The Various Kinds of Centres of Simplices

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Abstract. A triangle in the Euclidean plane has various kinds of centres such as the *centroid* G, the *circumcentre* O, the *incentre* I, the *orthocentre* H, and the *cleavance centre* J. We find higher dimensional analogues of these centres of simplices in Euclidean n-space and in spherical n-space. Each centre is described as the point of intersection of certain hyperplanes (or great hyperspheres in the spherical case). Several theorems relating the various kinds of centres for triangles are generalized to higher dimensions. For example, we show that the centres O, G, and H are collinear and that the centres J, G, and I are collinear for any simplex in Euclidean n-space.

Keyword: simplices.

Introduction

A triangle in the Euclidean plane has various kinds of centres such as the centroid G (the point of intersection of the medians), the circumcentre O (the point of intersection of the perpendicular bisectors, which is the centre of the circumcircle), the incentre I (the point of the intersection of the angle bisectors, and also the centre of the inscribed circle), the orthocentre H (which is the common point of the altitudes), the cleavance centre J (which is the intersection of the cleavers),

and the nine-point centre (which is the centre of the circle passing through the midpoints of the sides, the midpoints of the lines joining the orthocentre to the vertices, and the feet of the altitudes).
In this paper, we find higher dimensional analogs of these centres for simplices in Euclidean n-space and also in spherical n-space. Each centre is described as the point of intersection of certain hyperplanes (or great hyperspheres in the spherical case). Several theorems relating the various kinds of centres for triangles are generalized to higher dimensions.

The Centres of Simplices in $\mathbb{R}^N$

An n-*simplex* is the set of all convex combinations of a set of $n+1$ affinely independent points. A 1-simplex is called a *line segment*, a 2-simplex is called a *triangle* and a 3-simplex is called a *tetrahedron*. Note that, the n-simplex $[a_0, a_1, ..., a_n]$ is the set

$$[a_0, a_1, ..., a_n] \;=\; \{ \sum_{i=0}^{n} t_i a_i \;\mid\; \sum_{i=0}^{n} t_i = 1,\; t_i \geq 0 \;\text{ for all }\; i \}.$$

The Centroid

Given an n-simplex $T = [a_0, a_1, ..., a_n]$ in $\mathbb{R}^N$, and given $0 \leq i, j \leq n$ with $i \neq j$, the *medial plane* of T at the edge $[a_i, a_j]$ is the $(n-1)$-plane M_{ij} which passes through the midpoint of $[a_i, a_j]$ and through all the other vertices $a_k, k \neq i, j$. Note that if T is a triangle, then its medial planes are in fact its medians.

Theorem 1. *The medial planes M_{ij} of an n-simplex $T = [a_0, a_1, ..., a_n]$ have a unique point of intersection G, called the* centroid *(or the* barycentre*) of T. It is given by*

$$(n+1)G \;=\; \sum_{i=0}^{n} a_i.$$

Proof. Since $G - a_k = \sum_{l\neq i,j,k} \frac{1}{n+1}(a_l - a_k) + \frac{2}{n+1}(\frac{a_i + a_j}{2} - a_k), G \in M_{ij}$ for all $i \neq j$. Note that the intersection of the medial planes M_{0j} with $j = 1, 2, ..., m$ is the $(n-m)$-plane passing through $\frac{1}{m+1}(\sum_{j=0}^{m} a_j)$ and the other vertices $a_{m+1}, a_{m+2}, ..., a_n$. In particular, G is the unique point of intersection of the medial planes M_{ij}. $\square$

Given an n-simplex $T = [a_0, a_1, ..., a_n]$ in $\mathbb{R}^N$, we define the *medial line* at a_i to be the line M_i passing through a_i and the centroid $g_i = \frac{1}{n}(\sum_{k \neq i} a_k)$ of the opposite face $T_i = [a_0, a_1, ..., \hat{a}_i, ..., a_n]$, where $\hat{a}_i$ indicates omission of the vertex a_i. Since the medial line M_i is the intersection of the medial planes M_{jk} with $j, k \neq i$, the medial lines of an n-simplex T also meet at the centroid G of T.

