Single Beam Element Stiffness Matrix Formulation

Consider a prismatic beam of length L loaded by shear forces and moments at its two ends as shown in Fig. 1. Distance along the beam is measured with a coordinate x, starting at the left end. Deflection, $v(x)$, is measured positive down following the convention in Timoshenko and Gere.

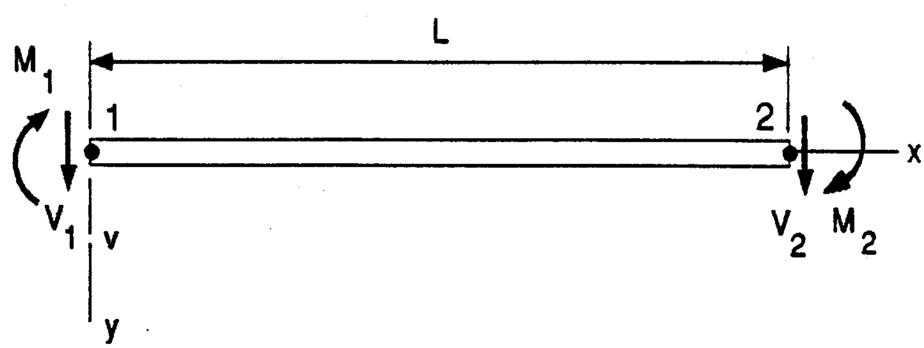

Fig. 1 – Forces and Moments on a Single Beam Element

The shear force and moment at the left end (end 1) are V_1 and M_1, respectively. Corresponding quantities at the right end (end 2) are V_2 and M_2.

Assuming zero transverse distributed load, the load-deflection differential equation can be integrated sequentially to yield expressions for shear force, bending moment, slope, and deflection.

$$
\begin{aligned}
EI\,\frac{d^4v}{dx^4} &= 0 \\
EI\,\frac{d^3v}{dx^3} &= c_1 = -V(x) \\
EI\,\frac{d^2v}{dx^2} &= c_1 x + c_2 = -M(x) \\
EI\,\frac{dv}{dx} &= c_1\frac{x^2}{2} + c_2 x + c_3 \\
EIv &= c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3 x + c_4
\end{aligned}
$$

The displacement and rotation at end 1 are denoted by v_1 and θ_1, respectively. Corresponding quantities at end 2 are v_2 and θ_2. These four kinematic variables may then be expressed in terms of the constants c_1, c_2, c_3, and c_4 using the equations above, as follows

$$
\begin{aligned}
v_1 &= v(0) = \frac{1}{EI}\,c_4 \\
\theta_1 &= \frac{dv}{dx}(0) = \frac{1}{EI}c_3 \\
v_2 &= v(L) = \frac{1}{EI}\left[c_1\frac{L^3}{6} + c_2\frac{L^2}{2} + c_3 L + c_4\right] \\
\theta_2 &= \frac{dv}{dx}(L) = \frac{1}{EI}\left[c_1\frac{L^2}{2} + c_2 L + c_3\right]
\end{aligned}
$$

These equations can be written in matrix form as

$$\begin{pmatrix} 0 & 0 & 0 & \frac{1}{EI} \\ 0 & 0 & \frac{1}{EI} & 0 \\ \frac{L^3}{6EI} & \frac{L^2}{2EI} & \frac{L}{EI} & \frac{1}{EI} \\ \frac{L^2}{2EI} & \frac{L}{EI} & \frac{1}{EI} & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix}$$

Solving this system of linear algebraic equations for c_1, c_2, c_3 and c_4 gives (the reader should verify)

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{-6EI}{L^2} & \frac{-4EI}{L} & \frac{6EI}{L^2} & \frac{-2EI}{L} \\ 0 & EI & 0 & 0 \\ EI & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix}$$

The shear forces and bending moments at the two ends of the beam can be expressed in terms of the constants c_1, c_2, c_3 and c_4. Then, using the result above, the shear forces and bending moments on the ends can be written in terms of the end displacements and rotations.

$$
\begin{aligned}
V(0) = -V_1 &= -c_1 \\
V_1 &= \frac{12EI}{L^3}v_1 + \frac{6EI}{L^2}\theta_1 - \frac{12EI}{L^3}v_2 + \frac{6EI}{L^2}\theta_2 \\
M(0) = M_1 &= -c_2 \\
M_1 &= \frac{6EI}{L^2}v_1 + \frac{4EI}{L}\theta_1 - \frac{6EI}{L^2}v_2 + \frac{2EI}{L}\theta_2 \\
V(L) = V_2 &= -c_1 \\
V_2 &= -\frac{12EI}{L^3}v_1 - \frac{6EI}{L^2}\theta_1 + \frac{12EI}{L^3}v_2 - \frac{6EI}{L^2}\theta_2 \\
M(L) = -M_2 &= -c_1 L - c_2 \\
M_2 &= \frac{6EI}{L^2}v_1 + \frac{2EI}{L}\theta_1 - \frac{6EI}{L^2}v_2 + \frac{4EI}{L}\theta_2
\end{aligned}
$$

These equations can be expressed in matrix form, as follows

$$\frac{EI}{L^3}\begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix} \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{pmatrix}$$

The vector on the left hand side of the equation is called the "element displacement vector," while the vector on the right hand side is called the "element force vector." The 4×4 matrix is often referred to as the "finite element stiffness matrix." This particular way of arranging the force and displacement quantities proves very useful in solving beam problems, i.e., for finding both displacements and reactions.

