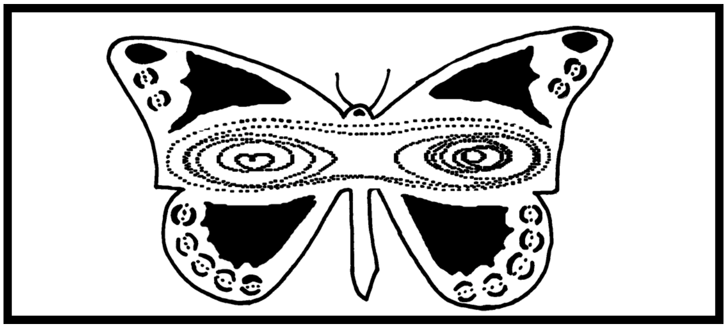
Name ______________________________ Section ____________ Date ____________

UNIT 15: OSCILLATIONS, DETERMINISM, AND CHAOS

Camels capable of carrying large loads brought exotic spices from Asia and the Middle East to Europe in ancient times. What happens to a camel's ability to carry goods as its load increases? Obviously a lightly loaded camel will not seem to notice if a mere straw is added to its load. But then where do expressions such as "that's the last straw" or "that's the straw that broke the camel's back" come from? One moment a camel is bearing its almost unbearable load and in another it has collapsed. Physicists would say that when a camel is heavily loaded its response is incredibly sensitive to small changes in load. In this unit we are going to study applications of a new interdisciplinary science, chaos, to some physical systems which can become so sensitive to initial conditions that they undergo strange and unpredictable motions even when the forces involved are completely understood.

UNIT 15: OSCILLATIONS, DETERMINISM, AND CHAOS



The modern study of chaos began with the creeping realization in the 1960s that quite simple mathematical equations could model systems every bit as violent as a waterfall. Tiny differences in input could quickly become overwhelming differences in output – a phenomenon given the name "sensitive dependence on initial conditions." In weather, for example, this translates into what is only half-jokingly known as the Butterfly Effect – the notion that a butterfly stirring the air today in Peking can transform storm systems next month in New York.

James Gleick (1987)
Chaos: Making a New Science, p. 8

OBJECTIVES

1. To investigate the phenomenon of chaos and some of its applications to dynamical systems in the natural sciences.

2. To show how the time series and phase diagram representations of a dynamical system can be used to display its behavior.

3. To explore how mathematical iterations based on Newton's second law can be used to model the behavior of any dynamical system for which the forces or interactions involved are known.

4. To recognize that a dynamical system can behave unpredictably even when the forces and interactions that govern the system are well understood.

5. To investigate some of the conditions under which a system can behave in an unpredictable and possibly chaotic manner.

AN INTRODUCTION TO CHAOS

15.1. DYNAMICAL SYSTEMS AND CHAOTIC BEHAVIOR

This unit is about the study of both simple and complex dynamical systems. A *dynamical system* is defined as a system that changes over time in such a way that the state of the system at one moment in time determines the state of the system at the next moment. For example, if we know the displacement and velocity of a mass oscillating on a spring and we also know the spring constant, we can find an analytic equation that allows us to calculate the displacement and velocity of the mass at any time in the future.

So far we have studied the motions of elements of relatively simple dynamical systems by developing and using Newton's laws. The most important of these is the second law, which states that an object's acceleration is determined by the net force on the object divided by its mass. This relationship can be symbolized by the equation

$$\vec{a} = \frac{\vec{F}^{\text{net}}}{m} \qquad (15.1)$$

In this unit you will be asked to study some conditions under which systems moving under the influence of known forces move in predictable ways. Then you will consider some conditions under which other systems that also move under the influence of known forces move in *unpredictable* ways.

Predictable Dynamical Systems

In principle, if we believe in the validity of the laws of motion and if the mass, position, and velocity of every particle in the universe and the nature of the forces of interaction between them were known, you could calculate the mass, position, and velocity of every particle in the universe at any time in the future. The ability to predict the motions of simple systems using Newton's laws of motion led some philosophers to develop a mechanistic view of nature that is commonly known as Laplacian determinism (after Pierre Laplace, an eighteenth-century French philosopher). According to Laplace,

> If an intellect were to know . . . all the forces that animate nature and the conditions of all the objects that compose her, and were capable of subjecting these data to analysis, then this intellect would encompass in a single formula the motions of the largest bodies in the universe as well as those of the smallest atom; and the future as well as the past would be present before its eyes.

15.1.1. Activity: Laplacian Determinism

Consider complex entities in the universe, including humans, computers, sun, rain, tides, and galaxies. Suppose that you could know the mass, shape, position, and velocity of every object in the universe to eight significant figures, how the forces and torques between them depend on these four quantities, and that the universe is governed only by Newton's laws of motion. How well could you predict the future? Explain.

Complex Dynamical Systems

Many complex dynamical systems, such as global weather patterns, behave in an irregular and unpredictable way over time. Certain unpredictable dynamical systems have recently been labeled by mathematicians as *chaotic*. The applications of the mathematical concept of Chaos are less than thirty years old. Chaos scientists are concerned with the analysis of unpredictable dynamical systems in which the forces determining the motions of the system elements are internal to the system. Such systems, though unpredictable, are still referred to as *deterministic* because they depend only on the state of the system from one moment to the next. The forces are not hard to determine or uncontrollable like the breezes acting on the leaf shown in Figure 15.1(a).

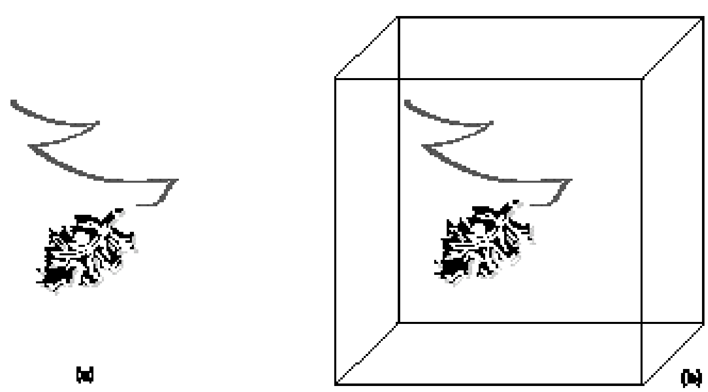


Fig. 15.1. An open dynamical system vs. a closed deterministic dynamical system. (a) A fluttering leaf whose motions are subject to external uncontrollable forces in the form of breezes. (b) A fluttering leaf falling in a closed container in which, in principle, all the forces of interaction between the leaf, the air, and the walls of the container, and the Earth's gravitational pull are known.

A fluttering leaf falling in a closed container under the influence of known forces as shown in Figure 15.1(b) is an example of a deterministic yet unpredictable dynamical system. The dynamical systems of interest to those studying Chaos, such as insect population cycles, the stock market, chemical reactions, global weather systems, or the time evolution of clusters of galaxies, are even more complex and unpredictable.

Many real systems display chaotic, or unpredictable, behavior in which tiny differences in the initial conditions of a system produce changes in its subsequent behavior that grow exponentially in time. This "sensitive dependence on initial conditions" is one hallmark of chaotic behavior. This sensitivity only occurs when the forces on a system are non-linear, and it was first described by the mathematician Henri Poincar_ (1852–1912).

> It may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible . . .

Poincar_ was unable to verify his assertion theoretically because without computers he was unable to perform calculations to describe complex systems.

More recently, a meteorologist, Edward Lorenz, "rediscovered" chaos while developing a model for forecasting weather. Using the method of *iterations* on a digital computer to help him solve complex equations numerically, Lorenz discovered by accident that tiny changes in atmospheric conditions could give rise later to large differences in predicted weather conditions. He referred to this as "The Butterfly Effect," and postulated whimsically that the beating of a butterfly's wings in one part of the world could cause after a month or so a storm halfway around the earth.

Creating an Complex Dynamical System

Why are the motions of some dynamical systems fairly predictable while the motions of others are very unpredictable? A typical physicist would approach this question by experimenting with relatively simple dynamical system and adding complications until the systems moves unpredictably. You will be asked to take this approach.

You will begin this unit by learning about some general aspects of the science of chaos on a layperson's level. Then you will begin a more scientific study of chaos by exploring the behavior of a pendulum system. What happens as you make your dynamical system more and more complex? In each case you should consider the following questions:

1. If you know the initial rotational positions and velocities of the elements of a pendulum system and the forces, how well can you predict the future motion of the system? Could it be a chaotic system?
2. Suppose the universe were made up of entities like a pendulum system that have elements that move in the presence of known forces. If we had enough large high-performance computers to use Newton's laws of motion to calculate the position of these elements at each time in the future, could Laplacian determinism exist?

To help you understand the motions you are studying you will learn to display position and velocity data that you take graphically in both time series and phase plot format. You will also learn to use the technique of computer iteration in conjunction with the rotational form of Newton's second law to create theoretical time series graphs and phase plots. Mathematical modeling using computer iterations is an extremely popular and powerful technique for studying many physical systems of interest in contemporary physics research and in engineering.

15.2. THE NOVA VIDEO ON THE STRANGE SCIENCE OF CHAOS

If it's available, you should view a videotape of an hour-long Nova program produced in 1988 entitled *The Strange New Science of Chaos*. This video shows many examples of chaotic systems of interest in different fields of study. It also provides you an overview of the emerging techniques for studying chaotic systems.

One of the common techniques for studying a chaotic system is to create a *phase plot* depicting changes on systems variables such as position and velocity. Phase plots of chaotic systems often look like strange figure eights. Since you will be creating phase plots of a couple of physical systems later in this unit, you might want to look for their depiction in the videotape. Thus, you will need the following videotape:

- 1 Nova video: *The Strange New Science of Chaos*[*]

Recommended Group Size:	All	Interactive Demo OK?:	Y

After you watch the tape you will be asked to answer the questions in Activity 15.2.1. You should keep these questions in mind as you watch the video.

15.2.1. Activity: Summarizing the Nova Video

a. This video was produced for a general audience, so chaotic behavior will not be defined in a technical manner as we did in section 15.1. How do the Nova producers define chaotic behavior?

b. List several examples of chaotic systems presented in the Nova video.

[*]Although this video is no longer available for purchase, a number of university libraries own it, and can make it available through interlibrary loan.

c. Can you think of any other systems of interest that might be chaotic?

15.3. CHAOTIC MOTION AND SENSITIVITY TO INITIAL CONDITIONS

The Simple Pendulum

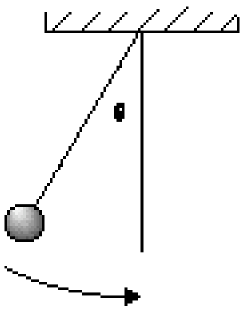
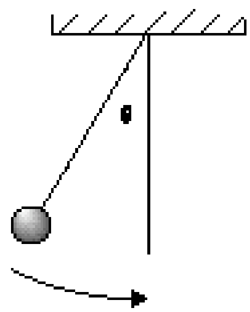


Fig. 15.2.

A steel paper clip hanging from a thread forms a simple pendulum that oscillates in a reproducible manner. What, theoretically, is the mathematical form of the force on the pendulum for such a reproducible motion? Answering this question is a good way to review the construction of a force diagram and the application of Newton's second law.