The Circumcentre

We define the *perpendicular bisector* of the edge $[a_i, a_j]$ to be the $(n-1)$-plane P_{ij} in $\langle T \rangle = a_k + \text{span}\{a_k - a_i \mid i \neq k\}$ which is perpendicular to the edge $[a_i, a_j]$ and passes through the midpoint of $[a_i, a_j]$.

Let $V(T) = \text{span}\{a_k - a_i \mid i \neq k\}$. We choose an orthonormal basis for the vector space $V(T)$. For fixed k with $0 \leq k \leq n$, let $A_k(T)$ denote the $n \times n$ matrix whose rows are the vector $a_i - a_k$ with respect to the chosen basis.

Theorem 2. *The perpendicular bisectors P_{ij} of an n-simplex T meet at a unique point O, called the* circumcentre *of T. If we fix k with $0 \leq k \leq n$, then O is given by*

$$O \quad = \quad a_k + \frac{1}{2} A^{-1} P$$

where $A = A_k(T)$ and P is the $n \times 1$ matrix whose rows are $|a_i - a_k|^2$.
Moreover, O is the centre of the $(n-1)$-sphere in $\langle T \rangle$, called the circumscribed sphere *of T which passes through each of the points a_i.*

Proof. Let $O = a_k + \frac{1}{2} A^{-1} P$. Since A is the matrix whose rows are $a_i - a_k$ and P is the matrix whose rows are $|a_i - a_k|^2$, this means that $\langle a_i - a_k, 2(O - a_k) \rangle = \langle a_i - a_k, a_i - a_k \rangle$. So $\langle a_i - a_k, 2O \rangle = \langle a_i - a_k, a_i + a_k \rangle$ for all $i \neq k$. For all $i \neq j$, we have $\langle a_j - a_i, 2O \rangle = \langle a_j - a_k + a_k - a_i, 2O \rangle = \langle a_j - a_k, a_j + a_k \rangle - \langle a_i - a_k, a_i + a_k \rangle = \langle a_j, a_j \rangle - \langle a_i, a_i \rangle = \langle a_j - a_i, a_j + a_i \rangle$. This shows that O is the point of intersection of the perpendicular bisectors P_{ij}. Fix k we shall show that O is the only point of intersection of the perpendicular bisectors $P_{ki}, i \neq k$ (hence also of all the $P_{ij}, i \neq j$). Suppose that $x \in P_{ki}, i \neq k$. Then $\langle a_i - a_k, 2x \rangle = \langle a_i - a_k, a_i + a_k \rangle$. So $\langle a_i - a_k, 2(x - a_k) \rangle = \langle a_i - a_k, a_i - a_k \rangle$ or equivalently, $2A(x - a_k) = P$. Thus $x = O$.
Note that the perpendicular bisectors P_{ij} is the set of all $x \in \langle T \rangle$ such that $|x - a_i| = |x - a_j|$. We see that O is the only point with the property that $|O - a_i| = |O - a_j|$ for all i, j. Then the $(n-1)$-sphere in $\langle T \rangle$, whose centre is O and radius is $|O - a_k|$ passes through each of the points a_i. $\square$

Given an n-simplex $T = [a_0, a_1, ..., a_n]$ in $\mathbb{R}^N$, let P_i be the line which is perpendicular to the face T_i and passes through the circumcentre o_i of T_i. Since P_i is the intersection of the perpedicular bisectors $P_{jk}, j \neq k$, the lines P_i also meet at the circumcentre O of T.

The Orthocentre

In $\mathbb{R}^2$, it can be shown that three altitudes of any triangle meet at a point H, called the *orthocentre* of the triangle. In $\mathbb{R}^N$, however the altitudes of an n-simplex do not always intersect, so we shall give an alternate definition for the orthocentre of an n-simplex.

Given an n-simplex $T = [a_0, a_1, ..., a_n]$ in $\mathbb{R}^N$, we define the *altitudinal plane* at the edge $[a_i, a_j]$ to be the $(n-1)$-plane A_{ij} in $\langle T \rangle$ which is perpendicular to the edge $[a_i, a_j]$ and passes through the centroid g_{ij} of the opposite $(n-2)$-face $T_{ij} = [a_0, a_1, ..., \hat{a}_i, ..., \hat{a}_j, ..., a_n]$. Note that if T is a triangle then the altitudinal planes are in fact its altitudes.

