Re} \left\{ \exp\left(-\frac{2\pi i k}{4}\right) \int\!\!\int_{|s|\leq R} p_k^{-s-\frac{3}{4}} \overline{\varphi(s)}\, \mathrm{d}\sigma\, \mathrm{d}t \right\}. \end{aligned} \tag{9.3}$$

This shows

$$\lim_{k\to\infty} \langle \eta_k, \varphi \rangle = 0.$$

Now define

$$\Delta(x) = \int\!\!\int_{|s|\leq R} \exp\left(-x\left(s + \frac{3}{4}\right)\right) \overline{\varphi(s)}\, \mathrm{d}\sigma\, \mathrm{d}t,$$

then the integral appearing on the right handside of (9.3) equals $\Delta(\log p_k)$. Further, let $\varphi(s) = \sum_{m=0}^{\infty} \alpha_m s^m$, then we may express $\Delta(x)$ in terms of the Taylor coefficients α_m as follows: obviously,

$$\begin{aligned} \Delta(x) &= \exp\left(-\frac{3x}{4}\right) \int\!\!\int_{|s|\leq R} \exp(-sx) \overline{\varphi(s)}\, \mathrm{d}\sigma\, \mathrm{d}t \\ &= \exp\left(-\frac{3x}{4}\right) \int\!\!\int_{|s|\leq R} \sum_{n=0}^{\infty} \frac{(-sx)^n}{n!} \sum_{m=0}^{\infty} \overline{\alpha_m s}^m\, \mathrm{d}\sigma\, \mathrm{d}t \\ &= \exp\left(-\frac{3x}{4}\right) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \overline{\alpha_m} \int\!\!\int_{|s|\leq R} \overline{s}^m s^n\, \mathrm{d}\sigma\, \mathrm{d}t. \end{aligned}$$

Using polar coordinates,

$$\int\!\!\int_{|s|\leq R} \overline{s}^m s^n\, \mathrm{d}\sigma\, \mathrm{d}t = \int_0^R \int_0^{2\pi} \varrho^{m+n} \exp(i\theta(n-m))\, \mathrm{d}\theta\, \mathrm{d}\varrho = \begin{cases} 2\pi \frac{R^{2m+2}}{2m+2} & \text{if} \quad m = n, \\ 0 & \text{if} \quad m \neq n. \end{cases}$$

This yields

$$\begin{aligned}\Delta(x) &= \pi R^2 \exp\left(-\frac{3x}{4}\right)\sum_{m=0}^{\infty}\frac{(-1)^m x^m \overline{\alpha_m}}{m!(m+1)}R^{2m}\\ &= \pi R^2 \exp\left(-\frac{3x}{4}\right)\sum_{m=0}^{\infty}\frac{\beta_m}{m!}(xR)^m, \end{aligned} \tag{9.4}$$

where $\beta_m = (-1)^m \frac{\overline{\alpha_m}R^m}{m+1}$. Since $\|\varphi\| = 1$, we get

$$1 = \int\int_{|s|\le R} |\varphi(s)|^2 \,\mathrm{d}\sigma\,\mathrm{d}t = \sum_{m=0}^{\infty}|\alpha_m|^2 \int\int_{|s|\le R}|s|^{2m}\,\mathrm{d}\sigma\,\mathrm{d}t = \pi R^2 \sum_{m=0}^{\infty}\frac{|\alpha_m|^2}{m+1}R^{2m}.$$

Hence,

$$0 < \sum_{m=0}^{\infty}|\beta_m|^2 \le 1, \tag{9.5}$$

which implies $|\beta_m| \le 1$. The function $F(z)$, given by

$$F(z) = \sum_{m=0}^{\infty}\frac{\beta_m}{m!}z^m,$$

defines an entire function in z.

Now we shall show that for any $\delta > 0$ there exists a sequence of positive real numbers z_j, tending to $+\infty$, for which

$$|F(z_j)| > \exp(-(1+2\delta)z_j). \tag{9.6}$$

Suppose the contrary. Then there is a $\delta \in (0,1)$ and a constant B such that $|F(z)| < B\exp(-(1+2\delta)z)$ for any $z \ge 0$. Consequently,

$$|\exp((1+\delta)z)F(z)| < B\exp(-(1+\delta)z) \qquad \text{for} \quad z \ge 0; \tag{9.7}$$

Since $|\beta_m| \le 1$, this estimate even holds for $z < 0$ by a suitable change of the constant B.

Here we have to apply the *theorem of Paley-Wiener* [44], Theorem X in Section 1, which states that *if $\alpha > 0$, then the identity*

$$G(z) = \int_{-\alpha}^{\alpha} g(\xi)\exp(i\xi z)\,d\xi$$

holds for some function $g(\xi)$ if and only if

$$\int_{-\infty}^{\infty}|G(z)|^2\,dz < \infty$$

and $G(z)$ has an analytic continuation throughout the complex plane satisfying $G(z) \ll \exp((\alpha+\varepsilon)z)$ for any $\varepsilon > 0$, and where the implicit constant may depend on ε (this characterizes all transcendent functions of fixed exponential type $\le \alpha$). Further, we have to make use of *Plancherel's theorem* [46] which states that *under the assumptions on $G(z)$ in the theorem of Paley-Wiener,*

$$g(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(z) \exp(-i\xi z)\, dz$$

almost everywhere in $\mathbb{R}$. The proofs can be found in [44] and [1]; they rely essentially on Fourier theory.

Application of the theorem of Paley-Wiener in our case with $G(z) = \exp(3z)F(z)$ yields, with regard to (9.7), the representation

$$\exp((1+\delta)z)F(z) = \int_{-3}^{3} f(\xi) \exp(i\xi z)\, \mathrm{d}\xi,$$

where $f(\xi)$ is a square integrable function with support on the interval $[-3,3]$. Further, Plancherel's theorem implies

$$f(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(z) \exp((1+\delta)z - i\xi z)\, \mathrm{d}z$$

almost everywhere. Hence, $f(\xi)$ is analytic in a strip covering the real axis. Since the support of $f(\xi)$ lies inside a compact interval, the integral above has to be zero outside this interval. Hence, $F(z)$ has to vanish identically, contradicting the existence of a sequence of positive real numbers z_j with (9.6).

Let $x_j = \frac{z_j}{R}$. Then it follows from (9.4) that

$$|\Delta(x_j)| > \pi R^2 \exp\left(-\frac{3x_j}{4}\right) F(x_j R) \geq \pi R^2 \exp\left(-x_j \left(\frac{3}{4} + R(1+2\delta)\right)\right).$$

Thus, with sufficiently small $\delta' > 0$ we obtain the existence of a sequence of positive real numbers x_j, tending to $+\infty$, for which

$$|\Delta(x_j)| > \exp(-(1-\delta')x_j). \tag{9.8}$$

Now we shall approximate F and Δ by polynomials. Let $N_j = [x_j]+1$ and assume that $x_j - 1 \leq x \leq x_j + 1$. Since $|\beta_m| \leq 1$,

$$\sum_{m=N_j^2+1}^{\infty} \frac{\beta_m}{m!}(xR)^m \ll \frac{(xR)^{N_j^2}}{(N_j^2)!} \sum_{n=0}^{\infty} \frac{(xR)^n}{n!} \ll \frac{N_j^{N_j^2} \exp(N_j)}{(N_j^2)!} \ll \exp(-2x_j),$$

by *Stirling's formula* $n! \sim \sqrt{2\pi n} n^n \exp(-n)$. Similarly,

$$\sum_{m=N_j^2+1}^{\infty} \frac{1}{m!} \left(-\frac{3x}{4}\right)^m \ll \exp(-2x_j).$$

Therefore,

$$F(xR) = \left\{ \sum_{m=0}^{N_j^2} + \sum_{m=N_j^2+1}^{\infty} \right\} \frac{\beta_m}{m!}(xR)^m = P_j(x) + O(\exp(-2x_j))$$

and analogously

$$\exp\left(-\frac{3x}{4}\right) = \tilde{P}_j(x) + O(\exp(-2x_j)),$$

where P_j and $\tilde{P}_j$ are polynomials of degree $\le N_j^2$. This yields in view of (9.4)

$$\Delta(x) = Q_j(x) + O(\exp(-x_j)) \qquad \text{for} \quad x_j - 1 \le x \le x_j + 1,$$

where $Q_j = P_j\tilde{P}_j$ is a polynomial of degree $\le N_j^4$.

In order to find lower bounds for $Q_j(x)$ we have to apply a classical *theorem of A.A. Markov* [43] which states that *if Q is a polynomial of degree N with real coefficients which satisfies the inequality*

$$\max_{-1\le x\le 1} |Q(x)| \le 1,$$

then

$$\max_{-1\le x\le 1} |Q'(x)| \le N^2;$$

for a proof see [43].

