GRAPHS, MATRICES, AND CIRCUIT THEORY

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1 Introduction

This is a set of notes I wrote while teaching courses in circuit theory for undergraduate electrical engineering students. They can be read in connection with any good standard book on the subject. They basically contain the essence of a large chunk of a circuit theory course. Circuit theory is, of course, obtained by making an idealization of Maxwell's equations in the absence of charges and when we can ignore magnetic fields. From a mathematical point of view, circuit theory can be seen as a topic in graph theory, a topic in linear algebra, but also a topic in discrete harmonic analysis. The benign set of Kirchoff's laws and Ohm's laws, familiar to any electrical engineer, in other words a mere set of linear equations, contains, in disguise, a lot of physical and mathematical beauty. The point of the notes is to first explain why a circuit can be solved, and, second, present the standard methods for doing so (variants of which exist in common software packages for circuit simulation).

2 From a circuit to its graph

The first thing we do here is to identify the *graph* of the circuit. That is, we identify the set of *vertices* V, and the set of *edges* E. The graph is $G = (V, E)$. Next, each edge is given an arbitrary *orientation*. If the edge between vertices k and ℓ is considered unoriented, it is denoted by $\{k, \ell\}$. If it is oriented it is designated as (k, ℓ), where k is the *start-vertex* and ℓ is the *end-vertex*. For example, here is a circuit and its graph.

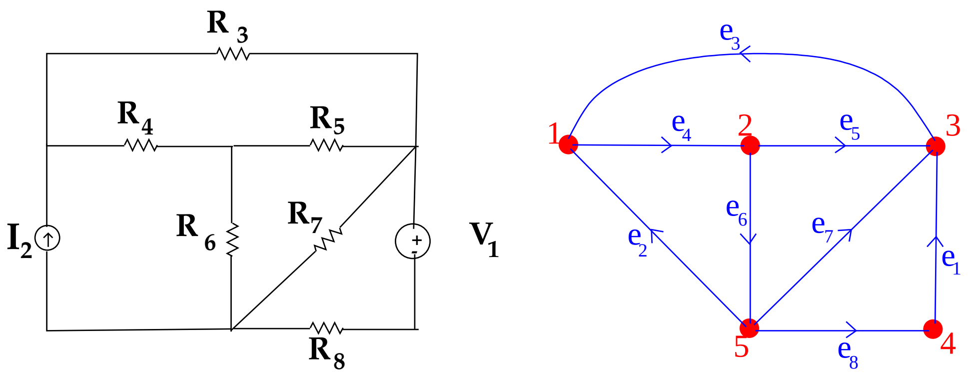


Figure 1: A CIRCUIT AND ITS GRAPH

The set of vertices is $V = \{1, 2, 3, 4, 5\}$. The set of edges is $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$. The edge orientation is entirely arbitrary. For example, if e_5 is considered as unoriented, it is denoted by $e_5 = \{2, 3\}$. Taking into account the orientation, we write $e_5 = (2, 3)$. We say that an edge $\{k, \ell\}$ is *incident* to a vertex if this vertex is k or ℓ. We do not consider self-edges, that is edges that start and end at the same vertex.

Much of what we say holds for circuits containing dependent sources, transformers, and other linear elements. However, to make the exposition simple, we imagine that we have only resistors (which could take complex values) and independent voltage and current sources. This is not a restriction of generalit! Also, we arrange so that there is at most one edge per pair of vertices (by introducing, if necessary, additional nodes). So, again, assuming that there is at most one edge per pair of vertices is no loss of generality. A word on terminology: an edge is sometimes called *branch*, and a vertex is called *node*, especially if we refer to the actual electrical circuit. The graph of Fig. 1 is *planar*, in that it can be drawn on the plane, without edge intersections. On the contrary, the graph below (Fig. 2) is non-planar. The reason is that, no matter how you redraw it, at least two of its edges will intersect.

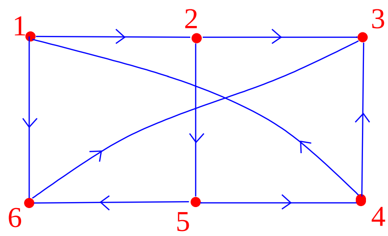


Figure 2: A NON-PLANAR GRAPH

The nice thing about graphs, is that they contain the minimal information about the "topological structure" of the circuit, which is responsible for the writing of the two circuit laws: Kirchhoff's Current Law (KCL) and Kirchhoff's Voltage Law (KVL). It is often nice and pleasing to reduce a model into its bare minimum so we can see exactly how much we can say about it, without getting confused by redundant information.

3 Currents and voltages

A *current configuration*, or simply current, for a graph $G = (V, E)$ is a collection $i = (i_e, e \in E)$ of numbers i_e associated to edges e so that they satisfy KCL. That is, for each vertex k look at all edges e incident to k, let $b_{k,e}$ be $+1$ is k is a start-vertex of e or -1 if k is an end-vertex of e and write

$$\sum_e b_{k,e} i_e = 0. \tag{1}$$

If we define $b_{k,e}$ to be 0 if e is not incident to vertex k, then the above summation can be extended over all $e \in E$.