Theorem 3. *The altitudinal planes A_{ij} have a unique point of intersection H, called the* orthocentre *of T. If we fix k with $0 \leq k \leq n$ and let $A = A_k(T)$ then H is given by*

$$H = a_k + A^{-1}K$$

where K is the $n \times 1$ matrix whose rows are $\langle a_i - a_k, g_{ik} - a_k \rangle$.

Proof. Let $H = a_k + A^{-1}K$. Since A is the matrix whose rows are $a_i - a_k$ and K is the matrix whose rows are $\langle a_i - a_k, g_{ik} - a_k \rangle$, this means that $\langle a_i - a_k, H - a_k \rangle = \langle a_i - a_k, g_{ik} - a_k \rangle$ for all $i \neq k$. So $\langle a_i - a_k, H - g_{ik} \rangle = 0$ for all $i \neq k$. Since $g_{ij} = g_{kj} - \frac{1}{n-1}(a_i - a_k)$, we have $\langle a_i - a_j, H - g_{ij} \rangle = 0$. This shows that H is the intersection of all altitudinal planes A_{ij}. Fix k, we shall show that H is the only point of intersection of the altitudinal planes $A_{ki}, i \neq k$ (hence also of all the $A_{ij}, i \neq j$). Suppose that $x \in A_{ki}, i \neq k$. Then $\langle a_i - a_k, x - g_{ik} \rangle = 0$. So $\langle a_i - a_k, x - a_k \rangle = \langle a_i - a_k, g_{ik} - a_k \rangle$. That is $A(x - a_k) = K$. Thus $x = H$. □

In $\mathbb{R}^2$, H, O and G of a triangle all lie on a line called the *Euler line* of a triangle, and $H + 2O = 3G$. We shall show more generally that the points H, O and G of an n-simplex in $\mathbb{R}^N$ all lie on a line, also called the Euler line.

Theorem 4. *H, O and G of an n-simplex in $\mathbb{R}^N$ are collinear and*

$$(n-1)H + 2O = (n+1)G.$$

Proof. Let $A = A_0(T)$.

$$\text{By Theorem 3, we have } H - a_0 = \begin{bmatrix} \langle a_1 - a_0, g_{01} - a_0\rangle \\ \langle a_2 - a_0, g_{02} - a_0\rangle \\ \vdots \\ \langle a_n - a_0, g_{0n} - a_0\rangle \end{bmatrix}$$

$$= \frac{1}{n-1}A^{-1}\begin{bmatrix} \langle a_1 - a_0, \sum_{i=0}^n a_i - a_1 - na_0\rangle \\ \langle a_2 - a_0, \sum_{i=0}^n a_i - a_2 - na_0\rangle \\ \vdots \\ \langle a_n - a_0, \sum_{i=0}^n a_i - a_n - na_0\rangle \end{bmatrix}.$$

$$\text{By Theorem 2, we have } O - a_0 = \frac{1}{2}A^{-1}\begin{bmatrix} \langle a_1 - a_0, a_1 - a_0\rangle \\ \langle a_2 - a_0, a_2 - a_0\rangle \\ \vdots \\ \langle a_n - a_0, a_n - a_0\rangle \end{bmatrix}.$$

$$\text{Thus } (n-1)(H - a_0) + 2(O - a_0) = A^{-1}\begin{bmatrix} \langle a_1 - a_0, \sum\limits_{i=0}^n a_i - (n+1)a_0\rangle \\ \langle a_2 - a_0, \sum\limits_{i=0}^n a_i - (n+1)a_0\rangle \\ \vdots \\ \langle a_n - a_0, \sum\limits_{i=0}^n a_i - (n+1)a_0\rangle \end{bmatrix}$$

$$= A^{-1}A(\sum_{i=0}^n a_i - (n+1)a_0)$$

$$= (n+1)(G - a_0).$$

Therefore $(n-1)H + 2O = (n+1)G$. □

Given an n-simplex $T = [a_0, a_1, ..., a_n]$ in $\mathbb{R}^N$, let g_i be the centroid of the face T_i. We have that $T' = [g_0, g_1, ..., g_n]$ is an n-simplex in $\mathbb{R}^N$. The n-simplex T' is called the *medial simplex* of T.