15.3.1. Activity: The Net Force on a Pendulum Bob

a. In the diagram below use labeled arrows to show the direction of the gravitational force, $\vec{F}^{\text{grav}}$, on the pendulum bob and the direction of the force due to the tension on the string, $\vec{F}^{\text{tens}}$. The tension force magnitude is the same as the magnitude of the gravitational force component parallel to the string. Why?



b. Since a pendulum bob only moves perpendicular to its supporting string, it is the *gravitational force component* perpendicular to the string that exerts a restoring force on the bob. What is the equation for the component of the net restoring force, $\vec{F}^{\text{net}}$, perpendicular to the direction of the string? Express this equation as a function of m, g, and θ.

$F_{\perp}^{\text{net}} =$

c. Suppose the amplitude of the pendulum is small (so that $\sin\theta \approx \theta$). Show that the perpendicular component of the unbalanced force, $F_{\perp}^{\text{net}}$, leading to changes in the pendulum bob's motion is given by $F_{\perp}^{\text{net}} \approx -mg\sin\theta$. **Note:** When the amplitude is small so that $F_{\perp}^{\text{net}} \approx mg\theta$, then a graph of $F_{\perp}^{\text{net}}$ vs. θ would be a straight line. In this case, the net restoring force is a linear function of θ.

Two aspects of the equation you just derived contribute to the fact that the simple pendulum is a predictable device: (1) For small amplitudes of oscillation the net force on the pendulum bob is a *linear function of its rotational displacement*, and (2) *the bob's motion can be completely specified by only two variables*—rotational velocity and rotational displacement.

The Magnetic Pendulum with Non-linear Forces

If small magnets are placed beneath a pendulum that has a steel paper clip as a bob, things are not so simple. The paper clip now experiences forces from the tension on the thread, gravity, and each of the magnets. In theory, if we understand the magnetic forces associated with each of the magnets, we can use Newton's laws to predict the motion of the clip. In practice, the motion may be chaotic because its motion can depend critically on the release of the clip, which is hard to reproduce exactly because of experimental uncertainty.

Your goal in this activity is to see if you can arrange the magnets below the paper clip pendulum in such a way that the subsequent motion of the paper clip is very sensitive to its initial position and velocity at the time of release. To do this activity you will need:

- 1 small steel paper clip (1.25" long)
- 1 thread (approx. 1 m)
- 2 bases
- 3 support rods
- 2 right angle clamps
- 3 small disk magnets
- 1 sheet of paper
- 3 pieces of Scotch tape (to affix magnets to the paper)

Recommended Group Size:	3	Interactive Demo OK?:	N

One possible setup is shown in the following diagram. Feel free to experiment with other arrangements of the magnets. The length of the thread should be adjusted so the clip can pass as close as possible to the magnets without touching them. The magnets can be placed several centimeters apart.

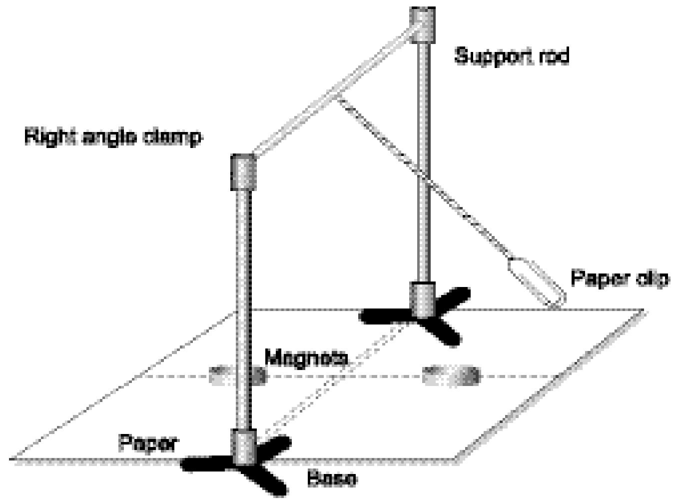


Fig. 15.3. The magnetic pendulum.

15.3.2. Activity: Magnetic Pendulum—Sensitivity

a. Set up the magnetic pendulum and play around with it. Can you find a set of initial conditions (that is, a point of release and an initial velocity) that, when repeated, leads to a reproducible motion? If you find something, show a sketch of your arrangement and the release point. Describe your initial velocity.

b. Describe an initial position (i.e., point of release) and initial velocity for the same setup that, when repeated, leads to a very unreproducible motion when the clip is released under the "same" initial conditions.

c. Take the magnets away and release the clip several times exactly as you had in part b. above. Describe the motion. Is the motion reproducible or unreproducible?

It turns out to be possible to measure both the gravitational and magnetic forces on the paper clip pendulum bob as a function of the location of the bob and its velocity. However, the net forces on the bob change in complex ways as the position and velocity of the bob change. These net forces cannot be described by a linear mathematical relationship.

About the Rest of This Unit

In Unit 14 and in the first part of this unit you studied the behavior of a simple pendulum consisting of a small mass attached to a string of negligible mass. In general, any pivoted object with an extended shape that oscillates naturally when displaced from equilibrium is known as a *physical pendulum*.

In the remaining activities in this unit you are going to work with a *physical pendulum* consisting of a mass mounted on the edge of a disk. Specifically, you are going to study the behavior of this physical pendulum both experimentally and theoretically as the forces you impress on it become progressively more complicated. Eventually, by driving the pendulum with a force that varies sinusoidally in time in the presence of drag forces, you will literally drive your pendulum crazy.

In the experimental phase of this pendulum study you will be introduced to the use of two different types of graphical representations of the motion of a dynamical system, the *time series graph* and the *phase plot*.

In the theoretical phase of this pendulum study you will be introduced to a general technique for solving equations that describes motions from any set of known forces, even when analytic motion equations are difficult or impossible to derive. The application of this general technique, which is called *iteration* or *numerical integration*, lies at the heart of analyzing complex, often chaotic, dynamical systems. Thus, learning how to use iterative numerical methods to describe motion is an essential part of the theoretical study of the complex systems considered in this unit.

As you study the pendulum motion you should ask two related questions: Is it the same when it is started in the same way? How sensitive is the motion to small changes in initial conditions?

LARGE ANGLE PENDULUM OSCILLATIONS

15.4. PHYSICAL PENDULUM OSCILLATIONS—EXPERIMENTAL

Since we would like you to study the behavior of a pendulum at rotational displacements of more than 90°, the mass must be attached to a rigid support. It is also helpful to eliminate large forces on the pendulum pivot during large angle oscillations. The system you are going to work with consists of a mass mounted on the edge of a disk like that shown in Figure 15.4b. Thus, this system is no longer a simple pendulum, but rather a physical pendulum.

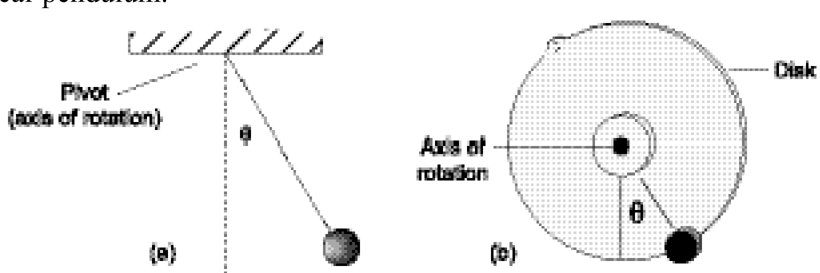


Fig. 15.4. (a) A simple pendulum with a point mass as a bob and **(b)** a physical pendulum consisting of an *edge mass* attached to a disk of radius *R*. An axle through the center of the disk is free to rotate on low-friction bearings.

In the next activity you will collect data on the changes in rotational position of the disk over time as it oscillates. Then the changes in the values of rotational position can be calculated to determine the rotational velocity of the edge mass at each time of interest. You will then use two methods of graphical data display: the familiar *time series graph* and a new type of graph known as the *phase plot*. Both of these graph types are commonly used in the study of dynamical systems.

For the activities in this section you will need a physical pendulum that can oscillate with a period of one second or more and a computer-based laboratory system that uses a rotary motion sensor to measure the rotational position of the disk as a function of time. You will need:

- 1 aluminum disk mounted on an axle (approx. 4” in dia. × 1/4” thick and threaded holes on its edge for added mass)
- 1 brass bolt , 1”, (to mount at the edge of the disk as a mass)
- 6 brass nuts (to add edge masses between 5 g and 20 g)
- 1 computer data acquisition system
- 1 rotary motion sensor
- 1 table clamp or rod base
- 1 rod, 20” long (to mount the rotary motion sensor)
- 1 small screw (to couple the disk axle to the rotary sensor)
- 1 electronic scale
- 1 ruler
- 1 stopwatch

Recommended Group Size:	3	Interactive Demo OK?:	N

A schematic of the physical pendulum is shown below.

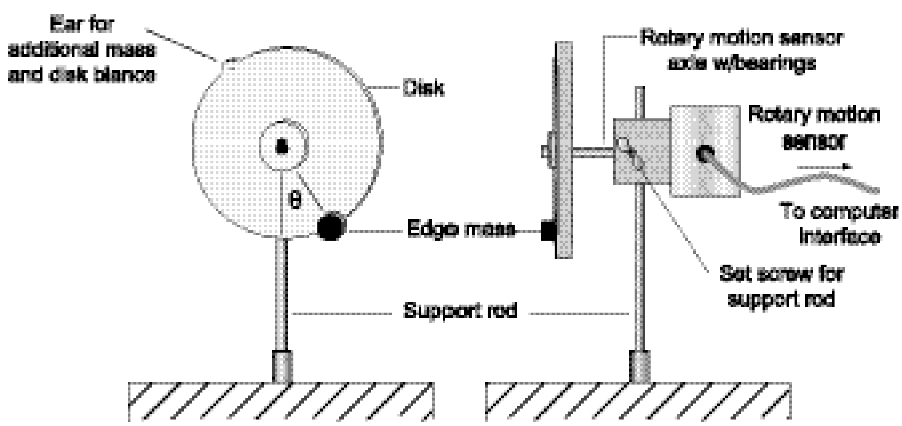


Fig. 15.5. A physical pendulum disk coupled to a rotary motion sensor. A disk and mass are mounted about an axis through the center of the disk on low-friction bearings that are attached to a rotary motion sensor. A version of the physical pendulum is available from PASCO scientific Co.

Note: The *rotary motion sensor* consists of an internal disk with lines scribed on it that rotates. The passing of the lines and their direction of motion are sensed by a pair of photogates. Signals from the sensor are transferred to the computer to record rotational motion. The software used with the rotary motion sensor is similar to that used with other sensors.

Observing the Behavior of the Physical Pendulum

Your physical pendulum might behave differently than a simple pendulum. Before taking data on its rotational position relative to its equilibrium position, you should play around with the system and contrast its behavior to what you should have learned already about simple pendulum behavior.

