A CONSISTENT EXTENSION OF CONDORCET'S ELECTION PRINCIPLE*

H. P. YOUNG† AND A. LEVENGLICK‡

Abstract. Condorcet's principle of choosing the majority alternative whenever one exists is violated not only by Borda's rule but by any scoring method; nevertheless the essential property of scoring functions—"consistency" of the outcome under aggregation of subgroups—is shown to be compatible with Condorcet's principle. Moreover these two properties, suitably interpreted, together with neutrality, determine a unique rule known as Kemeny's rule.

1. Introduction. In France during the latter half of the eighteenth century, the problem of designing voting rules for an assembly began to be studied systematically for the first time. Two of the major contributors were Jean-Charles de Borda and Marie J. A. N. Caritat, Marquis de Condorcet, both members of l'Académie Royale des Sciences. From these two men sprang two streams of thought on the problem of collective decision making that persist to the present day. In 1770, Borda [4] read a paper before the Academy in which he proposed the following method, which he called "election by order of merit": for each voter's announced (linear) preference order on the alternatives, a score of 0 is assigned to the least preferred alternative, 1 to the next-to-least preferred, and so forth; then the total score of each alternative is computed and the one with the highest score is declared the winner. Condorcet [6] proposed instead that if there is some alternative (the Condorcet alternative) that defeats every other in pairwise simple majority voting, then that alternative should be selected. A difficulty with Borda's rule is that it may not result in the choice of the Condorcet alternative (as in Example 1 where a_1 is the Condorcet alternative and a_2, the Borda choice); but a difficulty with Condorcet's principle is that, if a Condorcet alternative does *not* exist, it is unclear which alternative should be chosen.

Example 1. Number of voters

(6)	(4)	(1)
a_1	a_2	a_3
a_2	a_3	a_1
a_3	a_1	a_2

The challenge of combining the regularity of Borda's approach with Condorcet's principle into a unified method is a long-standing problem in the theory of elections. A variety of proposals have been made over the years; many of which are surveyed and compared in a recent paper by Fishburn [8]. One of the earliest is due to Black [3], who proposed that the Condorcet alternative be chosen when one exists, and otherwise that the Borda method be reverted to. This somewhat ad hoc proposal avoids the fundamental issue of choosing *properties* that are natural in the context of election, and then asking what (if any) methods have these properties.

In this paper we shall identify certain basic properties suggested by the Borda and the Condorcet approaches, respectively, and show that these properties uniquely determine a method first proposed by John Kemeny [10], [11].

Let $A = \{a_1, a_2, \cdots, a_m\}$ be a set of m alternatives under consideration, $\mathbb{N} = \{0, 1, 2, \cdots\}$ the names of the possible voters. A *preference order* on A is simply a *linear order*. We will denote such an order by the sequence $\sigma = (a_{i_1}, a_{i_2}, \cdots, a_{i_m})$ where a_{i_1} is most preferred, and so forth. $L(A)$ will denote the set of all $m!$ linear orders on A.

For any (finite *electorate* $\mathbf{M} \subsetneq \mathbb{N}$, a *profile* on $\mathbf{M}$ is a specification of the preferences of the voters in $\mathbf{M}$, i.e. a function $\phi : \mathbf{M} \to L(A)$. Let Φ be the set of all possible profiles for the given A. For any $\sigma \in L(A)$ and $\phi \in \Phi$, $n_\sigma(\phi)$ will denote the number of voters in ϕ having preference order σ.

A *preference function* (PF) is a function f that associates with each profile $\phi \in \Phi$ a nonempty set of preference orders $f(\phi) \subseteq L(A)$ representing the *consensus* preference orders for the given electorate. (Normally, if there are no ties, $f(\phi)$ will be a unique preference order.) A *choice function* (CF) is a function g from Φ to the nonempty subsets of A; $g(\phi)$ represents the "winning" (i.e. the most preferred) alternative(s) for the given electorate. To each PF f there corresponds in a natural way a CF $\hat{f}$ defined by

$$(1) \qquad \hat{f}(\phi) = \{a_i \in A : a_i \text{ is most preferred for } some \ \sigma \in f(\phi)\}.$$

If $f(\phi)$ depends only on the numbers $n_\sigma(\phi)$ for each ϕ, then f is said to be *anonymous*. Similarly, if f is symmetric in its treatment of alternatives, then f is *neutral*. Both of these are standard assumptions since the work of May [14] and Arrow [1]. A PF (or a CF) satisfying both is said to be *symmetric* [15].

2. Consistency and a generalization of Borda. Borda's "order of merit" method may be generalized to the situation where the scores awarded to alternatives are not necessarily evenly spaced. In general; given $\mathbf{s} = (s_1, s_2, \cdots, s_m) \in \mathbb{R}^m$ and profile ϕ we may assign a score of s_m to each voter's *least-preferred* alternative, a score of s_{m-1} to each *next-to-the-least* preferred alternative, and so forth. If the total score of alternative a_i is denoted by $s_i(\phi)$, then we may define a PF f^s by

$$(2) \qquad (a_{i_1}, a_{i_2}, \cdots, a_{i_m}) \in f^s(\phi) \quad \text{iff} \quad s_{i_1}(\phi) \geqq s_{i_2}(\phi) \geqq \cdots \geqq s_{i_m}(\phi).$$

Moreover, if ties occur relative to $\mathbf{s}$ (i.e., if several distinct preference orderings result from (2)) then we might use another scoring vector $\mathbf{t} \in \mathbb{R}^m$ to resolve these ties; thus we define the *composition* $f^t \circ f^s(\phi)$ by

$$(3) \qquad (a_{i_1}, a_{i_2}, \cdots, a_{i_m}) \in f^t \circ f^s(\phi) \quad \text{iff} \quad s_{i_1}(\phi) \geqq s_{i_2}(\phi) \geqq \cdots \geqq s_{i_m}(\phi)$$

$$\text{and for all } k < l, \quad s_{i_k}(\phi) = s_{i_l}(\phi) \quad \text{implies} \quad t_{i_k}(\phi) \geqq t_{i_l}(\phi).$$

Any PF f^s (or composed series of such functions) and the corresponding CF is called a *scoring function* [15], [17]. Borda's method is a particular scoring function in which no provision is made for resolving ties; another example is the method of *plurality voting* (take $\mathbf{s} = (1, 0, \cdots, 0)$).

A natural property of voting rules, which in particular is enjoyed by scoring rules, is the following. If two committees meeting separately arrive at the *same* consensus ordering (using some f), then meeting together this should still be their consensus. More precisely, we say that a PF (or a CF) f is *consistent* [15], [16], [17] if whenever ϕ and ϕ' are profiles on *disjoint* electorates such that $f(\phi) \cap f(\phi') \neq \phi$; then $f(\phi + \phi') = f(\phi) \cap f(\phi')$. (Here $\phi + \phi'$ is the profile defined by the union of the two electorates.)

Thus, if ties occur in the separate committees, then consensus rankings consist precisely of the ties *common to both*. It can be shown that any scoring PF f is consistent, as is the corresponding CF $\hat{f}$. In fact, consistency turns out to be the fundamental property characterizing scoring functions when regarded as choice functions.