The nine-point theorem says that for any triangle, the midpoints of the sides, the feet of the altitudes and the midpoints of the segments joining the vertices to the orthocentre all lie on a circle which is called the *nine-point circle* of a triangle. The centre of this circle lies midway between the orthocentre and the circumcentre. Note that this centre is the circumcentre of the medial triangle.

We shall generalize the nine-point theorem to the higher dimension.

Theorem 5. (The 3(n+1)-point Theorem) *Given an n-simplex $T = [a_0, a_1, ..., a_n]$ in $\mathbb{R}^N$, let g_i be the centroid of the face T_i, h_i the point which lies $(\frac{1}{n})^{th}$ of the way from H to a_i and k_i the point of the intersection of $\langle T_i \rangle$ with the line H_i which passes through h_i and is perpendicular to $\langle T_i \rangle$. Then the points g_i, h_i and k_i all lie on the circumscribed sphere $S(O', R')$ of T'.*

Proof. By Theorem 2, the circumscribed sphere $S(O', R')$ of T' passes through g_i for all i. Since $h_i = \frac{(n-1)H+a_i}{n}$ and $g_i = \frac{(n+1)G-a_i}{n}, O' = \frac{g_i+h_i}{2}$. So $[g_i, h_i]$ is a diameter of $S(O', R')$. Thus h_i lies on $S(O', R')$ for all i. Since $[h_i, k_i]$ is perpendicular to $\langle T_i \rangle$ and $[g_i, k_i] \subseteq \langle T_i \rangle, [h_i, k_i]$ is perpendicular to $[g_i, k_i]$. Thus the angle k_i in the triangle $[g_i, h_i, k_i]$ is $\pi/2$. Since $[g_i, h_i]$ is a diameter of $S(O', R')$, we have $k_i \in S(O', R')$. Hence the points g_i, h_i and k_i all lie on the circumscribed sphere $S(O', R')$ of T'. □

The Incentre

Given an n-simplex $T = [a_0, a_1, ..., a_n]$ in $\mathbb{R}^N$, there are two normal vectors $\pm m_i$ for the face T_i, that is there are two vectors $\pm m_i \in V(T)$ such that $|m_i| = 1$ and $\langle m_i, a_j - a_k \rangle = 0$ for all $k, j \neq i$. If m_i is the normal vector such that $\langle m_i, a_i - a_k \rangle > 0$ for all $k \neq i$, we call m_i the *inward normal vector* for the face T_i.
We define the angle between two $(n-1)$-faces of T as follows;
The *angle* between two faces T_i and T_j is the angle

$$\theta(T_i, T_j) \;=\; \arccos |\langle m_i, m_j \rangle| \in (0, \frac{\pi}{2})$$

where m_i and m_j are normal vectors of T_i and T_j, respectively. An *angle bisector* of T_i and T_j is an $(n-1)$-plane B in $\langle T \rangle$ which contains T_{ij} and $\theta(B, T_i) = \theta(B, T_j)$. Note that, for any n-simplex T in $\mathbb{R}^N$, there are two angle bisectors of T_i and T_j. They are the two $(n-1)$-planes in $\langle T \rangle$ with orthogonal vectors $m_i \pm m_j$.
We define the *internal angle bisector* of T_i and T_j to be the angle bisector B_{ij} of the faces T_i and T_j with orthogonal vector $m_i - m_j$.
For fixed k with $0 \leq k \leq n$, let $B_k(T)$ denote the $n \times n$ matrix whose rows are the vectors $m_i - m_k$, $i \neq k$ with respect to the chosen basis for $V(T)$.

Theorem 6. *The internal angle bisectors B_{ij} have a unique point of intersection I, called the* incentre *of T. If we fix k and let $B = B_k(T)$ then I is given by*

$$I \;=\; a_k + B^{-1}M$$

where M is an $n \times 1$ matrix whose rows are $\langle m_i - m_k, a_{l_i} - a_k \rangle$, $l_i \neq i, k$.
Moreover, I is the centre of $(n-1)$-sphere in T, called the inscribed sphere *of T whose radius is $d_R(I, \langle T_i \rangle)$.*