15.4.1. Activity: Observing System Oscillations

a. Suppose you mount the disk on an axle attached to low-friction bearings without adding a mass to one of the two edge holes. What do you predict will happen to the disk if you rotate its bottom edge through some initial angle θ_1 from its equilibrium position and release it? Will it oscillate or not? Explain the reasons for your prediction.

b. Set up the disk on its axis of rotation without the mass on the edge and observe what happens when you displace the disk through some angle and release it. How did the motion you observed compare with your prediction?

c. Suppose you add about 10 g to one of the edge holes (a brass bolt with 4 nuts is about right). What do you predict with happen to the disk if you rotate it through some angle θ from its equilibrium position and release it? Will it oscillate? Explain the reasons for your prediction.

d. Displace the mass through some angle and release it. Describe the resulting motion. How did the motion you observed compare with your prediction? You have created a physical pendulum!

e. Describe what happens to the rate of oscillation when you add more mass to the same place on the edge. (Don't add mass to the opposite edge!)

The nature of the motion of this type of disk/mass physical pendulum and that of a simple pendulum mounted on a thin rigid rod with negligible mass are quite similar in most respects. A surprising characteristic of the ideal simple pendulum is that its period of oscillation does not depend on its mass. This is because both the torque on the mass due to the gravitational force causing rotation and its rotational inertia resisting rotation are both proportional to the same mass.

You should have observed that the pendulum oscillates more rapidly as you add mass to the edge. As you study the motions of this physical pendulum in more detail, it might be useful to remember that the pendulum oscillates more rapidly when the amount of mass placed on the edge is increased. This reduction in the period occurs because even though the torque on the pendulum causing rotation is still proportional to the edge mass, the pendulum's rotational inertia depends on the rotational inertia of both the edge mass and the disk.

Measuring the Physical Pendulum Motion

As you may recall, the amplitude of a pendulum is the maximum rotational displacement from equilibrium. In Unit 14 you showed mathematically that at small amplitudes a simple pendulum ought to oscillate sinusoidally with simple harmonic motion just like a mass on a light spring. You will observe the motion of the disk pendulum at large amplitudes like those you will observe later when you endeavor to "drive the pendulum crazy." Thus, you should start your study of the physical pendulum by measuring the rotational displacement of the edge mass as a function of time for large angles using a computer-based rotary motion sensor. There are three different purposes for doing a study at large and small angles. We'd like you to:

1. Practice using *phase plot* and *time series* graphs to represent these motions, because these representations are used commonly by other scientists studying chaos and you should become familiar with them.
2. Compare the period of oscillation and the time series graph shapes for oscillations at large and small angles.
3. Obtain data that can be compared to mathematical models you will develop using spreadsheets in the next few sections of this unit.

Time Series Graphs

A time series graph is simply a plot of variables such as position or velocity as a function of time. You have been recording time series graphs all along, but we haven't bothered to define them as such.

In tracking the motion of the physical pendulum there are some inevitable uncertainties in the initial rotational position and velocity of the pendulum mass that will lead to uncertainty in the subsequent motion of the system. How reproducible is the pattern of motion? To answer this question, you can create two or more time series plots of the rotational position of the physical pendulum mass released in the same manner at a fairly large angle.

You can use the computer data acquisition system outfitted with a rotary motion sensor to record and display the rotational displacement of the pendulum mass as a function of time. You should create two plots in which the mass is started off in as close to the same way as possible. Figure out a way to estimate for what period of time the two series of motions are more or less alike. How long does it take for the uncertainties to accumulate so you can no longer predict the rotational position and velocity of the pendulum mass fairly accurately?

15.4.2. Activity: Motion Graphs and Reproducibility

a. Create two "identical" time series graphs of the rotational position of the physical pendulum bob over as many cycles as you can display clearly. (Use the experiment file L150402 or set up your own.) You should *release the mass from the same angle for each run as close to the same way as possible*. Affix an overlay graph of the two nearly identical releases in the space below. Cover up the hints if needed.

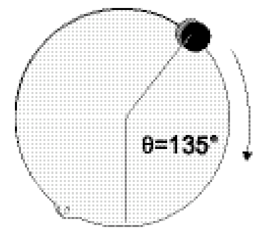


Fig. 15.6. Suggested angle for edge mass release at zero rotational velocity.

General Hints:

1. You should set the vertical scale to read in degrees (rather than revolutions or radians).
2. A release angle of +135° works quite well.
3. A run time to about 30 s at a data rate of 20 points/s is adequate.
4. Practice getting your release just right so the mass has close to the same rotational position and no rotational velocity each time it is released.

Problems Starting Two Motions the Same way?

1. If your software has a trigger mode, set the trigger to start when release angle falls below 130°.
2. If your software automatically interprets the first rotational position reading as zero, then:
   a. Let the edge mass hang vertically until the first data point is taken.
   b. Lift it quickly but steadily and release it. Take the same time to do this in each run.
   c. Obviously, ignore pre-release data on the graphs.

b. Were the two motions essentially the same over the full 30 seconds? If not, over what period of time are the two motions predictable? What technique did you use for determining this result? Please describe it!

Phase Plots

The Nova video on Chaos is full of jazzy looking unmarked elliptical traces flying around on the video screen. These are *phase diagrams* that were included by Nova producers to dazzle, but not illuminate, lay audiences. These strange looking phase diagrams are actually very useful in displaying the characteristics of dynamical systems oscillating with highly predictable harmonic motions as well as systems oscillating in a chaotic manner.

For example, for small angles, the motion of a physical pendulum can be described completely by specifying two quantities—the rotational position, θ, and rotational velocity, ω, of the mass added to the edge of the disk. It turns out that as long as we pick two quantities of the system that can be independently chosen, all the information of the system can be described by a graph of one quantity versus the other.

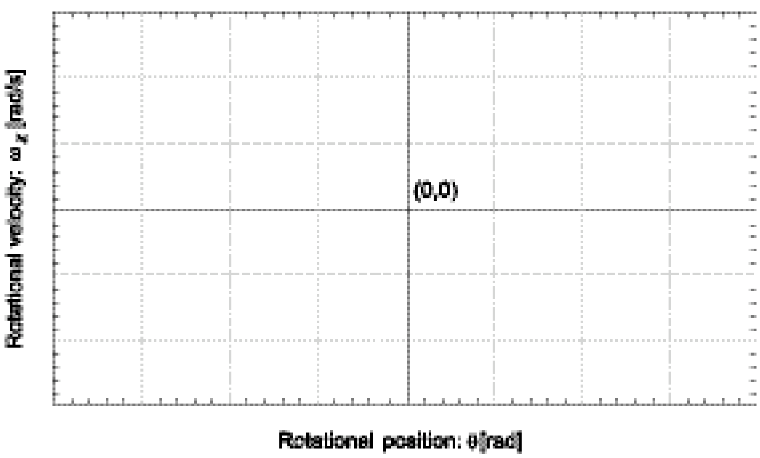
Let's explore how phase diagrams can represent the motion of a simple linear system by looking at those generated by the motion of the physical pendulum.

You will begin by predicting the shape of a *phase plot* in which the rotational position will be plotted on the horizontal axis and the rotational velocity on the vertical axis. Next you will observe the phase plot of the motion in real time using a computer-based motion detection system.

In the next few activities, we assume that the pendulum is rotating about the z-axis so that $\vec{\omega} = \omega_z \mathrm{k}$ and $\vec{\alpha} = \alpha_z \mathrm{k}$.

15.4.3. Activity: A Phase Plot of an Oscillation

a. Consider the following graph frame that displays the rotational velocity vs. the rotational position of the physical pendulum. Use a dotted line to predict the shape of the graph as the pendulum goes through one complete cycle of oscillation if it is released from an rotational position of 135° or 2.36 radians. **Hint:** What are the rotational positions when the rotational velocity is a maximum, a minimum, zero? The diagram below shows an edge mass oscillating with an amplitude of 135°. Indicating points on the graph corresponding to those shown in the diagrams of edge mass positions can help you to predict the shape of the phase plot.



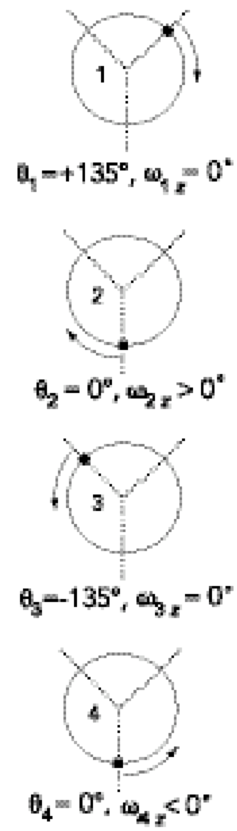


Fig. 15.7. Four different combinations of θ and ω_z during one cycle of a physical pendulum oscillation. The pendulum is released from rest with initial conditions $\theta_0 = +135°$, $\omega_0 = 0$.

b. Use a rotary motion sensor to record the rotational velocity vs. rotational position of the oscillating system for three or more complete cycles of oscillation. Set up the graph in the software so rotational velocity is plotted on the vertical axis and rotational position on the horizontal axis. Affix a printout of your best graph in the space below. Mark the following locations on your graph.

θ^{min}: the rotational position with respect to the equilibrium position is a minimum (1 location)

θ^{max}: the rotational position with respect to the equilibrium position is a maximum (1 location)

ω_z^{max}: the rotational velocity magnitude is a maximum (2 locations)

c. If you see a circle or ellipse spiraling inward after many cycles, can you explain why this is happening?

d. An *attractor* is defined as the point on the phase diagram to which the motion converges after a long period of time. Mark the predicted location of the attractor on your phase diagram in part a.

Large and Small Angle Motion

The differences in the behavior of the physical pendulum oscillating at large and at small amplitudes is of interest. In the next activity you will obtain data for both large and small angle oscillations. You should use a computer-based rotary motion sensing system to create a time series graph of rotational position for enough time so that the oscillation dies out completely. Start once more with an initial mass displacement of 135°. This will allow you to see the effects of drag forces that cause the pendulum to come to rest after about 100 cycles. If the drag forces are small, you'll need a time of about 130 seconds.

After you collect the data you can study the time series graph to compare the shape and period of oscillations at a large amplitude at the beginning of the run to small angle oscillations at the end of the run.

Later you will be studying the effects of drag in more detail, so the data you collect should be saved for use in the upcoming spreadsheet modeling exercises. **Note:** Be sure to save your best data set for comparison with spreadsheet models of the physical pendulum behavior you will be creating!

15.4.4. Activity: Large vs. Small Amplitudes

a. Set up the rotary motion software for a 130-second run at 20 data points/ second (or enough time for the pendulum to come to rest after being started at an amplitude of 135°) or use L150404. As usual, start the run with the pendulum mass hanging straight down and then quickly raise the mass through an angle of 135° and release it from "rest." Once you get a good data set, save it and include a printout of the time series graph in the space below. Also, be sure to record the values of the edge mass, m, the disk mass, M, and the radius of the disk R in the table to the left.

Edge Mass	m[kg]	
Disk Mass	M[kg]	
Disk Radius	R[kg]	

b. Figure out how to display one or two oscillations with an amplitude of about 135°. For example, you might want to change the horizontal axis and examine a 4-second time period between $t = 2$ s and $t = 6$ s. Affix a printout of this graph with the enlarged time scale in the space that follows. Use the analysis tools in your software to determine the amplitude and period of the oscillation and mark it on the graph.

c. Figure out how to display one or two oscillations with an amplitude of about 10°. For example, you might want to change the horizontal axis and examine a 4-second time period between $t = 120$ s and $t = 124$ s. Change the rotational position scale so the shape of the graph is as close as possible to that of the graph in part b. above. Affix a printout of this graph with your enlarged time and rotational position scales in the space that follows. Use the software analysis tools to determine the amplitude and period of the oscillation and mark it on the graph.

d. Are there noticeable differences between the oscillations at the two amplitudes? Are the periods the same? Theoretically, one of the graphs should have broader peaks showing the edge mass spending more time at the maximum angles. Can you tell which one? Explain!