EXAMPLES

(i.) Cantilever with a Tip Load

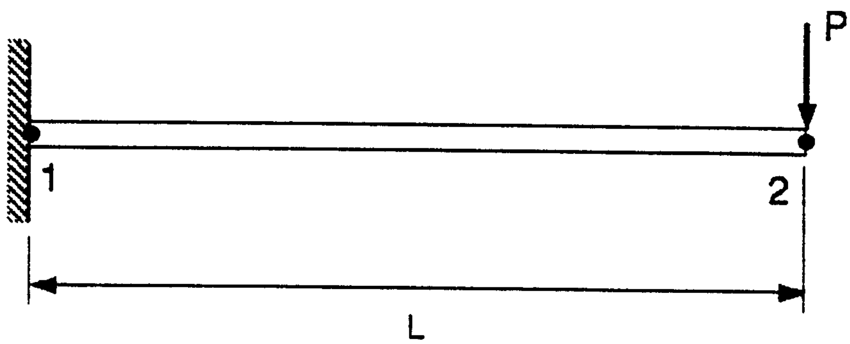

Fig. 2 - Cantilever with a Tip Load

It is common to refer to end 1 and end 2 as nodes. To solve for deflections and reactions, it is first necessary to consider any known kinematic boundary conditions and then any known external forces/moments.

BC's

The appropriate kinematic boundary conditions for the fixed end are

$$v_1 = \theta_1 = 0$$

Thus, for this problem the 4 × 4 system of equations reduces to

$$\frac{EI}{L^3}\begin{pmatrix} 12 & -6L \\ -6L & 4L^2 \end{pmatrix} \begin{pmatrix} v_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} V_2 \\ M_2 \end{pmatrix}$$

External Force and Moment

The external force and moment at node 2 are

$$V_2 = P$$

$$M_2 = 0$$

Thus, the following 2×2 system of equations must be solved for v_2 and θ_2,

$$\frac{EI}{L^3}\begin{pmatrix} 12 & -6L \\ -6L & 4L^2 \end{pmatrix} \begin{pmatrix} v_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} P \\ O \end{pmatrix}$$

The solution is (the reader should verify)

$$v_2 = \frac{PL^3}{3EI}$$

$$\theta_2 = \frac{PL^2}{2EI}$$

Reaction Force and Moment

The reaction force and moment at node 1 may be calculated using the original 4×4 system of equations,

$$V_1 = \frac{EI}{L^3}[-12v_2 + 6L\theta_2] = -P$$

$$M_1 = \frac{EI}{L^3}[-6Lv_2 + 2L^2\theta_2] = -PL$$

These results may be verified immediately by consideration of elementary statics.

(ii.) Cantilever with a Tip Moment

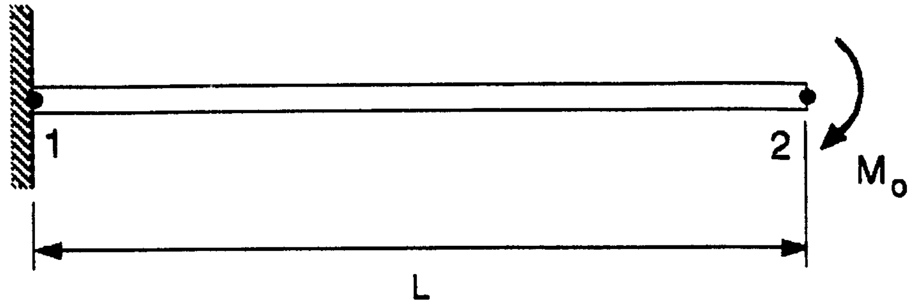

Fig. 3 - Cantilever with a Tip Moment

BC's

The appropriate kinematic boundary conditions for the fixed end are

$$v_1 = \theta_1 = 0$$

The 4 × 4 system of equations then reduces to

$$\frac{EI}{L^3}\begin{pmatrix} 12 & -6L \\ -6L & 4L^2 \end{pmatrix} \begin{pmatrix} v_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} V_2 \\ M_2 \end{pmatrix}$$

External Force and Moment

The external force and moment at node 2 are

$$V_2 = 0$$

$$M_2 = M_0$$

Thus, the following 2×2 system of equations must be solved for v_2 and θ_2

$$\frac{EI}{L^3}\begin{pmatrix} 12 & -6L \\ -6L & 4L^2 \end{pmatrix} \begin{pmatrix} v_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ M_0 \end{pmatrix}$$

The solution is (the reader should verify)

$$v_2 = \frac{M_0 L^2}{2EI}$$

$$\theta_2 = \frac{M_0 L}{EI}$$

Reaction Force and Moment

The reaction force and moment at node 1 are calculated by using the original 4×4 system of equations,

$$V_1 = \frac{EI}{L^3}\left[-12v_2 + 6L\theta_2\right] = 0$$

$$M_1 = \frac{EI}{L^3}\left[-6Lv_2 + 2L^2\theta_2\right] = -M_0$$

(iii.) Propped Cantilever with an End Moment (statically indeterminate)

