THE DELIGNE-ILLUSIE THEOREM

PIOTR ACHINGER

1. INTRODUCTION

The aim of this note is to discuss the now classical result of Deligne and Illusie [1] on ,,lifting modulo p^2 and decomposition of the de Rham complex". Before stating the theorem, let us explain the main motivations behind it.

1.1. **Algebraic geometry over $\mathbb{C}$.** Let $X/\mathbb{C}$ be a smooth proper complex variety of dimension d. To this data (the scheme X together with the morphism to $\operatorname{Spec}\mathbb{C}$) we can associate a complex manifold X^{an} together with a morphism of ringed spaces $\tau : X^{an} \to X$. Then the following results hold.

1.1.1. *Holomorphic Poincaré lemma.* The complex of sheaves on X^{an}

$$0 \to \underline{\mathbb{C}} \to \Omega^0_{X^{an}} \xrightarrow{d} \Omega^1_{X^{an}} \to \ldots \to \Omega^d_{X^{an}} \to 0$$

is exact. Therefore the singular cohomology groups $H^*(X,\mathbb{C}) = H^*(X,\underline{\mathbb{C}})$ equal the hypercohomology of the de Rham complex $H^*_{dR}(X/\mathbb{C}) = \mathsf{H}(X,\Omega^{\cdot}_{X/\mathbb{C}}) = \mathsf{H}^*(X^{an},\Omega^{\cdot}_{X^{an}})$. The last equality follows from GAGA and the spectral sequence below. Therefore, the singular cohomology groups of X^{an} can be defined in a purely algebraic way.

1.1.2. *Hodge to de Rham spectral sequence.* We have the hypercohomology ,,Hodge to de Rham" spectral sequence

$$E_1^{ij} = H^j(X,\Omega^i_{X/\mathbb{C}}) \quad \Rightarrow \quad H^*_{dR}(X/\mathbb{C}). \tag{1}$$

If moreover X is projective (hence Kähler), Hodge theory allows us to prove

1.1.3. (**DSS**). The spectral sequence (1) degenerates at E_1. That is, its differentials on all pages are zero.

Proof. Since all E_1^{ij} are finite-dimensional over $\mathbb{C}$, it suffices to show that

$$\sum_{i+j=k} \dim E_1^{ij} = \dim E_\infty^k.$$

This follows from the discussion above and the Hodge decomposition

$$H^k(X^{an},\mathbb{C}) = \bigoplus_{i+j=k} H^j(X^{an},\Omega^i_{X^{an}}). \qquad \square$$

1.1.4. (**KV**). We have the **Kodaira-Akizuki-Nakano vanishing**: for every ample line bundle L on X, we have

$$H^j(X, \Omega^i_X \otimes L) = 0 \quad \text{for} \quad i+j > d.$$

However, (**DSS**) and (**KV**) fail in characteristic $p > 0$ (with counterexamples first found by Mumford and Raynaud, respectively). We could also wonder whether we can find algebraic proofs of (**DSS**) and (**KV**) .

1.2. **Algebraic geometry in characteristic** $p > 0$**.** Now let X/k be a smooth proper variety of dimension d over an algebraically closed field k of characteristic $p > 0$. Instead of $X^{an} \to X$ as above, we should now look at the relative Frobenius $F : X \to X'$. Instead of the holomorphic Poincaré lemma, we have the

1.2.1. *Cartier isomorphism.* We have isomorphisms

$$C^k : \mathscr{H}^k(F_* \Omega^{\cdot}_{X/k}) \simeq \Omega^k_{X'/k}.$$

This means that the de Rham complex $(\Omega^{\cdot}_{X/k}, d)$ and the ,,Hodge complex" $(\Omega^{\cdot}_{X/k}, 0)$ have the same cohomology. We can therefore ask the following question:

1.2.2. (**QI**). Does there exist a quasi-isomorphism $\psi^{\cdot} : (F_* \Omega^{\cdot}_{X/k}, d) \simeq (\Omega^{\cdot}_{X'/k}, 0)$ inducing the Cartier isomorphisms C^i on cohomology?

1.3. **Preliminary observations.**

Lemma 1. (**QI**) *implies* (**DSS**) *and* (**KV**) .