15.5. PHYSICAL PENDULUM OSCILLATIONS—ANALYTIC THEORY

The frictional forces acting on the physical pendulum system don't seem to effect its motion very much over a single cycle or two. If you neglect friction, can you derive the equation of motion for the physical pendulum? Can you solve this equation of motion to get an analytic equation that describes how the edge mass oscillates as function of time?

Using the Rotational Equation

As usual, let's assume the physical pendulum rotates about a z-axis. In order to derive the equation of motion describing your physical pendulum recall that when a mass, m, constrained to move about a fixed axis experiences a net torque, it will undergo an rotational acceleration given by

$$\tau_z^{\mathrm{net}} = I\alpha_z$$

By deriving expressions for the torque on the edge mass and the rotational inertia of the physical pendulum, you can also derive an equation relating the rotational acceleration of the pendulum to the rotational position of the edge mass. Can this equation of motion be solved to yield an analytic expression for the changes in rotational position of the edge mass over time?

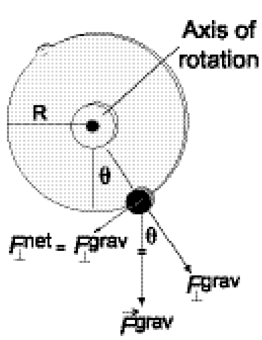


Fig. 15.8.

15.5.1. Activity: The Pendulum Equation of Motion

a. Refer to the diagram in Figure 15.8. and explain why the perpendicular component of the net restoring force, $F_{\perp}^{\mathrm{net}}$, on the edge mass is given by the equation $F_{\perp}^{\mathrm{net}} = mg \sin \theta$ where m represents the edge mass, g is the local gravitational constant, and θ is the angle between the vertical line and the edge mass. **Hint:** You did this for the simple pendulum in Activity 15.2.1b.

b. Show that the gravitational torque, $\vec{\tau}^{\text{grav}}$, about the pivot axis, taken as the *z*-axis, due to the gravitational torque on the edge mass is given by the expression

$$\vec{\tau}^{\text{grav}} = \vec{\tau}_z^{\text{grav}}\hat{\text{k}} = -mgR\sin\theta\,\hat{\text{k}} \tag{15.1}$$

where R is the radius of the disk and the gravitational torque is positive if the torque vector points out of the paper and $\hat{\text{k}}$ is the unit vector in the *z*-direction. **Note:** This particular relationship between θ and τ_z^{grav} assumes that θ = 0 rad when the edge mass is at its lowest point and θ is positive in the counterclockwise direction from the downward vertical axis.

c. Explain why, if drag and frictional forces are very small, there are no other significant unbalanced torques about the pivot on the disk/mass system. **Hint:** Consider the symmetry of the disk.

d. If the holes drilled at the edge of the disk to receive the added mass can be ignored, show that the rotational inertia of the disk/mass system is given by

$$I = mR^2 + \tfrac{1}{2}MR^2 \tag{15.2}$$

where R is the radius of the disk, M is the mass of the disk, and m is the edge mass. **Hint:** What is the rotational inertia of a disk of mass M and radius R rotating about an axis through its center perpendicular to its face? What is the rotational inertia of a point mass m rotating at a distance R from its axis of rotation?

e. Assuming that frictional and drag forces can be ignored, use the relationship between $\vec{\tau}^{\text{grav}}$, I, and $\vec{\alpha}$ to show that the z-component of the rotational acceleration of the pendulum can be described by the equation

$$\alpha_z(t) = \frac{\tau_z^{\text{net}}}{I} = \frac{\tau_z^{\text{grav}}}{I} = -\left(\frac{mgR\sin(\theta(t))}{I}\right) \qquad (15.3)$$

where $I = mR^2 + \frac{1}{2}MR^2$ and $g = 9.8\ \text{m/s}^2$

The Analytic Equation of Motion

The equation of motion you derived in part e. of the last activity, Equation 15.3, does not have an analytic solution that consists of simple familiar functions. Instead, there is a complicated function known as the elliptic integral that can be approximated in terms of the familiar cosine function for small angles of oscillation.

Unless you have mathematical training beyond the introductory level, you will not be able to obtain an analytic solution to the large angle problem. But don't give up. In the next section you will learn a powerful *iteration* technique for comparing theory with experiment for any dynamical system in which the forces on the system are known. This will allow you to construct a numerical rather than an analytical model to predict the motion of the physical pendulum.

For Small Angles but Not Large Ones

When the amplitude of the oscillation is small, it can be shown using methods explained in Unit 14 that an approximate solution to Equation 15.3 is given by

$$\theta(t) = \Theta\cos(\omega_z t + \phi) \qquad (15.4)$$

where Θ represents the pendulum's amplitude in radians and the z-component of rotational velocity is given by $\omega_z = \sqrt{\frac{mgR}{I}}$ (but only when θ is small so that $\sin\theta \approx \theta$.

USING ITERATIONS TO MODEL MOTIONS

15.6. GENERAL THEORY OF LARGE ANGLE PHYSICAL PENDULUM MOTION

You have just learned that it is not easy to obtain a theoretical equation describing the behavior of your physical pendulum for large oscillation angles, and we have noted that contemporary physicists now use a computer-based numerical technique to compare theory with experiment for systems experiencing large forces. Why is having a general method for comparing mathematical theory and experiment of such interest to scientists? Why is it worth learning?

Newton's second law and the related rotational law relating accelerations to forces are broadly applicable principles of motion for objects of ordinary sizes moving at ordinary speeds. Based on past experience, physicists have a profound belief in the validity of Newton's laws. They use this belief to figure out what the forces on systems that have not yet been studied are. This is how we claim to know about the mathematical nature of gravitational forces that we cannot "see" directly. In the following activities, you will learn how to use a belief in the laws of motion to determine the mathematical nature of the frictional forces on the physical pendulum you are studying.

Once the mathematical behaviors of various types of forces are understood, then engineers aided by high-performance computers can engage in *predictive engineering* by developing reliable mathematical models of the behaviors of new systems being designed. Predictive engineering saves years of time and millions of dollars compared to the process of building and testing real system prototypes over and over until a system is refined. In the last activity in this unit you will help to construct a reliable mathematical model of the behavior of the harmonically driven Chaotic Physical Pendulum. If you had been assigned the task of designing a Chaotic Physical Pendulum, as we were, then you could have done your predictive engineering using this model.

That's enough erudite digression. Let's get to the task at hand—to develop an iterative numerical modeling technique that can be used to explain your large angle physical pendulum motion theoretically.

An Iterative Model of the Physical Pendulum System

The word iterative means repetitive. In the next activity you are going to derive a set of equations for a dynamical system that allows you to calculate values of the system's linear or rotational acceleration, velocity, and position at one time based on a knowledge of these values at a slightly earlier time. In order to march through time in steps of Δt, you must know what the force or torque on a system is as a function of its velocity and position. The iterative equations that provide step-by-step marching orders are derived from: (1) the laws of motion relating forces and torques to accelerations and (2) the definitions of velocity and acceleration.

The values for positions, velocities, and accelerations are approximate but can be made more accurate, if necessary, by using smaller time steps.

A knowledge of the relationship between the torque and position relative to equilibrium of the physical pendulum, obtained from the law of rotational motion, is used to start deriving iterative equations that describe the edge mass motion. If we only wish to predict the motion of the physical pendulum for one or two cycles, we can neglect, for the time being, the influence of friction or drag forces that might be present. The net torque component for pendulum rotation about a z-axis is given by Equation 15.3 (that you derived in the last activity). **Note:** This equation assumes that the *only* contribution to the net torque is the gravitational torque. If there are additional torques due to bearing friction or misalignment *this model based on Equation 15.3 may not exactly match your data.*

Let's start by rewriting Equation 15.3:

$$\alpha_z(t) = \frac{\tau_z^{\text{net}}}{I} = -\left(\frac{mgR\sin\theta(t)}{I}\right)$$

The rotational inertia of the system is the sum of the edge mass, m, and the mass of the disk, M. So

$$I = mR^2 + \tfrac{1}{2}MR^2$$

Using the definition of the z-component of instantaneous rotational acceleration and rotational velocity we can see that:

$$\alpha_z(t) = \frac{d\omega_z}{dt} = \frac{d^2\theta}{dt^2} = -\left(\frac{mgR\sin\theta}{I}\right) \qquad (15.3)$$

where the local gravitational constant g is given by 9.8 N/kg. **Note:** Since α_z, ω_z and θ are all functions of time, in presenting the iterative equations we will use the notation $\alpha_z(t)$, $\omega_z(t)$, and $\theta(t)$ to represent the values of rotational acceleration, velocity, and position at time t.

Beware: The notation $\alpha(t)$ and so on, in this context, refers to α_z, ω_z, and θ being functions of time. It does *not* signify multiplication by t.

The First Iterative Equation

If we know the oscillator's rotational displacement, rotational velocity, and torque at any given time t, we can find the rotational acceleration, velocity, and position a short time $t + \Delta t$ later.

1. First we use the known value of the displacement, $\theta(t)$, at time t to calculate the initial value of the acceleration, $\alpha(t)$ using the following equation.

Iterative Equation One:

$$\alpha(t) = \frac{\tau}{I} = -\left(\frac{mgR\sin(\theta(t))}{I}\right)$$

2. Second, we use the known value of the rotational velocity at time t, $\omega(t)$, and the definition of rotational acceleration as the derivative of rotational velocity to calculate the rotational velocity at a new time $t + \Delta t$ that is just an instant, Δt, later.

$$\alpha(t) = \frac{d\omega}{dt} \approx \frac{\omega(t+\Delta t) - \omega(t)}{\Delta t} \qquad (15.5)$$

15.6.1. Activity: Deriving the Iterative Equations

a. Solve Equation 15.5 for $\omega_z(t+\Delta t)$ to find the velocity of the physical pendulum at the later time ($t+\Delta t$), in terms of the values of rotational velocity and acceleration at time t. Show the algebra needed to obtain the initial iterative equation two. (Ignore the $d\omega/dt$ term for now.)

Iterative Equation Two:

$$\omega_z(t+\Delta t) =$$

This equation for calculating the new value of ω from the *old* values of ω and α is called *Euler's method*.

b. We can use the fact that rotational velocity is the time derivative of the rotational position to obtain an expression for the rotational velocity an instant Δt later:

$$\omega(t+\Delta t) = \frac{d\theta}{dt} \approx \frac{\theta(t+\Delta t) - \theta(t)}{\Delta t}$$

Use this expression to find the position of the physical pendulum at the next instant in time in terms of the values of its rotational position, $\theta(t)$, at time t, and rotational velocity, $\omega_z(t+\Delta t)$, at the latest available time value at $t+\Delta t$. Show the algebra needed to obtain the initial iterative equation three. Ignore the $d\theta/dt$ term for now.