We return to the proof of Theorem 9.1. In view of (9.8) suppose that

$$\mu := \max_{x_j-1\le x\le x_j+1} |\mathrm{Re}\ Q_j(x)| > \frac{1}{2}\exp(-(1-\delta')x_j), \tag{9.9}$$

then there exists a $\xi \in [x_j - 1, x_j + 1]$ such that $\mathrm{Re}\ Q_j(\xi) = \mu$. In view of the mean-value theorem from real analysis there exists a κ in between x and ξ for which

$$|\mathrm{Re}\ Q_j(\xi) - \mathrm{Re}\ Q_j(x)| = |\mathrm{Re}\ Q_j'(\kappa)(\xi - x)|.$$

Define $\lambda = N_j^8|\xi - x|$. Markov's theorem, applied to $Q(x) = \frac{1}{\mu}\mathrm{Re}\ Q_j(x - x_j)$, implies

$$|\mathrm{Re}\ Q_j(\xi) - \mathrm{Re}\ Q_j(x)| \le \lambda\mu.$$

If $\lambda \le \frac{1}{2}$, then

$$|\mathrm{Re}\ Q_j(x)| \ge \frac{\mu}{2} \ge \frac{1}{16}\exp(-(1-\delta')x_j)$$

for $|\xi - x| \le \frac{1}{2}N^{-8}$. If (9.9) does not hold, we may argue analogously with $\mathrm{Im}\ Q_j$. In any case it follows that for sufficiently large x_j the intervals $[x_j - 1, x_j + 1]$ contains intervals $[\alpha, \alpha + \beta]$ of length $\ge \frac{1}{200}N_j^{-8}$ all of whose points satisfy at least one of the inequalities

$$|\mathrm{Re}\ \Delta(x)| > \frac{1}{200}\exp(-(1-\delta')x)\ , \quad |\mathrm{Im}\ \Delta(x)| > \frac{1}{200}\exp(-(1-\delta')x); \tag{9.10}$$

in particular

$$x_j - 1 \le \alpha \le x_j + 1 - \frac{1}{2x_j^8} \qquad \text{and} \qquad \frac{1}{2x_j^8} \le \beta \le 2.$$

In order to prove the divergence of a subseries of (9.2) we note that one of the inequalities in (9.10) is satisfied infinitely many often as $x \to \infty$; we may assume that it is the one with the real part. By the prime number theorem 6.2, the interval $[\exp(\alpha), \exp(\alpha+\beta)]$ contains

$$\begin{aligned}
&\int_{\exp(\alpha)}^{\exp(\alpha+\beta)} \frac{\mathrm{d}u}{\log u} + O\left(\exp\left(\alpha - c(\alpha+\beta)^{\frac{1}{9}}\right)\right) \\
= & \frac{\exp(\alpha+\beta)}{\alpha+\beta} - \frac{\exp(\alpha)}{\alpha} + O\left(\exp\left(\alpha - c(\alpha+\beta)^{\frac{1}{9}}\right)\right) \\
= & \frac{\exp(\alpha)}{\alpha}\left(\exp(\beta) - 1 + O\left(\alpha \exp\left(-c(\alpha+\beta)^{\frac{1}{9}}\right)\right)\right)
\end{aligned}$$

many primes, where $c > 0$ is some absolute constant. An easy computation shows

$$\pi(\exp(\alpha+\beta)) - \pi(\exp(\alpha)) \gg \frac{\exp(x_j)}{x_j^9}.$$

Under these prime numbers $p_k \in [\exp(\alpha), \exp(\alpha+\beta)]$ we choose those with $k \equiv 0 \bmod 4$. Since $\omega_{p_k} = \frac{k}{4}$, we get with view to (9.3)

$$\sum_{\substack{k \equiv 0 \bmod 4 \\ \alpha \le \log p_k \le \alpha+\beta}} \langle \eta_k, \varphi \rangle = \sum_{\substack{k \equiv 0 \bmod 4 \\ \alpha \le \log p_k \le \alpha+\beta}} \mathrm{Re}\ \Delta(\log p_k) \gg \exp(-(1-\delta')x_j)\frac{\exp(x_j)}{x_j^9} = \frac{\exp(\delta' x_j)}{x_j^9},$$

which diverges with $x_j \to \infty$. Analogously, one can create a subseries of (9.2) which diverges to $-\infty$.

Thus, we have shown that the series (9.2) satsifies the conditions of Theorem 8.1. Hence, there exists a rearrangement of the series (9.2) for which

$$\sum_{k=1}^{\infty} u_{j_k}(s) = g\left(\frac{s}{\kappa^2}\right). \tag{9.11}$$

Before we can finish the proof of Theorem 9.1 we have to prove the following

Lemma 9.2 *Suppose that $F(s)$ is continuous on $|s| \le R$. Suppose that there is a sequence of analytic functions $f_n(s)$ for which*

$$\lim_{n\to\infty} \int\int_{|s|\le r} |F(s) - f_n(s)|^2\, d\sigma\, dt = 0,$$

then for any $\varepsilon > 0$ there is an integer m such that for any fixed $r \in (0, R)$ and any $n \ge m$

$$\max_{|s|\le r} |F(s) - f(s)| < \varepsilon.$$

Proof. Define $G_n(s) = F(s) - f_n(s)$. By Cauchy's formula,

$$G_n(s)^2 = \frac{1}{2\pi i}\oint_{|s-z|=\varrho} \frac{G_n(z)^2}{z-s}\,\mathrm{d}z = \frac{1}{2\pi}\int_0^{2\pi} G_n^2(s + \varrho\exp(i\theta))\,\mathrm{d}\theta$$

for any $\varrho \leq R$. Fix $0 < r < R$. Taking the absolute modulus and integrating ϱ with $0 < R - r$, we arrive at

$$\begin{aligned} |G_n(s)|^2 \int_0^{R-r} \varrho \, \mathrm{d}\varrho &= \frac{1}{2\pi} \int_0^{R-r} \int_0^{2\pi} |G_n(s + \varrho \exp(i\theta))|^2 \varrho \, \mathrm{d}\theta \, \mathrm{d}\varrho \\ &= \frac{1}{2\pi} \int \int_{|s| \leq R} |G_n(s)|^2 \, \mathrm{d}\sigma \, \mathrm{d}t. \end{aligned}$$

This yields

$$|G_n(s)|^2 \leq \frac{1}{2\pi(R-r)^2} \int \int_{|s| \leq R} |G_n(s)|^2 \, \mathrm{d}\sigma \, \mathrm{d}t \qquad \text{for} \quad |s| \leq R < r.$$

Now the assumption on the limit implies the estimate of the lemma. •

We return to the proof of Theorem 9.1. According to (9.11),

$$\lim_{n \to \infty} \sum_{k=1}^{n} u_{j_k}(s) = g\left(\frac{s}{\kappa^2}\right)$$

in the norm of $\mathcal{H}_2^R$. This implies

$$\lim_{n \to \infty} \int \int_{|s| \leq R} \left| g\left(\frac{s}{\kappa^2}\right) - \sum_{k=1}^{n} u_{j_k}(s) \right|^2 \mathrm{d}\sigma \, \mathrm{d}t = 0$$

uniformly on $|s| \leq R$. Thus, application of Lemma 9.2 shows that for sufficiently large m

$$\max_{|s| \leq R} \left| g\left(\frac{s}{\kappa^2}\right) - \sum_{k=1}^{m} u_{j_k}(s) \right| < \frac{\varepsilon}{2}.$$

Now, by definition, there exists a finite set M, containing without loss of generality all primes $p \leq y$, such that

$$\log \zeta_M\left(s + \frac{3}{4}, \omega_0\right) := \sum_{k=1}^{m} u_{j_k}(s).$$

Hence, in view of (9.1) it follows that

$$\begin{aligned} &\max_{|s| \leq r} \left| \log \zeta_M\left(s + \frac{3}{4}, \omega_0\right) - g(s) \right| \\ &\quad \leq \max\nolimits_{|s| \leq r} \left| \log \zeta_M\left(s + \tfrac{3}{4}, \omega_0\right) - g\left(\tfrac{s}{\kappa^2}\right) \right| + \max\nolimits_{|s| \leq r} \left| g\left(\tfrac{s}{\kappa^2}\right) - g(s) \right| < \varepsilon. \end{aligned}$$

This finishes the proof of Theorem 9.1. •

10 Voronin's universality theorem

The next and main step in the proof of the universality theorem 1.2 is to switch from $\log \zeta_M$ to the logarithm of the zeta-function.

Theorem 10.1 *Let $0 < r < \frac{1}{4}$ and suppose that $g(s)$ is continuous on $|s| \leq r$ and analytic in the interior. Then, for any $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas \left\{ \tau \in [0,T] \, : \, \max_{s\in K} \left| \log \zeta \left(s + \frac{3}{4} + i\tau \right) - g(s) \right| < \varepsilon \right\} > 0.$$

Note that $\log \zeta(s)$ has singularities at the zeros of $\zeta(s)$. The truth of Riemann's hypothesis would imply that all such singularities lie to the left of the *strip of universality* $\frac{1}{2} < \sigma < 1$. However, unconditionally the set of such singularities has in view of the density theorem 4.4 zero density. Therefore the existence of these singularities is negligeable for our observations.

