INTEGRATION AND LINEAR OPERATIONS*

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1. Notation and introduction. By L_p $(1 \leqq p < \infty)$ will be understood the class of real valued measurable functions $\phi(P)$, $0 \leqq P \leqq 1$, for which $\int_0^1 |\phi(P)|^p dP < \infty$, by $S_p(X)$ the class of measurable† functions $f(P)$ on (0,1) to the space X of type B [see‡ 1, p. 53] for which $\int_0^1 \|f(P)\|^p dP < \infty$. With $\|f\| \equiv \{\int_0^1 \|f(P)\|^p dP\}^{1/p}$, $S_p(X)$ is a Banach space. In case $p > 1$, $L_{p'}$ is defined by the equality $p' = p/(p-1)$ and if $p = 1$ the symbol $L_{p'} = L_\infty = M$ will stand for the space of real, essentially bounded and measurable functions with $\|\phi\| = \text{ess. sup.} |\phi(P)|$. Similarly for $S_{p'}(X)$ and $S_\infty(X)$, and for brevity we write S_{pq} in place of $S_p(L_q)$. By L_{pq} we mean the class of real valued measurable functions $K(P, Q)$ that belong to L_q for each P and for which $\{\int_0^1 |K(P, Q)|^q dQ\}^{1/q}$ belongs to L_p. Finally the term *linear operation* is used in the sense of Banach, i.e., for an additive continuous function. In terms of this notation our results are described in the following paragraph.

It is easily shown that a kernel $K(P, Q)$ in $L_{p'q}$ defines a linear operation

$$T\phi = \int_0^1 K(P, Q)\phi(P)dP \tag{1}$$

on L_p to L_q. In case $p = 1$, $1 < q < \infty$, this is the expression for the most general linear transformation and its norm is

$$|T| = \text{ess. sup.} \left\{ \int_0^1 |K(P, Q)|^q dQ \right\}^{1/q}.$$

In case $1 < p \leqq \infty$, $1 \leqq q < \infty$ the operation (1) is completely continuous and the general linear operation is expressible as the limit of a sequence of operations of type (1). The general completely continuous linear operation on $L = L_1$ to L_q $(1 < q < \infty)$ is given by (1), where the kernel $K(P, Q)$ is in $L_{\infty q}$, vanishes outside the square $0 \leqq P$, $Q \leqq 1$, and satisfies the condition

$$\lim_{h=0} \text{ess. sup.} \int_0^1 |K(P, Q+h) - K(P, Q)|^q dQ = 0.$$

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† This concept is defined in §2 of this paper.

‡ References in brackets refer to the bibliography at the end of the paper.

Other representation theorems are given for operations on L to l_p (the space of sequences $\{a_i\}$ of real numbers with $\sum_{i=1}^{\infty}|a_i|^p$ convergent), and operations on the space of absolutely continuous functions to L_p, l_p, or Hilbert space. The abstract function $f(P)=K(P, Q)$ may be in S_{pq} without the real function $K(P, Q)$ being in L_{pq} but every function in S_{pq} has a unique representation in L_{pq} and conversely every function $K(P, Q)$ in L_{pq} defines uniquely a function $f(P)=K(P, Q)$ in S_{pq}.

2. **Preliminary remarks.** A function $f(P)$ on $0 \leqq P \leqq 1$ to a Banach space X is said to be *finitely valued** in case there exists a decomposition of the interval (0,1) into a finite number of disjoint measurable sets on each of which $f(P)$ is constant, and it is said to be *measurable* in case there exists a sequence $f_n(P)$ of finitely valued functions for which $f(P)=\lim_n f_n(P)$ almost everywhere. This definition of measurability which was used by Bochner is equivalent to saying that $f(P)$ satisfies the condition of Lusin, i.e., $f(P)$ is continuous on a closed set with measure arbitrarily near one. This fact is proved almost as in the case of real functions [13, p. 44].

S. Bochner has defined [3] a class of functions called summable on (0, 1) to X and an integral on this class. We [6] have done likewise and in both cases the space of summable functions is complete and has the set of finitely valued functions everywhere dense. This shows that the two notions of summable functions as well as the integrals coincide. Accordingly results from both papers will be used. The reader will find no difficulty in establishing the following theorems (the proofs are merely sketched) which are stated here and numbered for future reference.

2.1. *If $f(P)$ is measurable and $\phi(P)$ is a real finite valued measurable function then the functions $\|f(P)\|$ and $\phi(P)f(P)$ are measurable. $f(P)/\phi(P)$ is measurable if an arbitrary constant value is assigned to it on the set where $\phi(P)=0$. Also if a sequence of measurable functions $f_n(P)$ converges almost everywhere then $\lim_n f_n(P)$ is measurable.*

2.2. *The function $f(P)$ is summable if and only if $f(P)$ is measurable and $\|f(P)\|$ is summable.*

This is Bochner's definition of a summable function.

2.3. *Let X and Y be two spaces of type B, Tx linear on X to Y and $f(P)$ summable on $(0, 1)$ to X, then $Tf(P)$ is summable and $T\int_e f(P)dP=\int_e Tf(P)dP$ for every measurable subset e of $(0, 1)$.*

That $Tf(P)$ is summable follows from 2.2. For $Tf(P)$ is continuous on

* The reader should be careful to distinguish between finitely valued and finite valued, the latter term being used in the usual sense for real functions.

any set upon which $f(P)$ is continuous so that $Tf(P)$ is measurable. The norm $\|Tf(P)\|$ is summable since it is bounded by $|T|\cdot\|f(P)\|$. The equality in the theorem is proved by using a sequence of finitely valued functions converging to $f(P)$ in the mean. This theorem has been given by G. Birkhoff [2].

2.4. *If $f(P)$ is in $S_{p'}(X)$ then the transformation*

$$T\phi = \int_0^1 f(P)\phi(P)dP$$

is a linear operation on L_p to X, with $|T| \leqq \{\int_0^1 \|f(P)\|^{p'}dP\}^{1/p'}$ if $1 < p \leqq \infty$ and $|T| \leqq$ ess. sup. $\|f(P)\|$ if $p=1$.