Next, we turn to KVL. A *cycle* (or loop) ℓ of G is a collection of distinct vertices $k_1, \ldots, k_j$, such that $\{k_1, k_2\}$, $\{k_2, k_3\}, \ldots, \{k_{j-1}, k_j\}$, $\{k_j, k_1\}$ are all edges of the *unoriented* graph. The *cycle orientation* of cycle ℓ is the orientation at which we traverse its vertices. The actual orientation of a particular edge e of ℓ may or may not agree with that of ℓ. We define numbers $a_{\ell,e}$ as follows: if e does not belong to cycle ℓ then $a_{\ell,e} = 0$; if e belongs to ℓ and e has the same orientation as ℓ, then $a_e = +1$; otherwise, $a_e = -1$. A *voltage configuration* $v = (v_e, e \in E)$ is an assignment of numbers v_e to the edges e of G in a way that KVL is satisfied. KVL, for a given cycle ℓ, says that

$$\sum_e a_{\ell,e} v_e = 0. \tag{2}$$

4 From graphs to matrices

We can encode KCL by using the language of matrices. For this purpose, we define the *incidence matrix* B of the graph $G = (V, E)$ (whose edges have been given an arbitrary orientation) as a matrix whose *rows are* *indexed by vertices* and *columns by edges*. The entry of B corresponding to row (vertex) k and column (edge) e is simply taken to be $b_{k,e}$ as defined above. For instance, in the first example, we have

$$B = \begin{array}{cc} \begin{pmatrix} 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 1 \end{pmatrix} & \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \end{array} \\ \begin{array}{cccccccc} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} & \boldsymbol{e_4} & \boldsymbol{e_5} & \boldsymbol{e_6} & \boldsymbol{e_7} & \boldsymbol{e_8} \end{array} & \end{array}$$

Thus, column 3 of B has 1 at position 3 and -1 at position 1, meaning that edge e_3 starts at vertex 3 and ends at vertex 1. If we thus let $i = (i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8)^\intercal$ be a column of a current configuration, we can encode KCL as follows:

$$Bi = 0. \tag{3}$$

Indeed, if, e.g., we multiply the second row of B by i we get $-i_4 + i_5 + i_6 = 0$, which is KCL at vertex 2.

We can encode KVL (1) by introducing the <u>*cycle matrix*</u> A, whose *rows are indexed by oriented cycles* and *columns by oriented edges*, and whose typical element, corresponding to row (cycle) ℓ and column (edge) e is $a_{\ell,e}$, as defined above. For instance, in the example of Fig. 1, we have the cycles $(1,2,5)$, $(2,3,4,5)$, etc. There are too many cycles, so I'm not going to list them all. The cycle matrix is

$$A = \begin{array}{cc} \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} & \begin{array}{c} \text{cycle } \mathbf{(1,2,5)} \\ \text{cycle } \mathbf{(2,3,4,5)} \\ \cdots \end{array} \\ \begin{array}{cccccccc} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} & \boldsymbol{e_4} & \boldsymbol{e_5} & \boldsymbol{e_6} & \boldsymbol{e_7} & \boldsymbol{e_8} \end{array} & \end{array}$$

Again, I only listed its first two rows, corresponding to the cycles $(1,2,5)$ and $(2,3,4,5)$. KVL (2) now reads

$$Av = 0. \tag{4}$$

So, with the identification of a graph and the introduction of the matrices B and A, we managed to encode the relevant information of the circuit and express KCL and KVL in the form of (3) and (4). Let us see what we can say about these two laws, before we introduce any additional constraints, i.e., before we take into account the actual identity of the circuit elements sitting at the various branches of the circuit.

A WORD ON NOTATION: Suppose that A is a $m \times n$ matrix. We will use $A^\intercal$ to denote the <u>*transpose*</u> of A, i.e. the $n \times m$ matrix whose rows are the columns of A. If B is another $n \times k$ matrix then the rule for the transpose of the product says that $(AB)^\intercal = B^\intercal A^\intercal$. The set all n-vectors x such that $Ax = 0$ is denoted by $\mathfrak{N}(A)$:

$$\mathfrak{N}(A) := \{x : \ Ax = 0\}.$$

The set of all m-vectors y such that $y = Ax$ for some vector n-vector x is denoted by $\mathfrak{R}(A)$:

$$\mathfrak{R}(A) := \{y : \ y = Ax \text{ for some } x\}.$$

Note then, that every such y is a linear combination of the columns of A. In other words, $\mathfrak{R}(A)$ contains all linear combinations of the columns of A. In the same vein, the set containing all linear combinations of the rows of A is denoted by $\mathfrak{R}(A^\intercal)$ (because rows of A are columns of $A^\intercal$).

5 Spaces of circuit variables

We identify four spaces and use the following notation/terminology:

1. **CURR-SP** (current space): It is the set of all current configurations, i.e. all i satisfying (3). Thus, **CURR-SP** $= \mathfrak{N}(B)$.