Proof. Let $I = a_k + B^{-1}M$. Since B is the matrix whose rows are $m_i - m_k$ and M is the matrix whose rows are $\langle m_i - m_k, a_{l_i} - a_k\rangle$, this means that $\langle m_i - m_k, I - a_k\rangle = \langle m_i - m_k, a_{l_i} - a_k\rangle$ for all $l_i \neq i, k$. For $i \neq j, \langle m_i - m_j, I - a_k\rangle = \langle m_i - m_k, I - a_k\rangle - \langle m_j - m_k, I - a_k\rangle = \langle m_i - m_k, a_{l_p} - a_k\rangle - \langle m_j - m_k, a_{l_p} - a_k\rangle = \langle m_i - m_j, a_{l_p} - a_k\rangle, l_p \neq i, j, k$. This shows that I is the intersection of all internal angle bisectors B_{ij}. Fix k, we shall show that I is the only point of intersection of the internal angle bisectors $B_{ki}, i \neq k$ (hence also of all the $B_{ij}, i \neq j$). Suppose that $x \in B_{ki}, i \neq k$. Then $\langle m_i - m_k, x - a_{l_i}\rangle = 0$. So $\langle m_i - m_k, x - a_k\rangle = \langle m_i - m_k, a_{l_i} - a_k\rangle$ for all $l_i \neq i, k$. That is $B(x - a_k) = M$. Thus $x = I$.
Note that for $x \in \langle T\rangle$, we have $d_R(x, \langle T_i\rangle) = d_R(x, \langle T_j\rangle)$ if and only if x lies on one of the angle bisectors of T_i and T_j. Then I is the centre of $(n-1)$-sphere in T whose radius is $d_R(I, \langle T_i\rangle)$. □

The Cleavance Centre

Given an n-simplex $T = [a_0, a_1, ..., a_n]$ in $\mathbb{R}^N$, we define the *cleavance plane* at the edge $[a_i, a_j]$ to be the $(n-1)$-plane Q_{ij} which passes through the midpoint of the edge $[a_i, a_j]$ and is parallel to the internal angle bisector of T_i and T_j. Note that the cleavance planes of a triangle are its cleavers.

Theorem 7. *The cleavance planes Q_{ij} have a unique point of intersection J, called the* cleavance centre. *If we fix k, and let $B = B_k(T)$ then J is given by*

$$J \quad = \quad a_k + \frac{1}{2}B^{-1}Q$$

where Q is the $n \times 1$ matrix whose rows are $\langle m_i - m_k, a_i - a_k\rangle$.

Proof. Let $J = a_k + \frac{1}{2}B^{-1}Q$. Since B is the matrix whose rows are $m_i - m_k$ and Q is the matrix whose rows are $\langle m_i - m_k, a_i - a_k\rangle$, this means that $\langle m_i - m_k, 2(J - a_k)\rangle = \langle m_i - m_k, a_i - a_k\rangle$ for all $i \neq k$. For $i \neq j, \langle m_i - m_j, 2J\rangle = \langle m_i - m_k, a_i + a_k\rangle - \langle m_j - m_k, a_j + a_k\rangle = \langle m_i - m_k, a_i + a_j\rangle - \langle m_j - m_k, a_i + a_j\rangle = \langle m_i - m_j, a_i + a_j\rangle$. This shows that J is the intersection of all cleavance planes Q_{ij}. Fix k, we shall show that J is the only point of intersection of the cleavance planes $Q_{ki}, i \neq k$ (hence also of all the $Q_{ij}, i \neq j$). Suppose that $x \in Q_{ki}, i \neq k$. Then $\langle m_i - m_k, 2x\rangle = \langle m_i - m_k, a_i + a_k\rangle$. So $\langle m_i - m_k, 2(x - a_k)\rangle = \langle m_i - m_k, a_i - a_k\rangle$ for all $i \neq k$. That is $2B(x - a_k) = Q$. Thus $x = J$. □

Theorem 8. *I, J and G of an n-simplex in $\mathbb{R}^N$ are collinear and*

$$(n-1)I + 2J = (n+1)G.$$

Proof. Let $B = B_0(T)$. By Theorem 6,

$$I - a_0 = B^{-1}\begin{bmatrix} \langle m_1 - m_0, a_{j_1} - a_0\rangle \\ \langle m_2 - m_0, a_{j_2} - a_0\rangle \\ \vdots \\ \langle m_1 - m_0, a_{j_n} - a_0\rangle \end{bmatrix} \quad \text{where} \quad j_i \neq 0, i.$$

Since $\langle m_i - m_0, a_j\rangle = \langle m_i - m_0, a_l\rangle$ for all $j, l \neq i, 0$, we have $(n-1)\langle m_i - m_0, a_{j_i}\rangle = \langle m_i - m_0, \sum_{k\neq 0,i} a_k\rangle$.