Iterative Equation Three:

$$\theta(t+\Delta t) =$$

We have modified Euler's method here by using the *new* value of the rotational velocity, ω, to calculate the new value of the rotational position, θ, from its old value. This greatly improves the accuracy and stability of the resulting numerical solution. Such a mixed use of one old and one new value to calculate the rotational displacement, θ, as a function of time is called the *modified Euler's method.*

Using the Iterative Equations

Here are the marching orders for using the method of iterations or numerical integration to predict the motion of a dynamical system:

The General Idea

1. Write an equation showing how you think the z-component of the torque, τ_z, depends on values of θ, ω_z, and α_z.
2. Choose the length of the time step, Δt, to use for your step-by-step march through time. It is important to *choose a time interval*, Δt, *that is small enough so the observed rotational position and velocity of the object hasn't changed very much*.
3. Specify the initial values of rotational displacement, θ_1, and z-component of rotational velocity, $\omega_{1\ z}$, of the object of interest at some known initial time. (The initial time is usually zero.)
4. Use the initial value for θ_1, at $t = 0$ s in *Iterative Equation One* to calculate the object's z-component of rotational acceleration, $\alpha_{1\ z}$. This then gives us initial or starting values for all three motion variables θ, ω_z, and α_z.
5. Now you can calculate the next value of ω at a time Δt later in terms of the previous values of ω_z and α_z using *Iterative Equation Two* (Euler's method).
6. Then you can calculate the next value of θ at a time Δt later in terms of the *previous* value of θ and the *current* value of ω_z using *Iterative Equation Three*. (Modified Euler's method.)
7. Next you can calculate the system acceleration at a time Δt later using the equation for net torque and the new values of position and velocity.
8. Now that you have calculated the new "current" values for all three motion variables θ, ω_z, and α_z, you can repeat steps 4 through 7 above to find a whole series of rotational positions and velocities as we march through time.

You can set up a modeling spreadsheet like that shown in Figures 15.9 and 15.10 to construct your own iterative model of the behavior of the physical pendulum. An annotated version of this template is included to augment the specific instructions that follow.

Specific Instructions

(See the annotated spreadsheet that follows for more details and the Trouble Shooting section that follows Activity 15.6.2 for additional advice on overcoming pitfalls.)

1. For comparison with data it is useful to use the same Δt that you used when collecting data, provided the rotational position hasn't changed a lot from reading to reading. For example, if you collected 20 data points per second, then you should use a Δt of 1/20 s = 0.05 s. Enter the value of Δt using in taking your experimental data in cell F5.
2. Enter or paste in the experimental values of the rotational position as a function of time of the physical pendulum starting in cell B16.

3. Enter the first value of time, which is usually zero in cell A16 and the value of the time step for the data collection in cell F5. *Then you should create as many values of time as there are rotational positions* by using the equation A17 = A16 + F5 and copying it down to get A18 = A17 + F5, etc.
4. Enter the experimentally determined initial values of rotational position and rotational velocity in cells F6 and F7. **Reminder:** In order to reduce approximation errors that accumulate when using a finite Δt, *your first data point should be at a time when the rotational position is a maximum*.
5. Adjust the system constants to describe the disk and mass you actually used in your experiments. (The sample system constants shown in MKS units in Figures 15.9 and 15.10 are for an aluminum disk of approximate dimensions 4” dia. × 1/4” thick with a 10.7-g edge mass.)
6. Enter the equation needed to calculate the pendulum’s total rotational inertia, I, in cell F4.
7. Enter and copy down the iterative equations into as many rows as you have data.

15.6.2. Activity: An Iterative Model of Large Angle Physical Pendulum Motion

a. In preparation for modeling some of your large angle data, you should open up the Rotary Motion software data file created in Activity 15.4.4 and select about two cycles worth of large angle data for rotational position in radians. **Beware!** If you collected your data in degrees, be sure to configure the rotary motion software to report data in radians, not degrees, before you transfer your data to a modeling spreadsheet.

List the values of Δt and the initial values of θ and ω_z.

Δt = __________ [s]

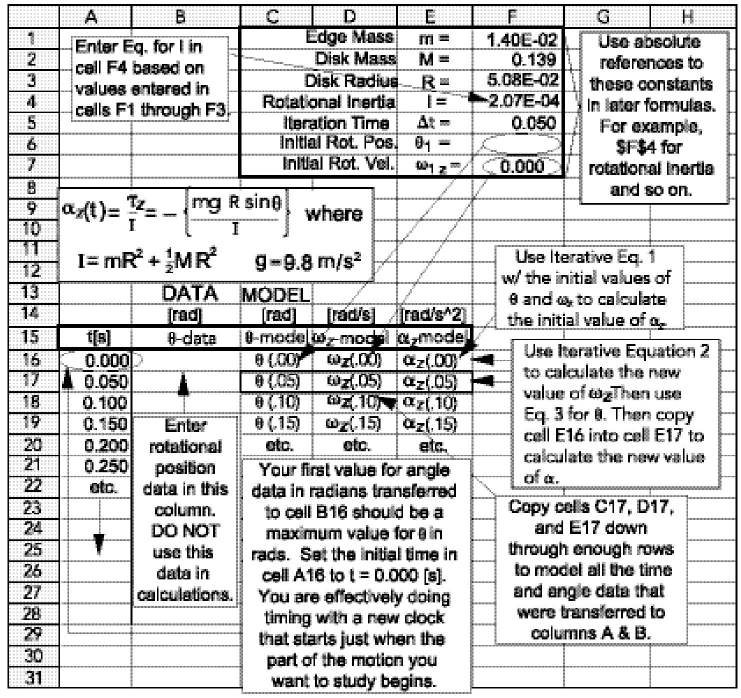
θ_1 = __________ [rad]

ω_{1z} = _________ [rad/s]

b. Open a blank spreadsheet file or an iterative spreadsheet template entitled S150602.XLT. Transfer just one or two cycles of the angle data to column B. The data for the angle should start in cell B16.

c. Next you should pretend you started a *new* clock that has t = 0.00 s when the pendulum was at its maximum angle of displacement. To do this, place the time 0.00 *s* in cell A16. Then set the contents of cell A17 to be “= A16 + Δt.” For example, if you have placed the value of Δt in cell F5, then the equation in cell A17 should be “= A16 + F5.” Copy this equation down through enough cells in column A so that there is a time to the left of each of the angle values.

d. In the following diagram, sample values for the constants needed in the model have been entered into cells F1 through F6. Use the sample constants to develop a modeling spreadsheet like the one that follows. Enter the Iterative Equations needed to track the theoretical motion of the physical pendulum for comparison with one or two cycles of data.



$$\alpha_z(t) = \frac{\tau_z}{I} = -\left\{\frac{mg\,R\sin\theta}{I}\right\}$$

$$I = mR^2 + \tfrac{1}{2}MR^2$$

Fig. 15.9. A sample iterative model spreadsheet layout with annotations, suggested iterative equations, and sample constants. Enter your own constants and data. **Note:** Remember ω_{1z} may not be zero for your data!

If your equations are entered correctly, your sample U15 Modeling Template spreadsheet should look similar to the one in Figure 15.10.

e. Create an overlay graph of θ (Data) and θ (Model) vs. *t*.

f. *Change any system constants that are different from those entered in the sample*. Once your system constants are entered and your iterative equations have been copied down to fill the same number of rows as your data, an overlay graph comparing the iterative model to your experimental values for rotational position will appear. How well does the approximate theoretical model based on iterations match your experimental result?

	A	B	C	D	E	F	G	H
1	System constants			Edge Mass	m =	1.40E-02	[kg]	
2				Disk Mass	M =	0.139	[kg]	
3				Disk Radius	R =	5.08E-02	[m]	
4				Rotational Inertia	I =	2.07E-04	[kg·m^2]	
5	Approximation Step:			Iteration Time	Δt =	0.050	[s]	
6	Initial Conditions:			Initial Rot. Pos.	θ_1 =	2.391	[rad]	
7				Initial Rot. Vel.	ω_{1z} =	0.000	[rad/s]	
8	$\alpha_z(t) = \frac{\tau_z}{I} = -\left\{\frac{mg R \sin\theta}{I}\right\}$ where							
9								
10	$I = mR^2 + \frac{1}{2}MR^2 \quad g = 9.8\ m/s^2$							
11								
12			ITERATIVE PENDULUM MODEL					
13		DATA	ITERATIVE EQUATIONS					
14		[rad]	[rad]	[rad/s]	[rad/s^2]			
15	t[s]	θ-data	θ-model	ω_z-model	α_z-model			
16	0.000	2.391	2.391	0.000	-22.966			
17	0.050	2.339	2.334	-1.148	-24.341			
18	0.100	2.234	2.215	-2.365	-26.916			
19	0.150	2.077	2.030	-3.711	-30.186			
20	0.200	1.833	1.769	-5.220	-33.013			
21	0.250	1.518	1.425	-6.871	-33.314			
22	etc.	1.134	0.998	-8.537	-28.303			
23	↓	0.681	0.501	-9.952	-16.165			
24		0.157	-0.037	-10.760	1.254			
25		-0.367	-0.572	-10.697	18.229			

Fig. 15.10. A sample iterative model spreadsheet showing the outcome of iterative calculations based on sample system constants and suggested iterative equations. **Warning:** *Be sure to enter your own system constants and initial conditions*! Also, remember ω_{1z} may not be zero for your data.

g. Since there is some uncertainty of your knowledge of some of the constants, you may want to change them within the limits of uncertainty to see if you can get a better fit. What factors did you need to adjust to get the theoretical iterative model and the experimental graphs to match so they have the same period and amplitude?

h. In the space that follows affix a printout of your best fit model graph showing the plotted experimental data and the approximate theoretical curve.

i. Save your best spreadsheet model for later use. But, don't close your spreadsheet—you'll be using it some more.

Software Troubleshooting

1. *Are you tired of waiting for slow automatic recalculations every time you enter data or are calculations getting in your way?* You can set spreadsheet calculations in a manual mode so recalculations are only done when you ask for them. To see results of changes you make in an Excel spreadsheet model that's set in manual mode, you must either press the command (⌘) key and equal sign (=) simultaneously for Macintosh Excel or the F9 key for PC-compatible Excel.

2. *What if nothing changes after you enter a new equation or constant?* Did you forget to fill the equation you entered down through all the columns? Did you forget to use $ signs in your equations to call on fixed constants?

Troubleshooting (continued)

3. *What if the model you created "blows up" or does not fit the data?*
   a. Did you adjust the time column after entering data so it updates itself by the amount of your time step (that is, A17=A16+E5) and did you remember to blank extra values of time so the number of entries in the time column (A) and the number of entries in the other columns exactly match.
   b. Did you convert all initial values and data from degrees to radians before entering it into the spreadsheet?
   c. Did you do the iterations in exactly the right order?

Limitations of the Basic Iteration Technique

The basic iterative technique you just used is known as the *Modified Euler Approximation*. It is relatively simple to use. However, there are other more accurate techniques. Once you master the Euler method, it is not too difficult to learn to use one of the more accurate approximation methods such as the Leapfrog method, the second order Runge Kutta method, or the fourth order Runge Kutta method.

Optional: Large vs. Small Angle Motion

At this point you can ask the question, how well would a model based on the force equation for small amplitudes fit your data? Answering this is easy once the basic model has been constructed. All you have to do is to use the small angle approximation and replace the sin θ term with the value of θ in radians in the equation for the rotational acceleration, α, in cell E16. Then you need to copy this equation *down through as many rows as needed to "match" all the data*.

