On a hierarchy of Borel additive subgroups of reals

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February 2009

1 Introduction

In his "A course of pure mathematics", G. H. Hardy considers the following limit (Example XXIV.14 in [1]):

$$\lim_{n\to\infty} \sin(n!\pi x)$$

He remarks that when x is rational, this limit is 0. Let G be the set of reals where this limit is 0. It is easily verified that G is an additive subgroup of $\mathbb{R}$. It is also not hard to see that Euler's constant e is also in G. Going a little further, one can show the following.

Lemma 1.1. *Suppose $x \in [0,1]$. Then x has a unique representation of the form*

$$x = \sum_{n\geq 2} \frac{x_n}{n!}$$

where $x_n \in \{0, 1, \ldots, n-1\}$. Under this representation, $x \in G$ iff $\lim_{n\to\infty} \frac{x_n}{n}$ is either 0 or 1.

Proof: Let $d(x,\mathbb{Z})$ denote the distance of x from the set of integers. First notice that for any sequence of reals $\langle x_n : n \geq 1\rangle$, $\lim_{n\to\infty} \sin(\pi x_n) = 0$ iff $\lim_{n\to\infty} d(x_n,\mathbb{Z}) = 0$. For any $x \in [0,1)$ with x_n's as above, $d(n!x,\mathbb{Z}) = d(b_n,\mathbb{Z})$, where $b_n = \frac{x_{n+1}}{n+1} + \varepsilon_n$ where $0 \leq \varepsilon_n < 1/n$. Hence if $\frac{x_n}{n} \to 0$ or 1, then $d(b_n,\mathbb{Z}) \to 0$. Conversely suppose, along some subsequence n_k, $\lim_{k\to\infty} \frac{x_{n_k}}{n_k} = a$ where $0 < a < 1$. Then for all large enough k, $d((n_k-1)!x,\mathbb{Z})$ is arbitrarily close to a so that $\sin(n!\pi x)$ is bounded away from zero on this subsequence.

2 A true $\mathbf{\Pi}^0_3$ group

As we remarked above, G can also be described as follows:

$$G = \{x \in \mathbb{R} : \lim_{n\to\infty} d(n!x,\mathbb{Z}) = 0\}$$

In this section, we'll show that G is a true $\mathbf{\Pi}^0_3$ additive subgroup of $\mathbb{R}$. Recall the definition of the pointclasses $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$:

- $\mathbf{\Sigma}^0_1$ is the family of open subsets of $\mathbb{R}$, $\mathbf{\Pi}^0_1$ is the family of closed sets.

- for each $1 < \alpha < \omega_1$, a set X is in $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) iff it is the union (resp. intersection) of a countable subfamily of $\bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta$ (resp. $\bigcup_{\beta<\alpha} \mathbf{\Sigma}^0_\beta$).

Remark: We never use effective versions of these pointclasses hence all our pointclasses are boldface even if they don't look bold.

One can also consider these pointclasses over other Polish spaces (separable completely metrizable spaces) like 2^ω (Cantor space), ω^ω (Baire space). A set in $\mathbf{\Pi}^0_\alpha$ is a true $\mathbf{\Pi}^0_\alpha$ set if it is not in $\mathbf{\Sigma}^0_\alpha$. A true $\mathbf{\Sigma}^0_\alpha$ set is defined similarly. We begin by recalling some basic facts about Wadge reductions. In what follows, all Polish spaces are uncountable and therefore the Borel hierarchies on such spaces do not terminate at any countable level.

Let $f : X \to Y$ be a continuous map between Polish spaces and let $A \subseteq X$ and $B \subseteq Y$. We write $f : (X, A) \xrightarrow{\text{Wadge}} (Y, B)$ (read "f Wadge reduces A to B") if $f^{-1}[B] = A$. The following are easily proved.

Lemma 2.1. *If $f : (X, A) \xrightarrow{\text{Wadge}} (Y, B)$ where B is $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) in Y then A is $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) in X.*

Lemma 2.2. *If $f : (X, A) \xrightarrow{\text{Wadge}} (Y, B)$ and A is true $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) in X, and if B is $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) in Y then B is true $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) in Y.*

Let A be a $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) set in X. We say A is $\mathbf{\Sigma}^0_\alpha$-complete (resp. $\mathbf{\Pi}^0_\alpha$-complete) if for every $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) set B in any Polish space Y there is a Wadge reduction $f : (Y, B) \xrightarrow{\text{Wadge}} (X, A)$. The following result in Wadge's thesis shows that any two true Σ^0_α (resp. Π^0_α) sets in Cantor space Wadge reduce to each other.

