On the Stopping of Fast Particles and on the Creation of Positive Electrons

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Introduction

The stopping power of matter for fast particles is at present believed to be due to three different processes: (1) the ionization; (2) the nuclear scattering; (3) the emission of radiation under the influence of the electric field of a nucleus. The first two processes have been treated in quantum mechanics by Bethe,† Møller,‡ and Bloch§ in a very satisfactory way. A provisional estimation of the order of magnitude to be expected in the third process has been given by Heitler.|| The result obtained was that the cross-section ϕ for the energy loss by radiation for very fast particles (if the primary energy $E_0 \gg mc^2$) is of the order

$$\phi \backsim \frac{Z^2}{137}\left(\frac{e^2}{mc^2}\right)^2, \tag{1}$$

where Z is the nuclear charge.

It is the aim of the present paper to discuss in greater detail the rate of loss of energy by this third process and its dependence on the primary energy; in particular we shall consider the effect of *screening*. The results obtained for very high energies ($> 137\ mc^2$) seem to be in disagreement with experiments made by Anderson (*cf.* § 7).

By an exactly similar calculation another process can be studied, namely, the "twin birth" of a positive and negative electron due to a light quantum in the presence of a nucleus. This process is the converse of the scattering of an electron with loss of radiation, if the final state has *negative energy*. The results are in exact agreement with recent measurements for γ-rays of 3–10 mc^2. A provisional estimate of the probability of this process has been given by Plesset and Oppenheimer,¶ who also obtain for the cross-section a quantity of the order of magnitude given by equation (1).

† 'Ann. Physik,' vol. 5, p. 325 (1930); 'Z. Physik,' vol. 76, p. 293 (1932).
‡ 'Ann. Physik,' vol. 14, p. 531 (1932).
§ 'Z. Physik,' vol. 81, p. 363 (1933); 'Ann. Physik,' vol. 16, p. 285 (1933).
|| 'Z. Physik,' vol. 84, p. 145 (1933). Referred to later as I.
¶ 'Phys. Rev.,' vol. 44, p. 53 (1933).

I. Theory

§ 1. *The Cross-section for the Energy-loss by Radiation*

1. *General Method.*—In order to obtain the rate of loss of energy of a particle by emission of radiation, we have to calculate the transition probability for the following process: a particle with momentum† $\mathbf{p}_0/c$ and energy E_0 makes a transition to a state with the momentum $\mathbf{p}/c$ and energy E, while a light quantum with the momentum $\mathbf{k}/c$ is emitted, the frequency ν being given by

$$k = h\nu = E_0 - E. \tag{2}$$

The perturbation causing this transition is the interaction of the particle with the nuclear field $V = Ze^2/r$ and with the radiation field $H = -e(\boldsymbol{\alpha}\mathbf{A})$, where $\mathbf{A}$ is the vector potential of the radiation field and $\boldsymbol{\alpha}$ is Dirac's three-dimensional matrix-vector signifying the velocity.

The wave functions of the electron describing the initial and the final state are supposed to be plane waves, the atomic field being only considered as a perturbation. This corresponds to the first approximation of Born's collision theory, which was shown by Bethe (*loc. cit.*) to hold for $Z/137 < v/c$. For fast particles ($v \backsim c$) this is always true, if Z is not too large. For lead, however, it is doubtful whether the calculations give quantitatively correct results. But from the experiments it seems that, at least for the twin birth, see § 8, the error is very small.