2. **VOLT-SP** (voltage space): It is the set of all voltage configurations, i.e. all v satisfying (4). Thus, **VOLT-SP** $= \mathfrak{N}(A)$.

3. **CYCL-SP** (cycle space): It is the set of all linear combinations of cycles, that is, of rows of A. Thus, **CYCL-SP** $= \mathfrak{R}(A^\intercal)$.

4. **VERT-SP** (vertex space): It is the set of all linear combinations of rows of B. Thus, **VERT-SP** $= \mathfrak{R}(B^\intercal)$.

These spaces are not unrelated. In this, and the following few sections, we will study their relations. This study will turn out to be quite fruitful because, as by-products, it will give us some general methods for solving circuits, it will tell us what energy conservation actually means, and it will essentially exhaust the study of circuits.

First, observe that every row of A (i.e. every cycle vector) is a valid current configuration, i.e., that every row of A satisfies (3). To see this, consider a vertex k and a loop ℓ. Either k does not belong to ℓ, in which case the contribution of ℓ to the KCL at k is zero, or k belongs to ℓ; in the latter case, there are two edges in ℓ Fig. to k, one ending at k and one starting at k; Hence the contribution of ℓ to the KCL at k is $-1+1=0$. In matrix notation,

$$BA^\intercal = 0. \tag{5}$$

A consequence of (5) is that the cycle space is contained in the current space:

$$\textbf{CYCL-SP} \subset \textbf{CURR-SP}\,. \tag{6}$$

Indeed, a vector in the cycle space is, by definition, of the form $A^\intercal x$. But then $B(A^\intercal x) = 0$. So $A^\intercal x$ is also in the current space. Now we can take transpose in (5) to get

$$AB^\intercal = 0, \tag{7}$$

which means that the node space is contained in the voltage space:

$$\textbf{VERT-SP} \subset \textbf{VOLT-SP}\,. \tag{8}$$

In fact, we claim that the opposite inclusions in (6) and (8) also hold. To show this, we need a bit more of graph theory.

6 Spanning trees: VOLT-SP = VERT-SP

Assume that the graph is *connected* (the general case follows easily), with n vertices and m edges. Since every vertex is connected by an edge to some other vertex, we have

$$m \geq n-1.$$

A *tree* is a connected graph that has no cycles at all. A *spanning tree* of G is a tree that contains all the vertices of G. Consider then a fixed spanning tree $T = (V, E_T)$ of G. Here is a spanning tree (Fig. 3) for our example of Fig. 1. Its edge set is $E_T = \{e_2, e_5, e_6, e_8\}$. These are the *tree branches*. The remaining edges are called *chords*.

It is easy to see that every spanning tree has $n-1$ branches. We are going to show that every voltage configuration v can be written as a linear combination of rows of B. This is nothing else but the familiar idea that every voltage configuration on the edges of G can be defined by means of a *potential configuration*

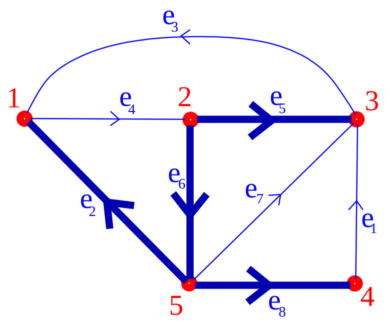


Figure 3: A SPANNING TREE

on the vertices of G. The spanning tree T helps to rigorously prove that. Pick a vertex of T and call it the *root* or the *ground node*. Now, for any vertex k, there is a unique path on T connecting k to the root. Define the orientation of this path to be that from the root to k. Define the potential p_k of vertex k to be the sum of the voltages over all edges of this path, with the correct sign. That is, if edge e of the path has the same orientation as the path, then add v_e with sign $+1$, or else, with sign -1. It is now easy to check that for any edge $e = (k, \ell)$ (not necessarily in T) the voltage v_e is just

$$v_e = p_k - p_\ell, \tag{9}$$

i.e., the difference of the potential of the start node minus that of the end node. Indeed, if e is in T, then this is obvious. If $e = (k, \ell)$ is a chord, then let s be the vertex at which the path of k and the path of ℓ first intersect, and apply KVL at the obvious cycle containing vertices s, k, ℓ. (See Fig. 4.)

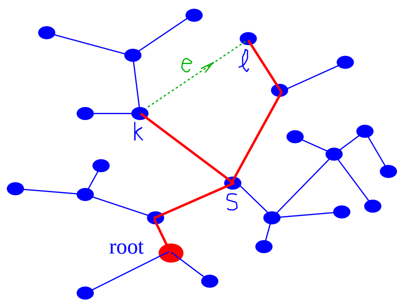


Figure 4: SHOWING THAT $v_e = p_k - p_\ell$

Now check that the above display (9) can be written, in matrix notation, as

$$v = B^\intercal p.$$

In other words, if v is a voltage configuration, then we showed that there is a vector p (a potential configuration) such that $v = B^\intercal p$. This means that every voltage configuration is a linear combination of rows of B, or that

$$\textbf{VOLT-SP} \subset \textbf{VERT-SP}\,. \tag{10}$$

Putting (8) and (10) together we arrive at the result that **VOLT-SP** is identical to **VERT-SP** .