15.6.3. Activity: The Small Angle Approximation

Use the small angle approximation in the torque term used to calculate rotational acceleration. Does the new model fit the large angle data? Why or what not? Are the shapes of the experimental and theoretical graphs the same? Are the periods the same? Discuss.

15.7. DAMPING IN THE PHYSICAL PENDULUM SYSTEM

Earlier in this unit you started the physical pendulum at a relatively large angle and watched its oscillations die out after a minute or two. This was probably due to mechanical friction in the bearings combined with other damping forces. In order to study the effects of additional damping on the physical pendulum experimentally, you will need:

- 1 mass/disk physical pendulum system (see Section 15.4)
- 1 computer data acquisition system
- 1 rotary motion sensor
- 1 disk-shaped neodymium magnet mounted on a rod (to damp the pendulum motion)
- 1 electronic scale
- 1 ruler

Recommended Group Size:	3	Interactive Demo OK?:	Y

15.7.1 Activity: Predicted Sources of Damping

What factors would you guess are causing the physical pendulum system to die out after a while?

In the next few activities you will add significantly more damping to the system by placing a small but powerful neodymium magnet near the surface of the disk as it rotates. This magnetic drag is known as *eddy damping*. You can modify your iterative model to take drag into account by simply changing the equation you use to calculate the torque and hence the rotational acceleration you expect the disk to experience. If you do this carefully you might be able to discover for yourself what mathematical relationship, if any, can be used to describe eddy damping.

Experiments with Eddy Damping

In the next activity you should create enough magnetic drag to cause the edge mass to come to rest within five or six cycles instead of the fifty or more cycles you observed earlier. You can apply the magnetic drag to the disk by placing a magnetic damping wand close to the side of the aluminum disk as shown in Figure 15.11.

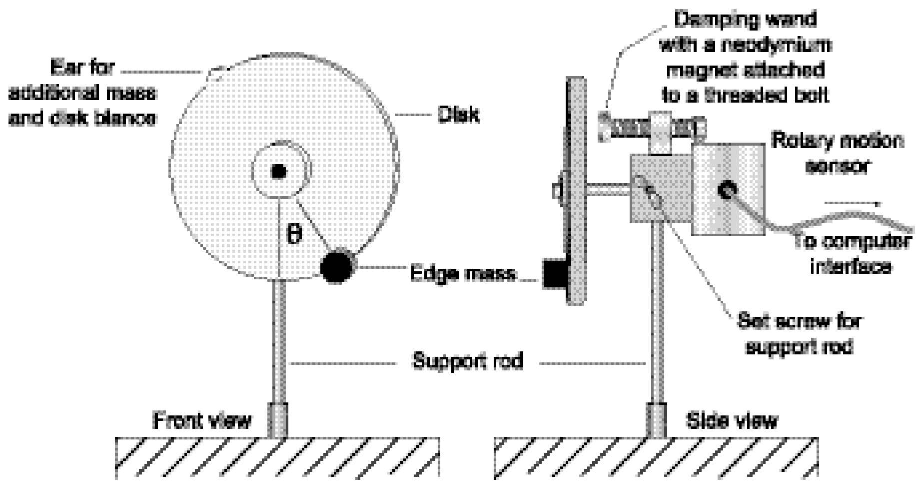


Fig. 15.11. A diagram showing the use of an eddy damping wand consisting of a magnet attached to a threaded bolt to provide an adjustable magnetic drag force.

15.7.2 Activity: Magnetic Drag Measurements

Arrange the magnet so that when the edge mass is released from an angle of about 135° (2.36 radians) above the vertical the motion dies out in about 15 seconds or less. Record the values of θ as a function of *t* using your computer data acquisition system. Set up your own experiment file or use the L150702 file. Print out a copy of your θ vs. *t* graph and affix it in the space that follows. Also save your electronic file for future use.

Describing Eddy Damping Mathematically

This is your chance to be a detective. There are two simple relationships that might describe the additional drag torque, $\vec{\tau}^{\,\mathrm{drag}}$, on the disk caused by the magnet. A fairly simple possibility is to assume that the torque increases linearly with the rotational velocity of the system. Another model is to assume that although the torque always opposes the motion, its magnitude is constant. The first model is the linear one shown next.

Linear Velocity Model

$$\tau_z^{\text{drag}} = -b\omega_z \tag{15.6}$$

where b is a positive constant known as the damping coefficient or damping factor. In this model the drag torque is assumed to be a linear function of the rotational velocity ω because if Equation 15.6 is valid, then a graph of the drag torque vs. rotational velocity would be *linear*. The second model does not depend on the magnitude of rotational velocity.

Velocity Independent Model

$$\tau_z^{\text{drag}} = -b \text{ if } \omega_z > 0$$

$$\tau_z^{\text{drag}} = +b \text{ if } \omega_z < 0 \tag{15.7}$$

In this model the drag torque changes its sign when the sign of ω_z changes but its magnitude $|\omega_z|$ is the same for all values of ω_z.

When drag forces are significant, then the net torque is no longer simply the gravitational torque but rather the sum of the gravitational and drag torques so that

$$\alpha_z = \frac{\tau_z^{\text{net}}}{I} = \frac{\tau_z^{\text{grav}} + \tau_z^{\text{drag}}}{I} = \frac{\tau_z^{\text{grav}}}{I} + \frac{\tau_z^{\text{drag}}}{I}$$

In order to build drag into your existing model all you need to do is open up the spreadsheet you saved in Activity 15.6.2 and add a term τ_z^{drag}/I to each equation you used to compute the rotational acceleration so that

$$\begin{aligned} \alpha_z(t+\Delta t) &= \frac{\tau_z^{\text{grav}}}{I} + \frac{\tau_z^{\text{drag}}}{I} \\ &= -\left(\frac{mgR\sin(\theta(t+\Delta t))}{I}\right) + \frac{-b\omega_z(t)}{I} \end{aligned} \tag{15.8}$$

The mixture of t and $t+\Delta t$ terms is a bit peculiar here, but the rotational velocity has not yet been calculated at time $t+\Delta t$ when it is time to calculate the new value of rotational acceleration at the $t+\Delta t$. You should use the value of ω_z available at the nearest time to $t+\Delta t$ and hope that this doesn't introduce too much error.

The Linear Velocity Dependent Eddy Damping Model

You should predict which of the two mathematical models might fit the data the best. Then you can begin testing the adequacy of the two models.

15.7.3. Activity: Velocity Dependent Damping Model

a. Which of the two relationships for eddy damping drag torque do you think will describe the data best? The velocity dependent one or the velocity independent one? Explain the reasons for your prediction.

b. Enter all of the five or six cycles of eddy damping data into column B of your spreadsheet for comparison with refinements in your model that take the drag into account. Be sure to copy down your time equations (Column A), and your iterative equations (Columns C, D, E) to match the number of data values you entered. Sketch the data graph that appears on the screen in the space below.

c. Change the physical pendulum model equations you entered and refined in Activity 15.6.2 to take eddy damping into account using the velocity dependent model presented in Equation 15.6 for different values of the damping factor, b. How well can this velocity dependent eddy damping model fit the data?

The best value of b is

$b =$

d. Affix a printout of the overlay graph for the theoretical and experimental values of rotational position vs. time for the system in the space below.

Optional: Testing the Velocity Independent Model

If you have time, you can give the second of the two models a try. This shouldn't take very long.

15.7.4 Activity: Velocity Independent Damping Model

a. Change the physical pendulum model equations you entered and refined in Activity 15.6.2 to take eddy damping into account using the velocity independent model presented in Equation 15.7 for different values of the damping factor, *b*. How well can this velocity dependent eddy damping model fit the data? **Hint:** In trying to enter Equations 15.7 you can use the absolute value function $\tau_z^{\text{drag}} = -b\omega_z / (\text{abs}(\omega_z) + 0.001)$ where abs( ) is the absolute value function that is available in your spreadsheet. (A negligible additional rotational velocity 0.001 rad/s has been added in just in case ω happens to be zero since computers don't know how to divide by zero!)

The best value of *b* is:

$b =$

b. Affix a printout of the overlay graph for the theoretical and experimental values of the system in the space below.

Final Comment: A Velocity Squared Model

Another mathematical relationship that works for air drag at high speeds that might also work for eddy damping relates the magnitude of the torque due to drag to the square of the rotational velocity. Since the torque vector should always be opposite the direction of rotational motion, this torque can be represented by the equation

$$\tau_z^{\text{drag}} = -\,b\omega_z \,\_\, |\omega_z| \tag{15.9}$$

Hint: In trying to enter Equation 15.9 you can use the absolute value function again in the form $\tau_z^{\text{drag}} = -\,b\omega_z \,\_\, \text{abs}(\omega_z)$ where abs( ) is the absolute value function that is available in your spreadsheet.

THE CHAOTIC PHYSICAL PENDULUM

15.8. DRIVING A PHYSICAL PENDULUM CRAZY

By making a few more modifications to the physical pendulum system you have been studying, you can create a *Chaotic Physical Pendulum*. This can be done by attaching a drive wheel with a small rotational inertia to the physical pendulum axle. It is then possible to drive the pendulum system at various frequencies by wrapping a string around the drive wheel and attaching the ends of the string to springs on either side of the wheel. One of the springs is then attached to another string that is threaded through a hole in a stabilizer. The end of the string is attached to an eccentric driven "off axis" by a variable speed DC motor.

If a torque is applied to the drive wheel by the motor, the disk should oscillate with simple harmonic motion. If an edge mass, m, is added to the disk, the extra gravitational force on it will cause the net torque on the disk to be a non-linear function of θ. In fact, the greater the edge mass, the more non-linear the torque becomes.

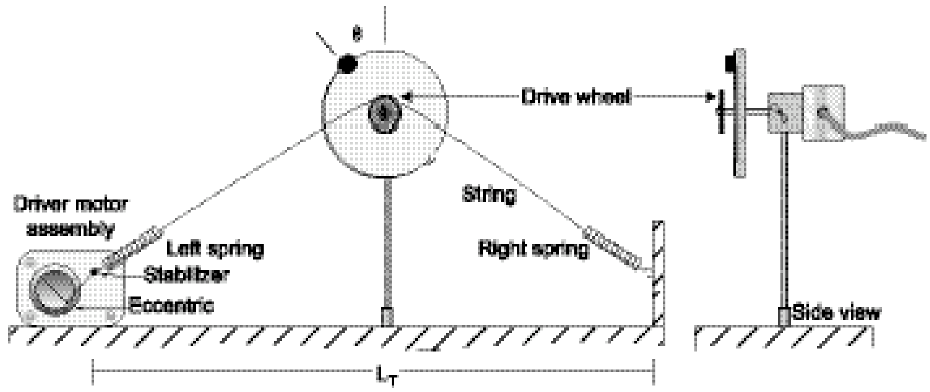


Fig. 15.12. The Chaotic Physical Pendulum. The driver motor assembly consists of a high torque 12 V dc motor and a variable 0-5 V dc power supply. An eccentric is attached to the motor axis off-center. The string that is attached to the eccentric passes through a hole drilled in a fixed stabilizer rod and is attached to the left spring. The voltage leads from the power supply can be connected to the analog input of a computer interface. (This setup can be purchased from PASCO scientific.)