(4) THEOREM 1 [17]. *A choice function is a scoring function if and only if it is symmetric and consistent.*

Notice, however, that if f is a consistent preference function, then $\hat{f}$ is not necessarily a consistent choice function. For example, with three alternatives a_1, a_2, a_3, it is possible that two committees agree on their *first* choice (say a_1) but differ on their ranking of the remaining two; in this situation consistency for a preference function makes no requirements on the consensus preference order for the two committees together, while a consistent choice function requires that the first choice for the merged committees be a_1.

Intuitively, consistency for preference functions seems to be a less stringent requirement than consistency for choice functions. However it should also be noticed that this is not, strictly speaking, a weaker concept, since it is possible that $\hat{f}$ is consistent while f is not. For example, define the PF f on three alternatives such that $\sigma = (a_{i_1}, a_{i_2}, a_{i_3}) \in f(\phi)$ iff a_{i_1} has the highest Borda score, and $n_\sigma(\phi)^2 + n_\tau(\phi)^2 \geqq n_{\sigma'}(\phi)^2 + n_{\tau'}(\phi)^2$ where $\sigma' = (a_{i_1}, a_{i_3}, a_{i_2})$, $\tau = (a_{i_2}, a_{i_3}, a_{i_1})$, and $\tau' = (a_{i_3}, a_{i_2}, a_{i_1})$.

3. The Condorcet property. Given a profile ϕ, let $n_{ij}(\phi)$ be the number of voters preferring a_i to a_j *minus* the number preferring a_j to a_i. Condorcet's proposal was that any alternative a_i such that $n_{ij}(\phi) > 0$ for all $j \neq i$ (i.e., that would defeat any other alternative in pairwise simple majority voting) should be the choice of the group, or, in the case of a preference function, the *most-preferred* alternative. A parallel concept is that if a_i is an alternative such that $n_{ij}(\phi) < 0$ for all $j \neq i$ (i.e., it would be defeated by every other alternative), then a_i should be the *least-preferred* alternative. Suppose now that a_i is an alternative such that $n_{ij}(\phi) = 0$ for all $j \neq i$. We call a_i a *quasi-Condorcet* alternative for ϕ. (Fishburn [8] uses this term somewhat differently.) By a natural extension of Condorcet's principle, a quasi-Condorcet alternative is evidently no better than, and also no worse than, any other alternative.

Formally, we say that a PF f is *quasi-Condorcet* if for any $\phi \in \Phi$ and any quasi-Condorcet alternative a_i for ϕ

(5) $(\cdots a_j, a_i \cdots) \in f(\phi)$ iff $(\cdots a_i, a_j, \cdots) \in f(\phi)$ for all $j \neq i$.

In other words, if a_i is a quasi-Condorcet alternative, then any consensus ordering in which a_i is just above an alternative a_j is just as plausible as the same ordering but with a_i just below a_j.

In general we define a *Condorcet* alternative to be any alternative a_i such that $n_{ij}(\phi) \geqq 0$ for all $j \neq i$.

If all of the alternatives are to be ordered in terms of their desirability (i.e. if a preference function is to be used), then the logic of ranking an alternative that achieves a *strict* majority over every other in first place is simply this: if it were ranked in any other position, then the alternative immediately preceding it (say a_j) would be ranked superior (and inferior) to exactly the same alternatives as a_i, while on the other hand a_i is surely better when compared just to a_j; hence a_i must be absolutely better than a_j and should be ranked before it. By this process we conclude that a_i must be ranked first. An extension of the same logic shows indeed that no alternative should enjoy a strict majority over its immediate predecessor in the consensus ranking. Thus

a preference function f is said to be *Condorcet* if

(6) $n_{ij}(\phi) > 0$ implies   not $(\cdots a_j, a_i, \cdots) \in f(\phi)$.

In the case of ties we have

(7)

$$n_{ij}(\phi) = 0 \quad \text{implies} \quad (\cdots a_i, a_j \cdots) \in f(\phi) \quad \text{iff}$$

$$(\cdots a_j, a_i \cdots) \in f(\phi).$$

Note that (6) and (7) speak only about relations between immediate successors a_i and a_j, since in this case their proper *relative* order can be determined without reference to the other alternatives.

We say that a choice function g is *Condorcet* if $g(\phi)$ is precisely the set of Condorcet alternatives whenever such exist. (This definition corresponds to what Fishburn [7] calls a "strongly Condorcet" CF.) Further, it should be noted that the Condorcet property defined above for preference functions is somewhat stronger than that used in [12].

Example 2. Consider the following profile on two voters and three alternatives.

$$\phi: \quad \begin{array}{l} (a_1, a_2, a_3), \\ (a_2, a_3, a_1), \end{array}$$

where a_1 is a quasi-Condorcet alternative. If f is any PF such that a_2 is always preferred to a_3 in a collective preference, then for f to be Condorcet we must have

$$f(\phi) = \{(a_1, a_2, a_3), (a_2, a_1, a_3), (a_2, a_3, a_1)\},$$

that is, a_1 is as good—and as bad—as any other alternative.

A very special case of the Condorcet situation occurs whenever all voters have the *same* preference order, say σ. Then σ is a *Pareto* ordering and a PF is said to be *Pareto* if it always chooses the Pareto ordering when one exists. (A still weaker condition, which implies the Pareto property in the presence of consistency, is the following: a PF is *faithful* if the consensus order is the same as the individual order when society consists of a single individual. This concept was first introduced in [16].)

We say that a PF (or a CF) f is *consistent* on a subdomain $\Phi' \subseteq \Phi$ if whenever $\phi', \phi'' \in \Phi'$, and $\phi' + \phi''$ is defined and contained in Φ'; then

(8) $f(\phi') \cap f(\phi'') \neq \phi$ implies $f(\phi' + \phi'') = f(\phi') \cap f(\phi'')$.

Let Γ be the subdomain of profiles for which a Condorcet alternative exists, that is

(9) $\Gamma = \{\phi \in \Phi : n_{ij}(\phi) \geqq 0 \text{ for some } i \text{ and all } j \neq i\}$.

It is easy to see that any Condorcet *choice* function is consistent on the domain Γ. One of the difficulties in trying to find a natural way to extend Condorcet's principle to the domain $\Phi - \Gamma$ (which is nonempty if $m \geqq 3$) is that it is impossible to find such an extension which is consistent as a choice function.

THEOREM 2. *There is no Condorcet choice function g and domain $\Delta \supsetneqq \Gamma$ such that g is consistent on Δ.*

Proof.[1] Let m be the number of alternatives. For $m = 1$ or 2, $\Gamma = \Phi$ and there is nothing to prove.

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[1] The authors wish to thank Bengt Hansson for suggesting this weakened version of consistency and simple way of proving Theorem 2.