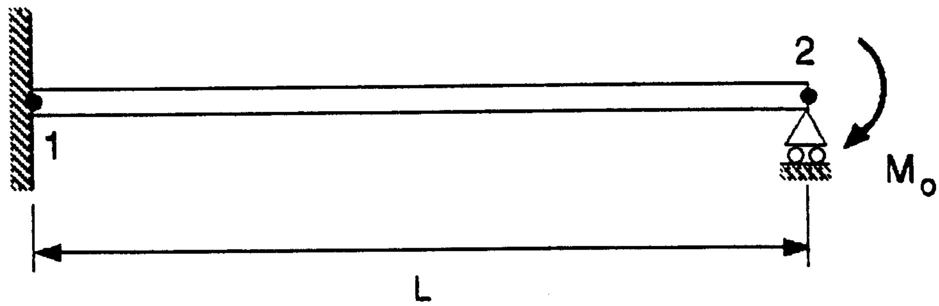

Fig. 4 - Propped Cantilever with End Moment

BC's

The appropriate kinematic boundary conditions for the fixed end and roller support are

$$v_1 = \theta_1 = v_2 = 0$$

The 4 × 4 system of equations then reduces to

$$\frac{4EI}{L}\,\theta_2 = M_2$$

External Force and Moment

The external moment at node 2 is

$$M_2 = M_0$$

The only unknown kinematic variable is θ_2, and it may be solved for easily

$$\theta_2 = \frac{M_0 L}{4EI}$$

Reaction Force and Moment

Exercise

Determine the reaction force and moment at node 1 and the reaction force at node 2.

Assembly Procedures

The real power of the method being developed here is when two or more beam "elements" are used together to solve more complex problems where: concentrated forces occur in midspan, discontinuities occur in EI, or where supports exist within the span. For example, consider the problem of a simply supported beam with a concentrated mid-span force, as shown in Fig. 5.

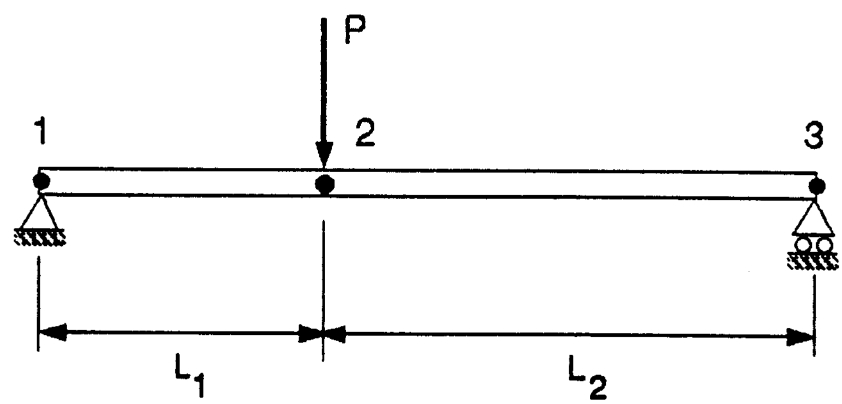

Fig. 5 - Simply Supported Beam - Two Elements

This problem can be solved by "assembling" two elements. The two elements and the forces/moments that act on them are shown in Fig. 6.

The equations relating forces/moments to displacements/rotations for each of the elements are

Element 1:

$$\begin{pmatrix} V_1^1 \\ M_1^1 \\ V_2^1 \\ M_2^1 \end{pmatrix} = \frac{EI}{L_1^3} \begin{pmatrix} 12 & 6L_1 & -12 & 6L_1 \\ 6L_1 & 4L_1^2 & -6L_1 & 2L_1^2 \\ -12 & -6L_1 & 12 & -6L_1 \\ 6L_1 & 2L_1^2 & -6L_1 & 4L_1^2 \end{pmatrix} \begin{pmatrix} v_1^1 \\ \theta_1^1 \\ v_2^1 \\ \theta_2^1 \end{pmatrix}$$

Element 2:

$$\begin{pmatrix} V_1^2 \\ M_1^2 \\ V_2^2 \\ M_2^2 \end{pmatrix} = \frac{EI}{L_2^3} \begin{pmatrix} 12 & 6L_2 & -12 & 6L_2 \\ 6L_2 & 4L_2^2 & -6L_2 & 2L_2^2 \\ -12 & -6L_2 & 12 & -6L_2 \\ 6L_2 & 2L_2^2 & -6L_2 & 4L_2^2 \end{pmatrix} \begin{pmatrix} v_1^2 \\ \theta_1^2 \\ v_2^2 \\ \theta_2^2 \end{pmatrix}$$

The superscripts 1 and 2 on the force/moment quantities and the displacement/rotation quantities refer to the element number, as do the subscripts on L.

It is clear that three nodal points will figure in this analysis. Now define

$$v_1 \equiv v_1^1 \qquad v_2 \equiv v_2^1 \qquad v_2 \equiv v_1^2 \qquad v_3 \equiv v_2^2$$

$$\theta_1 \equiv \theta_1^1 \qquad \theta_2 \equiv \theta_2^1 \qquad \theta_2 \equiv \theta_1^2 \qquad \theta_3 \equiv \theta_2^2$$

The subscripts 1,2,3 refer to node number.