The transition from the initial to the final state, however, only occurs under the simultaneous action of both the atomic field and that of the light wave. First, the electron goes, under the influence of one of the said perturbations, from the initial state to an intermediate state (conservation of energy does not necessarily hold here); then a second transition immediately happens to the final state, caused by the other perturbation. Since momentum is conserved in the emission of light, it can easily be seen (*cf.* I) that there exist two such intermediate states, where the electron has a momentum $\mathbf{p}'$, $\mathbf{p}''$ (energies E', E'') given by:

$$\left.\begin{array}{lll} \text{I} & \mathbf{p}' = \mathbf{p} + \mathbf{k} & \text{no light quantum present} \\ \text{II} & \mathbf{p}'' = \mathbf{p}_0 - \mathbf{k} & \text{a light quantum } \mathbf{k} \text{ present} \end{array}\right\}. \tag{3}$$

† It is convenient to express the momentum in energy units $p = c \times$ momentum. Throughout the rest of the paper we shall speak freely of p as the momentum instead of the strictly more correct p/c.

If we denote the initial state (momentum $\mathbf{p}_0$, no light quantum) by A, and the final state (momentum $\mathbf{p}$, $\mathbf{k}$) by E, the transition probability per unit time becomes (see I, equation (27))

$$w = \frac{2\pi}{\hbar}\,\rho_E\rho_k\,dk\left|\Sigma\frac{H_{EI}V_{IA}}{E'-E_0}+\Sigma\frac{V_{EII}H_{IIA}}{E''-E}\right|^2, \tag{4}$$

where ρ_E, ρ_k are the numbers of quantum states per unit volume, per unit solid angle and per unit energy interval for the electron and the light quantum in the final state; so that

$$\rho_E\,dE = \frac{\Omega_E pE\,dE}{h^3c^3}, \qquad \rho_k\,dk = \frac{\Omega_k k^2\,dk}{h^3c^3}, \tag{5}$$

(Ω_E, Ω_k being the elements of the solid angle).

The wave functions that occur in the matrix elements in (5) are normalized in such a way that there is one particle per unit volume. The summation in (4) has to be extended over both the spin directions and both signs of the energy of the intermediate states.

To obtain the differential cross-section from (4), we have, according to our method of normalization, to divide by the velocity of the incident electron, $v_0 = cp_0/E_0$. Putting in the values for the matrix elements for a pure Coulomb field (*cf.* I equations (18)–(21)) we obtain the differential cross-section

$$d\Phi = \frac{Z^2e^4}{137\pi^2}\frac{\Omega_E\,\Omega_k pEE_0k\,dk}{p_0q^4}\left|\Sigma\frac{(u^*\alpha_k u')\,(u'^*u_0)}{E'-E_0}+\Sigma\frac{(u^*u'')\,(u''^*\alpha_k u_0)}{E''-E}\right|^2, \tag{6}$$

where

$$\mathbf{q} = \mathbf{p}_0 - \mathbf{p} - \mathbf{k}, \tag{7}$$

denotes the momentum transferred to the nucleus in the process. u, u' are the amplitudes of the plane waves with momenta $\mathbf{p}$, $\mathbf{p}'$ each having four components. u refers to a definite spin direction. α_k is the component of $\boldsymbol{\alpha}$ in the direction of the polarization of the light quantum. $(u^*\alpha_k u')\,(u'^*u_0)$ depends only upon the angles in terms of which it can easily be expressed. This can be done by the usual method†: first, we carry out the summation Σ over the spin directions and both signs of the energy of the intermediate states (*i.e.*, over all four states having the same momentum $\mathbf{p}'$):

$$\Sigma\frac{(u^*\alpha_k u')\,(u'^*u_0)}{E_0-E'} = \frac{E_0\Sigma\,(u^*\alpha_k u')\,(u'^*u_0)}{E_0{}^2-E'^2}+\frac{\Sigma\,(u^*\alpha_k E'u')\,(u'^*u_0)}{E_0{}^2-E'^2}.$$

In the first term we have simply (Vollständigkeitsrelation)

$$\Sigma\,(u^*\alpha_k u')\,(u'^*Ou_0) = (u^*\alpha_k u_0). \tag{8}$$

† Casimir, 'Helv. Phys. Act.,' vol. 6, p. 287 (1933).