Proof. To prove (**DSS**) , observe that

$$\begin{aligned}\sum_{i+j=k} \dim E_1^{ij} = \sum_{i+j=k} h^j(X, \Omega^i_{X/k}) &= \dim \mathsf{H}^k(\Omega^{\cdot}_{X/k}, 0) = \dim \mathsf{H}^k(\Omega^{\cdot}_{X'/k}, 0) \\ &= \dim \mathsf{H}^k(F_* \Omega^{\cdot}_{X/k}, d) \\ &= \dim \mathsf{H}^k(\Omega^{\cdot}_{X/k}, d) = \dim H^k_{dR}(X/k).\end{aligned}$$

For (**KV**) , let L be an ample line bundle on X. Recall that the Picard groups of X and X' are canonically isomorphic, let L' be the line bundle on X' corresponding to L. Then

$$\begin{aligned}\sum_{i+j=k} h^j(X, \Omega^i_{X/k} \otimes L) &= \dim \mathsf{H}^k(\Omega^{\cdot}_{X/k} \otimes L, 0) = \dim \mathsf{H}^k(\Omega^{\cdot}_{X'/k} \otimes L', 0) \\ &= \dim \mathsf{H}^k((F_* \Omega^{\cdot}_{X/k}) \otimes L', d \otimes id) \\ &= \dim \mathsf{H}^k(F_*(\Omega^{\cdot}_{X/k} \otimes F^* L'), d \otimes \nabla_{can}) \\ &= \dim \mathsf{H}^k(\Omega^{\cdot}_{X/k} \otimes F^* L', d \otimes \nabla_{can}) \\ &= \dim \mathsf{H}^k(F_*(\Omega^{\cdot}_{X/k} \otimes L^{\otimes p}), d \otimes \nabla_{can}) \\ &\le \dim \mathsf{H}^k(F_*(\Omega^{\cdot}_{X/k} \otimes L^{\otimes p}), 0) = \sum_{i+j=k} h^j(X, \Omega^i_{X/k} \otimes L^{\otimes p}).\end{aligned}$$

(More generally, for any locally free sheaf E on X', we have $h^k_{Hodge}(X', E) \leq h^k_{Hodge}(X, F^*E)$ where $h^k_{Hodge}(E) = \sum_{i+j=k} h^j(X, \Omega^i_{X/k} \otimes E)$. It remains to observe that since L is ample, By Serre vanishing we have $h^j(X, \Omega^i_{X/k} \otimes L^{\otimes p^n})$ for $n \gg 0$. □

Recall that the inverse of the Cartier operator on one-forms is

$$C^{-1} : \Omega^1_{X'/k} \to \mathscr{H}^1(F_*\Omega^{\cdot}_{X/k} \quad C^{-1}(dg) = [g^{p-1}dg].$$

Here „$g^{p-1}dg$" is not really a well-defined cycle, but its image in cohomology is. Finding a quasi-isomorphism (**QI**) should in particular solve the problem of defining „$g^{p-1}dg = \frac{1}{p}dg^p = \frac{0}{0}$". The idea here is that $\frac{1}{p}dg^p$ should make sense „modulo p^2".

More precisely, assume that we have a smooth lifting $\widetilde{X}/W_2(k)$ of X/k, together with a lift of the Frobenius $\widetilde{F} : \widetilde{X} \to \widetilde{X}'$, where $\widetilde{X}' = \widetilde{X} \times W_2(k)$ (pull-back of X along the canonical lift of the Frobenius of k to $W_2(k)$) is a smooth lifting of X'/k. Given a local section ω of $\Omega^1_{X'/k}$, extend it to a section $\widetilde{\omega}$ of $\Omega^1_{\widetilde{X}'/W_2(k)}$. Then since $F^* : \Omega^1_{X'/k} \to F_*\Omega^1_{X/k}$ is zero, $d\widetilde{\omega}$ lies in $p\Omega^1_{\widetilde{X}'/W_2(k)}$ which is isomorphic to $F_*\Omega^1_{X/k}$ as $\widetilde{F}$ is flat.