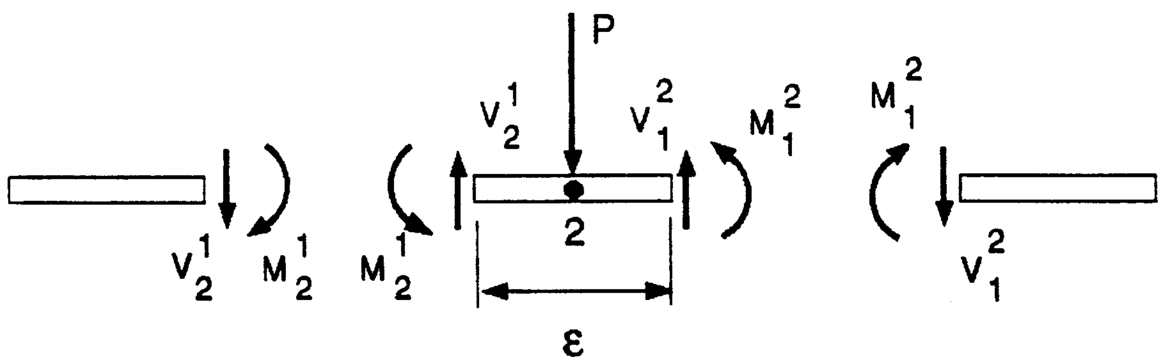

Fig. 6 - Static equilibrium at node 2

Fig. 6 shows a free-body diagram of an infinitesimal segment of the beam surrounding node 2. As the length of this segment $\epsilon \to 0$, the following equilibrium conditions must hold,

$$V_2^1 + V_1^2 = P$$

$$M_2^1 + M_1^2 = 0$$

These two equations can be written in matrix form as follows,

$$\begin{pmatrix} V_2^1 \\ M_2^1 \end{pmatrix} + \begin{pmatrix} V_1^2 \\ M_1^2 \end{pmatrix} = \frac{EI}{L_1^3} \begin{pmatrix} -12 & -6L_1 & 12 & -6L_1 \\ 6L_1 & 2L_1^2 & -6L_1 & 4L_1^2 \end{pmatrix} \begin{pmatrix} v_1^1 \\ \theta_1^1 \\ v_2^1 \\ \theta_2^1 \end{pmatrix}$$

$$+ \frac{EI}{L_2^3} \begin{pmatrix} 12 & 6L_2 & -12 & 6L_2 \\ 6L_2 & 4L_2^2 & -6L_2 & 2L_2^2 \end{pmatrix} \begin{pmatrix} v_1^2 \\ \theta_1^2 \\ v_2^2 \\ \theta_2^2 \end{pmatrix} = \begin{pmatrix} P \\ O \end{pmatrix}$$

or

$$\begin{pmatrix} V_2^1 + V_1^2 \\ M_2^1 + M_1^2 \end{pmatrix} = \begin{pmatrix} -\frac{12EI}{L_1^3} & -\frac{6EI}{L_1^2} & \frac{12EI}{L_1^3} + \frac{12EI}{L_2^3} & -\frac{6EI}{L_1^2} + \frac{6EI}{L_2^2} & -\frac{12EI}{L_2^3} & \frac{6EI}{L_2^2} \\ \frac{6EI}{L_1^2} & \frac{2EI}{L_1} & -\frac{6EI}{L_1^2} + \frac{6EI}{L_2^2} & \frac{4EI}{L_1} + \frac{4EI}{L_2} & -\frac{6EI}{L_2^2} & \frac{2EI}{L_2} \end{pmatrix} \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{pmatrix}$$

$$= \begin{pmatrix} P \\ O \end{pmatrix}$$

The next step is to combine these equilibrium equations with the element equations, as follows

$$\begin{pmatrix} \frac{12EI}{L_1^3} & \frac{6EI}{L_1^2} & -\frac{12EI}{L_1^3} & \frac{6EI}{L_1^2} & 0 & 0 \\ \frac{6EI}{L_1^2} & \frac{4EI}{L_1} & -\frac{6EI}{L_1^2} & \frac{2EI}{L_1} & 0 & 0 \\ -\frac{12EI}{L_1^3} & -\frac{6EI}{L_1^2} & \frac{12EI}{L_1^3}+\frac{12EI}{L_2^3} & -\frac{6EI}{L_1^2}+\frac{6EI}{L_2^2} & -\frac{12EI}{L_2^3} & \frac{6EI}{L_2^2} \\ \frac{6EI}{L_1^2} & \frac{2EI}{L_1} & -\frac{6EI}{L_1^2}+\frac{6EI}{L_2^2} & \frac{4EI}{L_1}+\frac{4EI}{L_2} & -\frac{6EI}{L_2^2} & \frac{2EI}{L_2} \\ 0 & 0 & -\frac{12EI}{L_2^3} & -\frac{6EI}{L_2^2} & \frac{12EI}{L_2^3} & -\frac{6EI}{L_2^2} \\ 0 & 0 & \frac{6EI}{L_2^2} & \frac{2EI}{L_2} & -\frac{6EI}{L_2^2} & \frac{4EI}{L_2} \end{pmatrix} \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} V_1^1 \\ M_1^1 \\ P \\ 0 \\ V_2^2 \\ M_2^2 \end{pmatrix}$$