For the second we use the wave equation

$$E'u' = [(\boldsymbol{\alpha}\mathbf{p}') + \beta\mu]\, u' \equiv H'u',$$

where $\mu = mc^2$. This equation holds for both signs of E′, since the operator H′ is independent of the sign of E′. Hence

$$\Sigma\, (u^*\alpha_k E'u')\,(u'^*u_0) = \Sigma\, (u^*\alpha_k H'u')\,(u'^*u_0) = (u^*\alpha_k H'u_0). \tag{9}$$

And finally

$$\Sigma\, \frac{(u^*\alpha_k u')\,(u'^*u_0)}{E_0 - E'} = \frac{E_0\,(u^*\alpha u_0) + (u^*\alpha_k H'u_0)}{E_0{}^2 - E'^2}. \tag{10}$$

(10) holds for definite spin directions of the initial and the final states. As we are not interested in the probability for special spin directions, we sum also over the spin directions of the final state. Using equation (10) we obtain then from (6) expressions of the form

$$S\,(u^*{}_0 A u)\,(u^* B u_0),$$

where the summation S has to be taken over the spin directions of the final state only, but not over the two signs of the energy. But this sum S can be reduced to a sum Σ over all four states of both spin and energy. For we have

$$Eu = Hu, \qquad \text{or} \qquad u = \frac{H + E}{2E}\, u. \tag{11}$$

For the states of *negative energy* with the wave function $\tilde{u}$ the expression $(H + E)\,\tilde{u}$ vanishes. Introducing the operator $(H + E)\, u/2E$ instead of u, we may now extend the summation also over both signs of the energy, obtaining

$$S\,(u^*{}_0 A u)\,(u^* B u_0) = \Sigma \left(u^*{}_0 A\, \frac{H + E}{2E}\, u\right)(u^* B u_0) = \left(u^*{}_0 A\, \frac{H + E}{2E}\, B u_0\right). \tag{12}$$

(12) is the average value of the operator $A\,(H + E)/2EB$ in the initial state. It can be evaluated by the usual methods.

2. *Differential and Integral Cross-sections.*—If one applies this method to (6) one can easily obtain the differential cross-section

$$d\Phi = \frac{Z^2 e^4}{137\cdot 2\pi}\,\frac{dk}{k}\,\frac{p}{p_0}\,\frac{\sin\theta\,\sin\theta_0\, d\theta\, d\theta_0\, d\phi}{q^4} \left\{ \frac{p^2 \sin^2\theta}{(E - p\cos\theta)^2}\,(4E_0{}^2 - q^2) \right.$$
$$+ \frac{p_0{}^2 \sin^2\theta_0}{(E_0 - p_0\cos\theta_0)^2}\,(4E^2 - q^2) - \frac{2p_0 p \sin\theta\,\sin\theta_0\,\cos\phi}{(E - p\cos\theta)\,(E_0 - p_0\cos\theta_0)}\,(4E_0E - q^2)$$
$$\left. + \frac{2k^2\,(p^2\sin^2\theta + p_0{}^2\sin^2\theta_0 - 2pp_0\sin\theta\,\sin\theta_0\,\cos\phi)}{(E - p\cos\theta)\,(E_0 - p_0\cos\theta_0)} \right\}. \tag{13}$$

θ, θ_0 are the angles between $\mathbf{k}$ and $\mathbf{p}$, $\mathbf{p}_0$ respectively, ϕ the angle between the $(\mathbf{pk})$ plane and the $(\mathbf{p}_0\mathbf{k})$ plane. The denominators arise from the resonance denominators

$$\left.\begin{aligned} E'^2 - E_0^2 &= (\mathbf{p} + \mathbf{k})^2 - \mathbf{p}_0^2 = -2k\,(E - p\cos\theta) \\ E''^2 - E^2 &= (\mathbf{p}_0 - \mathbf{k})^2 - \mathbf{p}^2 = 2k\,(E_0 - p_0\cos\theta_0) \end{aligned}\right\}. \tag{14}$$

In (14) the summation has already been taken over the directions of polarization, and the spin of the electron in the final state.