Proof. We choose $\kappa > 1$ and $\varepsilon \in (0,1)$ such that $\kappa r < \frac{1}{4}$ and

$$\max_{|s|\leq r} \left| g\left(\frac{s}{\kappa} \right) - g(s) \right| < \varepsilon_1.$$

Set $Q = \{p \leq z\}$ and let $\mathcal{E} = \{s \, : \, -\kappa r < \sigma \leq 2, -1 \leq t \leq t\}$. We shall estimate

$$\mathcal{I} := \int_T^{2T} \int\int_{\mathcal{E}} \left| \zeta_Q^{-1} \left(s + \frac{3}{4} + i\tau, \mathbf{0} \right) \zeta \left(s + \frac{3}{4} + i\tau \right) - 1 \right|^2 \mathrm{d}\sigma \, \mathrm{d}t \, \mathrm{d}\tau,$$

where $\mathbf{0} = (0, 0, \ldots)$. By Theorem 3.7,

$$\zeta(s + i\tau) = \sum_{n\leq T} \frac{1}{n^{s+i\tau}} + O(T^{-\sigma}).$$

This gives

$$\begin{aligned} \mathcal{I} &= \int\int_{\mathcal{E}+\frac{3}{4}} \int_T^{2T} |\zeta_Q^{-1}(s + i\tau, \mathbf{0})\zeta(s + i\tau) - 1|^2 \, \mathrm{d}\tau \, \mathrm{d}\sigma \, \mathrm{d}t \\ &\ll \int\int_{\mathcal{E}+\frac{3}{4}} \int_T^{2T} \left| \zeta_Q^{-1}(s + i\tau, \mathbf{0}) \sum_{n\leq T} \frac{1}{n^{s+i\tau}} - 1 \right|^2 \mathrm{d}\tau \, \mathrm{d}\sigma \, \mathrm{d}t \\ &\quad + \int\int_{\mathcal{E}+\frac{3}{4}} \int_T^{2T} T^{-\sigma} |\zeta_Q^{-1}(s + i\tau, \mathbf{0})|^2 \, \mathrm{d}\tau \, \mathrm{d}\sigma \, \mathrm{d}t, \end{aligned} \tag{10.1}$$

where $\mathcal{E} + \frac{3}{4}$ is the set of all s with $s - \frac{3}{4} \in \mathcal{E}$. By definition,

$$\zeta_Q^{-1}(s, \mathbf{0}) = \prod_{p\in Q} \left(1 - \frac{1}{p^s} \right) = \sum_{\substack{m=1 \\ p|m \Rightarrow p\in Q}}^{\infty} \frac{\mu(m)}{m^s}.$$

Obviously, we may bound the second term appearing on the right hand side of (10.1) by

$$T^{-2\left(\frac{3}{4}-\kappa r\right)} \max_{s\in\mathcal{E}+\frac{3}{4}} \int_T^{2T} |\zeta_Q^{-1}(s+i\tau,\mathbf{0})|^2 \,\mathrm{d}\tau \ll T^{2\kappa r-\frac{1}{2}} \left|\zeta_Q^{-1}\left(\frac{3}{4}-\kappa r,\mathbf{0}\right)\right|^2.$$

Furthermore, for $T > z$ a simple computation gives

$$\zeta_Q^{-1}(s,\mathbf{0}) \sum_{n\leq T} \frac{1}{n^s} = 1 + \sum_{z<k\leq z^z T} \frac{b_k}{k^s} \qquad \text{with} \quad b_k = \sum_{\substack{m|k \\ p|m\Rightarrow p\in Q;k\leq mT}} 1.$$

By a classical estimate for the divisor function from elementary number theory

$$\sigma_0(n) \ll n^\varepsilon \tag{10.2}$$

(for a proof see [25]). Thus, similarly as in (4.7) we have

$$|b_k| \leq \sigma_0(k) \ll k^\varepsilon \qquad \text{for any} \quad \varepsilon > 0. \tag{10.3}$$

Hence, for $T > z$

$$\begin{aligned}
\int_T^{2T} \left|\zeta_Q^{-1}(s+i\tau,\mathbf{0}) \sum_{n\leq T} \frac{1}{n^{s+i\tau}} - 1\right|^2 \mathrm{d}\tau &= \int_T^{2T} \left|\sum_{z<k\leq z^z T} \frac{b_k}{k^s}\right|^2 \mathrm{d}\tau \\
= \; T \sum_{z<k\leq z^z T} \frac{|b_k|^2}{k^{2\sigma}} &+ O\left(\sum_{0<l<k\leq z^z T} \frac{|b_k b_l|}{(kl)^\sigma} \left|\int_T^{2T} \left(\frac{k}{l}\right)^{i\tau} \mathrm{d}\tau\right|\right),
\end{aligned}$$

Using estimate (10.3) with $\varepsilon = \frac{\varepsilon_1}{2}$, the above is bounded by

$$T\sum_{k>z} \frac{\sigma_0^2(k)}{k^{2\sigma}} + \sum_{0<l<k\leq z^z T} \frac{\sigma_0(k)\sigma_0(l)}{(kl)^\sigma \log\frac{k}{l}} \ll Tz^{1-2\sigma+\varepsilon_1} + (z^z T)^{\varepsilon_1} \sum_{0<l<k\leq z^z T} \frac{1}{(kl)^\sigma \log\frac{k}{l}}.$$

The appearing sum can be estimated by $((z^z T)^{2-2\sigma}+1)\log^2(z^z T)$ as we did in the proof of Theorem 4.1. Thus, we finally arrive at

$$\begin{aligned}
&\int\int_{\mathcal{E}+\frac{3}{4}} \int_T^{2T} \left|\zeta_Q^{-1}(s+i\tau,\mathbf{0}) \sum_{n\leq T} \frac{1}{n^{s+i\tau}} - 1\right|^2 \mathrm{d}\tau \,\mathrm{d}\sigma \,\mathrm{d}t \\
\ll \; &\int\int_{\mathcal{E}+\frac{3}{4}} \left(Tz^{1-2\sigma+\varepsilon_1} + (z^z T)^{\varepsilon_1}((z^z T)^{2-2\sigma}+1)\log^2(z^z T)\right) \mathrm{d}\sigma \,\mathrm{d}t \\
\ll \; &z^{2\kappa r+\varepsilon_1-\frac{1}{2}} T.
\end{aligned}$$

In view of (10.1) we conclude that for any $\varepsilon_2 > 0$ there exists z_0 such that

$$\mathcal{I} \ll \varepsilon_2^4 T, \tag{10.4}$$

provided that $z > z_0$ and T sufficiently large, say $T > T_0$, depending on ε_2 and z. Define

$$\mathcal{A}_T = \left\{\tau \in [T,2T] \,:\, \int\int_{\mathcal{E}+\frac{3}{4}} |\zeta_Q^{-1}(s+i\tau,\mathbf{0})\zeta(s+i\tau) - 1|^2 \,\mathrm{d}\sigma \,\mathrm{d}t \ll \varepsilon_2^2\right\}.$$

Then it follows from (10.1) and (10.4) that for sufficiently large z and T

$$\operatorname{meas} \mathcal{A}_T > (1-\varepsilon_2)T, \tag{10.5}$$

which is surprisingly large; this idea goes back to Bohr [7]. Application of Lemma 9.2 gives for $\tau \in \mathcal{A}_T$

$$\max_{|s|\le r} |\zeta_Q^{-1}(s+i\tau,\mathbf{0})\zeta(s+i\tau)-1| < C\varepsilon_2,$$

where C is a positive constant, depending only on κ. For sufficiently small ε_2 we deduce

$$\max_{|s|\le r} \left|\log\zeta\left(s+\frac{3}{4}+i\tau\right) - \log\zeta_Q\left(s+\frac{3}{4}+i\tau,\mathbf{0}\right)\right| < 2C\varepsilon_2, \tag{10.6}$$

provided $\tau \in \mathcal{A}_T$; here we used a truncated Taylor expansion of the exponential function $\exp z = 1+z+O(|z|^2)$.

By Theorem 9.1 there exists a sequence of finite sets of prime numbers $M_1 \subset M_2 \subset \ldots$ such that $\cup_{k=1}^{\infty} M_k$ contains all primes and

$$\lim_{k\to\infty} \max_{|s|\le\kappa r} \left|\log\zeta_{M_k}\left(s+\frac{3}{4},\omega_0\right) - g\left(\frac{s}{\kappa}\right)\right| = 0. \tag{10.7}$$

Let $\omega' = (\omega_2', \omega_3', \ldots)$. By the continuity of $\log\zeta_M\left(s+\frac{3}{4},\omega_0\right)$, for any $\varepsilon_1 > 0$ there exists a positive δ for which, whenever

$$\|\omega_p - \omega_p'\| < \delta \qquad \text{for all} \quad p \in M_k, \tag{10.8}$$

then

$$\max_{|s|\le\kappa r} \left|\log\zeta_{M_k}\left(s+\frac{3}{4},\omega_0\right) - \log\zeta_{M_k}\left(s+\frac{3}{4},\omega'\right)\right| < \varepsilon. \tag{10.9}$$

Setting

$$\mathcal{B}_T = \left\{\tau \in [T,2T] \,:\, \left\|\tau\frac{\log p}{2\pi} - \omega_p\right\| < \delta\right\},$$

we get

$$\begin{aligned} &\frac{1}{T}\int_{\mathcal{B}}\int\int_{|s|\le\kappa r} \left|\log\zeta_Q\left(s+\frac{3}{4}+i\tau,\mathbf{0}\right) - \log\zeta_{M_k}\left(s+\frac{3}{4}+i\tau,\mathbf{0}\right)\right|^2 \mathrm{d}\sigma\,\mathrm{d}t\,\mathrm{d}\tau \\ &= \int\int_{|s|\le\kappa r}\frac{1}{T}\int_{\mathcal{B}_T} \left|\log\zeta_Q\left(s+\frac{3}{4}+i\tau,\mathbf{0}\right) - \log\zeta_{M_k}\left(s+\frac{3}{4}+i\tau,\mathbf{0}\right)\right|^2 \mathrm{d}\tau\,\mathrm{d}\sigma\,\mathrm{d}t. \end{aligned}$$