Theorem 2.3 (Wadge). *A subset A of 2^ω is Σ^0_α-complete if A is true Σ^0_α in 2^ω.*

Some examples follow.

- Any countable dense subset D of an uncountable Polish space X is $\mathbf{\Sigma}^0_2$-complete. E.g., $Q = \{x \in 2^\omega : \lim_{n\to\infty} x(n) = 0\}$. The trueness of D follows by Baire category theorem. Hence by Wadge's theorem, D is $\mathbf{\Sigma}^0_2$-complete.

- Let $\langle , \rangle : \omega^2 \to \omega$ be a pairing function; for example, $\langle m, n\rangle = \frac{1}{2}(m+n+1)(m+n)+n$. Let $P = \{x \in 2^\omega : \forall m(\lim_{n\to\infty} x(\langle m, n\rangle) = 0\}$. Then P is $\mathbf{\Pi}^0_3$-complete.

Proof: It is clear that P is $\mathbf{\Pi}^0_3$. To show that P is $\mathbf{\Pi}^0_3$-complete, fix an arbitrary $\mathbf{\Pi}^0_3$ set A in a Polish space X. Let $A = \bigcap_{n\in\omega} A_n$, where A_n's are $\mathbf{\Sigma}^0_2$. Since Q is $\mathbf{\Sigma}^0_2$-complete, there are Wadge reductions $f_m : (X, A_n) \xrightarrow{\text{Wadge}} (2^\omega, Q)$. Let $F : (X, A) \xrightarrow{\text{Wadge}} (2^\omega, P)$ be given by $F(x)(\langle m, n\rangle) = f_m(n)$.

Theorem 2.4. *The set $G = \{x \in \mathbb{R} : \lim_{n\to\infty} d(n!x, \mathbb{Z})) = 0\}$ is a true $\mathbf{\Pi}^0_3$ additive subgroup of reals.*

Proof: It is clear that G is an additive subgroup of reals and that it is $\mathbf{\Pi}^0_3$:

$$x \in G \Leftrightarrow \forall \varepsilon > 0(\exists n_0(\forall n \geq n_0(d(n!x, \mathbb{Z}) \leq \varepsilon)))$$

It suffices to construct a Wadge reduction from $P = \{x \in 2^\omega : \forall m(\lim_{n\to\infty} x(\langle m, n\rangle) = 0\}$ to G. Let $f : 2^\omega \to \mathbb{R}$ be defined as follows. Given $x \in 2^\omega$, let $y_x : \omega \to \omega$ be defined by letting $y_x(n)$ to be the least index $m < n$ such that $x(\langle m, n\rangle) = 1$. In case no such $m < n$ exists, we let $y_x(n) = n$. It is clear that the function $x \mapsto y_x$ is continuous and for every $x \in 2^\omega$, $x \in P \Leftrightarrow \lim_{n\to\infty} y_x(n) = \infty$. Set $f(x) = \sum_{n\geq 2} \frac{a_n}{n!}$, where $a_n = \lfloor \left(\frac{n}{2 + y_x(n)}\right) \rfloor$, and $\lfloor x \rfloor$ denotes the greatest integer not greater than x. Now if $\lim_{n\to\infty} y_x(n) = \infty$, then $\lim_{n\to\infty} \frac{a_n}{n} = 0$, hence $f(x) \in G$. On the other hand, if $x \notin P$, then along some subsequence $\langle n_k : k \in \omega\rangle$, $y_x(n_k)$ is constant so that $\frac{a_{n_k}}{n_k}$ does not go to either 0 or 1.