Lemma 2. *The map $\Omega^1_{X'/k} \to F_*\Omega^1_{X/k}$, associated to a smooth lifting of (X, F) to $W_2(k)$, discussed above, is well-defined and induces the Cartier isomorphism after passing to cohomology.*

2. The Main Theorem and its corollaries

Let k be a perfect field of characteristic $p > 0$ and let X/k be smooth of dimension d.

Theorem 1. *Any smooth lifting of X/k to $\widetilde{X}/W_2(k)$ induces a quasi-isomorphism*

$$\varphi : \tau_{<p}(\Omega^{\cdot}_{X'/k}, 0) \longrightarrow \tau_{<p}(F_*\Omega^{\cdot}_{X/k}, d) \tag{2}$$

which induces the inverse of the Cartier isomorphism on cohomology.

Corollary 3. *If X lifts to $W_2(k)$ and $\dim X < p$ then* (**QI**) *holds.*

Corollary 4. (**DSS**) *and* (**KV**) *hold for X/K smooth and proper over an algebraically closed field K of characteristic zero.*

Proof. We need to reduce X to positive characteristic in such a way that the new variety lifts modulo p^2.

To do this, first find a model $\mathscr{X}/A$ of X/K where A is a finitely generated domain (over $\mathbb{Z}$). We can pass to an affine open of $\operatorname{Spec} A$ over which $\mathscr{X}/A$ is smooth and the sheaves $R^j\pi_*\Omega^i_{\mathscr{X}/A}$ and $R^k\pi_*\Omega^{\cdot}_{\mathscr{X}/A}$ are locally free (hence commute with arbitrary base change, so their ranks are $h^j(X, \Omega^i_{X/K}$ and $\dim H^k_{dR}(X/K)$, respectively).

The second step is to reduce to a situation which is étale over $\operatorname{Spec}\mathbb{Z}$. For this, we reduce to a finite flat case by picking a homomorphism $A \otimes \mathbb{Q} \to \bar{\mathbb{Q}}$ and letting B be the image of A in $\bar{\mathbb{Q}}$. Then B is a quotient of A which is quasi-finite and torsion-free (hence flat) over $\operatorname{Spec}\mathbb{Z}$. There exists an affine open $\operatorname{Spec} B'$ of $\operatorname{Spec} B$ over which the projection to $\operatorname{Spec}\mathbb{Z}$ is étale.

In the third step, we pick maximal ideal $\mathfrak{m}$ of B' with residue field k of characteristic $p > d$. Since $\operatorname{Spec} B' \to \operatorname{Spec} \mathbb{Z}$ is unramified at $\mathfrak{m}$, we have $B'/\mathfrak{m}^2 \simeq W_2(k)$. Therefore $X_0 := \mathscr{X} \times_A k$ is of dimension $> p$ and has a smooth lift to $W_2(k)$ (namely, $\widetilde{X}_0 = \mathscr{X} \times_A B'/\mathfrak{m}^2$). By the Theorem, X_0 satisfies (**QI**), hence (**DSS**). But the Hodge and de Rham numbers of X and X_0 are the same, so by the usual argument we get (**DSS**) for X.

For (**KV**), we follow the above procedure for the pair (X, L) for an ample L (we have to ensure further that the sheaves $R^j\pi_*\Omega^i_{\mathscr{X}/A} \otimes \mathscr{L}$ are locally free and that $\mathscr{L}$ is relatively ample over A). □

3. Proof of the Main Theorem

Lemma 5. *Assume that there exists a morphism (in the derived category of X') $\varphi^1 : \Omega^1_{X'/k}[-1] \to F_*\Omega^\cdot_{X/k}$ inducing C^{-1} on $\mathscr{H}^1$. Then the Main Theorem holds, that is, φ^1 can be extended to a quasi-isomorphism* (2).