BC's

The appropriate kinematic boundary conditions for the simple supports are

$$v_1 = v_3 = 0$$

External Force and moment

The bending moments at the ends are zero for simple supports, i.e.,

$$M_1^1 = M_2^2 = 0$$

The system of equations from which the unknown $(\theta_1, v_2, \theta_2, \theta_3)$ kinematic variables are determined is then,

$$\begin{pmatrix} \frac{4EI}{L_1} & -\frac{6EI}{L_1^2} & \frac{2EI}{L_1} & 0 \\ -\frac{6EI}{L_1^2} & \frac{12EI}{L_1^3} + \frac{12EI}{L_2^3} & -\frac{6EI}{L_1^2} + \frac{6EI}{L_2^2} & \frac{6EI}{L_2^2} \\ \frac{2EI}{L_1} & -\frac{6EI}{L_1^2} + \frac{6EI}{L_2^2} & \frac{4EI}{L_1} + \frac{4EI}{L_2} & \frac{2EI}{L_2} \\ 0 & \frac{6EI}{L_2^2} & \frac{2EI}{L_2} & \frac{4EI}{L_2} \end{pmatrix} \begin{pmatrix} \theta_1 \\ v_2 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ P \\ 0 \\ 0 \end{pmatrix}$$

In practice, this assembly procedure can be carried out very systematically on the computer. To understand this, define the following,

$$\underline{f}^e = \begin{pmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{pmatrix} \qquad \underline{d}^e = \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix}$$

(continued)

and

$$\underline{k}^e = \frac{EI}{L^3}\begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix}$$

where the superscript e stands for the element number.

The assembly procedure for the simply supported beam discussed above begins by forming the "unconstrained system equations,"

$$\begin{pmatrix} k^1_{11} & k^1_{12} & k^1_{13} & k^1_{14} & 0 & 0 \\ k^1_{21} & k^1_{22} & k^1_{23} & k^1_{24} & 0 & 0 \\ k^1_{31} & k^1_{32} & k^1_{33}+k^2_{11} & k^1_{34}+k^2_{12} & k^2_{13} & k^2_{14} \\ k^1_{41} & k^1_{42} & k^1_{43}+k^2_{21} & k^1_{44}+k^2_{22} & k^2_{23} & k^2_{24} \\ 0 & 0 & k^2_{31} & k^2_{32} & k^2_{33} & k^2_{34} \\ 0 & 0 & k^2_{41} & k^2_{42} & k^2_{43} & k^2_{44} \end{pmatrix}\begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} P_1 \\ M_1 \\ P_2 \\ M_2 \\ P_3 \\ M_3 \end{pmatrix}$$

(continued)

where

$$
\begin{aligned}
v_1,\ \theta_1 &= \text{displacement and slope at node 1} \\
P_1,\ M_1 &= \text{external transverse load and moment at node 1} \\
v_2,\ \theta_2 &= \text{displacement and slope at node 2} \\
P_2,\ M_2 &= \text{external transverse load and moment at node 2} \\
v_3,\ \theta_3 &= \text{displacement and slope at node 3} \\
P_3,\ M_3 &= \text{external transverse load and moment at node 3}
\end{aligned}
$$

Notice that the unconstrained system stiffness matrix is formed by "nesting" the two element stiffness matrices. This is due to the fact that elements 1 and 2 share a common nodal point, node 2.

The next step is to impose the boundary conditions, and then impose any specified external forces and moments.

BC's

$$v_1 = v_3 = 0$$

External force and moment

$$M_1 = M_2 = M_3 = 0$$

$$P_2 = P$$

Imposing the boundary conditions has the effect of crossing out those rows and columns of the unconstrained system equations that correspond to the nodal displacements/rotations that are zero, i.e.,

$$\begin{pmatrix} k^1_{11} & k^1_{12} & k^1_{13} & k^1_{14} & 0 & 0 \\ k^1_{21} & k^1_{22} & k^1_{23} & k^1_{24} & 0 & 0 \\ k^1_{31} & k^1_{32} & k^1_{33}+k^2_{11} & k^1_{34}+k^2_{12} & k^2_{13} & k^2_{14} \\ k^1_{41} & k^1_{42} & k^1_{43}+k^2_{21} & k^1_{44}+k^2_{22} & k^2_{23} & k^2_{24} \\ 0 & 0 & k^2_{31} & k^2_{32} & k^2_{33} & k^2_{34} \\ 0 & 0 & k^2_{41} & k^2_{42} & k^2_{43} & k^2_{44} \end{pmatrix} \begin{pmatrix} 0 \\ \theta_1 \\ v_2 \\ \theta_2 \\ 0 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} P_1 \\ 0 \\ P \\ 0 \\ P_3 \\ 0 \end{pmatrix}$$

The four equations for the remaining four unknowns are

$$\begin{pmatrix} k^1_{22} & k^1_{23} & k^1_{24} & 0 \\ k^1_{32} & k^1_{33}+k^2_{11} & k^1_{34}+k^2_{12} & k^2_{14} \\ k^1_{42} & k^1_{43}+k^2_{21} & k^1_{44}+k^2_{22} & k^2_{24} \\ 0 & k^2_{41} & k^2_{42} & k^2_{44} \end{pmatrix} \begin{pmatrix} \theta_1 \\ v_2 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ P \\ 0 \\ 0 \end{pmatrix}$$

This matrix equation is called the "constrained system equation." The entries in it are exactly the same as the matrix equation that was derived earlier by considering equilibrium at the common node.