Putting $\omega(\tau) = \left(\tau\frac{\log 2}{2\pi}, \tau\frac{\log 3}{2\pi}, \ldots\right)$, we may rewrite the inner integral as

$$\int_{\mathcal{B}_T} \left|\log\zeta_Q\left(s+\frac{3}{4},\omega(\tau)\right) - \log\zeta_{M_k}\left(s+\frac{3}{4}+i\tau,\omega(\tau)\right)\right|^2 \mathrm{d}\tau.$$

Application of Theorem 7.1 to the curve $\gamma(\tau) = \left(\tau\frac{\log 2}{2\pi}, \tau\frac{\log 3}{2\pi}, \dots, \tau\frac{\log p_N}{2\pi}\right)$ (the logarithms of the prime numbers are linearly independent as we have seen in the proof of Theorem 7.3) yields

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T}\int_{\mathcal{B}_T} \left|\log\zeta_Q\left(s+\frac{3}{4},\omega(\tau)\right) - \log\zeta_{M_k}\left(s+\frac{3}{4}+i\tau,\omega(\tau)\right)\right|^2 \,\mathrm{d}\tau \\
&= \int_{\mathcal{D}} \left|\log\zeta_Q\left(s+\frac{3}{4},\omega\right) - \log\zeta_{M_k}\left(s+\frac{3}{4}+i\tau,\omega\right)\right|^2 \,\mathrm{d}\mu
\end{aligned}$$

uniformly in s for $|s| \le \kappa r$, where $\mathcal{D}$ is the subregion of the unit cube in $\mathbb{R}^N$ given by the inequalities (10.8) and $\mathrm{d}\mu$ is the Lebesgue measure. By the definition of $\zeta_M(s,\omega)$ it follows that for $M_k \subset Q$

$$\zeta_Q(s,\omega) = \zeta_{M_k}(s,\omega)\zeta_{Q\setminus M_k}(s,\omega),$$

and thus

$$\begin{aligned}
&\int_{\mathcal{D}} \left|\log\zeta_Q\left(s+\frac{3}{4},\omega\right) - \log\zeta_{M_k}\left(s+\frac{3}{4}+i\tau,\omega\right)\right|^2 \,\mathrm{d}\mu \\
&= \int_{\mathcal{D}} \left|\log\zeta_{Q\setminus M_k}\left(s+\frac{3}{4},\omega\right)\right|^2 \,\mathrm{d}\mu = \operatorname{meas}\mathcal{D}\cdot\int_{[0,1]^N} \left|\log\zeta_{Q\setminus M_k}\left(s+\frac{3}{4},\omega\right)\right|^2 \,\mathrm{d}\mu.
\end{aligned}$$

Since

$$\log\zeta_{Q\setminus M_k}\left(s+\frac{3}{4},\omega\right) = \sum_{p\in Q\setminus M_k}\sum_{n=1}^{\infty} \frac{\exp(-2\pi i\omega_p)}{np^{n\left(s+\frac{3}{4}\right)}},$$

we obtain

$$\int_{[0,1]^N} \left|\log\zeta_{Q\setminus M_k}\left(s+\frac{3}{4},\omega\right)\right|^2 \,\mathrm{d}\mu = \sum_{p\in Q\setminus M_k}\sum_{n=1}^{\infty} \frac{1}{n^2p^{2\sigma+\frac{3n}{2}}}.$$

If M_k contains all primes $\le y_k$, then

$$\sum_{p\in Q\setminus M_k}\sum_{n=1}^{\infty} \frac{\exp(-2\pi i\omega_p)}{n^2p^{2n\sigma+\frac{3n}{2}}} \ll y_k^{2\kappa r-\frac{1}{2}}.$$

Hence, we finally get

$$\begin{aligned}
&\frac{1}{T}\int_{\mathcal{B}_T}\int\int_{|s|\le\kappa r} \left|\log\zeta_Q\left(s+\frac{3}{4}+i\tau,\mathbf{0}\right) - \log\zeta_{M_k}\left(s+\frac{3}{4}+i\tau,\mathbf{0}\right)\right|^2 \,\mathrm{d}\sigma\,\mathrm{d}t\,\mathrm{d}\tau \\
&\quad \ll y_k^{2\kappa r-\frac{1}{2}}\operatorname{meas}\mathcal{D}.
\end{aligned}$$

A further application of Theorem 7.1 shows

$$\lim_{T\to\infty}\frac{1}{T}\operatorname{meas}\mathcal{B}_T = \operatorname{meas}\mathcal{D},$$

which implies for y_k sufficiently large

$$\begin{aligned}
\operatorname{meas}\Bigg\{\tau\in\mathcal{B}_T \,:\, \int\int_{|s|\le\kappa r} &\left|\log\zeta_Q\left(s+\frac{3}{4}+i\tau,\mathbf{0}\right) - \log\zeta_{M_k}\left(s+\frac{3}{4}+i\tau,\mathbf{0}\right)\right|^2 \,\mathrm{d}\sigma\,\mathrm{d}t \\
&\qquad < y_k^{\kappa r-\frac{1}{4}}\Bigg\} > \frac{\operatorname{meas}\mathcal{D}}{2}T.
\end{aligned}$$

The curve $\gamma(\tau)$ is uniformly distributed mod 1. Thus, application of Theorem 7.4 yields

$$\text{meas}\left\{\tau\in\mathcal{B}_T\,:\,\max_{|s|\le\kappa r}\left|\log\zeta_Q\left(s+\frac{3}{4}+i\tau,\mathbf{0}\right)-\log\zeta_{M_k}\left(s+\frac{3}{4}+i\tau,\mathbf{0}\right)\right|\right.$$
$$\left.<y_k^{\frac{1}{5}\left(\kappa r-\frac{1}{4}\right)}\right\}>\frac{\text{meas}\,\mathcal{D}}{2}T. \quad (10.10)$$

If we now take $0<\varepsilon_2<\frac{1}{2}\text{meas}\,\mathcal{D}$, then (10.5) implies

$$\text{meas }\mathcal{A}_T\cap\mathcal{B}_T>0.$$

Thus, in view of (10.7) we may approximate $g\left(\frac{s}{\kappa}\right)$ by $\log\zeta_{M_k}\left(s+\frac{3}{4},\mathbf{0}\right)$ (independent on τ), with (10.9) and (10.10) the latter function by $\log\zeta_Q\left(s+\frac{3}{4},\mathbf{0}\right)$, and finally with regard to (10.6) by $\log\zeta\left(s+\frac{3}{4}+i\tau\right)$ on a set of τ with positive measure. The theorem is proved. •

It is obvious what we have to do to get rid of the logarithm in Theorem 10.1. Let $f(s)=\exp(g(s))$; it is fundamental in complex analysis that *every analytic function $f(s)$ without zeros has an analytic logarithm $g(s)$*. Obviously,

$$f(s)-\zeta\left(s+\frac{3}{4}\right)=f(s)\left(1-\exp\left(\log\zeta\left(s+\frac{3}{4}\right)-g(s)\right)\right).$$

Let $\varepsilon>0$ and $\kappa>1$ such that $\kappa r<\frac{1}{4}$ and

$$\max_{|s|\le r}\left|f(s)-f\left(\frac{s}{\kappa}\right)\right|<\varepsilon.$$

Then, since $\exp z=1-z+O(|z|^2)$, we obtain

$$\begin{aligned}\max_{|s|\le r}\left|\zeta\left(s+\frac{3}{4}+i\tau\right)-f\left(\frac{s}{\kappa}\right)\right| &= \max_{|s|\le r}\left|\zeta\left(s+\frac{3}{4}+i\tau\right)-\exp\left(g\left(\frac{s}{\kappa}\right)\right)\right|\\ &\le \max_{|s|\le r}\left|f\left(\frac{s}{\kappa}\right)\right|\cdot\max_{|s|\le r}\left|\exp\left(\log\zeta_M\left(s+\frac{3}{4},\omega_0\right)-g\left(\frac{s}{\kappa}\right)\right)-1\right|\\ &\le \max_{|s|\le r}\left|f\left(\frac{s}{\kappa}\right)\right|\cdot\max_{|s|\le r}\left|\log\zeta_M\left(s+\frac{3}{4},\omega_0\right)-g\left(\frac{s}{\kappa}\right)\right|,\end{aligned}$$

which can be made sufficiently small by Theorem 10.1. Thus, we finally have proved Voronin's theorem:

Corollary 10.2 *Let $0<r<\frac{1}{4}$ and suppose that $f(s)$ is a continuous non-vanishing function on $|s|\le r$ which is analytic in the interior. Then, for any $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\textit{meas}\left\{\tau\in[0,T]\,:\,\max_{|s|\le r}\left|\zeta\left(s+\frac{3}{4}+i\tau\right)-f(s)\right|<\varepsilon\right\}>0.$$

Surprisingly, the set of translates on which $\zeta(s)$ approximate a given function $f(s)$ with arbitrary precision has a positive lower density! This improves Theorem 1.2 from the introduction significantly.