Consider then the case $m \geq 3$, and let $\Delta \supsetneq \Gamma$, $\phi^* \in \Delta - \Gamma$. Suppose, by way of contradiction, that there exists a Condorcet, consistent choice function g defined on Δ. Without loss of generality, let $a_1 \in g(\phi^*)$. Since $\phi^* \notin \Gamma$, a_1 is not a Condorcet alternative, and there is a $j \neq 1$ such that $n_{j1}(\phi^*) > 0$; say without loss of generality $j = 2$. Let n be the number of voters in ϕ^*. Define a new profile ϕ on $2n + n_{21}(\phi^*)$ voters such that $n + n_{21}(\phi^*)$ voters have preference order $(a_1, a_2, a_3, \cdots, a_m)$ and n voters have preference order $(a_2, a_1, a_3, \cdots, a_m)$. Evidently a_1 is a Condorcet alternative for ϕ, so $\phi \in \Gamma$ and $g(\phi) = \{a_1\}$. On the other hand, $(\phi^* + \phi) \in \Gamma$ and $g(\phi^* + \phi) = \{a_1, a_2\}$. But consistency implies that $g(\phi^* + \phi) = \{a_1\}$, a contradiction. Notice, moreover, that if Δ is homogeneous (i.e. $\phi^* + \phi^* = 2\phi^* \in \Delta$), then

$$a_1 \in g(2\phi^*) = g(\phi^*),$$

whereas $2\phi^* + \phi \in \Gamma$ and $g(2\phi^* + \phi) = \{a_2\}$, contradicting the following even weaker version of consistency: $g(\psi) \cap g(\psi') \subseteq g(\psi + \psi')$ for all ψ, ψ': $\quad\square$

(10) COROLLARY. *For $m \geq 3$, no scoring CF is Condorcet.*

Theorem 2 does *not* imply, however, that consistency for *preference functions* is incompatible with Condorcet's principle. We shall in fact show that there is a preference function with these two properties, and that subject to neutrality, it is unique.

4. The Kemeny function. As an example of axiomatic techniques applied to the social sciences Kemeny [10] introduced a PF (described in greater detail in Kemeny and Snell [11]) that turns out to play a special role here. The problem defined in [10] is that of finding a consensus preference order for alternatives being considered by a group of "experts". The approach is to reduce the problem "to one which is analogous to those of classical statistics" [11, p. 9] by introducing the idea of a distance measure between any two preference orders. The object then is to axiomatize some measure of distance uniquely. This is achieved by the usual sort of geometric conditions (including the triangle inequality), a certain notion of "betweenness," a kind of independence of irrelevant alternatives condition, and by a normalization assumption. The result of these axioms is the distance measure, which can be described as follows.

With each preference order $\sigma \in L(A)$ associate an $m \times m$ skew symmetric matrix X^σ, whose rows and columns are labeled by the alternatives, such that the (a_i, a_j) entry, x_{ij}^σ, is 1 if a_i is preferred to a_j in σ, -1 if a_j is preferred to a_i, and 0 if $i = j$. X^σ is called the *election matrix* for σ. Similarly, if ϕ is any nonnull profile, the *election matrix* for ϕ is defined to be

$$X^\phi = \sum_\sigma n_\sigma(\phi) X^\sigma.$$

If ϕ is null then set $X^\phi \equiv 0$. Notice that the (a_i, a_j) entry of X^ϕ is precisely $n_{ij}(\phi)$.

The Kemeny distance between any two permutations σ and τ is given by

(11) $$d(\phi, \tau) = \frac{1}{2} \sum_{i,j} |x_{ij}^\sigma - x_{ij}^\tau|.$$

The difficulty with this approach is: having achieved the distance measure, it is not clear how to use it to find a consensus. In other words, given a profile ϕ, what is the ordering that can be said to be "least-distant" from the individuals' orderings in ϕ? Kemeny proposes two alternate solutions from statistics: the median and the mean.

Given ϕ, a *median ranking* is defined to be any τ such that

(12) $$\tau \text{ minimizes } \sum_\sigma n_\sigma(\phi) \, d(\sigma, \tau).$$

A *mean ranking* is any τ such that

(13)
$$\tau \text{ minimizes } \sum_\sigma n_\sigma(\phi)\, d^2(\sigma, \tau).$$

The reader may verify that in Example 1 the mean is (a_2, a_1, a_3) (the same as the Borda result), whereas the median is (a_1, a_2, a_3).

Kemeny left the problem of which solution to choose unresolved. But from the standpoint of collective decision making there is ample reason to prefer the median, since it turns out that the median consensus leads to a Condorcet method, while the mean does not. The latter statement follows from Example 1. To see the former, we re-express the Kemeny function in the following way. For any preference orders σ and τ,

(14)
$$d(\sigma, \tau) = \tfrac{1}{2}(X^\sigma \cdot X^\sigma - X^\sigma \cdot X^\tau),$$

where, in general, $X \cdot X'$ is defined by $X \cdot X' = \sum_{i,j} x_{ij} x'_{ij}$; hence, since $X^\sigma \cdot X^\sigma = m(m-1)$ for any σ, minimizing $d(\sigma, \tau)$ with respect to τ is equivalent to maximizing $X^\sigma \cdot X^\tau$. Thus for any profile ϕ, τ is a median if and only if

(15)
$$\tau \text{ maximizes } \sum_\sigma n_\sigma(\phi) X^\sigma \cdot X^\tau = X^\phi \cdot X^\tau.$$

Kemeny's rule is the PF defined by

(16)
$$K(\phi) = \{\tau : X^\phi \cdot X^\tau = \max\}.$$

If for some i and ϕ, $n_{ij}(\phi) > 0$, then we could not have $\tau = (\cdots a_j, a_i \cdots) \in K(\phi)$ because $\tau' = (\cdots a_i, a_j \cdots)$ satisfies $X^\phi \cdot X^{\tau'} > X^\phi \cdot X^\tau$ so X^τ did not maximize $X^\phi \cdot X^\tau$. On the other hand, if $n_{ij}(\phi) = 0$ and $\tau \in K(\phi)$ then clearly $\tau' \in K(\phi)$. Hence K is a Condorcet PF and, in particular, a quasi-Condorcet PF. Notice that a_i is a Condorcet alternative for ϕ if and only if the a_ith row of X^ϕ is nonnegative. If a_i is the unique Condorcet alternative (i.e., the a_ith row of X^ϕ is positive except for position (a_i, a_i)) then a_i is necessarily most-preferred in any preference order in $K(\phi)$. The fact that Kemeny's median rule is Condorcet was not mentioned by Kemeny, though it provides an excellent reason to prefer the median over the mean. This fact was first pointed out in [12], where a characterization of Kemeny's rule for *two* and *three* alternatives was given in terms of six properties. In this paper we shall extend and generalize these results by showing that, for any number of alternatives, the Kemeny rule is the *unique* preference function that is neutral, consistent and Condorcet, thus reconciling the consistency aspect of Borda's rule with Condorcet's principle.

We have already noted above that K is Condorcet; it is also immediate that it is symmetric. We claim that K is consistent (as a PF). Indeed, if $\phi' + \phi''$ is defined and if there is a τ that maximizes both $X^{\phi'} \cdot X^\tau$ and $X^{\phi''} \cdot X^\tau$, then any such τ maximizes $X^{\phi'+\phi''} \cdot X^\tau = (X^{\phi'} + X^{\phi''}) \cdot X^\tau$ as well. Moreover, for any other τ' such that $(X^{\phi'} + X^{\phi''}) \cdot X^{\tau'} = (X^{\phi'} + X^{\phi''}) \cdot X^\tau$, we must have $X^{\phi'} \cdot X^{\tau'} = X^{\phi'} \cdot X^\tau$ and $X^{\phi''} \cdot X^{\tau'} = X^{\phi''} \cdot X^\tau$, proving that

(17)
$$K(\phi' + \phi'') = K(\phi') \cap K(\phi'')$$

whenever the latter is nonempty.