A byproduct of our discussion is that every voltage v can be defined by means of $n-1$ numbers, the potentials of the vertices other than the root vertex. Hence **VOLT-SP** is an $(n-1)$-dimensional linear space:

$$\dim \textbf{VOLT-SP} = n - 1.$$

Next, we want to show that the reverse inclusion in (6) also holds. We can do this in two ways. Either we can use another graph argument (and talk about chords and fundamental cycles) or use algebra and deduce it from (10). We prefer the latter. (We are going to talk about the former approach in connection to the general methods for solving circuits later.)

7 A bit of algebra: CURR-SP = CYCL-SP

We claim that this last inclusion (10) implies the opposite inclusion of (6). Let i be some current configuration and v some voltage configuration. Hence $v = B^\intercal p$ and $Bi = 0$. This gives

$$i^\intercal v = i^\intercal(B^\intercal p) = (Bi)^\intercal p = 0 \cdot p = 0.$$

By the way, this relationship between a current configuration i and a voltage configuration v is called *Tellegen's theorem* and it is a generalization of the *principle of energy conservation*.

We now do the following trick: pick set of linearly independent columns of A. Call these columns $a_1, a_2, \ldots, a_r$. (We can achieve this by starting with the first column of A, then picking the next column that is not a multiple of a_1 and call it a_2, then the next one that is not a linear combination of a_1, a_2, and so on.) Suppose that a_j occurs at the p-th position of A. Let u_j be a unit column, with 1 at the p-th position and 0 everywhere else. Then

$$a_j = Au_j, \quad j = 1, \ldots, r.$$

Next pick a maximal set of linearly independent voltage configurations. This is easy to do: every row of B is a voltage configuration. Any $n-1$ rows (for example all but the last row) are linearly independent. Call these rows $v_1^\intercal, \ldots, v_{n-1}^\intercal$. We claim that $u_1, \ldots, u_r$, $v_1, \ldots, v_{n-1}$ are linearly independent. Indeed, suppose that there are numbers $\lambda_1, \ldots, \lambda_r$, $\mu_1, \ldots, \mu_{n-1}$, such that

$$\sum_{j=1}^{r} \lambda_j u_j + \sum_{k=1}^{n-1} \mu_k v_k = 0.$$

Then, by applying A to both sides (and remembering that $Au_j = a_j$, and $Av_k = 0$), we find

$$\sum_{j=1}^{r} \lambda_j a_j = 0.$$

But the a_j's have been chosen to be linearly independent; thus $\lambda_j = 0$ for all $j = 1, \ldots, r$. But then $\sum_{k=1}^{n-1} \mu_k v_k = 0$, and, since the v_k are linearly independent, we have $\mu_k = 0$, for all $k = 1, \ldots, n-1$.

Now fix a column vector $x = (x_e, e \in E)$ and consider the product Ax. This is really a linear combination of the columns of A. Since every column is a linear combination of the columns $a_1, \ldots, a_r$ chosen above, we have that Ax itself is a linear combination of these columns. We can thus write

$$Ax = \sum_{j=1}^{r} \lambda_j a_j.$$

Using the same coefficients λ_j, define the column

$$y = \sum_{j=1} \lambda_j u_j.$$

We have $x = (x-y) + y$, and $A(x-y) = Ax - Ay = 0$. Hence $x - y \in \mathfrak{N}(A)$**VOLT-SP** . Thus, any m-vector x can be written as a sum of a vector in **VOLT-SP** and a vector y which is the linear combination of r unit vectors. Remembering that $\dim$ **VOLT-SP** $= n-1$, we have

$$m = (n-1) + r.$$

Instead of using the $u_1, \ldots, u_r$, we can, instead, use r linearly independent rows of A. Call these rows $w_1^\intercal, \ldots, w_r^\intercal$. The vectors $v_1, \ldots, v_{n-1}$, $w_1, \ldots, w_r$ are linearly independent and any m-vector x can be written as a linear combination of these $(n-1)+r = m$ vectors. In particular, this is true for a current configuration i: we can write $i = v + w$, where v is a linear combination of $v_1, \ldots, v_{n-1}$, and w is a linear combination of $w_1, \ldots, w_r$. Since $v \in$ **VOLT-SP** , we know that $i^\intercal v = 0$. Hence $0 = (v+w)^\intercal v = v^\intercal v + w^\intercal v$. But $w = A^\intercal u$, for some u, because $w \in$ **CYCL-SP** . So $w^\intercal v = (A^\intercal w)^\intercal v = w^\intercal A v = 0$. So $v^\intercal v = 0$, and this means that $v = 0$, so that $i = w$, and so $i \in$ **CYCL-SP** . We thus proved any i from **CURR-SP** is also contained in **CYCL-SP** .

$$\textbf{CURR-SP} \subset \textbf{CYCL-SP} . \tag{11}$$

Putting (6) and (11) together we arrive at the result that **CURR-SP** is the same as **CYCL-SP** . In particular, we have shown that

$$\dim \textbf{CURR-SP} = m - n + 1.$$

MNEMONIC RULE: The V-spaces (**VOLT-SP** and **VERT-SP**) are identical, and so are the C-spaces (**CURR-SP** and **CYCL-SP**).