Example

(iv.) Simply Supported Beam with Concentrated Load at Midspan

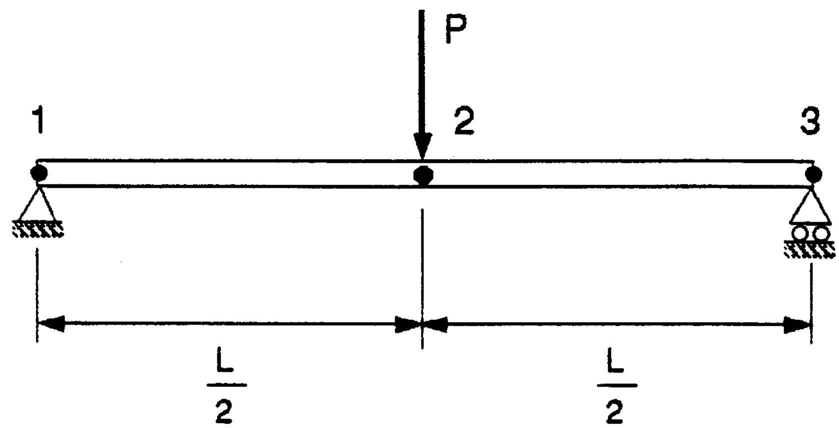

Fig. 7. Simply Supported Beam with Concentrated Load at Midspan

The matrix equation presented above may be used to solve this problem. To fix ideas, consider the case when $L_1 = L_2 = L/2$. Then take as data: $EI = 1$, $L = 1$, and $P = 1$.

The system of linear algebraic equations that must be solved is

$$\begin{pmatrix} 8 & -24 & 4 & 0 \\ -24 & 192 & 0 & 24 \\ 4 & 0 & 16 & 4 \\ 0 & 24 & 4 & 8 \end{pmatrix} \begin{pmatrix} \theta_1 \\ v_2 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

The solution is (obtained on HP-15c calculator)

$$\begin{pmatrix} \theta_1 \\ v_2 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 0.0625 \\ 0.02083 \\ 0.0 \\ -0.0625 \end{pmatrix}$$

The theoretical solution for this problem is (the reader should verify)

$$v_2 = \frac{PL^3}{48EI} = 0.02083$$

$$\theta_1 = -\theta_2 = \frac{PL^2}{16EI} = 0.0625$$

$$\theta_2 = 0$$

Accounting for Distributed Loading

Many beam deflection problems involve not only concentrated forces and moments, but distributed loads. The following development extends the results presented above to account for uniform distributed loads.

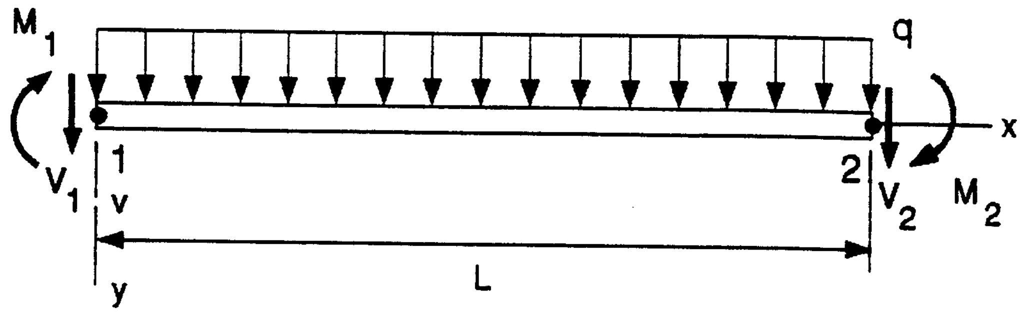

Fig. 8. Forces, Moments, and Distributed Load on Beam

The shear force and moment at the left end (end 1) are V_1 and M_1, respectively. Corresponding quantities at the right end (end 2) are V_2 and M_2. The load-deflection differential equation can be integrated sequentially to yield expressions for shear force, bending moment, slope, and deflection.

$$EI\frac{d^4v}{dx^4} = q$$

$$EI\frac{d^3v}{dx^3} = qx + c_1 = -V(x)$$

$$EI\frac{d^2v}{dx^2} = \frac{qx^2}{2} + c_1x + c_2 = -M(x)$$

$$EI\frac{dv}{dx} = \frac{qx^3}{6} + c_1\frac{x^2}{2} + c_2x + c_3$$

$$EIv = \frac{qx^4}{24} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3x + c_4$$

As before, the displacement and rotation at end 1 are denoted v_1 and θ_1, respectively. Corresponding quantities at end 2 are v_2 and θ_2. These four kinematic variables may be expressed in terms of the constants c_1, c_2, c_3, and c_4 using the equations above, as follows

$$
\begin{aligned}
v_1 &= v(0) = \frac{1}{EI}c_4 \\
\theta_1 &= \frac{dv}{dx}(0) = \frac{1}{EI}c_3 \\
v_2 &= v(L) = \frac{1}{EI}\left[\frac{qL^4}{24} + c_1\frac{L^3}{6} + c_2\frac{L^2}{2} + c_3 L + c_4\right] \\
\theta_2 &= \frac{dv}{dx}(L) = \frac{1}{EI}\left[\frac{qL^3}{6} + c_1\frac{L^2}{2} + c_2 L + c_3\right]
\end{aligned}
$$