Bagchi [3] extended Voronin's result significantly in different ways after some first progress due to Reich [47]. By that it was possible to replace the discs by arbitrary compact subsets of the strip $\frac{1}{2} < \sigma < 1$ with connected complement. Actually, Bagchi found a new and very transparent proof by using limit theorems for weakly convergent probability measures. This approach was completed and extended by Laurinčikas in various details and directions; see [28]. The strongest version of Voronin's universality theorem is

Theorem 10.3 *Let $\mathcal{K}$ be a compact subset of $\frac{1}{2} < \sigma < 1$ with connected complement and suppose that $f(s)$ is a continuous non-vanishing function on $\mathcal{K}$ which is analytic in the interior. Then, for any $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas \left\{\tau \in [0,T] \,:\, \max_{s\in\mathcal{K}} |\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0.$$

11 Functional independence

We state some consequences of the universality property of the zeta-function. We start with a classical result due to Bohr [7], namely that the set of values taken by $\zeta(s)$ on a vertical line $\sigma \in (\frac{1}{2}, 1)$ lies dense in the complex plane. This can be extended to

Theorem 11.1 *Let $\frac{1}{2} < \sigma < 1$ be fixed, then the sets*

$$\{(\log\zeta(s), (\log\zeta(s))', \ldots, (\log\zeta(s))^{(n-1)}) \,:\, t \in \mathbb{R}\}$$

and

$$\{(\zeta(s), \zeta'(s), \ldots, \zeta^{(n-1)}(s)) \,:\, t \in \mathbb{R}\}$$

lie everywhere dense in $\mathbb{C}^n$.

Proof. Suppose that we are given a vector $(b_0, b_1, \ldots, b_{n-1}) \in \mathbb{C}^n$. Let

$$r = \frac{1}{4} - \frac{1}{2}\min\left\{\sigma - \frac{1}{2}, 1 - \sigma\right\}$$

and define

$$g(s) = \sum_{k=0}^{n-1} \frac{b_k}{k!} s^k.$$

Obviously, $g^{(k)}(0) = b_k$ for $k = 0, 1, \ldots, n-1$. By Cauchy's formula, we have further

$$f^{(k)}(0) = \frac{k!}{2\pi i} \oint_{|s|=\varrho} \frac{f(s)}{s^{k+1}} \,\mathrm{d}s \tag{11.1}$$

for any $\varrho > 0$. With view to Theorem 10.1 the function $g(s)$ can be approximated to arbitrary precision on the disc $|s| \leq r$ by $\log\zeta\left(s + \frac{3}{4} + i\tau\right)$ for some τ. Hence, taking

$$f(s) = g(s) - \log\zeta\left(s + \frac{3}{4} + i\tau\right)$$

and $\varrho < r$ in (11.1), shows that $(\log\zeta(s), \log\zeta'(s), \ldots, \log\zeta(s)^{(n-1)})$ with $\frac{1}{2} < \sigma < 1$ lies somewhere arbtrarily close to $(g(0), g'(0), \ldots, g^{(n-1)}(0)) = (b_0, b_1, \ldots, b_{n-1})$. This implies the statement for the first set.

We use induction on m to prove that for any $(m+1)$-tuple $(a_0, a_1, \ldots, a_m) \in \mathbb{C}^{m+1}$, where $a_0 \neq 0$, there exists $(b_0, b_1, \ldots, b_m) \in \mathbb{C}^{m+1}$ for which

$$\exp\left(\sum_{k=0}^{m} b_k s^k\right) \equiv \sum_{k=0}^{m} \frac{a_k}{k!} s^k \mod s^{m+1}.$$

For $m = 0$ one only has to choose $b_0 = \log a_0$. By the induction assumption, we may assume that with some α

$$\exp\left(\sum_{k=0}^{m} b_k s^k\right) \equiv \sum_{k=0}^{m} \frac{a_k}{k!} s^k + \alpha s^{m+1} \mod s^{m+2}.$$

Thus,

$$\exp\left(\sum_{k=0}^{m} b_k s^k + \beta s^{m+1}\right) \equiv (1 + \beta s^{m+1})\left(\sum_{k=0}^{m} \frac{a_k}{k!} s^k + \alpha s^{m+1}\right) \mod s^{m+2}.$$

Hence, let $b_{m+1} = \beta$ be the solution of the equation

$$\beta a_0 + \alpha = \frac{a_{m+1}}{(m+1)!},$$

which exists by the restriction on a_0. This shows

$$\exp\left(\sum_{k=0}^{m+1} b_k s^k\right) \equiv \sum_{k=0}^{m+1} \frac{a_k}{k!} s^k \mod s^{m+2},$$

proving the claim.

Now

$$f(s) := \exp\left(\sum_{k=0}^{n-1} b_k s^k\right) \equiv \sum_{k=0}^{m+1} \frac{a_k}{k!} s^k \mod s^n.$$

By Voronin's universality theorem, Corollary 10.2, there exists a sequence τ_j, tending with j to infinity, such that

$$\lim_{j\to\infty} \max_{|s|\leq r} \left|\zeta\left(s + \frac{3}{4} + i\tau_j\right)\right| = 0$$

for some $r \in (0, \frac{1}{4})$. In view of (11.1) we obtain

$$\lim_{j\to\infty} \max_{|s|\leq r-\varepsilon} \left|\zeta^{(k)}\left(s + \frac{3}{4} + i\tau_j\right) - f^{(k)}(s)\right| = 0$$

for $k = 1, \ldots, n-1$ and any $\varepsilon \in (0, r)$. Arguing as above, this proves the theorem. •

Further, the universality result implies functional independence:

Theorem 11.2 *Let* $\mathbf{z} = (z_0, z_1, \ldots, z_{n-1}) \in \mathbb{C}$. *Suppose that* $F_0(\mathbf{z}), F_1(\mathbf{z}), \ldots, F_N(\mathbf{z})$ *are continuous functions, not all identically zero, then*

$$\sum_{k=0}^{N} s^k F_k(\zeta(s), \zeta(s)', \ldots, \zeta(s)^{(n-1)}) \neq 0$$

for some $s \in \mathbb{C}$,

In particular, we see that the zeta-function does not satisfy any algebraic functional equation. This solves one of Hilbert's famous problems which he posed at the International Congress of Mathematicians in Paris 1900. The first proof of this result was given by Ostrowski [41].

Proof. First, we shall show that if $F(\mathbf{z})$ is a continuous function and

$$F(\zeta(s), \zeta(s)', \ldots, \zeta(s)^{(n-1)})) = 0$$

identically in $s \in \mathbb{C}$, then F vanishes identically.

Suppose the contrary, i.e. $F(\mathbf{z}) \not\equiv 0$. Then there exists $\mathbf{a} \in \mathbb{C}^n$ for which $F(\mathbf{a}) \neq 0$. Since F is continuous, there exist a neighbourhood U of $\mathbf{a}$ and a positive ε such that

$$|F(\mathbf{z})| > \varepsilon \qquad \text{for} \quad \mathbf{z} \in U.$$

Choosing an arbitrary $\sigma \in (\frac{1}{2}, 1)$, application of Theorem 11.1 yields the existence of some t for which

$$(\zeta(s), \zeta(s)', \ldots, \zeta(s)^{(n-1)}) \in U,$$

which contradicts our assumption. This proves our claim, resp. the assertion of the theorem with $N = 0$.

Without loss of generality we may assume that $F_0(\mathbf{z})$ is not identically zero. As above there exist an open bounded set U and a positive ε such that

$$|F_0(\mathbf{z})| > \varepsilon \qquad \text{for} \quad \mathbf{z} \in U.$$

Denote by M the maximum of all indices m for which

$$\sup_{\mathbf{z} \in U} |F_m(\mathbf{z})| \neq 0.$$

If $M = 0$, then the assertion of the theorem follows from the result proved above. Otherwise, we may take a subset $V \subset U$ such that

$$\inf_{\mathbf{z} \in V} |F_M(\mathbf{z})| > \varepsilon$$

for some positive ε. By Theorem 11.1, there exists a sequence t_j, tending with j to infinity, such that

$$(\zeta(\sigma + it_j), \zeta(\sigma + it_j)', \ldots, \zeta(\sigma + it_j)^{(n-1)}) \in V.$$

This implies

$$\lim_{j \to \infty} \left| \sum_{k=0}^{M} (\sigma + it_j)^k F_k(\zeta(\sigma + it_j), \zeta(\sigma + it_j)', \ldots, \zeta(\sigma + it_j)^{(n-1)}) \right| = \infty.$$

This proves the theorem. •

12 Self-similarity and the Riemann hypothesis

In view of Voronin's universality theorem a natural question arises: is it possible to approximate functions with zeros? The answer is more or less negative but of special interest. We give an heuristic argument which however can be made waterproof with a bit more effort and the same techniques which we shall use later on.

In order to see that the zeta-function cannot approximate uniformly a function with zeros recall *Rouche's theorem*, which states that *if $f(s)$ and $g(s)$ are analytic inside and on a contour $\mathcal{C}$, and $|f(s)| < |g(s)|$ on $\mathcal{C}$, then $g(s)$ and $f(s)+g(s)$ have the same number of zeros inside $\mathcal{C}$*; for a proof see [54], §3.42.