In order to obtain the total probability for the emission of a light-quantum of a given frequency ν, one has to integrate (14) over the angles, both of the electron and of the light quantum. This integration is elementary but rather tedious. We shall give only the result.†

$$\begin{aligned} \Phi = \frac{Z^2}{137}\left(\frac{e^2}{mc^2}\right)^2 \frac{p}{p_0}\frac{dk}{k}\Big\{ &\tfrac{4}{3} - 2E_0E\,\frac{p^2 + p_0^2}{p^2p_0^2} + \mu^2\left(\frac{\varepsilon_0 E}{p_0^3} + \frac{\varepsilon E_0}{p^3} - \frac{\varepsilon\varepsilon_0}{pp_0}\right) \\ &+ \left[\tfrac{8}{3}\frac{E_0E}{p_0p} + \frac{k^2}{p_0^3p^3}(E_0^2E^2 + p_0^2p^2)\right].\log + \frac{\mu^2 k}{2pp_0}\left[\frac{E_0E + p_0^2}{p_0^3}\,\varepsilon_0\right. \\ &\left. - \frac{E_0E + p^2}{p^3}\,\varepsilon + \frac{2kE_0E}{p^2p_0^2}\right]\log\Big\}, \end{aligned} \tag{15}$$

with

$$\left.\begin{aligned} \mu &= mc^2, \qquad \varepsilon = \log\frac{E + p}{E - p} = 2\log\frac{E + p}{\mu}, \\ \varepsilon_0 &= \log\frac{E_0 + p_0}{E_0 - p_0} = 2\log\frac{E_0 + p_0}{\mu}, \\ \log &= \log\frac{p_0^2 + p_0p - E_0k}{p_0^2 - p_0p - E_0k} = 2\log\frac{E_0E + p_0p - \mu^2}{\mu k} \end{aligned}\right\}. \tag{15A}$$

For energies large compared with mc^2, *i.e.*, for

$$E_0 \gg mc^2, \qquad E \gg mc^2, \qquad k \gg mc^2,$$

(15) reduces to

$$\Phi = \frac{Z^2}{137}\left(\frac{e^2}{mc^2}\right)^2 \frac{dk}{k}\frac{4}{E_0^2}(E_0^2 + E^2 - \tfrac{2}{3}E_0E)\left(\log\frac{2E_0E}{k\mu} - \tfrac{1}{2}\right). \tag{16}$$

The result will be discussed in § 5, 6, and 7.

† The same formulæ (13), (15), have been obtained by F. Sauter. We are indebted very much to Dr. Sauter for the communication of his results. The comparison with his results has made it possible to avoid some mistakes in the calculations. The same formula has also been obtained by G. Racah. We wish to thank him also for sending us his results.

§ 2. *Creation of Positive Electrons*

The creation of a pair of electrons of opposite charge is considered as a kind of photoelectric process: an electron which is initially in a state of negative energy $E = -|E|$ is excited by a light quantum $h\nu$ to a state of positive energy

$$E_0 = h\nu - |E|.$$

Then it is observed that a negative electron of energy E_0 and a positive one of energy $E_+ = |E|$ are created, the light quantum being absorbed.

The reverse of this process would be the transition of an electron from the state E_0 to a state of negative energy, the energy $E_0 + |E|$ being radiated. This reverse process will not usually occur because the states of negative energy are occupied.† The process is identical with that treated in § 1, the only difference being the sign of the final energy. It is clear that this cannot make any difference in the calculation up to formula (10). One might, however, expect a different value for the quantities $(u^* \alpha_k u_0)$ and $(u^* \alpha_k H' u_0)$.