To make observations with a Chaotic Physical Pendulum you will need:

- 1 mass/disk physical pendulum system (see Section 15.4)
- 1 magnetic damping wand (see Section 15.7)
- 1 computer data acquisition system
- 1 rotary motion sensor
- 1 drive wheel attached to the disk axle
- 1 driver motor assembly (with an adjustable eccentric and stabilizer)
- 1 variable DC power supply, 0–5 V. @ 300 mA
- 2 springs (with $k \approx 2.0$ N/m)
- 1 string ($\approx$ 1 m)
- 1 electronic scale
- 1 ruler
- 1 stopwatch

Recommended Group Size:	3	Interactive Demo OK?:	Y

Exploring the Natural Frequencies of the System

You will observe the natural oscillation frequencies of various parts of the Chaotic Physical Pendulum to help you understand why the system motion becomes chaotic when it is driven at certain frequencies.

First, you will observe and determine the frequency of oscillation of the disk without the edge mass added while it moves under the influence of torques caused by springs wrapped around the drive wheel shown in Figure 15.12. Next you will configure the system as a pendulum by adding a small edge mass to it and measure the natural frequency of the pendulum without the springs. Then you will re-attach the springs to the driver wheel of the pendulum, re-balance the system so the springs are stretched equally when the mass is perched straight up on the top of the disk, and see where the mass ends up when it falls to the left of the vertical line and to the right of the vertical line. Finally you will measure the natural frequency of oscillation of the spring-pendulum system when the mass has fallen to the right or left of its highest possible position.

Changing the Polar Axis and Angle Measure

You can see from Figure 15.12 that we have chosen a different rotational coordinate system for the measurement of the rotational position of the edge mass. In your basic investigations of the Chaotic Physical Pendulum you will be asked to start observing the system when the edge mass is up in its highest position. From this position the mass has an equal probability of falling to the right or left, so we would like you to define θ as 0 rad when the mass is straight up, positive if it moves in a counterclockwise direction to the left, and negative if it moves in a clockwise direction to the right.

Before making your observations and measurements, you need to adjust the physical pendulum system in various ways. There is some fiddling involved. Even though we offer some suggestions to save you some time, you may come up with better techniques for adjusting the pendulum.

Suggested Physical Pendulum Adjustments

1. Wrap the string around the drive wheel two times.
2. Attach each end of the string to a spring.
3. Clamp the free end of one spring to the table top on one side of the disk. Attach the free end of the other spring to a string.
4. Thread the end of this string through the stabilizer and attach it to the eccentric driven by the DC motor.
5. Adjust the length of the string, the height of the pendulum disk, and the spacing between the springs so the springs remain stretched at all times as the pendulum oscillates. (A total string length of about 90 cm, equilibrium spring lengths of L_{sp} = 25 cm, and a pendulum disk height of about h = 15 cm work well.)
6. The disk is coupled with a rotary motion sensor. This sensor should be plugged into a computer data acquisition system so you can make real-time measurements of the rotational position and velocity of the disk.
7. Attach voltage leads from the variable dc power supply output to an analog input on the computer interface. Set up the rotary motion software to transform dc voltage readings to motor frequency values.
8. Set the driver motor eccentric at its neutral position between a maximum distance from the disk and the minimum distance from the disk.
9. For observations in which an edge mass is used, hold the drive wheel and slide the string over it one way or the other until the springs are balanced and each one has about 35 cm of stretch. The whole setup can be clamped to a track or table edge of about L_T = 1.20 m in length.
10. To check the accuracy of the vertical balance, allow the edge mass to fall to the left and settle into a new location when the gravitational torque of the mass and the torque from the now "unbalanced springs" are equal. Now allow it to fall to the right and fall to a new location. If you did adjustment 9 well, the left and right angles will have equal magnitudes. If they don't, then repeat steps 8 and 9 until the left and right angles are within 5 or 10 degrees of each other.

15.8.1. Activity: Disk-Spring, Pendulum, and Pendulum-Spring Motions

a. Start with the edge mass removed from the disk, the string and springs attached and balanced, and the driver turned off. Does the disk appear to have a "natural" frequency of oscillation? If so, what is that frequency and how did you measure it?

b. Remove the springs from the disk, create a physical pendulum again by adding a mass of 10 g or 15 g to the edge of the disk, and measure the natural frequency of oscillation for the physical pendulum at small angles (<20°). What is the frequency and how did you measure it?

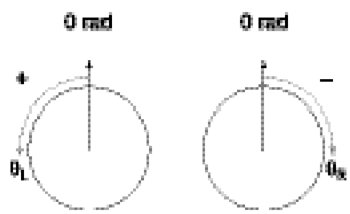
c. Which natural frequency is greater—that of the spring-disk system or that of the physical pendulum system? Based on your answer, which exerts the largest torque—the spring torque on the driver wheel or the gravitational torque on the edge mass?

d. If the springs are re-attached to the driver wheel, do you expect the natural frequency of the new pendulum-spring system to be closer to that of the pendulum system or the disk-spring system? Explain the reasons for your prediction.

e. To test your prediction, re-attach the springs and balance the system so that each spring is stretched by the same amount when the edge mass is at its highest point. (See steps 7 and 8.) Release the mass and use the computer-based laboratory system to observe the motion of the spring-physical pendulum system as the edge mass falls to one side. What is the natural frequency?

f. How does the natural frequency of the spring-physical pendulum system compare to that of the spring-disk system or the physical pendulum system? How good was your prediction in part d?

g. Measure the angle in radians that the edge mass settles into when it falls to the left and to the right. To get the positive angle for the fall to the left, measure in a counterclockwise direction from an axis pointing vertically upward. To get the negative angle for the fall to the right, measure in a clockwise direction from an axis pointing vertically upward. Sketch these angles in the space that follows.



Driving the Spring-Physical Pendulum System

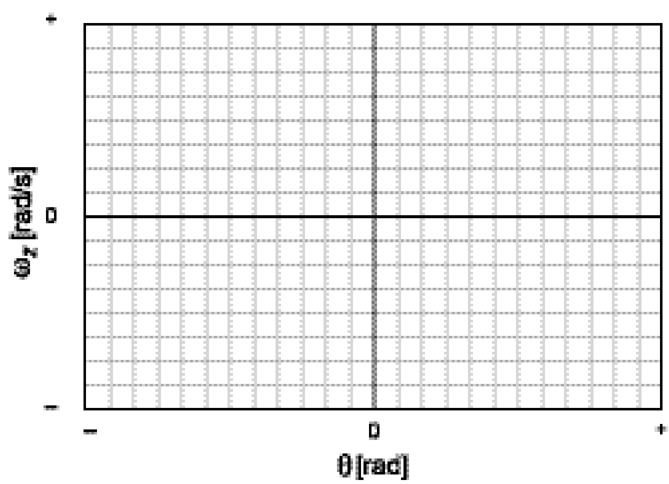
Now you are ready to observe the behavior of the system when it is driven at frequencies comparable to some of the natural frequencies you have been measuring. Before starting the next activity, you should balance the system again. To do this, set the driver frequency close to the natural frequency of the spring-physical pendulum system and then turn it off. Re-balance the spring-physical pendulum by following steps 7 and 8 in the adjustment suggestions. (If the driver arm is set at its midpoint and released at $t = 0$, this sets the phase of the driver to either 0 rad or 6.28 rad at $t = 0$.)

15.8.2. Activity: Driving the Spring–Physical Pendulum System

a. Now, start the computer-based rotational position measurements with the mass lifted straight up. Once the reading starts on the computer, release the edge mass just slightly to the left of 0 rad and turn on the driver at the same time. Wait a couple of minutes until the system "settles in." Describe the behavior of the system. Do you suspect that the motions are chaotic? Why or why not? Think of the definition of chaos. How could you test to see if the motions might be chaotic?

Note: The system behavior should look unpredictable. If it is too regular, try adjusting the driver frequency up or down at bit or adding eddy damping (using the magnetic wand).

b. Can you predict what the phase diagram of the Chaotic Physical Pendulum might look like when its drag forces and driving frequency combine to make its motion appear unpredictable? Sketch the predicted phase diagram in the space below. **Hints:** Look at the motion of your driven system carefully and think about the angles that the spring-physical pendulum settles into to the left and the right when it's not being driven. (See Activity 15.8.1e). Look at the motion when the edge mass flips from one side of the vertical axis to the other.



Observing the Motion of the Chaotic Physical Pendulum

In the next few activities you will actually record some phase diagrams and time series graphs of the motion of the Chaotic Physical Pendulum when it appears to be acting unpredictably. You will also be asked to test the system experimentally for sensitivity to initial conditions. Finally, you will summarize what torques are acting on the Chaotic Physical Pendulum in theory. These torques have been used in an iterative model to predict the pattern of behavior of the oscillator as a function of different initial values of the rotational displacement and velocity. Thus, you will end this exploration of the young science of chaos by observing the *theoretical* sensitivity of the oscillator to small changes in initial conditions.

It is interesting to observe how *reproducible* the motion of the Chaotic Physical Pendulum is. If the pendulum motion is started as close to the same way as possible several times, the motion is reproducible if the time series graphs of the rotational positions have similar shapes. Thus, to check the reproducibility you will collect data twice using approximately the same initial conditions. To reproduce the initial conditions you should turn the driver motor off and re-balance the system with the mass pointed straight up as per steps 8, 9, and 10 in the adjustment instructions. If you follow these steps, the initial rotational velocity and position of the pendulum and the phase of the driver should be approximately zero. However, you should shift the initial position of the lifted edge mass just slightly so it falls to the left each time.

15.8.3. Activity: Phase Plots, Time Series, and Reproducibility

a. Create an overlay plot of two "identical" time series graphs of the rotational displacement of the Chaotic Physical Pendulum that is set up in such a way that its motion is sensitive to initial conditions. Include as many cycles as you can display clearly. A one- or two-minute run might work. You can use L150803 or set up your own experiment file. Affix the overlay plot in the space below. *Try to make the initial conditions as identical as possible!*

b. For how long a time period are the two motions reproducible? What technique did you devise for determining this result?

c. As discussed in Section 15.3, chaotic systems are sensitive to initial conditions. They behave differently after a short time if the initial conditions are slightly different. Does the Chaotic Physical Pendulum live up to its name and behave like a system that is sensitive to initial conditions? Discuss.

d. Use the rotary motion software to create a phase diagram using the experimental values of ω_z and θ. It should look a bit like the unmarked plots in the Nova video on chaos. Affix the phase diagram below.

e. What can you tell about the motion by examining its phase plot? Pretend you hadn't seen the original motion. Are there any attractors? If so, where are they? What is happening when the angles are negative? Positive?

15.9. MODELING THE CHAOTIC PENDULUM–THEORY

Can the erratic unreproducible behavior of the Chaotic Physical Pendulum be explained in terms of known torques? Or should we assume that the forces on the system need to be changing in random or unknown ways? One of the ways of answering these questions is to see if you can create a realistic mathematical model that describes the characteristic behavior of the system.