3 A few more groups

Let $G_0 = G$, $G_{k+1} = \{x \in \mathbb{R} : \lim_{n\to\infty} d(n!x, \mathbb{Z}) \in G_k\}$. Then one can easily check that, for each $k \in \omega$, G_k is an additive subgroup of $\mathbb{R}$. Next we show that

Lemma 3.1. *G_k is $\mathbf{\Pi}^0_{k+3}$.*

Proof: Let $W = \{x \in \mathbb{R} : \lim_{n\to\infty} d(n!x, \mathbb{Z}) \text{ exists}\}$. Then W is $\mathbf{\Pi}^0_3$ since

$$x \in W \Leftrightarrow \forall \varepsilon > 0(\exists n_0(\forall m, n \geq n_0(|d(m!x, \mathbb{Z}) - d(n!x, \mathbb{Z})| \leq \varepsilon)))$$

Let $h : W \to \mathbb{R}$ be defined by $h(x) = \lim_{n\to\infty} d(n!x, \mathbb{Z})$. For every open interval (a, b), and for every $x \in W$,

$$h(x) \in (a, b) \Leftrightarrow \exists n_0 \forall n \geq n_0(d(n!x, \mathbb{Z}) \in [a + 1/n, b - 1/n])$$

This implies that for every open set U, $h^{-1}[U]$ is the intersection of a $\mathbf{\Sigma}^0_2$ set with W. This implies that, if $G_k \in \mathbf{\Pi}^0_{k+3}$, then $G_{k+1} = h^{-1}[G_k]$ is the intersection of a $\mathbf{\Pi}^0_{k+4}$ set with W hence is also $\mathbf{\Pi}^0_{k+4}$.

In the remaining part of this section we will show that G_k is a true $\mathbf{\Pi}^0_{k+3}$ set. First we need a nice family of $\mathbf{\Pi}^0_k$-complete sets for $k \geq 3$. The following construction appears in [**?**].

Let $\phi : 2^\omega \to 2^\omega$ be defined by $\phi(x)(m) = 1$ iff $\forall n(x(\langle m, n\rangle)) = 0$. Extend ϕ to $2^{\leq\omega}$ by defining $\phi(\sigma) = \phi(\sigma\overline{0})$, where $\sigma \in 2^{<\omega}$ and $\sigma\overline{0}$ is σ followed by 0's. Note that although ϕ is not continuous, (e.g., $0^n\overline{1}$ converges to $\overline{0}$, $\phi(0^n\overline{1}) = \overline{0}$ does not converge to $\phi(\overline{0}) = \overline{1}$), $\phi(x \restriction n)$ does converge to $\phi(x)$. Let $H_1 = \{\overline{0}\}$, $H_{k+1} = \phi^{-1}[H_k]$. Then H_k is $\mathbf{\Pi}^0_k$-complete.

Theorem 3.2. *For every $k \in \omega$, G_k is a true $\mathbf{\Pi}^0_{k+3}$ additive subgroup of reals.*

Proof: When $k = 0$, this was proved above. Suppose $f : (2^\omega, H_{k+3}) \xrightarrow{\text{Wadge}} (\mathbb{R}, G_k)$, where H_{k+3} is the $\mathbf{\Pi}^0_{k+3}$-complete set defined above. For $x \in 2^\omega$, let $a_n = f(\phi(x \restriction n))$ and

$a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} f(\phi(x \restriction n)) = f(\lim_{n\to\infty} \phi(x \restriction n)) = f(\phi(x))$. Put $b_n = \lfloor n(d(a_n, \mathbb{Z}))\rfloor$ and define $g : 2^\omega \to \mathbb{R}$ by $g(x) = \sum_{n\geq 2} \frac{b_n}{n!}$. Then g is continuous. Also $g(x) \in G_{k+1}$ iff $\lim_{n\to\infty} \frac{b_n}{n} = \lim_{n\to\infty} d(a_n, \mathbb{Z}) = d(a, \mathbb{Z}) \in G_k$. Hence $x \in H_{k+4} \Leftrightarrow \phi(x) \in H_{k+3} \Leftrightarrow f(\phi(x)) \in G_k \Leftrightarrow a \in G_k \Leftrightarrow d(a, \mathbb{Z}) \in G_k \Leftrightarrow g(x) \in G_{k+1}$.

As a corollary, $G_\omega = \bigcup\{G_k : k \in \omega\}$ is in $\mathbf{\Sigma}^0_\omega$ but not in $\mathbf{\Sigma}^0_k$ for any $k \in \omega$ since $G_\omega = \{x \in \mathbb{R} : \lim_{n\to\infty} d(n!x, \mathbb{Z}) \in G_\omega\}$.

References

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[3] W. Wadge: Reducibility and determinateness on the Baire space, PhD thesis Univ. of California Berkeley (1983)