The operation is defined on L_p, for by 2.1, $f(P)\phi(P)$ is measurable and since $\|f(P)\phi(P)\| = \|f(P)\|\cdot|\phi(P)|$ is summable it follows from 2.2 that $f(P)\phi(P)$ is summable. $T\phi$ is linear since $\|T\phi\| \leqq \int_0^1 \|f(P)\|\cdot|\phi(P)|dP \leqq \|f\|\cdot\|\phi\|$.

3. **The representation of summable and of measurable functions.** A few preliminary remarks are in order. It is easily seen that if the abstract function $f(P)=K(P, Q)$ is in S_{pq}, $1\leqq p, q<\infty$, then

$$\int_0^1 dP\left\{\int_0^1 |K(P, Q)|^q dQ\right\}^{p/q} = \int_0^1 \|f(P)\|^p dP < \infty,$$

but this does not mean that $K(P, Q)$ is in L_{pq} for $K(P, Q)$ may not be measurable. In fact it may be that $f(P)=0$ for all P and that $K(P, Q)$ is a non-measurable function. This is the case if $K(P, Q)$ is the characteristic function of a Sierpiński set [15]. Such a set is bounded and has positive exterior measure and has no more than two points in common with any straight line. Also one might ask if every function $K(P, Q)$ for which

$$\int_0^1 dP\left\{\int_0^1 |K(P, Q)|^q dQ\right\}^{p/q} < \infty$$

defines a function $f(P)=K(P, Q)$ in S_{pq}. That such is not the case is seen by the example $K(P, Q)=\phi(P)\psi(Q)$, where $\psi(Q)$ is any function in L_q and $\phi(P)$ is a non-measurable function with $|\phi(P)|$ in L_p. However, as will be seen, every $K(P, Q)$ in L_{pq} defines a function in S_{pq} and conversely every $f(P)$ in S_{pq} has a representation in L_{pq} and two such representations must (by the theorem of Fubini) be equal for all P, Q except for a set of two-dimensional measure zero. In fact for a large class of Banach spaces Y a measurable function $f(P)$ on (0,1) to Y has a measurable representation. For L_2, this fact as well as 3.2 has been established by S. Bochner and J. von Neumann [4]. Let Y be a Banach space composed of real almost everywhere finite measurable functions $\psi=\psi(Q)$ defined for $0\leqq Q\leqq 1$. Let it be supposed that Y satisfies the following conditions:

(a) If $\|\psi_n\|\to 0$ then $\psi_n(Q)\to 0$ in measure.

(b) If $\psi(Q)=0$ almost everywhere then $\psi=0$.

Then we have

3.1. *If the function $f(P)$ on $(0,1)$ to Y is measurable, there exists a function $K(P,Q)$ measurable on the square $0\leqq P, Q\leqq 1$ such that $f(P)=K(P,Q)$. Any two measurable representations of $f(P)$ differ only on a set of two-dimensional measure zero.*

First suppose that $f(P)=K^*(P,Q)$ is finitely valued, and let $e_1, e_2, \cdots, e_n$ be the sets upon which $f(P)$ is constant. Let P_i be a point of e_i and define

$$K(P,Q) = K^*(P_i,Q) \text{ for } P \text{ in } e_i \qquad (i=1,2,\cdots,n).$$

So that $K(P,Q)$ is a measurable representation of $f(P)$. Now suppose $f(P)=K^*(P,Q)$ is an arbitrary measurable function, and let $f_n(P)$ be a sequence of finitely valued functions approaching $f(P)$ almost everywhere. Let $K_n(P,Q)$ be a measurable representation of $f_n(P)$, then

$$\lim_{n,m}\|K_n(P,Q)-K_m(P,Q)\|_Y = \lim_{n,m}\|f_n(P)-f_m(P)\| = 0 \text{ for almost all } P.$$

Thus by (a), $K_n(P,Q)$ converges in measure for almost all P in $(0,1)$ and so we have

$$\lim_{n,m}\int_0^1 \frac{|K_n(P,Q)-K_m(P,Q)|}{1+|K_n(P,Q)-K_m(P,Q)|}dQ = 0 \text{ for almost all } P,$$

and hence (using the theorem of Fubini)

$$\lim_{n,m}\int_0^1\int_0^1 \frac{|K_n(P,Q)-K_m(P,Q)|}{1+|K_n(P,Q)-K_m(P,Q)|}dPdQ = 0.$$

There is therefore a function $K'(P,Q)$ measurable on the square $0\leqq P, Q\leqq 1$ such that

$$K_m(P,Q)\to K'(P,Q) \text{ in measure on } 0\leqq P, Q\leqq 1.$$

Hence there is a subsequence $K_{m_i}(P,Q)$ and a set E_1 in $(0, 1)$ with $m(E_1)=1$ such that

$$\lim_{i=\infty}\int_0^1 \frac{|K_{m_i}(P,Q)-K'(P,Q)|}{1+|K_{m_i}(P,Q)-K'(P,Q)|}dQ = 0 \text{ for } P \text{ in } E_1,$$

that is, for P in E_1

$$K_{m_i}(P, Q) \to K'(P, Q) \text{ in measure.}$$

Now since $\|f_n(P) - f(P)\| \to 0$ for almost all P, say for P in E_2 where $m(E_2) = 1$, we have (by (a)) for P in E_2

$$K_m(P, Q) \to K^*(P, Q) \text{ in measure.}$$

Thus for P in E_1E_2

$$K'(P, Q) = K^*(P, Q) \text{ for almost all } Q,$$

and so by (b) $f(P) = K'(P, Q)$ for P in E_1E_2. By defining

$$\begin{aligned} K(P, Q) &= K'(P, Q) \text{ for } P \text{ in } E_1E_2, \\ &= K^*(P, Q) \text{ elsewhere,} \end{aligned}$$

we have $f(P) = K(P, Q)$ for $0 \leqq P \leqq 1$. Hence since K differs from K' on a set of two-dimensional measure zero it is a measurable representation of $f(P)$.

Now suppose $K(P, Q)$ and $K'(P, Q)$ are two different measurable representations of $f(P)$. From (a) it follows that $\psi = 0$ implies $\psi(Q) = 0$ almost everywhere and hence for all P

$$K(P, Q) = K'(P, Q) \text{ for almost all } Q,$$

so that by the theorem of Fubini $K(P, Q) = K'(P, Q)$ almost everywhere on $0 \leqq P, Q \leqq 1$.