8 The node and loop methods

If a spanning tree T is chosen then the $n-1$ potentials p_k on every vertex k other than the root define all voltages. This was explained above. This actually leads to the *node method* for circuit analysis. In this method, we consider the $n-1$ potentials as the unknowns and, for each vertex k which is not root, we write KCL.

The dual to this method is the *loop method*. In class, we explained how to pick meshes in a planar graph. If the graph is not necessarily planar, then the way we pick "linearly independent" cycles (loops) is again by considering a spanning tree T. Each edge e which is not in T (called chord) defines a cycle, in the following manner: if we add e to T then a unique cycle is formed, called the *fundamental cycle* of the chord e. For example, in Fig. 3, if we add e_1 to the tree we obtain the cycle $(4, 3, 2, 5)$. We give the fundamental cycle the orientation of the chord that defines it. We showed earlier (11) that **CURR-SP** $\subset$ **CYCL-SP** . This means that every current configuration i is a linear combination of cycles. But we never said which is this linear combination. This is achieved by letting the currents j_e on each chord (on each fundamental cycle) e undefined, and by expressing the vector i as a linear combination of these j_e's. These variables are called *fundamental loop currents*. For instance, Fig. 3 can be redrawn as follows (Fig. 5). This figure shows the fundamental cycles and their currents.

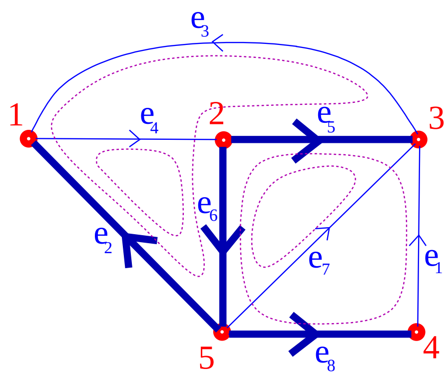


Figure 5: FUNDAMENTAL CYCLES AND CURRENTS

Now, a current on a branch of the tree can be expressed as combination of the loop currents for each fundamental loop containing this branch. So we have

$$\begin{aligned} i_1 &= j_1 & i_5 &= -j_1 + j_3 - j_7 \\ i_2 &= j_4 - j_3 & i_6 &= j_1 - j_3 + j_4 + j_7 \\ i_3 &= j_3 & i_7 &= j_7 \\ i_4 &= j_4 & i_8 &= j_1 \end{aligned}$$

Another way to think of these is as follows. Each branch of the tree separates it into two upon its removal. Let V_1, V_2 be the set of vertices of each part. For instance, by removing e_6 we have $V_1 = \{1,4,5\}$ and $V_2 = \{2,3\}$. The removed edge together with the chords from V_1 to V_2 form a *fundamental cut*, and each of the above equations is nothing else but KCL for this fundamental cut. For example, the fundamental cut of e_6 contains e_1, e_7, e_4, e_3 and e_6. The equation $i_6 = j_1 - j_3 + j_4 + j_7$ is KCL for this cut. Next observe that we can write the equations above in matrix form as follows:

$$\begin{pmatrix} i_1 \\ i_2 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \end{pmatrix} = j_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + j_3 \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + j_4 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + j_7 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

In matrix notation,

$$i^{\mathsf{T}} = j^{\mathsf{T}} \widetilde{A},$$

where $\widetilde{A}$ is the *reduced cycle matrix*

$$\widetilde{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 \end{pmatrix}$$

In general, the reduced cycle matrix $\widetilde{A}$ is obtained from the cycle matrix A by eliminating all rows except a maximal set of linearly independent ones. It thus has $m - n + 1$ rows and m columns.

One then considers the $m - n + 1$ loop currents as unknowns and requires that, for each fundamental loop, a KVL be written. It leads to $m - n + 1$ linearly independent equations in $m - n + 1$ unknowns.

In practice, choosing the right method, or picking a spanning tree T in a "correct" way are both matters that require some experience.

9 Tellegen's theorem

This was proven earlier: any current configuration i is orthogonal to any voltage configuration:

$$i^{\mathsf{T}} v = 0.$$

Notice that the only requirement is that both i and v refer to the same graph. We have not used the identity of the circuit elements. And, although this can be interpreted as the energy conservation principle, it is a fact that reflects simply the geometry of the spaces of variables.