These equations cn be written in matrix form as

$$
\begin{pmatrix}
0 & 0 & 0 & \frac{1}{EI} \\
0 & 0 & \frac{1}{EI} & 0 \\
\frac{L^3}{6EI} & \frac{L^2}{2EI} & \frac{L}{EI} & \frac{1}{EI} \\
\frac{L^2}{2EI} & \frac{L}{EI} & \frac{1}{EI} & 0
\end{pmatrix}
\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}
=
\begin{pmatrix} v_1 \\ \theta_1 \\ v_2 - \frac{qL^4}{24EI} \\ \theta_2 - \frac{qL^3}{6EI} \end{pmatrix}
$$

Solving this system of linear algebraic equations for c_1, c_2, c_3 and c_4 gives (the reader should verify)

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{-6EI}{L^2} & \frac{-4EI}{L} & \frac{6EI}{L^2} & \frac{-2EI}{L} \\ 0 & EI & 0 & 0 \\ EI & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 - \frac{qL^4}{24EI} \\ \theta_2 - \frac{qL^3}{6EI} \end{pmatrix}$$

The shear forces and bending moments at the two ends of the beam can be expressed in terms of the constants c_1, c_2, c_3, c_4, and the distributed load q. Then, using the result above, the shear forces and bending moments on the ends can be written in terms of the end displacements and rotations.

$$V(0) = -V_1 = -c_1$$

$$V_1 + \frac{qL}{2} = \frac{12EI}{L^3} v_1 + \frac{6EI}{L^2}\theta_1 - \frac{12EI}{L^3} v_2 + \frac{6EI}{L^2}\theta_2$$

$$M(0) = M_1 = -c_2$$

$$M_1 + \frac{qL^2}{12} = \frac{6EI}{L^2} v_1 + \frac{4EI}{L}\theta_1 - \frac{6EI}{L^2} v_2 + \frac{2EI}{L}\theta_2$$

(continued)

$$
\begin{aligned}
V(L) = V_2 &= -qL - c_1 \\
V_2 + \frac{qL}{2} &= -\frac{12EI}{L^3}v_1 - \frac{6EI}{L^2}\theta_1 + \frac{12EI}{L^3}v_2 - \frac{6EI}{L^2}\theta_2
\end{aligned}
$$

$$
\begin{aligned}
M(L) = -M_2 &= -\frac{qL^2}{12} - c_1 L - c_2 \\
M_2 - \frac{qL^2}{12} &= \frac{6EI}{L^2}v_1 + \frac{2EI}{L}\theta_1 - \frac{6EI}{L^2}v_2 + \frac{4EI}{L}\theta_2
\end{aligned}
$$

These equations can be expressed in matrix form,

$$
\frac{EI}{L^3}\begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix}\begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{pmatrix} + \frac{qL}{12}\begin{pmatrix} 6 \\ L \\ 6 \\ -L \end{pmatrix}
$$

This matrix equation contains one additional term as compared with the corresponding equation developed earlier. The term accounting for the distributed load q is called the vector of "equivalent nodal loads."

Examples – continued

(v.) Propped Cantilever with a Uniform Load

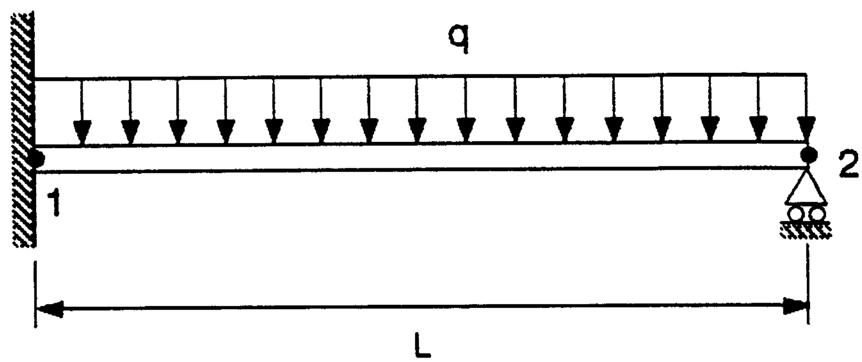

Fig. 9. Propped Cantilever with a Uniform Load.

BC's

The appropriate boundary conditions for the fixed and propped ends are

$$v_1 = \theta_1 = v_2 = 0$$

The 4 × 4 system of equations reduces to

$$\frac{4EI}{L}\theta_2 = M_2 - \frac{qL^2}{12}$$

External force and moment

The external moment at node 2 is

$$M_2 = 0$$

Thus, the solution is simply

$$\theta_2 = -\frac{qL^3}{48EI}$$

Reaction force and moment

The reaction force and moment at the nodes may be calculated using the original 4×4 system of equations,