The converse of the theorem is not true in general; i.e., if $K(P, Q)$ is a measurable function of the two variables P and Q, and if for each P, $K(P, Q)$ is in Y then it does not follow that the abstract function $f(P) = K(P, Q)$ on $(0, 1)$ to Y is measurable. For take $Y = M$ the space of real, essentially bounded and measurable functions defined on $(0, 1)$, and define

$$\begin{aligned} K(P, Q) &= 0, \qquad P \leqq Q, \\ &= 1, \qquad P > Q. \end{aligned}$$

Then the abstract function $f(P) = K(P, Q)$ has the property that for $P < P'$, $\|f(P) - f(P')\| = 1$ and consequently does not satisfy the condition of Lusin. As we shall see in 3.2 this phenomenon does not occur if M is replaced by L_p.

Hereafter when we speak of a function $f(P)$ in S_{pq} and write $f(P) = K(P, Q)$ it will be understood, without explicit mention of the fact that $K(P, Q)$ is measurable. It should be noted that if $K(P, Q)$ is in L_{pq} then for every finite valued $\phi(P)$ in $L_{p'}$, $K(P, Q)\phi(P)$ is in L_{1q} and is a measurable representation of $f(P)\phi(P)$.

3.2. *The function $f(P)$ is in S_{pq}, $1 \leqq p, q < \infty$, if and only if there exists a function $K(P, Q)$ in L_{pq} such that $f(P) = K(P, Q)$. If $f(P)$ is in S_{pq} then*

$$\int_e f(P)dP = \int_e K(P, Q)dP$$

for every measurable subset e of (0, 1).

First suppose that $K(P, Q)$ is in L_{pq}, then

$$\|f(P)\|^p = \left\{ \int_0^1 | K(P, Q) |^q dQ \right\}^{p/q}$$

is summable and it only remains to show that $f(P)$ is measurable. Let $\psi_n(Q)$ be the orthonormal set of Haar [8]. Then for each P

$$K(P, Q) = f(P) = \sum_{n=1}^{\infty} \phi_n(P)\psi_n(Q),$$

where

$$\phi_n(P) = \int_0^1 K(P, Q)\psi_n(Q)dQ.$$

This means that $\sum_1^n \phi_n(P)\psi_n(Q)$ converges in the norm of L_q to $K(P, Q)$ [14]. Since $K(P, Q)$ is measurable so is $\phi_n(P)$ and thus by 2.1 the abstract function

$$f_n(P) = \sum_1^n \phi_n(P)\psi_n$$

is measurable. Since $f_n(P) \to f(P)$ it follows again by 2.1 that $f(P)$ is measurable and thus $f(P)$ is in S_{pq}.

Now conversely suppose $f(P)$ is in S_{pq}. By 3.1 there is a measurable function $K(P, Q)$ such that $f(P) = K(P, Q)$ and $K(P, Q)$ is obviously in L_{pq}. To establish the equality stated in the theorem we again use the Haar base and recall [1, p. 111] that for each x in L_q

$$x = \sum_1^{\infty} \psi_i T_i x,$$

where T_i is a linear function on L_q. If we write

$$\phi_i(P) = T_i f(P), \qquad |\phi_i(P)| \leqq |T_i| \cdot \|f(P)\|$$

it is seen that $\phi_i(P)$ is a finite valued function in L_p and

$$K(P, Q) = f(P) = \sum_1^{\infty} \phi_i(P)\psi_i(Q).$$

From 2.3 it follows that

$$\int_e f(P)dP = \sum_1^\infty \psi_i T_i \int_e f(P)dP = \sum_1^\infty \psi_i \int_e T_i f(P)dP = \sum_1^\infty \psi_i \int_e \phi_i(P)dP,$$

which means that

$$\xi_n(Q) \equiv \sum_1^n \psi_i(Q) \int_e \phi_i(P)dP$$

approaches $\int_e f(P)dP$ in L_q as $n \to \infty$. To complete the proof it is only necessary to show that this same sequence approaches $\int_e K(P, Q)dP$ in L_q. Since $\xi_n(Q)$ converges in L_q it is sufficient to show that $\xi_n(Q) \to \int_e K(P, Q)dP$ in measure. Now writing $K_n(P, Q) = \sum_1^n \phi_i(P)\psi_i(Q)$, we have for each P [14]

$$\left\{ \int_0^1 | K_n(P, Q) - K(P, Q) |^q dQ \right\}^{1/q} \to 0$$

and

$$\left\{ \int_0^1 | K_n(P, Q) - K(P, Q) |^q dQ \right\}^{1/q} \leqq \left\{ \int_0^1 | K_n(P, Q) |^q dQ \right\}^{1/q} + \left\{ \int_0^1 | K(P, Q) |^q dQ \right\}^{1/q} \leqq 2 \left\{ \int_0^1 | K(P, Q) |^q dQ \right\}^{1/q}$$

Thus the sequence on the left side of this inequality is bounded by a function in L_p, and so

$$\int_0^1 dQ \int_0^1 | K_n(P, Q) - K(P, Q) | \, dP \leqq \left(\int_0^1 dP \left\{ \int_0^1 | K_n(P, Q) - K(P, Q) |^q dQ \right\}^{p/q} \right)^{1/p} \to 0,$$

from which the desired conclusion is easily deduced. This completes the proof of 3.2.

3.3. *The function* $f(P)$ *on* $(0, 1)$ *to* L_q, $1 \leqq q < \infty$, *is measurable and essentially bounded if and only if there exists a measurable function* $K(P, Q)$ *in* L_q *for each* P *and satisfying the conditions*

$$\text{ess. sup.} \int_0^1 | K(P, Q) |^q dQ < \infty,$$

$$f(P) = K(P, Q),$$

and in case the conditions are satisfied

$$\int_e f(P)dP = \int_e K(P, Q)dP$$

for every measurable subset e of (0, 1).

This is an immediate corollary of 3.2.

3.4. *The function* $f(P) = \{k_i(P)\}$ *is in* $S_p(l_q)$ $(1 \leqq p \leqq \infty, 1 \leqq q < \infty)$ *if and only if*

(i) $k_i(P)$ *is measurable* $(i = 1, 2, 3, \cdots)$;

(ii) $\left\{ \sum_1^\infty | k_i(P) |^q \right\}^{1/q}$ *is in* L_p.