10 General solution

We present the solution to "any" linear circuit. First, we assume that the circuit is well-defined. We leave this notion vague, but what we mean is that the circuit should not contain, for example, current sources connected in a way that they violate KCL, neither voltage sources violating KVL. By source transformations we can then reduce the circuit so that each branch e contains a resistor r_e in series with a voltage source g_e. (See Fig. 6.) We then have

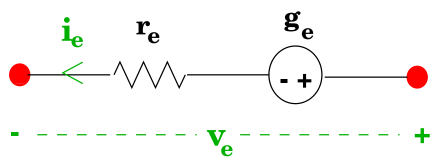


Figure 6: A TYPICAL BRANCH

$$v_e = r_e i_e + g_e,$$

for each edge e. In matrix notation,

$$v = Ri + g, \tag{12}$$

where R is a diagonal matrix containing the resistances. Now let $\widetilde{B}$ be the reduced incidence matrix, that is, the matrix B with the last row omitted. This has now the property that all its rows are linearly independent. Then choose a spanning tree T, as explained earlier. Finally, arrange the edges in two sets, those in T, and those not in T, and permute the columns of both $\widetilde{B}$ and $\widetilde{A}$ to reflect that splitting. For instance, in the earlier exaple, $\widetilde{A}$ is rewritten as

$$\widetilde{A} = \begin{pmatrix} 0 & -1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = (\widetilde{A}_T \quad \widetilde{A}_N)$$

Notice that $\widetilde{A}_N$ is the identity matrix. Similarly, write $\widetilde{B} = (\widetilde{B}_T \quad \widetilde{B}_N)$, $i = (i_T, i_N)$, $v = (v_T, v_N)$. We write (5) as:

$$0 = \widetilde{B}_T \widetilde{A}_T^\intercal + \widetilde{B}_N \widetilde{A}_N^\intercal = \widetilde{B}_T \widetilde{A}_T^\intercal + \widetilde{B}_N.$$

Since $\widetilde{B}_T$ has $n-1$ rows and $n-1$ columns, and its rows are linearly independent, it is invertible. Hence

$$\widetilde{A}_T^\intercal = -\widetilde{B}_T^{-1} \widetilde{B}_N.$$

Next, KCL (3) is written as

$$0 = \widetilde{B}i = (\widetilde{B}_T \quad \widetilde{B}_N) \begin{pmatrix} i_T \\ i_N \end{pmatrix} = \widetilde{B}_T i_T + \widetilde{B}_N i_N.$$

Hence

$$i_T = -\widetilde{B}_T^{-1} \widetilde{B}_N i_N = \widetilde{A}_T^\intercal i_N.$$

This means that

$$i = \begin{pmatrix} \widetilde{A}_T^\intercal \\ \widetilde{A}_N^\intercal \end{pmatrix} i_N = \widetilde{A}^\intercal i_N.$$

Now write KVL (4) together with the branch equations (12).

$$0 = \widetilde{A}v = \widetilde{A}(Ri + g) = \widetilde{A}R\widetilde{A}^\intercal i_N + Ag.$$

Observe that $\widetilde{A}R\widetilde{A}^\intercal$ is a square matrix of size $m-n+1$ with linearly independent rows. Hence it is invertible. So

$$i_N = -(\widetilde{A}R\widetilde{A}^\intercal)^{-1}\widetilde{A}g.$$

And so, the complete solution of the circuit is

$$i = -\widetilde{A}^\intercal(\widetilde{A}R\widetilde{A}^\intercal)^{-1}\widetilde{A}g.$$

Actually, in this section, not only we have produced a solution, but we have also shown that a solution always exists and is unique.

11 Minimum norm interpretation

If all resistances are equal to 1, then KVL together with (12) yields

$$Ai = -Ag.$$

The solution $i = -\widetilde{A}^\intercal(\widetilde{A}\widetilde{A}^\intercal)^{-1}\widetilde{A}g$ is the minimum Euclidean norm solution. In fact, $\widetilde{A}^\intercal(\widetilde{A}\widetilde{A}^\intercal)^{-1}$ is the pseudo-inverse of A. See exercises.

12 Circuits with one independent source: combinatorial solution

The *principle of superposition* says that the general solution to a circuit containing k independent sources is obtained by adding the solutions of k circuits. Each circuit is formed by setting all but one independent source to zero.

It is thus not a great restriction to consider a circuit with just a single source. Suppose that this source is a current source and suppose that current enters at vertex s (the source) and leaves at vertex t (the sink), as in the example of Figure 7.