Before proving that these conditions uniquely characterize K, we turn to a consideration of the polytope whose extreme points are the electron matrices X^σ, $\sigma \in L(A)$. This polytope turns out to have important applications to a variety of combinatorial optimization problems [18].

5. Permutation polytopes. Let

$$(18) \qquad \mathbf{P} = \left\{ \sum_{\sigma} \lambda_{\sigma} X^{\sigma} : \sum_{\sigma \in L(A)} \lambda_{\sigma} = 1, \lambda_{\sigma} \geqq 0 \right\}.$$

To compute $K(\phi)$ for any ϕ, it suffices to consider the linear programming problem: maximize $X^{\phi} \cdot X$ over all $X \in \mathbf{P}$. The optimum extreme points X^{σ} give the consensus ranking(s) $\sigma \in K(\phi)$. This means in particular that the Kemeny rule is considerably more difficult to compute, and is less "intuitively obvious", than some of the other more widely known methods. Nevertheless in terms of its properties it may actually be fairer.

The polytopes of form (18) are linear transformations of so-called *permutation polytopes* [5], [18]: if J is the $m \times m$ matrix of all 1's, and I is the $m \times m$ identity matrix, then the transformation

$$(19) \qquad X \rightarrow \tfrac{1}{2}(X + J - I)$$

defines the *permutation polytope* $\mathbf{P}'$ associated with $\mathbf{P}$. The set of inequalities defining $\mathbf{P}'$ (and hence $\mathbf{P}$) are not explicitly known (Bowman having proposed an insufficient set in [5]; a counterexample is due to A. J. Hoffman). However, various characterizations of *neighbors* on these polytopes are known that give some computational assistance in finding optimal extreme points [18].

For any permutation σ let $S(\sigma)$ be the set of all *unordered* pairs of symbols $\{a_i, a_j\}$ such that σ reverses the order of a_i and a_j (relative to the natural order):

$$(20) \qquad S(\sigma) = \{\{a_i, a_j\} : i < j \text{ and } a_j \text{ precedes } a_i \text{ in } \sigma\}.$$

(21) Let Γ_{σ} be the graph whose vertex set is $S(\sigma)$ such that $\{a_i, a_j\} \in S(\sigma)$ is adjacent to $\{a_i, a_k\} \in S(\sigma)$, $k \neq j$, if and only if $\{a_j, a_k\} \notin S(\sigma)$.

The following characterization of neighbors is due to Hoffman. (For a proof of this and other facts about permutation polytopes, see [18].)

$$(22) \qquad X^e \text{ and } X^{\sigma} \text{ are neighbors on } \mathbf{P} \text{ if and only if } \Gamma_{\sigma} \text{ is connected.}$$

6. The principal theorem.

(23) THEOREM 3. *Kemeny's rule is the unique preference function that is neutral, consistent, and Condorcet.*

We shall in fact prove a stronger result.

Define, for any PF f, the *dual* of f, $-f$, to be the PF that inverts the orders of f:

$$(24) \qquad (a_{i_1}, a_{i_2}, \cdots, a_{i_m}) \in -f(\phi) \quad \Leftrightarrow \quad (a_{i_m}, a_{i_{m-1}}, \cdots, a_{i_1}) \in f(\phi).$$

The *trivial* PF, $T(\phi)$, is the PF defined by

$$(25) \qquad T(\phi) = L(A), \quad \text{for all } \phi \in \Phi.$$

We shall show that

(26)   the only PF's that are neutral, consistent, and quasi-Condorcet are T, K, and $-K$.

Theorem 3 follows from this since of the three only K is Condorcet—in fact, only K is faithful—when $m \geqq 2$ (they are all equal when $m = 1$).

To prove (26), we begin by establishing two lemmas.

Let $\mathbf{X}$ denote the vector space of all $m \times m$, real skew-symmetric matrices whose rows and columns are labeled by the alternatives. The (a_i, a_j) entry of $X \in \mathbf{X}$ is

interpreted as the *net* majority (possibly fractional or irrational) that a_i has over a_j. Clearly, the notions of consistent, neutral, quasi-Condorcet, etc. all have their natural extensions to functions defined on the domain $\mathbf{X}$. The subset of $\mathbf{X}$ whose entries are all rational will be denoted by $\mathbf{X}^Q$.

LEMMA 1. *Let f be a neutral, consistent, quasi-Condorcet PF. Then for every $\phi \in \Phi$, $f(\phi)$ depends only on the election matrix X^ϕ. Moreover, there exists a unique extension of f to $\mathbf{X}^Q$ that is neutral, consistent, and quasi-Condorcet.*

Proof. The proof of Lemma 1 is similar to results in [16] and [17]. With f as given, suppose ϕ, ϕ' are two profiles on *disjoint* voter sets such that $X^\phi = X^{\phi'}$. Choose a profile ψ on a voter set disjoint from those of ϕ and ϕ' such that $X^{\phi+\psi} = X^{\phi'+\psi} = 0$. f quasi-Condorcet implies $f(\phi + \psi) = f(\phi' + \psi) = L(A)$; hence by consistency

$$f(\phi) = f(\phi) \cap L(A) = f(\phi + (\psi + \phi')) = f((\phi + \psi) + \phi') = L(A) \cap f(\phi') = f(\phi').$$

From this it follows that whenever $X^\phi = X^{\phi'}$ then $f(\phi) = f(\phi')$ (since we can always find ϕ'' disjoint from ϕ and ϕ' such that $X^\phi = X^{\phi''} = X^{\phi'}$). We may therefore view f as defined on the domain of election matrices. Notice that any matrix in $\mathbf{X}^Q$ can be expressed as a rational linear combination of election matrices. Indeed, for any $i \neq j$ the profile with two voters having preference orders $(a_i, a_j, a_1, a_2, \cdots, a_m)$ and $(a_m, a_{m-1}, \cdots, a_1, a_i, a_j)$ has $n_{ij} = -n_{ji} = 2$, and $n_{hk} = 0$ for all other h, k, and any matrix in $\mathbf{X}^Q$ can be expressed as a rational linear combination of these. For any positive integer $n > 0$ and election matrix X define $f((1/n)X) = f(X)$. This is well-defined, because if $(1/n)X = (1/n')X'$ then by consistency $f(nX') = f(X')$ and $f(n'X) = f(X)$; so $f((1/n)X) = f((1/n')X')$. It is easy to verify that this extended f is neutral, consistent, and quasi-Condorcet, and this is clearly the only extension of f to $\mathbf{X}^Q$ that has these properties. □

A particular consequence of the hypotheses of Lemma 1 is that f is anonymous, hence symmetric. In the sequel, any f satisfying the three hypotheses (in particular, K) will be considered as acting in the domain $\mathbf{X}^Q$.

For the next lemma we need to introduce the notion of derived PF's. For any subset $B \subseteq A$, $|B| = k > 0$, and $\sigma \in L(B)$, let $\sigma\langle A - B\rangle$ denote the set of all $m!/k!$ linear orderings of A that agree with σ when restricted to B. Further, let $\mathbf{X}_B^Q$ denote the set of all rational $k \times k$ skew-symmetric matrices whose rows and columns are indexed by the elements of B, and for any $Y \in \mathbf{X}_B^Q$ let $X = Y\langle A - B\rangle$ denote the matrix in $\mathbf{X}^Q$ obtained from Y by adjoining a *zero* row and a *zero* column for every $a \in A - B$.