For functions $f(P) = \{k_i(P)\}$ *in* $S_p(l_q)$ *we have*

$$\int_e f(P)dP = \left\{ \int_e k_i(P)dP \right\}$$

for every measurable subset e of (0, 1).

This theorem can be established by a method quite analogous but much simpler than the method used in the proof of 3.2. In fact if we take $\psi_i = \{\delta_k^i\}$ (the Kronecker delta) as the base for l_q, then

(iii) $$f(P) = \sum_1^\infty T_i f(P)\psi_i = \sum_1^\infty k_i(P)\psi_i,$$

which shows that $k_i(P)$ is measurable providing $f(P)$ is. Condition (ii) is merely a statement of the fact that $\|f(P)\|$ is in L_p. Conversely if (i) and (ii) are satisfied, (iii) combined with 2.1 shows that $f(P)$ is measurable and (ii) shows that $\|f(P)\|$ is in L_p. The equality $\int_e f(P)dP = \{\int_e k_i(P)dP\}$ is merely an application of 2.3, for we have

$$\int_e f(P)dP = \sum_1^\infty \psi_i T_i \int_e f(P)dP = \sum_1^\infty \psi_i \int_e T_i f(P)dP$$
$$= \sum_1^\infty \psi_i \int_e k_i(P)dP = \left\{ \int_e k_i(P)dP \right\}.$$

4. **Applications of the preceding results.** In view of 3.2 each theorem regarding functions in S_{pq} may be translated into a theorem concerning functions in L_{pq}. Since it is merely a translation of symbols, we shall content ourselves here with one illustration. For example, Theorem 9 of reference 6 reads as follows:

4.1. *If the functions* $K(P, Q)$, $K_m(P, Q)$, $m=1, 2, 3, \cdots$, *are in* L_{pq} $(1 \leqq p, q < \infty)$, *and if* $\int_0^1 |K_m(P, Q) - K(P, Q)|^q dQ$ *approaches zero in measure, then the following assertions are equivalent*:

$$(1) \quad \lim_m \int_0^1 dQ \left| \int_e \left\{ K_m(P,Q) - K(P,Q) \right\} dP \right|^q = 0$$

for every measurable set e *in* $(0, 1)$.

$$(2) \quad \lim_m \int_0^1 dQ \left| \int_e \left\{ K_m(P,Q) - K(P,Q) \right\} dP \right|^q = 0 \textit{ uniformly for } e \textit{ in } (0, 1).$$

$$(3) \quad \lim_{n,m} \int_0^1 dQ \left| \int_e \left\{ K_m(P,Q) - K_n(P,Q) \right\} dP \right|^q = 0 \textit{ for each } e \textit{ in } (0, 1).$$

$$(4) \quad \lim_{m(e)=0} \limsup_m \int_0^1 dQ \left| \int_e K_m(P,Q) dP \right|^q = 0.$$

$$(5) \quad \lim_{m(e)=0} \int_0^1 dQ \left| \int_e K_m(P, Q) dP \right|^q = 0 \textit{ uniformly with respect to } m.$$

5. **The representation of linear operations.** We have

5.1. *If* X *is of type* B *with a base, then* $T\phi$ *is a linear operation on* L_p $(1 \leqq p \leqq \infty)$ *to* X *if and only if there exists a sequence of functions* $f_n(P)$ *in* $S_{p'}(X)$ *such that* $\int_0^1 f_n(P)\phi(P)dP$ *converges for each* $\phi(P)$ *in* L_p *and*

$$T\phi = \lim_n \int_0^1 f_n(P)\phi(P)dP \textit{ for } \phi(P) \textit{ in } L_p.$$

Let T be a linear operation on L_p to X, and let $\{x_i\}$ be a base for X, then

$$T\phi = \sum_1^\infty a_i(\phi) x_i,$$

where $a_i(\phi) = T_i T\phi$ is a linear functional on L_p. Thus there is a finite valued function $\alpha_i(P)$ in $L_{p'}$ such that [1, p. 64]

$$a_i(\phi) = \int_0^1 \alpha_i(P)\phi(P)dP.$$

Since $\alpha_i(P)x_i$ is in $S_{p'}(X)$, $f_n(P) = \sum_1^n \alpha_i(P)x_i$ is likewise and the sequence

$$\int_0^1 f_n(P)\phi(P)dP = \sum_1^n a_i(\phi)x_i$$

converges for each ϕ in L_p to $T\phi$. The converse statement follows from 2.4 and a well known result [1, p. 23, Theorem 4].

5.2. *The function $T\phi$ is a linear operation on $L_p(1 \leqq p \leqq \infty)$ to $L_q(1 \leqq q < \infty)$ if and only if there exists a sequence of measurable functions $K_n(P, Q)$ belonging to L_q for each P, and satisfying the conditions*

$$\left\{ \int_0^1 | K_n(P, Q) |^q dQ \right\}^{1/q} \text{ is in } L_{p'} \text{ for each } n,$$

and

$$T\phi = \lim_n \int_0^1 K_n(P, Q)\phi(P)dP \text{ for } \phi \text{ in } L_p.$$

This follows immediately from 5.1, 3.2, and 3.3.

In the case of transformations $T\phi$ on L to X, where X is a Banach space satisfying one of the conditions:

(A) X has a uniformly convex norm [5];

(B) X has a base $\{x_i\}$ such that $\sum_1^\infty a_i x_i$ converges whenever $\{a_i\}$ is a sequence of constants for which $\|\sum_1^n a_i x_i\|$ is bounded,

further results may be given. The spaces satisfying one or the other of these conditions include Hilbert space, $l_p(1 \leqq p < \infty)$, and $L_p(1 < p < \infty)$. The fundamental fact [5, 7] in this connection is that a function $F(P)$ on (0, 1) to such a space, which satisfies a Lipschitz condition $\|F(P) - F(P')\| \leqq M|P - P'|$ (or even a more general condition) has a derivative $F'(P)$ almost everywhere which is summable and such that

$$F(P) = \int_0^P F'(P)dP + F(0).$$

We shall first give a generalization of this result which is embodied in 5.3. The corresponding theorem for real functions has been given for special cases by J. Radon and P. J. Daniell and in the general case by O. Nikodym [13, pp. 255–257]. We shall use the terminology of Saks [13, pp. 247–257] and denote by $m(e)$ a completely additive non-negative function defined on an additive family A of sets of abstract elements P belonging to a space E. The sets of A are called measurable sets and the number $m(e)$ is called the measure of the set e. It is assumed that all subsets of a set of measure zero are measurable. An additive set function $F(e)$ on A to a Banach space is said to be of bounded variation in case $\sum_{i=1}^n \|F(e_i)\|$ is bounded with respect to all choices of a finite number of disjoint measurable sets $e_1, e_2, \cdots, e_n$. The l.u.b. of $\sum_{i=1}^n \|F(e_i)\|$, where $e_1, e_2, \cdots, e_n$ are disjoint measurable subsets of a measurable set e is called the total variation of F on e and is denoted by $T(F, e)$.