To create a model, you can use what you already know about the various torques to derive an expression for the net force on the physical pendulum system as it oscillates. The following forces must be taken into account: (1) the gravitational torque on the edge mass that is added to the disk, (2) the eddy damping drag torque due to the action of the springs and damping magnet, (3) the torque exerted by the springs, and (4) the driving torque contributed by the motor through the springs. The net torque on the pendulum is thus a combination of the four torques

$$\vec{\tau}^{\,\text{net}} = \vec{\tau}^{\,\text{grav}} + \vec{\tau}^{\,\text{damping}} + \vec{\tau}^{\,\text{spring}} + \vec{\tau}^{\,\text{driver}} \tag{15.9}$$

Since it is conventional to define the z-axis along the axis of rotation, an alternative equation is

$$\tau_z^{\text{net}} = \tau_z^{\text{grav}} + \tau_z^{\text{damping}} + \tau_z^{\text{spring}} + \tau_z^{\text{driver}}$$

An iterative computer model of the Chaotic Physical Pendulum can be developed using the torque equations.

Gravitational Torque: Recall that the first torque term, τ_z^{grav}, in Equation 15.9 is simply the torque exerted on the edge mass, m, when it is displaced by an angle θ. Since the rotational position, θ, is defined as zero when the mass is straight up, the minus sign in Equation 15.9 can be dropped so

$$\tau_z^{\text{grav}} = + mg\,R \sin(\theta)$$

Damping Torque: The second term describing the eddy and spring damping torques was found experimentally to be

$$\tau_z^{\text{damping}} = - b\omega_z$$

Spring Torque: The third torque, τ_z^{spring}, results from the unbalanced stretch of the springs. When the disk is displaced by an angle θ from its equilibrium point, one spring stretches more and the other relaxes more as shown in the following diagram.

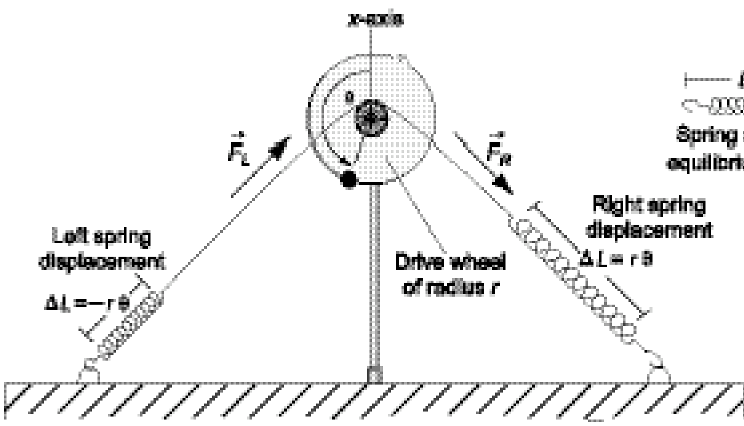


Fig. 15.13. Illustration of the unbalanced springs exerting forces on the drive wheel of radius *r* when the Chaotic Physics Pendulum is displaced from its balance point by an angle θ. (Not to scale.)

L_0 is the *total stretch* of each spring at equilibrium with no rotational displacement of the drive wheel ($\theta = 0$). Whenever a spring is compressed, the other is stretched by the same amount. This means the *magnitude* of the force exerted by each spring is the same.

Thus $$\left|\vec{F}_L\right| = -kr\theta \qquad (15.10)$$

and $$\left|\vec{F}_R\right| = +kr\theta \qquad (15.11)$$

According to the right-hand rule, the z-component of the net spring torque on the drive wheel due to the two springs is given by

$$\tau_z^{\text{spring}} = -\left(\left|\vec{F}_L\right| + \left|\vec{F}_R\right|\right) r = -(2kr\theta)\, r \qquad (15.12)$$

where k is the spring constant for each of the system springs and r is the radius of the drive wheel (not the disk!).

Driver Torque: The off-center attachment to the circular disk on the motor provides additional stretch and contractions to the springs so that there is another contribution to the net torque due to the driver. This can be expressed by the equation

$$\tau_z^{\text{driver}} = rkA_d \sin(2\pi f_d t + \phi_0) \qquad (15.13)$$

where:

f_d = the frequency of the driver motor

A_d = the amplitude of the driver motor
k = the spring constant for the system springs
ϕ_0 = the initial phase of the driver

15.9.1. Activity: Reviewing the Force Equation

a. Draw together all the terms for z-components of torque into one giant expression about a z-axis through its center.

$$\tau_z^{\text{net}} = \tau_z^{\text{grav}} + \tau_z^{\text{damping}} + \tau_z^{\text{spring}} + \tau_z^{\text{driver}}$$

b. Define each of the variables (and their units) listed below:

$m =$	
$\theta =$	
$\theta_1 =$	
$\omega_z =$	
$\omega_{1z} =$	
$\alpha_z =$	
$g =$	
$R =$	
$b =$	
$t =$	
$L =$	
$r =$	
$k =$	
$f_d =$	
$A_d =$	

c. Which of the four torques depends on either θ or ω_z in a non-linear way?

A Mathematical Model for the Pendulum

In total we have four different torques acting on the Chaotic Physical Pendulum. It is possible but tedious to derive the iterative equations for modeling the Chaotic Physical Pendulum. It is time consuming to measure all the system constants that need to be put in the model such as the spring constants, the mass and radius of the rotating disk, the edge mass, the frequency and amplitude of the driver, etc. Some of these constants have already been entered for the PASCO physical pendulum into a spreadsheet model that is available for you to use and experiment with.

Before you run the model you will need to modify any system constants that are different for the system you are using, enter the time step you are employing for taking your data, and enter the initial values for the driver initial phase, rotational position of the edge mass, and rotational velocity of the edge mass. We suggest that you use zero for all three of these initial conditions.

Now you should run the model and devise a method for describing how sensitive it is to small changes in the initial conditions of the oscillating system (that is, the rotational displacement and the rotational velocity at time t = 0 s).

An example of a Runge Kutta calculation for another Chaotic Pendulum system is shown in Figure 15.14.

	A	B	C	D	E	F	G	H	I	J	K	L
1		CHAOTIC PHYSICAL PENDULUM										
2		MATHEMATICAL MODEL:										
3		Second order Runge-Kutta Integration						Second order Runge-Kutta Integration				
4		Theoretical Iterative Equations (full step)						Theoretical Iterative Equations (half step)				
5		t	θ	ω_z	τ_z	α_z		t	θ	ω_z	τ_z	α_z
6	n	[s]	[rad]	[rad/s]	[N m]	[rad/s^2]		[s]	[rad]	[rad/s]	[N m]	[rad/s^2]
7	0	0.00	0.00	0.00	-0.01	-3.68		0.01	0.00	-0.04	-0.01	-2.75
8	1	0.02	0.00	-0.06	-0.01	-2.75		0.03	0.00	-0.08	-0.01	-2.75
9	2	0.04	0.00	-0.11	-0.01	-2.75		0.05	0.00	-0.14	-0.01	-2.74
10	3	0.06	-0.01	-0.16	-0.01	-2.74		0.07	-0.01	-0.19	-0.01	-2.73
11	4	0.08	-0.01	-0.22	-0.01	-2.73		0.09	-0.01	-0.25	-0.01	-2.72

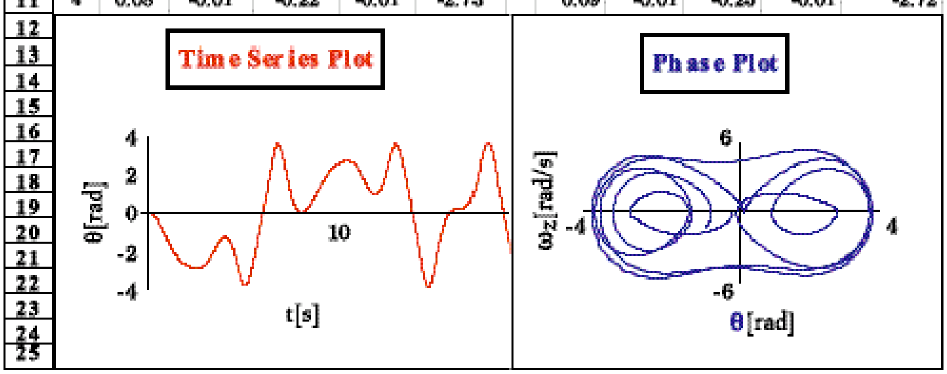


Fig. 15.14. Example of a Second Order Runge Kutta model of a possible motion of a Chaotic Physical Pendulum. System constants are not shown.

A Note about Iteration Methods: Euler vs. Second Order Runge Kutta

The *Chaotic Physical Pendulum Model* was written using a more elaborate set of iterations called the *Second Order Runge Kutta* method. This was necessary to keep errors in the model values from accumulating from step to step, causing the model to "blow up." The Runge Kutta method involves setting up calculations for values of θ, ω_z, and α at both times $t + \Delta t$ and times $t + \Delta t/2$. Then when calculating a new ω_z from an old ω_z and an old α_z, the α_z that was taken to be that at the time halfway between the time of the old ω_z and the new ω_z. The same was done when calculating a new θ from an old θ using a value of ω_z. This half-step method is more work to set up, but is considerably more accurate.

15.9.2. Activity: Theoretical Reproducibility

a. Use the Chaotic Physical Pendulum Model with the file name T150902.XLS to make two "identical" time series graphs of the rotational displacement of the Chaotic Physical Pendulum; the term "identical" means that the initial conditions of the runs should be only slightly different (on the order of 0.01 radians or radians per second) from each other. Affix the graphs below.

b. For how long a time period are the two motions remain close to one another? What technique did you devise for determining this result?

15.10. CHAOS AND LAPLACIAN DETERMINISM

Physicists have studied the conditions required for a dynamical system to be chaotic. There are two requirements: (1) It takes three or more independent dynamical variables to describe the state of the system at any given time, and (2) the equation describing the net force or torque on the system must have a non-linear term that couples several of the variables.

If a pendulum system, like the one you studied, is driven at large amplitudes with a force that varies periodically in the presence of drag or damping forces, it can undergo chaotic motion. This is because the natural restoring force on the pendulum bob is a non-linear function of its rotational displacement from equilibrium. In fact, this force is given by $\tau_z = -mgR \sin\theta$ where m is the edge mass and θ is its rotational displacement. Indeed, three independent dynamical variables are needed to describe the state of the chaotic physical pendulum: (1) θ, its rotational position relative to equilibrium; (2) ω_z, the z-component of its rotational velocity; and (3) ϕ_0, the initial phase of the driving force. Thus, when some driving force amplitudes and frequencies combined with certain drag forces, it is possible for the uncertainties in the independent dynamical variables to grow exponentially.

Now that you have studied one chaotic dynamical system, you should reconsider the feasibility of Laplacian Determinism.

15.10.1. Activity: Laplacian Determinism Revisited

Consider complex systems, including humans, computers, sun, rain, tides, and galaxies. Based on what you have learned by using Newton's laws of motion and a set of known torques to model and predict the motion of your chaotic pendulum, what changes, if any, would you make to your answer to Activity 15.1.1 on page 390 about how predictable the future the future of the universe is? Please re-read your previous answer and give reasons for any changes you have made.

The chaotic dynamical system you have experimented with is one of many. The new science of chaos has fostered a deeper appreciation of the similarities in the behavior of complex non-linear dynamical systems. It has brought together mathematicians, physicists, meteorologists, economists, ecologists, chemists, and astronomers in a quest to render unpredictable phenomena more comprehensible. You have acquired some understanding of chaos and have learned techniques that can help you to contribute to this strange new science of chaos.