For any quasi-Condorcet PF f defined on $\mathbf{X}^Q$, let f^B, the *derived* PF *on* alternative set B, be defined as follows:

(27)   For every $Y \in \mathbf{X}_B^Q$, $\sigma \in f^B(Y)$ iff $\sigma\langle A - B\rangle \subseteq f(Y\langle A - B\rangle)$.

LEMMA 2. *If f is a neutral, consistent, quasi-Condorcet PF for alternative set A, then*

(i) *for any $B \subseteq A$, $|B| > 0$, f^B is neutral, consistent, and quasi-Condorcet;*

(ii) *either f^B is Pareto   for all $|B| \geq 2$,*
    or $-f^B$ is Pareto   for all $|B| \geq 2$,
    or f^B is trivial   for all $|B| \geq 2$.

Proof. The verification of (i) is left to the reader; it may also be found in [13]. (ii) proceeds by showing that it holds for all B, $2 \leq |B| \leq k$, inducting on k.

Let $B = \{a_1, a_2\}$, and let X be the 2×2 election matrix corresponding to the

preference order (a_1, a_2). Then there are three possibilities:

$$(28) \qquad\qquad f^{\{a_1,a_2\}}(X) = \{(a_1, a_2)\},$$

$$(29) \qquad\qquad f^{\{a_1,a_2\}}(X) = \{(a_2, a_1)\},$$

$$(30) \qquad\qquad f^{\{a_1,a_2\}}(X) = \{(a_1, a_2), (a_2, a_1)\}.$$

If (28) holds then by neutrality $f^{\{a_1,a_2\}}$ is Pareto; if (29) holds, then by neutrality $-f^{\{a_1,a_2\}}$ is Pareto; if (30) holds, then by neutrality $f^{\{a_1,a_2\}}(-X) = f^{\{a_1,a_2\}}(X)$ and since all elements in $\mathbf{X}_B^Q$ are expressible as nonnegative linear combinations of X and $-X$ it follows by consistency that in this case $f^{\{a_1,a_2\}}$ is trivial. Thus (ii) holds for the particular $B = \{a_1, a_2\}$; hence by neutrality it holds whenever $|B| = 2$.

Suppose now that (ii) has been shown to hold whenever $2 \le |B| \le k$ for some $k < m$; we will show it holds whenever $2 \le |B| \le k + 1$.

Let $B = \{a_1, a_2, \cdots, a_{k+1}\}$, and for each h, $1 \le h \le k + 1$ let $Y^h = (y_{ij}^h)$ be the $(k+1) \times (k+1)$ skew-symmetric matrix such that the hth row and column are zero and

$$y_{ij}^h = \begin{cases} 1 & \text{if } i < j, \quad i, j \ne h, \\ -1 & \text{if } i > j, \quad i, j \ne h. \end{cases}$$

(Here the ith row and column are associated with alternative a_i, etc.)

Y^h, with the hth row and column deleted, is the election matrix $Z^h \in \mathbf{X}_{B-\{a_h\}}^Q$ corresponding to the preference order $(a_1, a_2, \cdots, \hat{a}_h, \cdots, a_{k+1}) = \sigma_h$. In the above notation

$$Y^h = Z^h \langle a_h \rangle.$$

Case 1. Suppose (28) holds. Then by induction $f^{B-\{a_h\}}$ is Pareto, so

$$f^{B-\{a_h\}}(Z^h) = \{\sigma_h\} \quad \text{for all } h, \quad 1 \le h \le k + 1,$$

and

$$f^B(Y^h) = \sigma_h \langle a_h \rangle.$$

By consistency of f^B,

$$f^B\left(\sum_{h=1}^{k+1} Y^h\right) = \bigcap_{1 \le h \le k+1} \sigma_h \langle a_h \rangle = \{\sigma\}.$$

But

$$\sum_{h=1}^{k+1} Y^h = kX^\sigma,$$

where X^σ is the $(k+1) \times (k+1)$ election matrix corresponding to σ; hence by consistency

$$f^B(X^\sigma) = \{\sigma\}.$$

Thus by neutrality f^B is Pareto for $B = \{a_1, a_2, \cdots, a_{k+1}\}$; hence f^B is Pareto whenever $|B| = k + 1$. Therefore (28) implies that f^B is Pareto whenever $2 \le |B| \le k + 1$.

Case 2. If (29) holds, we can apply the same argument as above to show that $-f^B$ is Pareto for $2 \le |B| \le k + 1$.

Case 3. If (30) holds, applying a similar argument to that of Case 1 shows that $f^B(X^\sigma) = L(B)$ for all permutations σ of $B = \{a_1, a_2, \cdots, a_{k+1}\}$; hence by consistency f^B is trivial. Therefore, in fact, f^B is trivial whenever $2 \le |B| \le k + 1$. $\quad\square$

A subset $C \subseteq Q^n$ (Q is the set of rationals) is said to be *Q-convex* if for all $\mathbf{x}, \mathbf{y} \in C$ and rational $\lambda, 0 \leqq \lambda \leqq 1, \lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in C$.

One of the peculiar difficulties in providing results about consistent preference functions (and CF's) is that, while under various natural conditions it is easy to extend their domain uniquely to the rationals (i.e. to include "fractional" voters) in such a way that consistency and the other conditions are still satisfied, there is in general no *immediate* way to obtain such an extension to the reals, even though in certain cases a unique extension may exist. (See for example [16], [17].) As a practical matter errors can develop if, during the proofs, one is not careful to restrict the analysis to Q-convex sets and rational-valued vectors. Once the desired characterization is obtained then the extension to the reals is usually obvious. (On this point a difficulty is encountered in certain results in [12], where for example Theorem 2 asserts incorrectly that a certain class of functions may be uniquely extended to the reals.)

To be able to work with Q-convex sets conveniently in the present context, we need the following results. (Proofs may be found in [17].) The following abbreviations will be used: "cvx" for convex hull "int" for interior, "ri" for relative interior "aff" for affine hull, and "cl" for closure. Sometimes it will also be notationally convenient to write $\bar{S}$ rather than cl (S) for the closure of S.

LEMMA 3 [17]. $C \subseteq Q^n$ *is Q-convex if and only if* $C = Q^n \cap$ cvx C.

LEMMA 4 [17]. *If* $C \subseteq Q^n$ *is Q-convex then* cl C *is convex.*

LEMMA 5 [17]. *If* $C = \bigcup_{i=1}^{k} S_i$, *where* $C \subseteq \mathbb{R}^n$ *is convex and* k *is finite, then for some* i, dim $C =$ dim S_i.

A particular consequence of Lemma 4 is that $C \subseteq$ cvx $C \subseteq$ cl C, hence cl $C \subseteq$ cl (cvx C) $\subseteq$ cl (cl C) so

$$\text{cl (cvx } C) = \text{cl } C.$$

From this we deduce ri(cl C) = ri(cvx C), so by Lemma 3, $Q^n \cap$ ri (cl C) $\subseteq C$, and $Q^n \cap$ ri (cl C) is dense in ri (cl (C)).

Proof of (26). Let f be a neutral, consistent, quasi-Condorcet SPF. To show that f is K, $-K$, or T, it suffices, by Lemma 2, to assume that f is Pareto and to show that $f = K$. (If f is not Pareto then either $-f$ is Pareto or else $f = T$.)