5.3. *Every completely additive set function $F(e)$ defined on A, which is of bounded variation and vanishes on sets of measure zero and whose values lie in a Banach space X which satisfies condition* (B) *is representable in the form*

$$F(e) = \int_e f(P)dm.$$

The function $f(P)$ is uniquely determined except for a set of measure zero.

It is no loss of generality to assume [7]

$$\text{(C)} \qquad \left\| \sum_{i=1}^{n} a_i x_i \right\| \leqq \left\| \sum_{i=1}^{n+1} a_i x_i \right\|.$$

Now suppose

$$F(e) = \sum_{i=1}^{\infty} a_i(e) x_i = \sum_{i=1}^{\infty} x_i \int_e a_i(P)dm,$$

where in the last equality we have used the theorem of Nikodym. Set

$$F_n(e) = \sum_{i=1}^{n} a_i(e) x_i, \qquad f_n(P) = \sum_{i=1}^{n} a_i(P) x_i.$$

By a well known computation and the inequality (C) we get

$$T(F_n, e) = \int_e \|f_n(P)\| dm \leqq T(F, E)$$

$$\|f_n(P)\| \leqq \|f_{n+1}(P)\|.$$

Thus if $b(P) \equiv \lim_n \|f_n(P)\|$ it is seen that $b(P)$ is summable. Hence by (B)

$$f(P) = \sum_{i=1}^{\infty} a_i(P) x_i$$

is convergent for almost all P and $\|f(P)\| = b(P)$, so that the function $f(P)$ is summable and

$$\int_e f(P)dm = \sum_{i=1}^{\infty} x_i \int_e a_i(P)dm = \sum_{i=1}^{\infty} a_i(e) x_i = F(e).$$

Suppose we have also $F(e) = \int_e f^*(P)dm$, where $f^*(P) = \sum_{i=1}^{\infty} a_i^*(P) x_i$. Then $\int_e a_i^*(P)dm = \int_e a_i(P)dm$ for each e and so $a_i^*(P) = a_i(P)$ almost everywhere and thus the same is true of $f^*(P)$ and $f(P)$.

In the case of an absolutely continuous additive function $F(R)$ of bounded variation on elementary figures in a fundamental interval of Euclidean n-space to a Banach space with a uniformly convex norm Clarkson [5] has shown that

$$F(R) = \int_R F'(P)dP,$$

where R is an elementary figure. Hence in case the Banach space X satisfies either condition (A) or (B) we can say that an additive function $F(e)$ on Lebesgue measurable sets in (0, 1) to X which satisfies the Lipschitz condition $\|F(e)\| \leqq M|e|$ also satisfies the conditions

$$F(e) = \int_e F'(P)dP, \qquad \|F'(P)\| \leqq M.$$

Such a function $F(e)$ is defined by a linear operator T on L to X by placing $F(e) = T\phi(e)$, where $\phi(e)$ is the characteristic function of the set e. Here $M = |T|$. Thus for all finitely valued functions $\phi(P)$

$$T\phi = \int_0^1 F'(P)\phi(P)dP.$$

By taking a sequence of finitely valued functions converging in the mean to an arbitrary summable function it is readily seen that the above equality holds for all ϕ in L and that

$$|T| \leqq \sup \|F'(P)\| \leqq M = |T|.$$

Thus we have

5.4. *The function $T\phi$ is a linear operation on L to X (where X satisfies* (A) *or* (B)) *if and only if there exists an essentially bounded and measurable function $f(P)$ on* (0, 1) *to X such that*

$$T\phi = \int_0^1 f(P)\phi(P)dP.$$

The norm of T is $|T| = \text{ess. sup. } \|f(P)\|$.

As a corollary we have (using 3.2)

5.5. *The function $T\phi$ is a linear operation on L to L_q* $(1 < q < \infty)$ *if and only if there exists a measurable function $K(P, Q)$ in L_q for each P and satisfying the conditions*

$$\text{ess. sup. } \int_0^1 |K(P, Q)|^q dQ < \infty,$$

$$T\phi = \int_0^1 K(P, Q)\phi(P)dP \textit{ for } \phi \textit{ in } L.$$

The norm of T is $|T| = \text{ess. sup. } \{\int_0^1 |K(P, Q)|^q dQ\}^{1/q}$.

As would be expected, the method of constructing the absolutely continuous function $F(P) = T\phi(0, P)$ from the operation T does not work in case the domain of T is L_p, $p>1$, instead of L. In this case

$$\sum_1^n \|F(b_i) - F(a_i)\| \leqq |T| \sum_1^n |b_i - a_i|^{1/p}.$$

But now it is not in general true that

$$\sum_1^n |b_i - a_i|^{1/p} \to 0 \quad \text{as} \quad \sum_1^n |b_i - a_i| \to 0,$$

for suppose ϵ is any positive number, then for $n>(2^p\epsilon)^{1/(1-p)}$ and $|b_i-a_i| = \epsilon/n$ we have $\sum_1^n |b_i-a_i| = \epsilon$ and $\sum_1^n |b_i-a_i|^{1/p} > 1/2$.

Using 3.4 we have

5.6. *The function $T\phi$ is a linear operation on L to l_q $(1 \leqq q < \infty)$ if and only if there exists a sequence $k_i(P)$ of measurable functions such that*

$$\text{ess. sup.} \sum_1^\infty |k_i(P)|^q < \infty,$$

$$T\phi = \left\{ \int_0^1 k_i(P)\phi(P)dP \right\}.$$

The norm of T is $|T| =$ ess. sup. $\{\sum_1^\infty |k_i(P)|^q\}^{1/q}$.