$$V_1 = -\frac{5}{8}qL$$

$$M_1 = -\frac{qL^2}{8}$$

$$V_2 = -\frac{3}{8}qL$$

(vi.) Cantilever with a Uniform Load

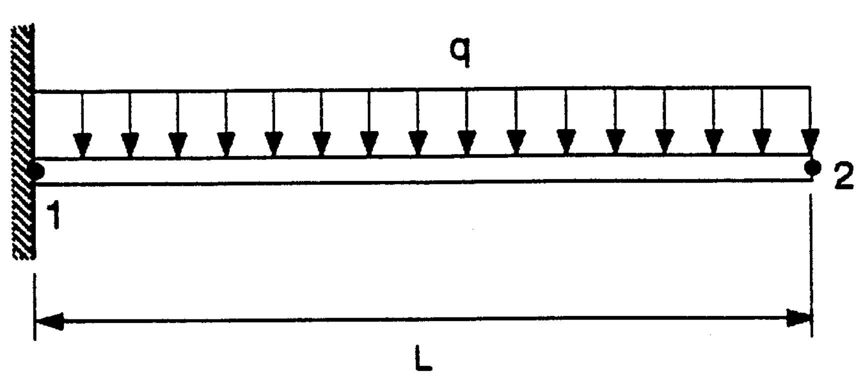

Fig. 10. Cantilever with a Uniform Load

BC's

The appropriate boundary conditions for the fixed end are

$$v_1 = \theta_1 = 0$$

The 4 × 4 system of equations reduces to

$$\frac{EI}{L^3}\begin{pmatrix} 12 & -6L \\ -6L & 4L^2 \end{pmatrix}\begin{pmatrix} v_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} V_2 \\ M_2 \end{pmatrix} + \frac{qL}{12}\begin{pmatrix} 6 \\ -L \end{pmatrix}$$

External force and moment

The external force and moment at node 2 are

$$V_2 = M_2 = 0$$

For the following data: $q = 0.1$, $L = 100$, $EI = 100{,}000$, the linear system of algebraic equations that must be solved is

$$\begin{pmatrix} 1.2 & -60 \\ -60 & 4000 \end{pmatrix}\begin{pmatrix} v_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 5.0 \\ -83.333 \end{pmatrix}$$

The solution is

$$\begin{pmatrix} v_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 12.5 \\ 0.16667 \end{pmatrix}$$

The theoretical solution for this problem is

$$v_2 = \frac{qL^4}{8EI} = 12.5$$

$$\theta_2 = \frac{qL^3}{6EI} = 0.16667$$

Reaction force and moment

The reader is encouraged to calculate the reaction force and moment and compare with simple statics.

(vii.) Cantilever with a Concentrated Moment and Partial Uniform Load

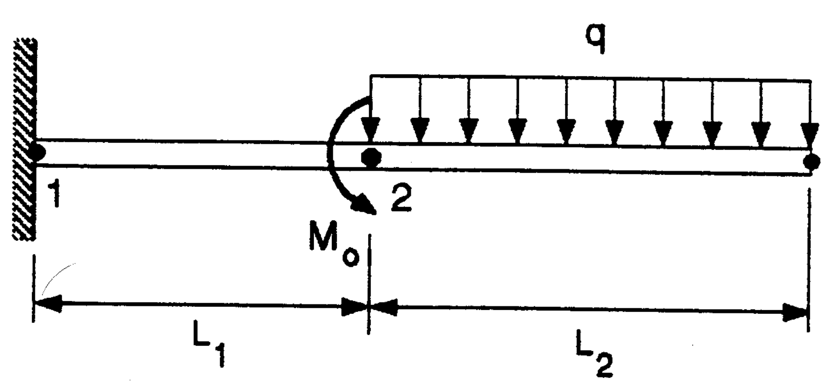

Fig. 11. Cantilever with a Concentrated Moment and Partial Uniform Load

The unconstrained system equations for this case are

$$\begin{pmatrix} k^1_{11} & k^1_{12} & k^1_{13} & k^1_{14} & 0 & 0 \\ k^1_{21} & k^1_{22} & k^1_{23} & k^1_{24} & 0 & 0 \\ k^1_{31} & k^1_{32} & k^1_{33}+k^2_{11} & k^1_{34}+k^2_{12} & k^2_{13} & k^2_{14} \\ k^1_{41} & k^1_{42} & k^1_{43}+k^2_{21} & k^1_{44}+k^2_{22} & k^2_{23} & k^2_{24} \\ 0 & 0 & k^2_{31} & k^2_{32} & k^2_{33} & k^2_{34} \\ 0 & 0 & k^2_{41} & k^2_{42} & k^2_{43} & k^2_{44} \end{pmatrix} \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} P_1 \\ M_1 \\ P_2 \\ M_2 \\ P_3 \\ M_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{qL_2}{2} \\ \frac{qL_2^2}{12} \\ \frac{qL_2}{2} \\ \frac{-qL_2^2}{12} \end{pmatrix}$$

The appropriate boundary conditions for the fixed end are

$$v_1 = \theta_1 = 0$$

External force and moment

The external forces and moments are

$$P_2 = P_3 = M_3 = 0$$
$$M_2 = -M_0$$

The constrained system equations are then

$$\begin{pmatrix} k^1_{33}+k^2_{11} & k^1_{34}+k^2_{12} & k^2_{13} & k^2_{14} \\ k^1_{43}+k^2_{21} & k^1_{44}+k^2_{22} & k^2_{23} & k^2_{24} \\ k^2_{31} & k^2_{32} & k^2_{33} & k^2_{34} \\ k^2_{41} & k^2_{42} & k^2_{43} & k^2_{44} \end{pmatrix} \begin{pmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} \frac{qL_2}{2} \\ -M_0 + \frac{qL_2^2}{12} \\ \frac{qL_2}{2} \\ \frac{-qL_2^2}{12} \end{pmatrix}$$

For a given set of data, these equations can be solved for v_2, θ_2, v_3 and θ_3.

Reaction force and moment

Reaction forces and moments could be calculated after the displacements and rotations are computed for a given set of data.