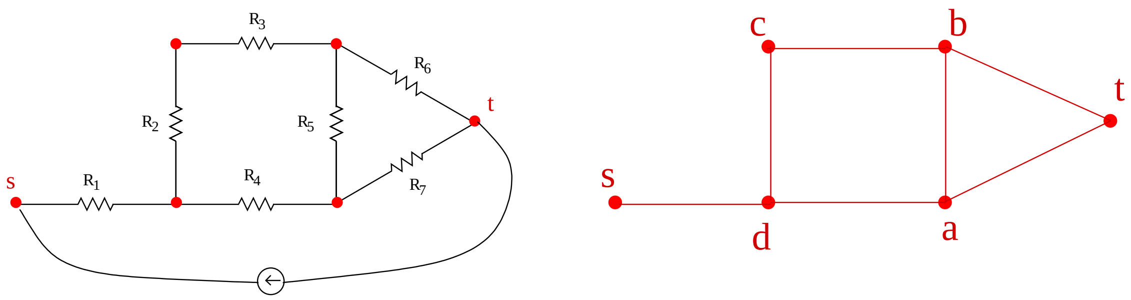


Figure 7: A CIRCUIT WITH A SINGLE CURRENT SOURCE

Let us consider the graph G of the circuit, omitting the current source. We assume that the graph is connected. We also assume that *all resistances are equal to one Ohm* and that *the current source sends current of one Ampere*. Let N be the total number of spanning trees. For each edge $\{a, b\}$, where a, b are two distinct vertices, we *let $N(s, a, b, t)$ be the number of spanning trees of G that contain a, b and, in addition, the path that starts at s and ends at t contains a and b in this order*. Then the following is a beautiful formula that gives the current $i_{a,b}$ on each edge (a, b):

$$i_{a,b} = \frac{N(s, a, b, t) - N(s, b, a, t)}{N}.$$

Since resistances are all 1, we have $v_{a,b} = i_{a,b}$. To prove that this is correct, we need to check that KCL are satisfied and then that KVL are satisfied.

Consider KCL at s. We need to show that $\sum_{x\sim s} i_{s,x} = 1$, where the sum is over all neighbors x of s, i.e., that $N = \sum_{x\sim s} N(s,s,x,t) - \sum_{x\sim s} N(s,x,s,t)$. Clearly, $N(s,x,s,t) = 0$ because no tree can contain (s,x) and (x,s). On the other hand, every spanning tree T contains exactly one neighbor x of s. Hence $N = \sum_{x\sim s} N(s,s,x,t)$.

To check KCL at t amounts to the same reasoning.

To check KCL at a vertex a other than s or t, we need to show that $\sum_{x\sim a} i_{a,x} = 0$, or that

$$\sum_{x\sim a} N(s,a,x,t) = \sum_{x\sim a} N(s,x,a,t).$$

Let $\mathcal{T}(s,a,t)$ be the set of spanning trees T such that the unique path from s to t of T contains a. Each $T \in \mathcal{T}(s,a,t)$ there is exactly one $x \sim a$ such that (a,x) is on the path of T from s to t and with the same orientation. Hence,

$$|\mathcal{T}(s,a,t)| = \sum_{x\sim a} N(s,a,x,t).$$

Similarly,

$$|\mathcal{T}(s,a,t)| = \sum_{x\sim a} N(s,x,a,t).$$

To check KVL, we need to consider an oriented cycle C and show that $\sum_{(a,b)\in C} i_{a,b} = 0$, or that

$$\sum_{(a,b)\in C} N(s,a,b,t) = \sum_{(a,b)\in C} N(s,b,a,t),$$

which is further equivalent to

$$\sum_{T\in\mathcal{T}} \big|\{(a,b)\in C:\ T\in\mathcal{T}(s,a,b,t)\}\big| = \sum_{T\in\mathcal{T}} \big|\{(a,b)\in C:\ T\in\mathcal{T}(s,b,a,t)\}\big|.$$

(Here, $\mathcal{T}$ is the set of all spanning trees.) Fix a tree T and an oriented cycle C, and let (a,b) be en edge which belongs to C and to the path of T from s to t, respecting both orientations. If we remove the edge (a,b), then the tree is split into two components, one containing s–call it T_s, and one containing t–call it T_t. The edge (a,b) is oriented from T_s to T_t. The cycle C must contain another edge, say (x,y), oriented from T_t to T_s. If we pick the "earliest possible such edge" after (a,b) on C, then, by replacing $\{a,b\}$ by $\{x,y\}$ we obtain another tree T' such that $T' \in \mathcal{T}(s,y,x,t)$. This proves the validity of the formula.

Let us now consider the example of Figure 7. Fig. 8 below depicts all spanning trees in the first column. There are 11 of them. Each of the other columns concerns a particular edge and lists the appearances of this edge on the trees. For example, (a,b) appears twice and (b,a) once. Thus, $N(s,a,b,t) = 2$, $N(s,b,a,t) = 1$. The current on the edge (a,b) is thus equal to $(2-1)/11 = 1/11$. Similarly, we find

$$i_{s,d} = 1,\ i_{d,c} = 4/11,\ i_{c,b} = 4/11,\ i_{d,a} = 7/11,\ i_{a,b} = 1/11,\ i_{b,t} = 5/11,\ i_{a,t} = 6/11.$$

We can extend the formula to the case where the resistances are not necessarily equal. In this case, define the weight of a spanning tree to be the product of the inverses of the resistances of its edges. Then define $N(s,a,b,t)$ to be the sum of the weights of all trees at which (a,b) appears. With this new definition, the previous formula remains valid. See exercises.