That this is not so can easily be seen from Casimir's method of evaluating quantities of this kind, which has been used already in § 1, equations (8)–(13). If the momentum **p** is given, all considerations leading to (13) remain unchanged, except that we have for the wave amplitude $\tilde{u}$ of a *negative energy state*, instead of (11):

$$\frac{H - |E|}{-2|E|}\,\tilde{u} = \tilde{u},$$

while for a state of positive energy $(H - |E|)\,u = 0$. Therefore, in (13) nothing is changed except the sign of the energy, viz.,

$$S\,(u^*{}_0 A \tilde{u})\,(\tilde{u}^* B u_0) = \left(u^*{}_0 A \frac{H - |E|}{-2|E|} B u_0\right) = \left(u^*{}_0 A \frac{H + E}{2E} B u_0\right). \tag{17}$$

We conclude that formula (13) holds for the reverse process of the creation of pairs as well as for the "normal" emission of radiation. To calculate the probability of the creation itself, one has only to consider that now there exist *two* electrons in the final state instead of one electron and one light quantum. Instead of $\rho_\nu\, d\nu$, we must therefore write in (5)

$$\rho_{E_0}\, dE_0 = \frac{\Omega_0 E_0 p_0\, dE_0}{h^3\, c^3},$$

i.e., the number of electronic states with energy between E_0 and $E_0 + dE_0$

† It may happen that a positive electron is annihilated in this way by an inner electron of a heavy atom.

per unit volume and per solid angle Ω_0. Furthermore, one has to divide by the number of incident light quanta per cm^2 and sec., *i.e.*, by c, the density of light quanta being normalized to unity. For the emission of radiation, we had to divide by the velocity $v_0 = cp_0/E_0$ of the primary electron; therefore we get an additional factor $p_0 E_0/k^2 \cdot v_0/c = p_0{}^2/k^2$ in the cross-section (6) or (13).

There are, however, two points which have to be mentioned. Firstly, one should consider the interaction energy between the two created electrons. Fortunately, it can be seen that this interaction energy V_{+-} does not affect the calculation to our approximation. For the matrix element of $(V_{+-})_{p_+p_-}$ corresponding to the creation is probably the matrix element of a Coulomb interaction belonging to a transition from a positive energy state $\mathbf{p}_-$ to a state with negative energy and momentum $-\mathbf{p}_+$, viz.,

$$(V_{+-})_{p_+p_-} = \int d\tau \left(\exp \frac{i}{\hbar}(\mathbf{p}_-\mathbf{r}_-) \exp\left(\frac{i}{\hbar}(\mathbf{p}_+ \cdot \mathbf{r}_+)\right)\right) \Big/ |\mathbf{r}_+ - \mathbf{r}_-| . \qquad (18)$$

But this matrix element vanishes except when momentum is conserved, *i.e.*, $\mathbf{p}_+ + \mathbf{p}_- = 0$. It follows that, if we add to the Coulomb potential V the interaction energy V_{+-}, the latter will not contribute anything to the matrix elements occurring in (4). There will, of course, be a contribution in the next approximation which is only of the order $e^2/\hbar v$ compared with the result of our approximation, while the application of Born's approximation means an error of the order $Ze^2/\hbar v$.

The second point is, that in (13) the momentum and the energy E are the momentum and energy of the hole in the " sea of negative energy electrons " which corresponds to the positive electron. The momentum and energy of the positive electron itself are $-\mathbf{p}$ and $-E$. It is, therefore, physically more significant to introduce

$$E_+ = -E, \quad \mathbf{p}_+ = -\mathbf{p}, \quad \theta_+ = \pi - \theta, \quad \phi_+ = \pi + \phi, \quad p_+ = p, \qquad (19)$$