The space $A.C.$ of real functions absolutely continuous on (0, 1) which vanish at the origin, with the norming operation $\|\phi\| = V(\phi) =$ total variation of $\phi(P)$ on (0, 1) is a space of type B. Furthermore the linear operation $d\phi/dP = \phi'$ establishes the fact that $A.C.$ and L are isometric, isomorphic, and thus equivalent in the sense of Banach [1, p. 180]. The following theorems are therefore corollaries of 5.4, 5.5, and 5.6.

5.7. *The function $T\phi$ is a linear operation on $A.C.$ to the space X (X satisfying* (A) *or* (B)) *if and only if there exists an essentially bounded and measurable function $f(P)$ on* (0, 1) *to X such that*

$$T\phi = \int_0^1 f(P)\phi'(P)dP.$$

If T is linear then $|T| =$ ess. sup. $\|f(P)\|$.

5.8. *The function $T\phi$ is a linear operation on $A.C.$ to L_q $(1<q<\infty)$ if and only if there exists a measurable function $K(P, Q)$ in L_q for each P and satisfying the conditions*

$$\text{ess. sup.} \int_0^1 | K(P, Q) |^q dQ < \infty,$$

$$T\phi = \int_0^1 K(P, Q)\phi'(P)dP \text{ for } \phi \text{ in } A.C.$$

If T is linear then $|T| = \text{ess. sup. } \{\int_0^1 | K(P, Q) |^q dQ\}^{1/q}$.

5.9. *The function $T\phi$ is a linear operation on A.C. to l_q $(1 \leqq q < \infty)$ if and only if there exists a sequence $k_i(P)$ of measurable functions such that*

$$\text{ess. sup.} \sum_1^\infty | k_i(P) |^q < \infty,$$

$$T\phi = \int_0^1 k_i(P)\phi'(P)dP \text{ for } \phi \text{ in } A.C.$$

The norm of T is $|T| = \text{ess. sup. } \{\sum_1^\infty | k_i(P) |^q\}^{1/q}$.

6. **Completely continuous transformations and further properties of the general integral.** In the last part of the proof of 3.2, put

$$T_n\phi = \int_0^1 K_n(P, Q)\phi(P)dP, \qquad T\phi = \int_0^1 K(P, Q)\phi(P)dP.$$

It is seen that

$$| T_n - T | \leqq \left(\int_0^1 dP\left\{\int_0^1 | K_n(P, Q) - K(P, Q) |^q dQ\right\}^{p/q}\right)^{1/p} \to 0,$$

and thus since T_n is a completely continuous linear operation on $L_{p'}$ to L_q, it follows from a well known result [1, p. 96, Theorem 2] that T is also. This fact has been established by E. Hille and J. D. Tamarkin [10] and by T. H. Hildebrandt.* Writing p' in place of p we may state

6.1. *For $1 < p \leqq \infty$, $1 \leqq q < \infty$ the transformation on L_p to L_q defined by*

$$T\phi = \int_0^1 K(P, Q)\phi(P)dP$$

with $K(P, Q)$ in $L_{p'q}$ is completely continuous.

Combined with 3.1 this gives

6.2. *For $1 < p \leqq \infty$, $1 \leqq q < \infty$ the function $f(P)$ in $S_{p'q}$ defines a completely continuous transformation*

* The proof given here is due to Hildebrandt.

$$T\phi = \int_0^1 f(P)\phi(P)dP$$

on L_p to L_q.

This result gives the integral that we are using more properties in common with the Lebesgue integral of numerical functions. The ones we have in mind center around the fact that a completely continuous linear transformation takes weakly convergent sequences into convergent sequences* [1, p. 143].

6.3. *Let $\phi_n(P)$, $n=0, 1, 2, \cdots$, be real functions and $1<p<\infty$, $1\leqq q<\infty$, then for every $f(P)$ in $S_{p'q}$ the integrals $\int_0^1 f(P)\phi_n(P)dP$ are defined and approach $\int_0^1 f(P)\phi_0(P)dP$ if and only if $\phi_n(P)$, $n=0, 1, 2, \cdots$, is in L_p and*

$$\int_0^1 |\phi_n(P)|^p dP \leqq M, \qquad n = 0, 1, 2, \cdots, \textit{for some } M,$$

$$\int_0^P \phi_n(P)dP \to \int_0^P \phi_0(P)dP, \qquad 0 \leqq P \leqq 1.$$

That the last conditions imply the convergence of the integrals follows from the statement preceding Theorem 6.2 and a theorem of F. Riesz [1, p. 135]. Conversely, if $f(P)\phi_n(P)$ is summable for every $f(P)$ in $S_{p'q}$ it is seen first, by taking $f(P)=k(P, Q)=1$ $(0\leqq P, Q\leqq 1)$ and using 3.2, that there is a summable function $K_n(P, Q)$ such that for every P

$$K_n(P, Q) = \phi_n(P) \text{ for almost all } Q.$$

It follows from the theorem of Fubini that $\phi_n(P)$ is measurable. Now let $\psi(P)$ be an arbitrary finite valued function in $L_{p'}$, and define $f(P)$ to be the function on (0, 1) to L_q that takes each point P into the function in L_q which is constant and equal to $\psi(P)$, i.e., $f(P)=\psi(P)$. Then by 3.1, $\psi(P)\phi_n(P)$ is summable and

$$\int_0^1 f(P)\phi_n(P)dP = \int_0^1 \psi(P)\phi_n(P)dP.$$

From the summability of $\psi(P)\phi_n(P)$ it follows that the function $\phi_n(P)$ is in L_p, and the remaining conditions on the sequence follow from the convergence of the sequence $\int_0^1 f(P)\phi_n(P)dP$ and the theorem of Riesz. In case $p=\infty$ we have

* It is easily seen that this property characterizes completely continuous linear transformations between arbitrary Banach spaces X and Y in case the domain X is weakly complete and the conjugate space $\overline{X}$ is separable, or in case every bounded set in X is weakly compact.