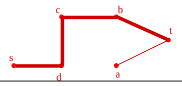
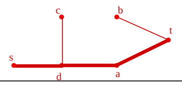
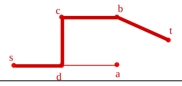
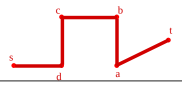
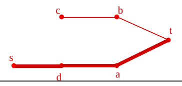
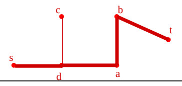
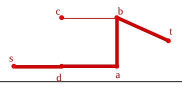
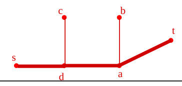
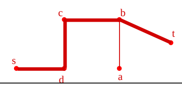
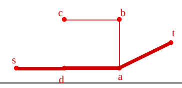
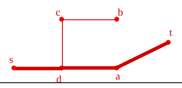
	(s,d)	(d,s)	(d,c)	(c,d)	(c,b)	(b,c)	(d,a)	(a,d)	(a,b)	(b,a)	(b,t)	(t,b)	(a,t)	(t,a)
	●		●		●						●			
	●						●						●	
	●		●		●						●			
	●		●		●					●			●	
	●						●						●	
	●						●		●		●			
	●						●		●		●			
	●						●						●	
	●		●		●						●			
	●						●						●	
	●						●						●	
	11	0	4	0	4	0	7	0	2	1	5	0	6	0

Figure 8: COMPUTING CURRENTS USING THE COMBINATORIAL FORMULA

Exercises

To solve these exercises, you need to have understood the material here, but also have the physical intuition obtained in the circuit theory class.

1. Show that every connected graph has a spanning tree (easy).
2. If a circuit graph is planar then it can be embedded (=drawn) in the plane. A face is a cycle which contains no edges inside. Show that the rows of the cycle matrix A corresponding to cycles which are not faces are linear combinations of the rows which correspond to faces.
3. We claimed that considering graphs where there is at most one edge per pair of vertices is no loss of generality. Explain how more general circuits (with two or more edges corresponding to at least one pair of vertices) can be reduced to the previous case. Hint: introduce new vertices.
4. Prove that the rank of the incidence matrix is never equal to the number of edges.

5. Explain why $AB^{\intercal} = 0$ implies that **VERT-SP** $\subset$ **VOLT-SP** .
6. Suppose that the circuit contains other linear, but dynamic, elements, such as capacitors and inductors. If the system is in sinusoidal steady state (that is, if all the sources are sinusoids with the same frequency) and if the system is in steady state, then explain how, by using complex, rather than real, resistances, one can reduce the study of this circuit to the one considered here.
7. Explain how, by using operators instead of resistances, one can actually extend our discussion to even more general circuits (not necesserily sinusoidal sources).
8. Show, by considering the circuit directly, that every row of B is a valid voltage configuration. Then observe that this means that $BA^{\intercal} = 0$.
9. Since KVL, KCL and Ohm's law are linear, the principle of superposition follows: if there circuit is driven by a number of sources, then, by considering a family of circuits, one for each source (when the rest are removed), we can find the general solution by, simply, adding the individual solutions. Prove this.
10. Use the principle of superposition to give another proof of the fact that, for every circuit, there is at most one solution. Hint: if there are two, then we can find a nontrivial solution for a circuit that has no external sources. Physically this is impossible. Show that it is also mathematically (logically) impossible.
11. Generalize the combinatorial formula for the case of general resistances.
12. Show how the combinatorial formula, together with the principle of superposition, imply uniqueness of solution.
13. Explain why it is no loss of generality to consider circuits with sources of one kind only (either current or voltage sources, but not both).
14. For every pair of vertices a, b, define the equivalent resistance $R_{ab} = R_{ba}$ as the voltage induced at the edge ab, when a unit current source is applied. Observe that the symmetric matrix $[R_{ab}]$ consisting of all equivalent resistances provides the full solution to the circuit (every quantity can be computed in terms of this matrix).
15. Consider a graph whose vertices and edges are those of a cube. Each edge contains a resistance in series with (a possibly zero) voltage source. Explain why it is better to use the loop method, rather than the node method, to solve the circuit.
16. Find a circuit for which the node method is easier than the loop method.
17. Consider an arbitrary circuit with unit resistances. Show that

$$i = -\widetilde{A}^{\intercal}(\widetilde{A}\widetilde{A}^{\intercal})^{-1}\widetilde{A}g$$

   achieves the minimum in

$$\min\{\|i\|^2 : \ Ai = -Ag\},$$

   where $\|i\|^2 := \sum_e i_e^2$. In other words, among all possible current configurations, the one that nature chooses is the one which has least energy.
18. Generalize the above to an arbitrary circuit, i.e., formulate and prove a minimum energy theorem.

References

[1] Béla Bollabás. *Graph Theory*. Springer-Verlag, New York, 1979.

[2] Charles Desoer and Ernest Kuh. *Basic Circuit Theory.* McGraw-Hill, 1969.