The proof that $f = K$ is by induction on the number of alternatives, m.

If $m = 1$ there is nothing to prove.

Suppose that $m = 2$. Then, by Lemma 1, f is symmetric and consistent; hence (for two alternatives) the associated choice function $\hat{f}$ is also symmetric and consistent, so Theorem 1 implies that $\hat{f}$ is a scoring function. But clearly the only distinct scoring functions on two alternatives are $f^{(1,0)}$, $f^{(0,1)}$, and $f^{(0,0)}$, which are the same as K, $-K$, and T respectively for two alternatives.

Suppose that $m = 3$. For any $X \in \mathbf{X}^Q$, $X = (x_{ij})$, such that $e = (a_1, a_2, a_3) \in K(X)$. It is easy to see that the following inequalities must hold:

(31)
$$x_{12} + x_{23} \geqq 0,$$
$$x_{13} + x_{23} \geqq 0,$$
$$x_{12} + x_{13} \geqq 0.$$

Now any 3×3 rational skew-symmetric X satisfying (31) must be a *positive, rational*, linear combination of the five skew-symmetric 3×3 matrices $X^1, \cdots, X^5$

determined by

$$(0, 0, 0) : X^1$$

$$(1, 0, 0) : X^2$$

$$(x_{12}, x_{13}, x_{23}) \quad = \quad (0, 1, 0) : X^3$$

$$(0, 0, 1) : X^4$$

$$(1, -1, 1) : X^5$$

(32)   We claim that $f(X^i) = K(X^i)$ for $1 \le i \le 5$.

Indeed $f(X^1) = K(X^1) = L(a_1, a_2, a_3)$ by neutrality; also, $f(X^2) = (a_1, a_2)(a_3) = \{(a_1, a_2, a_3), (a_1, a_3, a_2), (a_3, a_1, a_2)\} = K(X^2)$, by the fact that f^B is Pareto for all $|B| \ge 2$ and f is quasi-Condorcet. Similarly $f(X^3) = K(X^3)$, $f(X^4) = K(X^4)$. Finally, note that X^5 is symmetric under *any* 3-cycle permutation of a_1, a_2, and a_3; hence (since $f(X^5) \ne \phi$ and f is neutral) either

(33) $$f(X^5) = \{(a_1, a_2, a_3), (a_2, a_3, a_1), (a_3, a_1, a_2)\},$$

or

(34) $$f(X^5) = \{(a_3, a_2, a_1), (a_2, a_1, a_3), (a_1, a_3, a_2)\},$$

or

(35) $$f(X^5) = L(\{a_1, a_2, a_3\}).$$

Now let

$$Y = X^{(a_2, a_1, a_3)} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

Because f is Pareto, $f(Y) = \{(a_2, a_1, a_3)\}$. But $Y + X^5 = 2X^4$, so if either (34) or (35) were true consistency would imply $f(X^4) = f(Y) \cap f(X^5) = \{(a_2, a_1, a_3)\}$, a contradiction. Therefore $f(X^5)$ is as in (33), so $f(X^5) = K(X^5)$ and (32) holds.

Now for any X such that $K(X) = \{e\}$ we have $X = \sum_i \lambda_i X^i$ for some rational $\lambda_i > 0$; so by consistency and the fact that $e \in f(X^i) = K(X^i)$ for all i,

$$f(X) = \bigcap_i f(X^i) = \bigcap_i K(X^i) = K(X).$$

Thus $f(X) = K(X)$ whenever $e \in K(X)$; hence by symmetry $f(X) = K(X)$ for all X, proving the theorem for $m = 3$.

Suppose that $m \ge 4$. The domain of f and of K is $\mathbf{X}^Q \subseteq \mathbf{X}$, and, by consistency, the sets

(36) $$f_\sigma^{-1} = \{x \in \mathbf{X}^Q : \sigma \in f(X)\}$$

are Q-convex for each $\sigma \in L(A)$.

In outline, the idea of the proof is now as follows. We first show that the Q-convex sets f_σ^{-1} and f_τ^{-1} can be separated in the space $\mathbf{X}$ by a nonzero vector. This involves showing that the interiors of the *convex* sets cl (f_σ^{-1}) and cl (f_τ^{-1}) are disjoint (40). Next we consider the case where σ and τ are neighbors, it being sufficient (by neutrality) to consider the case where τ is the identity e. The structure of the separating vector is then analyzed by constructing a collection of vectors that must all

be orthogonal to it (43)–(45). The conclusion of the analysis is that the separating vector is identical to the corresponding separating vector for the Kemeny function. From this it follows that f must be a *refinement* of K in the sense that $f(X) \subseteq K(X)$ for every X (see (51)). Finally, it is shown in the following manner that f must be equal to K: if not, then there is a pair of neighbors σ and τ such that $\sigma, \tau \in K(X)$ and $\tau \in f(X)$ but $\sigma \notin f(X)$ (see (52)); but this is shown to lead to a contradiction by again using the convex separation theorem.

The set

$$(37) \qquad\qquad K_\sigma^{-1} = \{X \in \mathbf{X}^Q : \sigma \in K(X)\}$$

is Q-convex for each $\sigma \in L(A)$.

Note that $f_\sigma^{-1} \supseteq f^{-1}(\sigma) = \{X \in \mathbf{X}^Q : f(X) = \{\sigma\}\}$ and $K_\sigma^{-1} \supseteq K^{-1}(\sigma) = X \in \mathbf{X}^Q : K(X) = \{\sigma\}\}$.

By Lemma 4, cl $(f_\sigma^{-1}) = \overline{f_\sigma^{-1}}$ is convex for each σ. Since $\bigcup_\sigma \overline{f_\sigma^{-1}} = \mathbf{X}$, Lemma 5 implies that int $\overline{f_\sigma^{-1}} \neq \phi$ for some σ; hence by neutrality int $\overline{f_\sigma^{-1}} \neq \phi$ for *all* σ. Now int $\overline{f_\sigma^{-1}} \subseteq \operatorname{cvx} f_\sigma^{-1} \subseteq \overline{f_\sigma^{-1}}$, so by Lemma 3,

$$(38) \qquad\qquad \mathbf{X}^Q \cap \operatorname{int} \overline{f_\sigma^{-1}} \subseteq f_\sigma^{-1}.$$

We claim that

$$(39) \qquad\qquad \mathbf{X}^Q \cap \operatorname{int} \overline{f_\sigma^{-1}} \subseteq f^{-1}(\sigma),$$

that is, $f(X) = \{\sigma\}$ for any $X \in \mathbf{X}^Q \cap \operatorname{int} \overline{f_\sigma^{-1}}$. Indeed, $X^\sigma \in f_\sigma^{-1}$, and for any $X \in \mathbf{X}^Q \cap \operatorname{int} f_\sigma^{-1}$ and sufficiently small *rational* $\varepsilon > 0$ we have $Y = (1 + \varepsilon)X - \varepsilon X^\sigma \in \mathbf{X}^Q \cap \operatorname{int} f_\sigma^{-1}$ and

$$X = \left(\frac{\varepsilon}{1 + \varepsilon}\right)X^\sigma + \left(\frac{1}{1 + \varepsilon}\right)Y;$$

so by consistency

$$f(X) = f(X^\sigma) \cap f(Y) = \{\sigma\},$$

proving (39).