Now assume that $f(s)$ is an analytic function on $|s| \leq r$, where $0 < r < \frac{1}{4}$, which has a zero λ with $|\lambda| < r$ but which is non-vanishing on the boundary. For $\varepsilon > 0$ sufficiently small we may assume that $\max_{|s|=r} |f(s)| > \varepsilon$. Hence, whenever the inequality

$$\max_{|s|\leq r}\left|\zeta\left(s+\frac{3}{4}+i\tau\right)-\zeta(s)\right| < \varepsilon < \min_{|s|\leq r}|\zeta(s)|; \tag{12.1}$$

holds, $\zeta\left(s+\frac{3}{4}+i\tau\right)$ has to have a zero inside $|s| \leq r$ (since by the *maximum principle* the maximum on the left hand side is actually taken on the boundary). Note that the second inequality in (12.1) holds for sufficiently small ε (since the zeros of an analytic function lie discrete or the function vanishes identically). Therefore, if for any $\varepsilon > 0$

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T]\,:\,\max_{|s|\leq r}\left|\zeta\left(s+\frac{3}{4}+i\tau\right)-f(s)\right|<\varepsilon\right\}>0,$$

then we expect $\gg T$ many complex zeros of $\zeta(s)$ in the strip $\frac{3}{4}-r<\sigma<\frac{3}{4}+r$ (for a rigorous proof one has to consider exactly the densities of values τ satisfying (12.1); this can be done along the lines of the proof of Theorem 12.1 below). This contradicts the density theorem 4.4, which gives

$$N\left(\frac{3}{4}-r,T\right)=o(T).$$

Thus, an approximation of a function with a zero on a sufficiently rich set cannot be done!

The above reasoning shows that the location of the complex zeros of the zeta-function is closely connected with its universality property. We can go a little bit further.

Theorem 12.1 *The Riemann hypothesis is true if and only if for any compact subset $\mathcal{K}$ of $\frac{1}{2}<\sigma<1$ with connected complement and any $\varepsilon>0$*

$$\liminf_{T\to\infty}\frac{1}{T}\mathit{meas}\left\{\tau\in[0,T]\,:\,\max_{s\in\mathcal{K}}|\zeta(s+i\tau)-\zeta(s)|<\varepsilon\right\}>0. \tag{12.2}$$

This theorem is due to Bagchi [3]; in [4] he generalized this result in various directions. Since Bagchi's proof relies mainly on the theory of topolgical dynamics he speaks about the property (12.2) as *strong recurrence*. However, we call it *self-similarity*. It should be noted that Bohr [8] detected in 1922 a similar result. Therefore, we have to introduce an important class of zeta-functions.

For a character $\chi \bmod q$ (i.e. a non-trivial group homomorphism on the prime residue class $\bmod q$) the **Dirichlet L-function** is for $\sigma > 1$ given by

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Dirichlet L-functions have similar behaviour and properties as the Riemann zeta-function; actually, $\zeta(s)$ may be regarded as Dirichlet L-function associated to the principal character $\chi_0 \bmod 1$. As for the Riemann zeta-function it is conjectured that $L(s,\chi)$ does not vanish in $\sigma > \frac{1}{2}$, this is the so-called **Generalized Riemann hypothesis**. Harald Bohr introduced the fruitful notion of almost periodicity into analysis. We say that a function $L(s)$ is **almost periodic** in $\mathcal{K}$ if for all $\varepsilon >$ there exists a sequence of values $\ldots, \tau_{-1} < 0 < \tau_1 < \tau_2 < \ldots\ldots$ with

$$\liminf_{m\to\pm\infty} (\tau_{m+1} - \tau_m) > 0 \qquad \text{and} \qquad \limsup_{m\to\pm\infty} \frac{\tau_m}{|m|} < \infty$$

for which

$$|L(s + i\tau_m) - L(s)| < \varepsilon \qquad \text{for all} \quad s \in \mathcal{K}.$$

Bohr proved that *Dirichlet series are almost periodic in their half-plane of absolute convergence*. Moreover, he discovered an interesting relation between Riemann's hypothesis and almost periodicity: *if χ is a non-trivial character, then the Riemann hypothesis for $L(s,\chi)$ is true if and only if $L(s,\chi)$ is almost periodic for $\sigma > \frac{1}{2}$*. Note that self-similarity implies almost periodicity (but not vice versa). The condition on the character seems somehow unnatural but Bohr's argument does not apply to $\zeta(s)$. However, by Voronin's universality theorem this gap can be filled. We shall give a simple proof of Bagchi's theorem which actually combines Bohr's idea with the one of Bagchi.

Proof. If Riemann's hypothesis is true, then we can apply the universality theorem 10.3 with $g(s) = \zeta(s)$, which implies the self-similarity. The idea for the proof of the other implication is that if there is at least one proof to the right of the critical line, then the self-similarity property implies the existence of *many* zeros, *too many* with regard to well-known density theorems).

Suppose that the Riemann hypothesis is not true, then there exists a zero λ of $\zeta(s)$ with $\mathrm{Re}\,\lambda > \frac{1}{2}$. Further, we We have to show that there exists a disc $|s| \le r < \frac{1}{4}$ and $\varepsilon > 0$ such that

$$\lim_{T\to\infty} \frac{1}{T} \mathrm{meas} \left\{\tau \in [0,T] \,:\, \max_{s\in\mathcal{K}} |\zeta(s + i\tau\Big) - \zeta(s)| < \varepsilon\right\} = 0. \tag{12.3}$$

Locally, the zeta-function has the expansion

$$\zeta(s) = c(s-\lambda)^m + O\left(|s-\lambda|^{m+1}|\right) \tag{12.4}$$

with some non-zero $c \in \mathbb{C}$ and $m \in \mathbb{N}$. Now assume for a neighbourhood $\mathcal{K}_\delta := \{s \in \mathbb{C} : |s - \lambda \leq \delta\}$ of λ that

$$\max_{s\in\mathcal{K}_\delta} |\zeta(s+i\tau) - \zeta(s)| < \varepsilon < \min_{|s|=\delta} |\zeta(s)|; \tag{12.5}$$

this formula should be compared with (12.1). Then Rouche's theorem implies the existence of a zero ϱ of $\zeta(s+i\tau)$ in $\mathcal{K}_\delta$. We may say that the zero λ of $\zeta(s)$ **generates** the zero ϱ of $\zeta(s+i\tau)$. With regard to (12.4) and (12.5) the zeros λ and ϱ are intimately related, more precisely:

$$\begin{aligned} \varepsilon &> |\zeta(\varrho) - \zeta(\varrho - i\tau)| = |\zeta(\varrho - i\tau)| \\ &\geq |c| \cdot |\varrho - i\tau - \lambda|^m + O\left(|\varrho - i\tau - \lambda|^{m+1}\right). \end{aligned}$$

Hence,

$$|\varrho - i\tau - \lambda| \leq \left(\frac{\varepsilon}{|c|}\right)^{\frac{1}{m}} + O\left(\delta^{1+\frac{1}{m}}\right),$$

and in particular

$$\frac{1}{2} < \mathrm{Re}\ \lambda - 2\left(\frac{\varepsilon}{|c|}\right)^{\frac{1}{m}} < \mathrm{Re}\ \varrho < 1,$$

$$|\mathrm{Im}\ \varrho - (\tau + \mathrm{Im}\ \lambda)| < 2\left(\frac{\varepsilon}{|c|}\right)^{\frac{1}{m}}$$

for sufficiently small $\delta = o(\varepsilon^{m+1})$. It may happen that different values of τ for which (12.5) hold lead to the same zero ϱ. Therefore, we have to consider the densities for such τ. If we now write

$$\mathcal{I} := \bigcup_j \mathcal{I}_j := \{\tau \in [0,T] : \max_{s\in\mathcal{K}_\delta} |\zeta(s+i\tau) - \zeta(s)| < \varepsilon\},$$

where the $\mathcal{I}_j$ are disjoint intervals, then it follows that there are

$$\geq \left[\frac{1}{4}\left(\frac{|c|}{\varepsilon}\right)^{\frac{1}{m}} \mathrm{meas}\,\mathcal{I}_j\right] + 1 > \frac{1}{4}\left(\frac{|c|}{\varepsilon}\right)^{\frac{1}{m}} \mathrm{meas}\,\mathcal{I}_j$$

many zeros according to $\tau \in \mathcal{I}_j$, and thus

$$N\left(\mathrm{Re}\ \lambda - 2\left(\frac{\varepsilon}{|c|}\right)^{\frac{1}{m}}, T + \mathrm{Im}\ \lambda + 2\left(\frac{\varepsilon}{|c|}\right)^{\frac{1}{m}}\right) \geq \frac{1}{4}\left(\frac{|c|}{\varepsilon}\right)^{\frac{1}{m}} \mathrm{meas}\,\mathcal{I};$$

recall that the zero-counting function $N(\sigma, T)$ was defined in Section 4. By the density theorem 4.4 we obtain $\mathrm{meas}\,\mathcal{I} = o(T)$, which implies (12.3). The theorem is proved. •

The critical line is a natural borderline for self-similarity since Levinson [36] proved that the frequency of c-values of the zeta-function, i.e. solutions of $\zeta(s) = c$, close to the critical line increases rapidly with increasing imaginary part; similar as the Riemann von-Mangold formula (2.5) in connection with Theorem 4.3 show for the particular case of zeros.

Assuming Riemann's hypothesis the self-similarity property (12.2) has an interesting interpretation. The amplitude of light waves is a physical bound for the size of objects which human beings can see (even with microscopes), or take the Planck constant 10^{-33} which is the smallest size of objects in quantum mechanics. Thus, if we assume that ε is less than this quantity, then we cannot distinguish between $\zeta(s)$ and $\zeta(s+i\tau)$ whenever

$$\max_{s\in\mathcal{K}} |\zeta(s+i\tau) - \zeta(s)| < \varepsilon.$$

This shows that even if one would have all knowledge on the zeta-function, one could not decide wherever one actually is in the analytic landscape of $\zeta(s)$ above the right half of the critical strip without moving to the boundary. The zeta-function as an amazing maze!