So $(n-1)(I - a_0) = B^{-1}\begin{bmatrix} \langle m_1 - m_0, \sum_{i=0}^{n} a_i - a_1 - na_0\rangle \\ \langle m_2 - m_0, \sum_{i=0}^{n} a_i - a_2 - na_0\rangle \\ \vdots \\ \langle m_n - m_0, \sum_{i=0}^{n} a_i - a_n - na_0\rangle \end{bmatrix}$.

By Theorem 7, $2(J - a_0) = B^{-1}\begin{bmatrix} \langle m_1 - m_0, a_1 - a_0\rangle \\ \langle m_2 - m_0, a_2 - a_0\rangle \\ \vdots \\ \langle m_1 - m_0, a_n - a_0\rangle \end{bmatrix}$.

$$\text{Thus } (n-1)(I - a_0) + 2(J - a_0) = B^{-1}\begin{bmatrix} \langle m_1 - m_0, \sum_{i=0}^{n} a_i - (n+1)a_0\rangle \\ \langle m_2 - m_0, \sum_{i=0}^{n} a_i - (n+1)a_0\rangle \\ \vdots \\ \langle m_n - m_0, \sum_{i=0}^{n} a_i - (n+1)a_0\rangle \end{bmatrix}$$
$$= B^{-1}B(\sum_{i=0}^{n} a_i - (n+1)a_0)$$
$$= (n+1)(G - a_0).$$

Hence $(n-1)I + 2J = (n+1)G$. □

The Centres of Simplices in $\mathbb{S}^n$

The *n-sphere* $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ is the set of all points u in $\mathbb{R}^{n+1}$ such that $\|u\| = 1$. Given $u_1, u_2, ..., u_{k+1}$ in $\mathbb{S}^n$, we define

$$(u_1, u_2, ..., u_{k+1}) \quad = \quad \{ \sum_{i=1}^{k+1} t_i u_i \in \mathbb{S}^n \mid t_i \geq 0 \text{ for all } i \}.$$

If $\{u_1, u_2, ..., u_{k+1}\}$ is linearly independent in $\mathbb{R}^{n+1}$ then $(u_1, u_2, ..., u_{k+1})$ is called a *k-simplex* in $\mathbb{S}^n$. A 1-simplex is called an *arc* and a 2-simplex is called a *spherical triangle*.

Given a k-simplex $S = (u_1, u_2, ..., u_{k+1})$ in $\mathbb{S}^n$, we write $[0, S]$ for the $(k+1)$-simplex in $\mathbb{R}^{n+1}$ given by $[0, S] = [0, u_1, u_2, ..., u_{k+1}]$.

The centroid

Given an n-simplex $S = (u_1, u_2, ..., u_{n+1})$ in $\mathbb{S}^n$, we define the *medial great hypersphere* of the edge (u_i, u_j) to be the great hypersphere SM_{ij} in $\mathbb{S}^n$ which passes through the midpoint of the edge (u_i, u_j) and through the points $u_k, k \neq i, j$.

Theorem 9. *The medial great hyperspheres SM_{ij} of an n-simplex $S = (u_1, u_2, ..., u_{n+1})$ in $\mathbb{S}^n$ have two points of intersection $\pm\frac{G}{|G|}$ where G is the centroid of the simplex $[0, S]$ in $\mathbb{R}^{n+1}$.*

Proof. Since $G = \frac{1}{n+2}(\sum_{i=1}^{n+1} u_i)$, and $SM_{ij} = \mathbb{S}^n \cap M_{ij}$ where M_{ij} is the medial plane at the edge $[u_i, u_j]$ of $[0, S]$ in $\mathbb{R}^{n+1}$, the medial great hyperspheres meet at the two points $\pm\frac{G}{|G|}$. □

Note that $\frac{G}{|G|} \in S$ but $-\frac{G}{|G|} \notin S$. We call the point $\frac{G}{|G|}$ the *centroid* of S and denote it by G_s.