θ_+ being the angle between the direction of motion of the positive electron and that of the incident light quantum, etc. If we introduce these quantities into (13), all terms involving the first power of E or p change sign, thus:

$$d\Phi = -\frac{Z^2}{137}\frac{e^4}{2\pi}\frac{p_0 p_+}{k^3}\, dE_0 \frac{\sin\theta_0\, d\theta_0 \sin\theta_+\, d\theta_+\, d\phi_+}{q^4} \left\{ \frac{p_+{}^2 \sin^2\theta_+ (4E_0{}^2 - q^2)}{(E_+ - p_+\cos\theta_+)^2} \right.$$
$$+ \frac{p_0{}^2 \sin^2\theta_0 (4E_+{}^2 - q^2)}{(E_0 - p_0\cos\theta_0)^2} + \frac{2p_0 p_+ \sin\theta_0 \sin\theta_+ \cos\phi_+ (4E_0E_+ + q^2)}{(E_0 - p_0\cos\theta_0)(E_+ - p_+\cos\theta_+)}$$
$$\left. - \frac{2k^2 (p_+{}^2 \sin^2\theta_+ + p_0{}^2\sin^2\theta_0 + 2p_0p_+ \sin\theta_0 \sin\theta_+ \cos\phi_+)}{(E_0 - p_0\cos\theta_0)(E_+ - p_+\cos\theta_+)} \right\}. \qquad (20)$$

The integration over the angles is naturally also identical with that for the radiation case. Formula (15) has, therefore, only to be multiplied by p_0^2/k^2, dk to be replaced by dE_0, and E put equal to minus the energy of the positive electron. The cross-section for the creation of a positive electron with energy E_+ and a negative one with energy E_0 by a light quantum $k = h\nu$ then becomes

$$\Phi(E_0)\,dE_0 = \frac{Z^2}{137}\left(\frac{e^2}{mc^2}\right)^2 \frac{p_0 p_+}{k^3}\,dE_0 \left\{-\tfrac{4}{3} - 2E_0E_+\frac{p_0^2+p_+^2}{p_0^2p_+^2}\right.$$
$$+\mu^2\left(\frac{\varepsilon_0 E_+}{p_0^3}+\frac{\varepsilon_+ E_0}{p_+^3}-\frac{\varepsilon_+\varepsilon_0}{p_0p_+}\right)+\left[\frac{k^2}{p_0^3p_+^3}(E_0^2E_+^2+p_0^2p_+^2)-\tfrac{8}{3}\frac{E_0E_+}{p_0p_+}\right]\log$$
$$\left.+\frac{\mu^2 k}{2p_0p_+}\left[\frac{E_0E_+-p_0^2}{p_0^3}\,\varepsilon_0+\frac{E_0E_+-p_+^2}{p_+^3}\,\varepsilon_+ +\frac{2kE_0E_+}{p_0^2p_+^2}\right]\log\right\}, \quad (21)$$

with

$$\varepsilon_+ = 2\log\frac{E_++p_+}{\mu}, \quad \log = \log\frac{E_0k-p_0^2+p_0p_+}{E_0k-p_0^2-p_0p_+} = 2\log\frac{E_0E_++p_0p_++\mu^2}{\mu k}. \quad (21\text{A})$$

This formula is, of course, symmetrical in E_0 and E_+. An asymmetry would only arise in higher approximations and is small for high energies (*cf*, § 7). If all energies involved are large compared with mc^2, the formula reduces to

$$\Phi(E_0)\,dE_0 = \frac{Z^2}{137}\left(\frac{e^2}{mc^2}\right)^2 4\,\frac{E_{0+}^2E_+^2+\tfrac{2}{3}E_0E_+}{(h\nu)^3}\,dE_0\left(\log\frac{2E_0E_+}{h\nu mc^2}-\tfrac{1}{2}\right). \quad (22)$$

(21) and (22) will be discussed in § 8.

§ 3. *Effect of Screening*

It could be expected that the screening of the atomic potential by the outer electrons would have a considerable effect on the cross-section for the radiation phenomena considered in this paper, because it may be seen that a large part of the processes take place at big distances from the nucleus of the field-producing atom, *i.e.*, at places where the atomic field is no longer a Coulomb field.