6.4. *Let $\phi_n(P)$ $(n=0, 1, \cdots)$ be real functions and $1 \leqq q < \infty$, then for every summable function $f(P)$ on $(0, 1)$ to L_q the integrals $\int_0^1 f(P)\phi_n(P)dP$ are defined and approach $\int_0^1 f(P)\phi_0(P)dP$ if and only if $\phi_n(P)$ are measurable,*

$$\int_0^P \phi_n(P)dP \to \int_0^P \phi_0(P)dP, \qquad 0 \leqq P \leqq 1,$$

$$\operatorname{ess.}\sup_P |\phi_n(P)| \leqq M, \qquad n = 0, 1, \cdots, \textit{for some } M.$$

Suppose the last conditions are satisfied and $f_0(P)$ is finitely valued with

$$\int_0^1 \|f_0(P) - f(P)\| dP < \epsilon.$$

Then by 6.3

$$\int_0^1 f_0(P)\phi_n(P)dP \to \int_0^1 f_0(P)\phi_0(P)dP,$$

and since

$$\left\| \int_0^1 f(P)(\phi_n(P) - \phi_0(P))dP \right\| \leqq \left\| \int_0^1 (f(P) - f_0(P))(\phi_n(P) - \phi_0(P))dP \right\| + \left\| \int_0^1 f_0(P)(\phi_n(P) - \phi_0(P))dP \right\|,$$

it follows that

$$\overline{\lim_n} \left\| \int_0^1 f(P)\phi_n(P)dP - \int_0^1 f(P)\phi_0(P)dP \right\| \leqq 2M\epsilon.$$

Now conversely if the integrals $\int_0^1 f(P)\phi_n(P)dP$ are defined for every summable $f(P)$ it follows as before that the functions $\phi_n(P)$ are measurable and that each $\phi_n(P)$ is essentially bounded. Also as in 6.3 if we let $\psi(P)$ be an arbitrary finite valued summable function and define $f(P) = \psi(P)$ and consider $f(P)$ as a function on $(0, 1)$ to L_q then

$$\int_0^1 f(P)\phi_n(P)dP = \int_0^1 \psi(P)\phi_n(P)dP.$$

Now the boundedness of the sequence $\int_0^1 \psi(P)\phi_n(P)dP$ for every function $\psi(P)$ in L insures the uniform boundedness of ess. $\sup_P |\phi_n(P)|$, and the convergence of the sequence $\int_0^1 \psi(P)\phi_n(P)dP$ insures by the theorem of Riesz the condition

$$\int_0^P \phi_n(P)dP \to \int_0^P \phi_0(P)dP.$$

Theorem 6.4 may be regarded as a generalization of the Riemann-Lebesgue theorem.

6.5. *The function* $T\phi$ *is a completely continuous linear operation on* L *to* L_q $(1<q<\infty)$ *if and only if there exists a measurable kernel* $K(P, Q)$ *which vanishes outside the unit square and satisfies the conditions*

$$\text{ess. sup.} \int_0^1 |K(P,Q)|^q dQ < \infty,$$

$$\lim_{h=0} \text{ ess. sup.} \int_0^1 |K(P,Q+h) - K(P,Q)|^q dQ = 0,$$

$$T\phi = \int_0^1 K(P,Q)\phi(P)dP \textit{ for } \phi(P) \textit{ in } L.$$

The norm of T *is* $|T| = \text{ess. sup. } \{\int_0^1 |K(P,Q)|^q dQ\}^{1/q}$.

If the conditions are satisfied, 5.5. shows that $T\phi$ is linear. Now let S be the unit sphere in L and

$$\psi(Q) = \int_0^1 K(P,Q)\phi(P)dP.$$

It is only necessary to show that

$$\lim_{h=0} \int_0^1 |\psi(Q+h) - \psi(Q)|^q dQ = 0$$

uniformly with respect to ψ in TS [see 10, p. 445, or 12]. We have for ϕ in S

$$\left(\int_0^1 |\psi(Q+h) - \psi(Q)|^q dQ\right)^{1/q}$$

$$= \left(\int_0^1 \left|\int_0^1 \{K(P,Q+h) - K(P,Q)\}\phi(P)dP\right|^q dQ\right)^{1/q}$$

$$\leqq \int_0^1 dP|\phi(P)| \left\{\int_0^1 |K(P,Q+h) - K(P,Q)|^q dQ\right\}^{1/q}$$

$$\leqq \text{ess. sup.} \left\{\int_0^1 |K(P,Q+h) - K(P,Q)|^q dQ\right\}^{1/q}.$$

(The first of the above inequalities is a generalization of Minkowski's inequality [9, p. 148, §202].) Thus the conditions are sufficient to insure the complete continuity of $T\phi$. Now conversely if $T\phi$ is a completely continuous linear operation on L to L_q, it follows from 5.5 that the first and third conditions are satisfied by some measurable kernel $K(P, Q)$. If $K(P, Q)$ is defined to be zero outside the square, then

$$\int_0^1 | K(P,Q+h) |^q dQ \leqq \int_0^1 | K(P,Q) |^q dQ,$$

and it follows again from 5.5 that

$$T_h\phi = \int_0^1 \{K(P, Q+h) - K(P, Q)\}\phi(P)dP$$

is a linear operation on L to L_q with

$$| T_h | = \text{ess. sup.} \left\{ \int_0^1 | K(P,Q+h) - K(P,Q) |^q dQ \right\}^{1/q}.$$

Since $T\phi$ is completely continuous, there is a $\delta_\epsilon > 0$ depending only upon an arbitrary positive number ϵ such that

$$\left\{ \int_0^1 | \psi(Q+h) - \psi(Q) |^q dQ \right\}^{1/q} < \epsilon \text{ for } | h | < \delta_\epsilon, \ \psi \text{ in } TS.$$

Fix h with $|h| < \delta_\epsilon$, then there is a ϕ in S such that

$$| T_h | \leqq \|T_h\phi\| + \epsilon;$$

that is,

$$\text{ess. sup.} \left\{ \int_0^1 | K(P,Q+h) - K(P,Q) |^q dQ \right\}^{1/q}$$
$$\leqq \left\{ \int_0^1 | \psi(Q+h) - \psi(Q) |^q dQ \right\}^{1/q} + \epsilon < 2\epsilon$$

which completes the proof of 6.5.