Theorem 12.1 offers an interesting approach towards Riemann's hypothesis. However, we shall only prove, following Bohr's argument, that the zeta-function is self-similar in the half-plane of absolute convergence.

Theorem 12.2 *Let $\mathcal{K}$ be any compact subset in the half-plane $\sigma > 1$. Then, for any $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas \left\{\tau \in [0,T] \,:\, \max_{s\in\mathcal{K}} |\zeta(s+i\tau) - \zeta(s)| < \varepsilon\right\} > 0.$$

Proof. Since $\zeta(s)$ is regular and zero-free in $\sigma > 1$, we may define the logarithm (by choosing any one of the values of the logarithm). In view of the Euler product representation it is easily shown that

$$\log\zeta(s) = \sum_{p,k} \frac{1}{kp^{ks}} \qquad \text{for} \quad \sigma > 1,$$

where the sum is taken over all prime numbers p and all positive integers k. Hence,

$$\log\zeta(s) - \log\zeta(s+i\tau) = \sum_{p,k\geq 1} \frac{1}{kp^{ks}}\left(1 - \frac{1}{p^{ik\tau}}\right). \tag{12.6}$$

We shall use diophantine approximation to find values of τ for which $p^{-ik\tau}$ lies sufficiently close to 1. We apply Theorem 7.1 with $a_n = \frac{1}{2\pi}\log p_n$ where, as usual, p_n denotes the n-th prime number. Further, we choose

$$\gamma = \left\{(z_1,\ldots,z_N) \in \mathbb{R}^N \,:\, \|z_n\| < \frac{1}{\omega} \quad \text{for} \quad 1 \leq n \leq N\right\},$$

where the parameter $\omega > 0$ will be chosen later and where $\|z\|$ denotes the distance of z to the next integer. Then we find about

$$\Gamma = T\left(\frac{4\pi}{\omega}\right)^N$$

many values of $\tau \in [0,T]$ such that

$$\|\tau \log p_n\| < \frac{2\pi}{\omega} \qquad \text{for} \quad 1 \leq n \leq N$$

as T tends to infinity. Consequently,

$$\cos(k\tau \log p) = 1 + O\left(\frac{k^2}{\omega^2}\right), \quad \sin(k\tau \log p) = O\left(\frac{k}{\omega}\right),$$

resp.

$$|1 - p^{-ik\tau}|^2 = (1 - \cos(k\tau \log p))^2 + \sin^2(k\tau \log p) \ll \frac{k^2}{\omega^2}$$

for $p \leq p_N$, provided that $\omega > 4k$; here and in the sequel all implicit constants are absolute. This yields

$$\sum_{p \leq x, k \leq y} \frac{1}{p^s}\left(1 - \frac{1}{p^{i\tau}}\right) \ll \frac{xy}{\omega},$$

where $x = p_N$. Furthermore, we have

$$\sum_{p \leq x, k > y} \frac{1}{kp^{ks}}\left(1 - \frac{1}{p^{ik\tau}}\right) \ll \frac{1}{y}\sum_{p \leq x} \frac{1}{p^{\sigma y}} \ll \frac{x}{y2^y},$$

and

$$\sum_{p > x, k \geq 1} \frac{1}{kp^{ks}}\left(1 - \frac{1}{p^{ik\tau}}\right) \ll \sum_{n > x} \frac{1}{n^\sigma} \ll \frac{x^{1-\sigma}}{\sigma - 1}.$$

In view of (12.6) we obtain

$$\log \zeta(s) - \log \zeta(s + i\tau) \ll \frac{xy}{2^\sigma \omega} + + \frac{x}{y2^y} + \frac{x^{1-\sigma}}{\sigma - 1} \tag{12.7}$$

for a set of values of $\tau \in [0,T]$ with positive lower density as $T \to \infty$ and any $\sigma > 1$; note that the estimates can easily be improved. Since $\mathcal{K}$ is compact, there exists $\kappa := \min\{\sigma \,:\, s \in \mathcal{K}\}$. Then

$$\max_{s \in \mathcal{K}} |\zeta(s)| \leq \zeta(\kappa).$$

Now for any given $\varepsilon' > 0$ we can find values x, y and ω such that the right hand side of (12.7) is $< \varepsilon'$. It remains to get rid of the logarithm. Obviously,

$$\zeta(s) - \zeta(s + i\tau) = \zeta(s)(1 - \exp\{\log \zeta(s + i\tau) - \log \zeta(s)\}).$$

Hence,

$$\max_{s \in \mathcal{K}} |\zeta(s) - \zeta(s + i\tau)| \leq \zeta(\kappa) \cdot \max_{s \in \mathcal{K}} |\log \zeta(s + i\tau) - \log \zeta(s)| < \varepsilon' \zeta(\kappa).$$

The choice $\varepsilon' = \frac{\varepsilon}{2M}$ proves the theorem. •

Taking Theorems 12.1 and 12.2 into account, we might conjecture that the zeta-function has the above self-similarity property for $\sigma > \frac{1}{2}$.

13 Effective bounds

However, the known proofs of Voronin's universality theorem are ineffective, giving neither an estimate for the first approximating translate τ nor lower bounds for the positive lower density. This is caused by Pechersky's ineffective theorem 8.1 on the rearrangement of series. In this section we shall prove for a sufficiently large class of functions upper bounds for the upper density; for more on this topic see [52].

Denote by B_r the closed disc of radius $r > 0$ with center in the origin. For our purpose we consider **analytic isomorphisms** $g : B_r \to B_1$, i.e. the inverse g^{-1} exists and is analytic. Obviously, such an analytic isomorphisms g has exactly one simple zero λ in the interior of B_r; moreover, it can be shown (by the Schwarz lemma, see [54], §7.2) that $g \in \mathcal{A}_r$ has the representation

$$g(s) = r\exp(i\varphi)\frac{\lambda - s}{r^2 - \overline{\lambda}s} \qquad \text{with} \quad \varphi \in \mathbb{R},\ |\lambda| < r. \tag{13.1}$$

Denote by $\mathcal{A}_r$ the class of analytic isomorphisms from B_r onto the unit disc. Further, define for an analytic isomorphisms $g \in \mathcal{A}_r$ with fixed $r \in \left(0, \frac{1}{4}\right)$ and a positive ε the upper density

$$\overline{d}(\varepsilon, g) := \limsup_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] \,:\, \max_{|s|\le r}\left|\log\zeta\left(s + \frac{3}{4} + i\tau\right) - g(s)\right| < \varepsilon\right\}$$

and the lower density $\underline{d}(\varepsilon, g)$ analogously; note that ($\underline{d}(\varepsilon, g) > 0$ implies the universality property of Corollary 10.2 with respect to g. Finally, let $N(\sigma_1, \sigma_2, T)$ count the number of zeros of $\log\zeta(s)$ in $\frac{1}{2} < \sigma_1 < \sigma < \sigma_2 < 1$ (according multiplicities). Then

Theorem 13.1 *Suppose that $g \in \mathcal{A}_r$ and $\underline{d}(\varepsilon, g) > 0$ for all $\varepsilon > 0$. Then, for any $\varepsilon \in \left(0, \frac{1}{2r}\left(\frac{1}{4} + Re\,|\lambda|\right)\right)$,*

$$\begin{aligned}\overline{d}(\varepsilon, g) &\le \frac{8r^4\varepsilon}{r^2 - |\lambda|^2}\limsup_{T\to\infty}\frac{1}{T}N\left(\frac{3}{4} + Re\,\lambda - 2r\varepsilon, \frac{3}{4} + Re\,\lambda + 2r\varepsilon, T\right) \\ &= o(\varepsilon).\end{aligned} \tag{13.2}$$

Therefore, the decay of $\overline{d}(\varepsilon, g)$ with $\varepsilon \to 0$ is more than linear in ε.

Proof. The idea of proof is (as in the proof of Theorem 12.1) that the zero λ of g generates some zeros of $\log\zeta(s)$ in $\frac{1}{2} < \sigma < 1$. Since g maps the boundary of B_r onto the unit circle, Rouche's theorem implies the existence of one simple zero ϱ of $\log\zeta(z)$ in

$$\mathcal{K}_\tau := \left\{z = \frac{3}{4} + s + i\tau \,:\, s \in B_r\right\}$$

whenever (12.1) holds.