Since $\mathbf{X}^Q$ is dense in int $\overline{f_\sigma^{-1}}$ for all σ, it follows from (39) that

$$(40) \qquad\qquad \operatorname{int} \overline{f_\sigma^{-1}} \cap \operatorname{int} \overline{f_\tau^{-1}} = \phi \quad \text{for all } \sigma \neq \tau.$$

Therefore by the separation theorem for convex sets there exists, for every $\sigma \neq \tau$, a *nonzero* matrix $U^{\sigma\tau} \in \mathbf{X}$ such that

$$(41) \qquad\begin{aligned} U^{\sigma\tau} \cdot X &\geqq 0 \quad \text{for all } X \in \overline{f_\sigma^{-1}}, \\ U^{\sigma\tau} \cdot X &\leqq 0 \quad \text{for all } X \in \overline{f_\tau^{-1}}. \end{aligned}$$

Here " $\cdot$ " denotes the inner product of $U^{\sigma\tau}$ and X, regarded as vectors.

Let $e = (a_1, a_2, \cdots, a_m) \in L(A)$ denote the identity permutation, and let σ be any *neighbor* of e (i.e., X^σ is a neighbor of X^e on the polytope **P**). Let $U^{e\sigma} = (u_{ij})$ and recall from § 5 that $S(\sigma)$ is the set of pairs $p = \{a_i, a_j\}$ such that $i < j$ and a_j precedes a_i in σ. We claim that for some $\lambda > 0$ and for all $i < j$,

$$(42) \qquad\begin{aligned} u_{ij} &= -u_{ji} = \lambda \quad \text{whenever } \{a_i, a_j\} \in S(\sigma), \\ u_{ij} &= -u_{ii} = 0 \quad \text{whenever } \{a_i, a_j\} \notin S(\sigma). \end{aligned}$$

First, given any pair $\{a_i, a_j\} \notin S(\sigma)$, let $i < j$ and define a matrix $X \in \mathbf{X}^Q$ by $x_{ij} = -x_{ji} = 1$, and 0 elsewhere. Since $f^{\{a_i, a_j\}}$ is Pareto, the quasi-Condorcet property of

f implies that $e, \sigma \in f(X)$; hence $U^{e\sigma} \cdot X = 0$ and

(43) $$u_{ij} = 0 \quad \text{whenever } \{a_i, a_j\} \notin S(\sigma).$$

Now let $p = \{a_i, a_j\}$, $q = \{a_i, a_k\} \in S(\sigma)$ be any *adjacent* pairs in the graph Γ_σ, where, without loss of generality, $j < k$. Then $\{a_j, a_k\} \notin S(\sigma)$ and either

$$e = (- a_i - a_j - a_k -) \quad \text{and} \quad \sigma = (- a_j - a_k - a_i -),$$

or

$$e = (- a_j - a_k - a_i -) \quad \text{and} \quad \sigma = (- a_i - a_j - a_k -).$$

Define a matrix $X = (x_{ij}) \in \mathbf{X}^Q$ such that

(44)
$$x_{ij} = -x_{ji} = 1,$$
$$x_{jk} = -x_{kj} = 1,$$
$$x_{ik} = -x_{ki} = -1,$$
$$x_{hl} = 0 \quad \text{for all other } h, l.$$

Letting $B = \{a_i, a_j, a_k\}$ we know by Lemma 2 that f^B satisfies our hypotheses, so if $\hat{X}$ is X restricted to the rows and the columns corresponding to a_i, a_j, a_k, then $(a_i, a_j, a_k) \in f^B(\hat{X})$ and $(a_j, a_k, a_i) \in f^B(\hat{X})$, by the preceding analysis for $m = 3$ (i.e. (33) above). Since f is quasi-Condorcet, we therefore have $e, \sigma \in f(X)$; whence $U^{e\sigma} \cdot X = 0$ and

$$2u_{ij} + 2u_{jk} - 2u_{ik} = 0.$$

By (43), $u_{jk} = 0$; whence

(45)
$$u_{ij} = u_{ik} \quad \text{whenever } \{a_i, a_j\} \in S(\sigma) \text{ is adjacent}$$
$$\text{in } \Gamma_\sigma \text{ to } (a_i, a_k) \in S(\sigma).$$

(46)   Notice, moreover, that if $\{a_i, a_j\} \in S(\sigma)$ is adjacent in Γ_σ to $\{a_i, a_k\} \in S(\sigma)$, then $i < j$ implies $i < k$ (because if not then we have $e = (- a_k - a_i - a_j -)$ so $\sigma = (- a_j - a_i - a_k -)$ and $\{a_j, a_k\} \in S(\sigma)$, a contradiction).

Since σ is a neighbor of e, Γ_σ is connected (22); hence (45) and (46) imply that for some λ

(47)
$$u_{ij} = -u_{ji} = \lambda \quad \text{whenever } i < j \text{ and } \{a_i, a_j\} \in S(\sigma),$$
$$u_{ij} = u_{ji} = 0 \qquad \text{whenever } i < j \text{ and } \{a_i, a_j\} \notin S(\sigma).$$

Since f is Pareto, $X^e \in f_e^{-1}$, and we must have $U^{e\sigma} \cdot X^e \geq 0$, that is, $\lambda \geq 0$, and in fact $\lambda > 0$ because $U^{e\sigma} \neq 0$, proving (42). Without loss of generality, let $\lambda = 1$; then

(48) $$U^{e\sigma} = \tfrac{1}{2}(X^e - X^\sigma).$$

For any given $Y \in \mathbf{X}^Q$, if $e \in f(Y)$ then

$$U^{e\sigma} \cdot Y \geq 0, \text{ that is,}$$

(49) $$X^e \cdot Y \geq X^\sigma \cdot Y \text{ for all neighbors } \sigma \text{ of } e.$$

It follows that

(50) $$X^e \cdot Y \geq X^\tau \cdot Y \quad \text{for all } \tau \in L(A),$$

implying that $e \in K(Y)$. Thus $e \in f(Y)$ implies $e \in K(Y)$, so by neutrality f is a refinement of K:

$$(51) \qquad f(X) \subseteq K(X) \quad \text{for all } X \in \mathbf{X}^Q \subseteq \mathbf{X}.$$

We claim that

(52)   whenever σ is a neighbor of τ, if both $\sigma, \tau \in K(X)$ and $\tau \in f(X)$ then $\sigma \in f(X)$.