Proof. We exploit the multiplicative structures on both the source and the target, defining φ^i, $i < p$ to be the composition

$$\Omega^i_{X'/k} \xrightarrow{a} (\Omega^1_{X'/k})^{\otimes i} \xrightarrow{(\varphi^1)^{\otimes i}} (F_*\Omega^\cdot_{X/k})^{\otimes i} \xrightarrow{prod} F_*\Omega^\cdot_{X/k}.$$

The map a is the antisymmetrization

$$a(\omega_1 \wedge \ldots \wedge \omega_i) = \frac{1}{i!}\sum_\sigma (-1)^\sigma \omega_{\sigma(1)} \otimes \ldots \otimes \omega_{\sigma(i)}$$

which makes sense only for $i < p$. (This is the only place where we use the assumption that $i < p$). □

Proposition 6. *Under the assumptions of the Theorem, there exists a map (in the derived category) $\varphi^1 : \Omega^1_{X'/k}[-1] \to F_*\Omega^\cdot_{X/k}$ inducing the Cartier isomorphism C^{-1} on $\mathscr{H}^1$.*

We sketch three proofs of this statement.

Original proof. By Lemma 2, we know that φ^1 exists locally on X, since the pair $(U, F|_U)$ lifts to $W_2(k)$ for all U affine. We would like to glue the various φ^1 obtained this way to a global φ^1. This is tricky because we have to glue in the derived category.

Recall that a morphism $C^\cdot \to D^\cdot$ in the derived category can be represented as a „roof" of honest maps of complexes $C^\cdot \to E^\cdot \leftarrow D^\cdot$ with $D^\cdot \to E^\cdot$ a quasi-isomorphism. In our case $C^\cdot = \Omega^\cdot_{X'/k}$ and $D^\cdot = F_*\Omega^\cdot_{X/k}$. Since we know how to construct our morphism locally on an affine open cover $\mathfrak{U} = \{\bigcup U_i\}_{i \in I}$, it makes sense to take for $E^\cdot$ the Cech complex

$$F_*\mathscr{C}(\mathfrak{U}, \Omega^\cdot_{X/k}) = F_* \operatorname{Tot}(\mathscr{C}^\cdot(\mathfrak{U}, \Omega^\cdot_{X/k}))$$

where $\mathscr{C}^\cdot(\mathfrak{U}, \Omega^\cdot_{X/k})$ is the bicomplex with

$$\mathscr{C}^{ij} = \prod_{s_1, \ldots, s_j \in I} \iota_{s_1\cdots s_j *}\iota^*_{s_1\cdots s_j}\Omega^i_{X/k}$$

(here $\iota_{s_1\cdots s_j}$ is the inclusion of $U_{s_1} \cap \ldots \cap U_{s_j}$) and with one differential coming from $\Omega^\cdot_{X/k}$ and the other the simplicial one.

The natural map $F_*\Omega^{\cdot}_{X/k} \to F_*\mathscr{C}(\mathfrak{U}, \Omega^{\cdot}_{X/k})$ is a quasi-isomorphism. Now we need to construct a map $\Omega^1_{X'/k}[-1] \to F_*\mathscr{C}(\mathfrak{U}, \Omega^{\cdot}_{X/k})$ inducing C^{-1} on cohomology. Note that

$$F_*\mathscr{C}(\mathfrak{U}, \Omega^{\cdot}_{X/k})_1 = F_*\mathscr{C}^1(\mathfrak{U}, \mathscr{O}_X) \oplus F_*\mathscr{C}^0(\mathfrak{U}, \Omega^1_{X/k}).$$

The obvious idea is to let the map $\Omega^1_{X'/k} \to F_*\mathscr{C}^0(\mathfrak{U}, \Omega^1_{X/k})$ be formed of all the maps $f_i : \Omega^1_{X'/k} \to \Omega^1_{U_i'/k} \to F_*\Omega^1_{U_i/k}$. We need to find the second component of this map, $\Omega^1_{X'/k} \to F_*\mathscr{C}^1(\mathfrak{U}, \mathscr{O}_X)$ so that the image has zero differential. This means that we need to $h_{ij} : \Omega^1_{U_i'\cap U_j'/k} \to F_*\mathscr{O}_{U_i'\cap U_j'}$ such that

$$dh_{ij} = f_i - f_j \quad \text{and} \quad h_{ij} + h_{jk} + h_{ki} = 0.$$

On $\widetilde{U}_i \cap \widetilde{U}_j$, we have chosen two lifts $\widetilde{F}_i$ and $\widetilde{F}_j$ of Frobenius. Their difference is then a derivation $d_{ij} : \mathscr{O}_{U_i'\cap U_j'} \to F_*\mathscr{O}_{U_i\cap U_j}$, hence giving a map h_{ij} as required. □