Universality (as self-similarity) is a phenomenon that happens in intervalls. Now we have to prove an upper bound for the distance of different translates generating the same zero ϱ of $\log\zeta(s)$: suppose that a zero ϱ of $\log\zeta(s)$, generated by λ, lies in two different translates $\mathcal{K}_{\tau_1}$ and $\mathcal{K}_{\tau_2}$, then

$$|\tau_1-\tau_2|<\frac{8r^4\varepsilon}{r^2-|\lambda|^2}. \tag{13.3}$$

Suppose that there exist

$$s_j=\mathrm{Re}\ \varrho-\frac{3}{4}+it_j\in B_r,\quad \text{and}\quad \tau_j\in\mathbb{R}\quad\text{with}\quad \log\zeta\left(s_j+\frac{3}{4}+i\tau_j\right)=0,$$

for $j=1,2$, such that

$$\varrho=s_1+\frac{3}{4}+i\tau_1=s_2+\frac{3}{4}+i\tau_2.$$

In view of (13.1),

$$|g(s_2)-g(s_1)|=\frac{r^2-|\lambda|^2}{|r^2-\overline{\lambda}s_1||r^2-\overline{\lambda}s_2|}|s_2-s_1|.$$

We deduce from (12.1) that $|g(s_j)|<\varepsilon$ for $j=1,2$, and therefore

$$|\tau_1-\tau_2|=|t_2-t_1|\le\frac{4r^4}{r^2-|\lambda|^2}|g(s_2)-g(s_1)|<\frac{8r^4\varepsilon}{r^2-|\lambda|^2},$$

which proves estimate (13.3).

Now, denote by $\mathcal{I}_j(T)$ the disjoint intervalls in $[0,T]$ such that (12.1) is valid exactly for

$$\tau\in\bigcup_j\mathcal{I}_j(T)=:\mathcal{I}(T).$$

Using (13.3), in every intervall $\mathcal{I}_j(T)$ lie at least

$$1+\left[\frac{r^2-|\lambda|^2}{8r^4\varepsilon}\mathrm{meas}\,\mathcal{I}_j(T)\right]\ge\frac{r^2-|\lambda|^2}{8r^4\varepsilon}\mathrm{meas}\,\mathcal{I}_j(T)$$

zeros ϱ of $\log\zeta(s)$ in the strip $\frac{1}{2}<\sigma<1$. Therefore, the number $\mathcal{N}(T)$ of such zeros ϱ satisfies the estimate

$$\frac{8r^4\varepsilon}{r^2-|\lambda|^2}\mathcal{N}(T)\ge\mathrm{meas}\,\mathcal{I}(T). \tag{13.4}$$

The next step is to replace $\mathcal{N}(T)$ by the zero counting function appearing in the theorem.

Obviously, the value distribution of $\log\zeta(z)$ in $\mathcal{K}_\tau$ is ruled by that of $g(s)$ in B_r. As we shall see below, this gives a restriction on the real parts of zeros ϱ. Let $s\in B_r$. If $|g(s)|\ge\varepsilon$, then, in view of (12.1),

$$\left|\log\zeta\left(s+\frac{3}{4}+i\tau\right)\right|\ge|g(s)|-\left|g(s)-\log\zeta\left(s+\frac{3}{4}+i\tau\right)\right|>0.$$

Since (13.1) implies

$$|g(s)| \geq \frac{|\lambda - s|}{2r},$$

we obtain the estimate

$$\left|\mathrm{Re}\ \varrho - \frac{3}{4} - \mathrm{Re}\ \lambda\right| < 2r\varepsilon \tag{13.5}$$

by taking the real parts.

In view of (13.5) we find

$$\mathcal{N}(T) \leq N\left(\frac{3}{4} + \mathrm{Re}\ \lambda - 2r\varepsilon, \frac{3}{4} + \mathrm{Re}\ \lambda + 2r\varepsilon, T\right). \tag{13.6}$$

On the other side, since $\underline{d}(\varepsilon, g) > 0$, there exists an incresing sequence (T_k) with $\lim_{k\to\infty} T_k = \infty$ such that for any $\delta > 0$

$$\mathrm{meas}\, \mathcal{I}(T_k) \geq (\overline{d}(\varepsilon, g) - \delta)T_k.$$

Consequently, this together with (13.6), leads in (13.4) to

$$\frac{8r^4\varepsilon}{r^2 - |\lambda|^2} N\left(\frac{3}{4} + \mathrm{Re}\ \lambda - 2r\varepsilon, \frac{3}{4} + \mathrm{Re}\ \lambda + 2r\varepsilon, T_k\right) \geq (\overline{d}(\varepsilon, g) - \delta)T_k.$$

Sending $\delta \to 0$, yields the estimate (13.2) of the theorem. As it was shown by Bohr and Jessen the limit

$$\lim_{T\to\infty} \frac{1}{T} N\left(\frac{3}{4} + \mathrm{Re}\ \lambda - \delta, \frac{3}{4} + \mathrm{Re}\ \lambda + \delta, T\right)$$

exists, and tends with ε to zero (see Hilfssatz 6, [9]). Further, one has

$$\begin{aligned}
&\max_{s\in B_r} \left|\zeta\left(s + \frac{3}{4} + i\tau\right) - \exp g(s)\right| \\
&\leq \max_{s\in B_r} |\exp g(s)| \times \max_{s\in B_r} \left|\exp\left(\log\zeta\left(s + \frac{3}{4} + i\tau\right) - g(s)\right) - 1\right| \\
&\leq e \max_{s\in B_r} \left|\zeta\left(s + \frac{3}{4} + i\tau\right) - \exp g(s)\right|.
\end{aligned}$$

The Theorem is shown. •

Recently, Garunkštis [17] proved a first effective universality theorem along the lines of Voronin's proof in addition with some old ideas due to Good [19] and new insights. In particular, his remarkable result shows that *if* $f(s)$ *is analytic in* $|s| \leq 0.05$ *with* $\max_{|s|\leq 0.06} |f(s)| \leq 1$, *then for any* $0 < \varepsilon < \frac{1}{2}$ *there exists a*

$$\tau \leq \exp\left(\exp\left(10\varepsilon^{-13}\right)\right)$$

such that

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas}\left\{\tau \in [0, T] : \max_{|s|\leq 0.0001} \left|\zeta\left(s + \frac{3}{4} + i\tau\right) - f(s)\right| < \varepsilon\right\} \geq \exp\left(-\varepsilon^{-13}\right).$$

Another approach via the rate of convergence of weak convergent probability measures is due to Laurinčikas [31].

14 Other zeta-functions

It is natural to ask whether other functions with similar properties as the Riemann zeta-function are *universal*. Meanwhile, it is known that there exists a rich zoo of Dirichlet series having the universality property; we mention only some further significant examples:

- *Joint* universality for Dirichlet L-functions, i.e. for a collection of Dirichlet L-functions attached to pairwise non-equivalent characters $\chi \bmod q$ (nontrivial group homomorphisms on the group of prime residue classes), proved by Voronin [58];

- Dirichlet L-functions with respect to the modulus of the characters, by Eminyan [15];

- Dedekind zeta-functions associated to a number field $\mathbb{K}$ over $\mathbb{Q}$

$$\zeta_{\mathbb{K}}(s) = \sum_{\mathcal{A}}^{\infty} \frac{1}{(N\mathcal{A})^s} = \prod_{\mathcal{P}} \left(1 - \frac{1}{(N\mathcal{P})^s}\right)^{-1},$$

  where the sum is taken over all non-zero integral ideals, the product is taken over all prime ideals of the ring of integers of $\mathbb{K}$ and where $N\mathcal{A}$ is the norm of the ideal $\mathcal{A}$, obtained by Reich [48];

- Dirichlet series with multiplicative coefficients, by Laurinčikas [27], Laurinčikas and Šleževicienė [35];

- Matsumoto zeta-functions, obtained by Laurinčikas [30];

- L-functions associated to cusp forms, resp. new forms (or elliptic curves by Wiles et al. celebrated proof of Fermat's last theorem), by Laurinčikas and Matsumoto [33], resp. Laurinčikas, Matsumoto and Steuding [34];

- Hurwitz zeta-functions

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s} \tag{14.1}$$

  with parameter $\alpha \in (0, 1]$ where α is rational $\neq \frac{1}{2}, 1$ or transcendental, proved by Gonek [18], Bagchi [3];

- Lerch zeta-functions, proved by Laurinčikas [29];

This list could be continued with Hecke L-functions, Artin L-functions, Rankin-Selberg convolution L-functions to mention only some more important examples.

Some of the examples given above, e.g. Hurwitz zeta-functions, have the **strong** universality property, i.e. that they can approximate functions with zeros. It seems that the restriction to a *conditional* universality property is intimately linked to the property of the Dirichlet series in question to have an Euler product, and by that, to satisfy Riemann's hypothesis. It should be noted that the strong universality leads by the arguments used in the proof of Theorem 12.1 to the existence of *many* zeros off the critical line; see [16] for more details.

All known proofs of universality for zeta-functions depend on a certain *independence*, namely that the logarithms of the prime numbers are linearly independent (as we have used it in the proof of Voronin's universality theorem when we applied Kronecker's approximation theorem 7.1), resp. that the numbers $\log(n+\alpha)$ for $n \in \mathbb{N}$ are linearly independent if α is transcendental (which is necessary to deal with (14.1)). The **Linnik-Ibragimov conjecture** states that all functions given by Dirichlet series and analytically continuable to the left of the half plane of absolute convergence are universal. However, this is better to understand as a program than a conjecture. For example, define $a(n) = 1$ if $n = 2^k, k \in \mathbb{N} \cup \{0\}$, and $a(n) = 0$ otherwise, then

$$A(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \sum_{k=0}^{\infty} \frac{1}{2^{sk}} = \frac{1}{1 - 2^{-s}},$$

and obviously, this function is far away from beeing universal. However, in [53] a universality theorem for the so-called Selberg class, which covers all from a number theoretical point of view interesting Dirichlet series known so far (i.e. with Euler product), was proved.

More detailed surveys on the value distribution of zeta-functions are the excellent written papers [32] and [40].

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