In view of the equivalence of $A.C.$ and L, 6.5 translates into

6.6. *The function $T\phi$ is a completely continuous linear operation on $A.C.$ to L_q $(1 < q < \infty)$ if and only if there is a measurable kernel $K(P, Q)$ which vanishes outside the unit square and satisfies the conditions*

$$\text{ess. sup.} \int_0^1 | K(P, Q) |^q dQ < \infty,$$

$$\lim_{h=0} \ \text{ess. sup.} \int_0^1 | K(P, Q+h) - K(P, Q) |^q dQ = 0,$$

$$T\phi = \int_0^1 K(P, Q)\phi'(P)dP \ \textit{for } \phi \textit{ in } A.C.$$

The norm of T is $|T| = \text{ess. sup. } \{\int_0^1 | K(P, Q) |^q dQ\}^{1/q}$.

For a linear operation on L to L_q $(1<q<\infty)$ the adjoint $\overline{T}$ on $\overline{L}_q$ to $\overline{L}$ may be regarded as a linear operation on $L_{q'}$ to M (since $\overline{L}_q$ and $\overline{L}$ are equivalent to $L_{q'}$ and M respectively). Recalling [1, p. 100, Theorems 3 and 4] that $|\overline{T}|=|T|$ and that $\overline{T}$ is completely continuous providing T is, the following theorem then follows from 5.5 and 6.5.

6.7. *Let* $1<q<\infty$ *and suppose the measurable kernel* $K(P, Q)$ *satisfies the condition*

$$\text{ess. sup.} \int_0^1 |K(P, Q)|^{q'} dQ < \infty,$$

then the operation

$$T\psi = \int_0^1 K(P, Q)\psi(Q)dQ$$

is a linear operation on L_q *to* M *with* $|T| = \text{ess. sup.} \left\{\int_0^1 |K(P, Q)|^{q'} dQ\right\}^{1/q'}$. *The operation* T *is completely continuous provided*

$$\lim_{h=0} \text{ ess. sup.} \int_0^1 |K(P, Q+h) - K(P, Q)|^{q'} dQ = 0.$$

7. **Conclusion.** The problem of representation of operations on X to C (the space of continuous functions) where X is a Banach space for which the form of the general linear functional (numerically valued operator) is known, is much more easily handled than that of operations on L to L_q. This has been discussed in the case of C to C by J. Radon [11]. The fact that gives the hold on the problem is that for $\psi(Q)$ in C, $\psi(Q)=0$ for each Q provided $\|\psi\|=0$. This shows that by fixing Q the operator $\psi = T\phi$ on X to C can be interpreted as a functional. Thus, for example, in the case of L_p $(1<p<\infty)$ to C

$$\psi(Q) = \int_0^1 K(P, Q)\phi(P)dP, \tag{1}$$

where for each Q, $K(P, Q)$ is in $L_{p'}$ and thus

$$|T| \leqq \sup \left\{\int_0^1 |K(P, Q)|^{p'} dP\right\}^{1/p'}.$$

But now for each ϕ in L_p, and each Q in $(0, 1)$

$$\lim_{Q'\to Q} \int_0^1 K(P, Q)\phi(P)dP = \int_0^1 K(P, Q)\phi(P)dP,$$

which is equivalent to saying that

$$\lim_{Q' \to Q} \int_0^t K(P, Q)dP = \int_0^t K(P, Q)dP, \qquad 0 \leqq t \leqq 1, \tag{2}$$

and

$$\sup \left\{ \int_0^1 | K(P, Q) |^{p'} dP \right\}^{1/p'} \equiv M < \infty. \tag{3}$$

Now fix Q so that

$$\left\{ \int_0^1 | K(P, Q) |^{p'} dP \right\}^{1/p'} \geqq M - \epsilon,$$

then there is a ϕ in L_p with $\|\phi\| = 1$ such that

$$| \psi(Q) | = \left| \int_0^1 K(P, Q)\phi(P)dP \right| \geqq M - \epsilon,$$

which shows that $\|\psi\| \geqq M$. Hence the norm of T is given by

$$| T | = \sup \left\{ \int_0^1 | K(P, Q) |^{p'} dP \right\}^{1/p'}. \tag{4}$$

To summarize: The function $\psi = T\phi$ is a linear operation on L_p $(1 < p < \infty)$ to C if and only if there exists a function $K(P, Q)$ in $L_{p'}$ for each Q and satisfying (1), (2), and (3). The norm of T is given by (4). Similar results can be obtained for other Banach spaces.

References

1. S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932.

2. G. Birkhoff, *Integration of functions with values in a Banach space*, these Transactions, vol. 38 (1935), pp. 357–378.

3. S. Bochner, *Integration von Funktionen, deren Werte die Elemente eines Vektorraumes sind*, Fundamenta Mathematicae, vol. 20 (1933), pp. 262–276.

4. S. Bochner and J. von Neumann, *On compact solutions of operational differential equations*, I, Annals of Mathematics, (2), vol. 36 (1935), pp. 255–291.

5. J. A. Clarkson, *Uniformly convex spaces*, these Transactions, vol. 40 (1936), pp. 396–414.

6. N. Dunford, *Integration in general analysis*, these Transactions, vol 37 (1935), pp. 441–453.

7. N. Dunford and A. P. Morse, *Remarks on the preceding paper of James A. Clarkson*, these Transactions, vol. 40 (1936), pp. 415–420.

8. A. Haar, *Zur Theorie der Orthogonalen Funktionensysteme*, Inaugural-Dissertation, Göttingen, 1909, Chapter III, §§1–3.

9. Hardy, Littlewood and Pólya, *Inequalities*, Cambridge, 1934.

10. E. Hille and J. D. Tamarkin, *On the theory of linear integral equations*, II, Annals of Mathematics, (2), vol. 35 (1934), pp. 445–455.

11. J. Radon, *Über lineare Funktionaltransformationen und Funktionalgleichungen*, Sitzungsberichte, Akademie der Wissenschaften, Wien, Mathematische-Naturwissenschaftliche Klasse, vol. 128, 2a (1929), pp. 1083–1122.

12. M. Riesz, *Sur les ensembles compact de fonctions sommables*, Acta Szeged, vol. 6 (1933), pp. 136–142.

13. S. Saks, *Théorie de l'Intégrale*, Warsaw, 1933.

14. J. Schauder, *Eine Eigenschaft des Haarschen Orthogonalsystem*, Mathematische Zeitschrift, vol. 28 (1928), pp. 317–320.

15. W. Sierpiński, *Sur un problème concernant les ensembles mesurables superficiellement*, Fundamenta Mathematicae, vol. 1 (1920), pp. 112–115.

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