Proof due to Srinivas [3]. Let $\xi \in \mathrm{Ext}^1(\Omega^i_{X'/k}, F_*B^1_X)$ be the class of the extension

$$0 \to F_*B^1_X \to F_*Z^1_X \to \Omega^i_{X'/k} \to 0.$$

The short exact sequence $0 \to \mathscr{O}_{X'} \to F_*\mathscr{O}_X \to F_*B^1_X \to 0$ gives, after applying $\mathrm{Hom}(\Omega^1_{X'/k}, -)$, the long exact sequence

$$\ldots \to \mathrm{Ext}^1(\Omega^1_{X'/k}, F_*\mathscr{O}_X) \xrightarrow{\alpha} \mathrm{Ext}^1(\Omega^1_{X'/k}, F_*B^1_X) \xrightarrow{\delta} \mathrm{Ext}^2(\Omega^1_{X/k}, \mathscr{O}_{X'}) \to \ldots$$

It is not difficult to check that ξ equals the obstruction $o(X, F, W_2(k))$ to lifting (X, F) to $W_2(k)$, and that $\delta(\xi)$ is the obstruction $o(X, W_2(k))$ to lifting X to $W_2(k)$.

The latter class vanishes by assumption, hence there is a $\zeta \in \mathrm{Ext}^1(\Omega^1_{X'/k}, F_*\mathscr{O}_X)$ with $\alpha(\zeta) = \xi$. If ζ corresponds to an extension $0 \to F_*\mathscr{O}_{X'} \to \mathscr{E} \to \Omega^1_{X'/k} \to 0$, we have a push-out diagram:

$$\begin{array}{lccccccccc}
\zeta: & 0 & \longrightarrow & F_*\mathscr{O}_{X'} & \longrightarrow & \mathscr{E} & \longrightarrow & \Omega^1_{X'/k} & \longrightarrow & 0 \\
 & & & \downarrow d & & \downarrow & & \| & & \\
\xi: & 0 & \longrightarrow & F_*B^1_X & \longrightarrow & F_*Z^1_X & \longrightarrow & \Omega^1_{X'/k} & \longrightarrow & 0
\end{array}$$

The top row is a quasi-isomorphism $\Omega^1_{X'/k}[-1] \to \{F_*\mathscr{O}_{X'} \to \mathscr{E}\}$ and the vertical maps give a morphism $\{F_*\mathscr{O}_{X'} \to \mathscr{E}\} \to F_*\Omega^{\cdot}_{X/k}$ inducing C^{-1} on $\mathscr{H}^1$ as required. □

Proof by Ogus and Vologodsky [2]. This proof relies on a construction of a vector bundle with integrable connection $(\mathscr{E}, \nabla)$ on X, fitting into a short exact sequence (of bundles with an integrable connection)

$$0 \to \mathscr{O}_X \to \mathscr{E} \to F^*\Omega^1_{X'/k} \to 0.$$

The p-curvature of $\mathscr{E}$ is 0 on $\mathscr{O}_X$ and 0 on $F^*\Omega^1_{X'/k}$ and the part $F^*\Omega^1_{X'/k} \to \mathscr{O}_X \otimes F^*\Omega^1_{X'/k}$ is the identity.

The bundle $\mathscr{E}' = \mathrm{Sym}^{p-1}\mathscr{E}$ has an induced connection. We can build a double complex

$$\mathscr{C}^{ij} = \mathscr{E}' \otimes_{\mathscr{O}_X} \Omega^i_{X/k} \otimes_{\mathscr{O}_X} F^*\Omega^j_{X'/k}$$

with vertical differentials $d_H^{ij} : \mathscr{C}^{ij} \to \mathscr{C}^{i,j+1}$ coming from the p-curvature and the horizontal ones d_V^{ij} coming from the connection. Then the total complex $\mathrm{Tot}(\mathscr{C}^{\cdot\cdot})$ is quasi-isomorphic to both complexes $\Omega^{\cdot}_{X'/k} = \ker d_V^{0\cdot}$ and $\Omega^{\cdot}_{X/k} = \ker d_H^{\cdot 0}$. □

References

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