Suppose (52) is false for some neighbors σ and τ. Say without loss of generality that $\tau = e$. Then for some $X^0 \in \mathbf{X}^Q$,

$$(53) \qquad \sigma, e \in K(X^0), \qquad e \in f(X^0), \qquad \sigma \notin f(X^0).$$

Since $0 \in K_{e\sigma}^{-1} = \{X \in \mathbf{X}^Q : e, \sigma \in K(X)\}$, $\bar{\mathbf{Y}} = \text{aff } K_{e\sigma}^{-1}$ is a *sub-space* of $\mathbf{X}$. Let

$$\mathbf{Y}_e = f_e^{-1} \cap K_{e\sigma}^{-1},$$

$$\mathbf{Y}_\sigma = f_\sigma^{-1} \cap K_{e\sigma}^{-1}.$$

Further, let $\bar{\mathbf{Y}}_e = \text{cl}(\mathbf{Y}_e) \subseteq \bar{\mathbf{Y}}$, $\bar{\mathbf{Y}}_\sigma = \text{cl}(\mathbf{Y}_\sigma) \subseteq \bar{\mathbf{Y}}$. Since e and σ are neighbors, there is an $X \in \mathbf{X}$ such that

$$X \cdot X^e = X \cdot X^\sigma > X \cdot X^\pi \quad \text{for all } \pi \neq e, \sigma.$$

Hence there is a *rational* $X^* \in \mathbf{X}^Q$ with this property, and

$$K(X^*) = \{e, \sigma\}.$$

Let $\mathbf{W} = \text{ri}(\text{cvx } K_{e\sigma}^{-1})$. Since $X^* \in \text{cvx } K_{e\sigma}^{-1}$, it follows that for every rational $X \in \mathbf{W}$ and sufficiently small rational $\lambda > 0$,

$$X' = (1 + \lambda)X - \lambda X^* \in \mathbf{W}.$$

Since X is rational, we have $X' \in K_{e\sigma}^{-1}$ (see the remarks after Lemma 5). Hence $\{e, \sigma\} \subseteq K(X') \cap K(X^*) = K(X)$. But $K(X^*) = \{e, \sigma\}$; so

$$(54) \qquad K(X) = \{e, \sigma\} \quad \text{for all } \textit{rational } X \in \mathbf{W}.$$

For any *rational* $X \in \mathbf{W}$, (51) combined with (54) shows that $e \in f(X)$ or $\sigma \in f(X)$, hence $X \in \mathbf{Y}_e \cup \mathbf{Y}_\sigma$. If $X \in \mathbf{W}$ is *not rational*, then X is a limit point of a sequence of rational $X^i \in K_{e\sigma}^{-1}$. Hence there is a subsequence in $\mathbf{Y}_e$ converging to X, or a subsequence in $\mathbf{Y}_\sigma$ converging to X. In any case

$$(55) \qquad \bar{\mathbf{Y}}_e \cup \bar{\mathbf{Y}}_\sigma \supseteq \mathbf{W}.$$

By Lemma 5, at least one of the subspaces aff $\bar{\mathbf{Y}}_e$, aff $\bar{\mathbf{Y}}_\sigma$ is equal to $\bar{\mathbf{Y}}$; let the other be $\bar{\mathbf{Y}}' \subseteq \bar{\mathbf{Y}}$.

We claim that

$$(56) \qquad \text{ri } \bar{\mathbf{Y}}_e \cap \text{ri } \bar{\mathbf{Y}}_\sigma = \phi.$$

Suppose (56) is false. Then ri $\bar{\mathbf{Y}}_e \cap \text{ri } \bar{\mathbf{Y}}_\sigma = S \neq \phi$ is open with respect to the subspace $\bar{\mathbf{Y}}'$. Since every point in $\bar{\mathbf{Y}}_e$ is a limit point of $\mathbf{Y}_e \subseteq \mathbf{X}^Q$, and similarly for $\bar{\mathbf{Y}}_\sigma$, S contains a rational $X' \in \mathbf{X}^Q$, and by the remarks after Lemma 5,

$$(57) \qquad X' \in \mathbf{Y}_e \cap \mathbf{Y}_\sigma.$$

The X^0 of (53) is in $\bar{\mathbf{Y}}_e$, so for sufficiently small rational $\lambda > 0$,

$$X'' = (1 + \lambda)X' - \lambda X^0 \in \text{ri } \bar{\mathbf{Y}}_e,$$

whence $e \in f(X'')$. Since

$$X' = X''/(1+\lambda) + \lambda X^0/(1+\lambda),$$

f consistent implies

$$e \in f(X') = f(X'') \cap f(X^0),$$

while

$$\sigma \notin f(X^0) \text{ implies } \sigma \notin f(X').$$

Thus $X' \notin \mathbf{Y}_\sigma$, contradicting (57). This shows that (56) is true.

The separation theorem for convex sets implies that there is a *nonzero* skew-symmetric real matrix U, $U \in \bar{\mathbf{Y}}$, such that

$$U \cdot X \geqq 0 \quad \text{for all } X \in \bar{\mathbf{Y}}_e,$$

$$U \cdot X \leqq 0 \quad \text{for all } X \in \bar{\mathbf{Y}}_\sigma.$$

Then, just as in the derivation of (47), we conclude that for some real number λ,

$$u_{ij} = -u_{ji} = \lambda \quad \text{whenever } i < j \text{ and } \{a_i, a_j\} \in S(\sigma),$$

$$u_{ij} = -u_{ji} = 0 \quad \text{whenever } i < j \text{ and } \{a_i, a_j\} \notin S(\sigma).$$

But $U \in \bar{\mathbf{Y}} = \text{aff } K_{e\sigma}^{-1}$ implies

$$U \cdot (X^e - X^\sigma) = 4 \sum_{\substack{i < j \\ \{a_i, a_j\} \in S(\sigma)}} u_{ij} = 0.$$

Thus $\lambda = 0$, so $U = 0$, contradicting the choice of U. This proves (52).

Now suppose $f \neq K$, that is, by (51) there is an X^* such that $f(X^*) \subsetneqq K(X^*)$. Say

$$\sigma, \tau \in K(X^*), \quad \sigma \in f(X^*), \quad \tau \notin f(X^*).$$

Let $\mathbf{P}^*$ be the polytope $\{X \in \mathbf{P} : X^* \cdot X = \max\}$.

Since $|K(X^*)| \geqq 2$, $\dim P^* \geqq 1$, so by a theorem of Balinski [2] there is a path $\sigma = \pi^1, \pi^2, \cdots, \pi^k = \tau$ such that for $1 \leqq i < k$, X^{π^i} is a neighbor of $X^{\pi^{i+1}}$ on $\mathbf{P}$ and $\pi^i \in K(X^*)$ for $1 \leqq i \leqq k$. Now $\pi^1, \pi^2 \in K(X^*)$ and $\pi^1 \in f(X^*)$, so by (52), $\pi^2 \in f(X^*)$. Continuing in this manner, we conclude that $\pi^k = \tau \in f(X^*)$, a contradiction. Thus f is identical to K.

7. Conclusion. In the social choice literature two divergent streams of thought are apparent. One, which stems from Borda's work, emphasizes scoring methods; the fundamental property enjoyed by these methods is that they are consistent, which amounts to saying that they satisfy a kind of Pareto principle for subgroups. This condition seems very natural and desirable in the context of collective decision making, and it is difficult to see how a preference function not satisfying this property would be able to find general acceptance. The other, which grows out of Condorcet's proposal that an alternative able to get a majority over any other should be the most-preferred alternative, is the problem of extending this idea in a natural way to cases when a Condorcet alternative does not exist.

As we have shown, these two ideas, suitably interpreted for preference functions, are resolved in the method known as Kemeny's rule; moreover subject to the basic requirement that all alternatives be treated in an unbiased manner, Kemeny's rule is

the *only* preference function satisfying the three properties.[2]

The present development, compared to Kemeny's, also provides an interesting example of the use of axioms in the social sciences. Kemeny axiomatized a distance measure by invoking certain *mathematically appealing* properties. But by concentrating on the distance minimizing aspect of the problem, the point was missed that it is the decision function itself, rather than the distance, that possesses the important properties.

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[2] Of course, these are not the only desirable properties it enjoys; for example, it is also Pareto and anonymous.