The Circumcentre

Let SP and SQ be two great hyperspheres in $\mathbb{S}^n$, say $SP = P \cap \mathbb{S}^n$ and $SQ = P \cap \mathbb{S}^n$ for some hyperspaces P and Q in $\mathbb{R}^{n+1}$. The *angle* between SP and SQ in $\mathbb{S}^n, \theta_s(SP, SQ)$ is given by $\theta_s(SP, SQ) := \theta(P, Q)$. Let $S = (u_1, u_2, ..., u_{n+1})$ be an n-simplex in $\mathbb{S}^n$. The *perpendicular bisector* of the edge (u_i, u_j) is the great hypersphere SP_{ij} in $\mathbb{S}^n$ which is perpendicular to the edge (u_i, u_j) and passes through the midpoint of the edge (u_i, u_j).

Theorem 10. *The perpendicular bisectors SP_{ij} of any n-simplex in $\mathbb{S}^n$ meet at two points $\pm\frac{O}{|O|}$ where O is the circumcentre of the simplex $[0, S]$ in $\mathbb{R}^{n+1}$.*

Proof. Since $SP_{ij} = \mathbb{S}^n \cap P_{ij}$ where P_{ij} is the perpendicular bisector of the edge $[u_i, u_j]$ of $[0, S]$ in $\mathbb{R}^{n+1}$ and $\bigcap_{i \neq j} P_{ij}$ is the line passing through 0 and O, we have the perpendicular bisectors SP_{ij} meet at the two points $\pm\frac{O}{|O|}$. □

Let $O_s = \frac{O}{|O|}$. Since the perpendicular bisector of (u_i, u_j) is the set of all points $u \in \mathbb{S}^n$ such that $d_s(u, u_i) = d_s(u, u_j)$ for all $i, j, d_s(O_s, u_i) = d_s(O_s, u_j)$ for all i, j. Let $R_s = d_s(O_s, u_i) = \theta(O_s, u_i) = \arccos\langle O_s, u_i\rangle \in (0, \frac{\pi}{2})$. The sphere $S(O_s, R_s) = \{u \in \mathbb{S}^n \mid d_s(O_s, u) = R_s\}$ is called the *circumscribed sphere* of S. Note that $S(O_s, R_s)$ passes through each of the points u_i.

The Incentre

Given a point u and a great hypersphere SP in $\mathbb{S}^n$, the *distance* between u and SP, denoted by $d_s(u, SP)$, is given by

$$d_s(u, SP) := \inf \{d_s(u, v) \mid v \in SP\} = \arcsin d(u, P).$$

An *angle bisector* of two great hypersphere SP and SQ is a great hypersphere SB in $\mathbb{S}^n$ which contains $SP \cap SQ$ and $\theta_s(SB, SP) = \theta_s(SB, SQ)$. In other words, SB is the intersection of $\mathbb{S}^n$ with an angle bisector B of P and Q at $P \cap Q$ in $\mathbb{R}^{n+1}$.

We define the *inward pole* of the face S_i to be the pole m_i such that $\langle m_i, u_i \rangle > 0$. Equivalently, m_i is the inward normal vector for the n-face $[0, S]_i$ which is opposite to the vertex u_i of the simplex $[0, S]$ in $\mathbb{R}^{n+1}$.

Let m_i and m_j be the inward poles of S_i and S_j, respectively. The *internal angle bisector* of S_i and S_j is the great hypersphere SB_{ij} which passes through $u_k, k \neq i, j$ with the pole $\frac{m_i - m_j}{|m_i - m_j|}$.

Theorem 11. *The angle bisectors SB_{ij} of an n-simplex in $\mathbb{S}^n$ meet at two points $\pm\frac{I}{|I|}$ where I is the incentre of the simplex $[0, S]$ in $\mathbb{R}^{n+1}$.*

Proof. Since $SB_{ij} = \mathbb{S}^n \cap B_{ij}$ where B_{ij} is the internal angle bisector of the edge $[u_i, u_j]$ of $[0, S]$ in $\mathbb{R}^{n+1}$ and $\bigcap_{i \neq j} B_{ij}$ is the line passing through 0 and I, we have the internal angle bisectors SB_{ij} meet at the two points $\pm\frac{I}{|I|}$. □

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