1. *Differential Cross-section.*—The potential which the atom exerts on the electron occurs in formula (4) inside the matrix elements V_{AI} and V_{EII}. Both these matrix elements have the value (besides a factor $(u'u_0)$ and $(u''u)$ respectively)

$$V_{AI} = V_{EII} = \int V \exp\frac{i}{\hbar c}(\mathbf{qr})\,d\tau, \quad (23)$$

where q is the momentum transferred to the atom in the process. Now (23) can be brought into the form

$$V_{AI} = V_{EII} = \frac{4\pi\hbar^2 c^2}{q^2}[Z - F(q)]^2, \tag{24}$$

where F is the well-known atomic form-factor

$$F(q) = \int \rho(r) \exp \frac{i}{\hbar c}(\mathbf{qr})\, d\tau, \tag{25}$$

$\rho(r)$ being the density of the atomic electrons at the distance r from the nucleus. Therefore, we can take account of the screening simply by writing $(Z - F)^2$ instead of Z^2 in formula (13), a change which is familiar from the theory of electron scattering.†

The atomic form-factor F depends on the distribution of the atomic electrons. We assume in our calculations that $\rho(r)$ is the Fermi distribution, which, especially for heavy atoms, should be very accurate. We can, then, write

$$F = Z\mathscr{F}(qZ^{-\frac{1}{3}}), \tag{26}$$

where $\mathscr{F}$ is a general atomic form-factor valid for all atoms. It is given numerically in several papers.‡

We can easily get an idea under which conditions the screening will have an appreciable effect. The atomic form-factor F becomes comparable with Z, if $q/\hbar c$ is of the order (or smaller than) the reciprocal atomic radius. Now, the radius of the Fermi atom is approximately $a_0 Z^{-\frac{1}{3}}$, a_0 being the hydrogen radius. Therefore screening is effective if

$$q \ll \alpha = \frac{\hbar c}{a_0} Z^{\frac{1}{3}} = \frac{mc^2}{137} Z^{\frac{1}{3}}. \tag{27}$$

On the other hand, q takes its minimum value if the momentum of the electron is parallel to that of the emitted light quantum both before and after the radiation. Here q is equal to

$$q_{\min} = \delta = p_0 - p - k = E - p - (E_0 - p_0). \tag{28}$$

For energies E_0 and E large compared with mc^2 this reduces to

$$\delta = \frac{(mc^2)^2 h\nu}{2E_0 E}. \tag{29}$$

† *Cf.*, for instance, Mott and Massey, "Atomic Collisions," Oxford Univ. Press, 1934, p. 89.

‡ For instance, Bethe, 'Ann. Physik,' vol. 5, p. 385 (1930).

This formula shows that the minimum momentum transferred to the atom decreases with increasing energy and becomes smaller than α, if

$$E_0 E/h\nu > \tfrac{1}{2}\,137 mc^2 Z^{-\frac{1}{3}}, \tag{30}$$

that is, about $15mc^2$ for heavy atoms. (30) is the condition for the *screening to be effective.* We see from this condition that the screening will only be important for energies large compared with mc^2, and we can, therefore, assume throughout this section that mc^2 is negligible in comparison with E_0, E and $h\nu$. This assumption greatly simplifies the calculation.

2. *Integral Cross-section.*—The integration over the angles θ_0, θ, ϕ is carried out in another paper†. The last part of this integration can only be carried out numerically, since the atomic form factor F of the Fermi atom is only known numerically. The result, *i.e.*, the integral cross-section, can conveniently be written in the form

$$\Phi(\nu)\,d\nu = \frac{Z^2}{137} r_0^2 \frac{1}{E_0^2} \frac{d\nu}{\nu} \left[(E_0^2 + E^2)\left(\phi_1(\gamma) - \tfrac{4}{3}\log Z\right) - \tfrac{2}{3}E_0 E\left(\phi_2(\gamma) - \tfrac{4}{3}\log Z\right)\right], \tag{31}$$

where

$$r_0 = e^2/mc^2$$

is the electronic radius and

$$\gamma = 100\,\frac{mc^2 h\nu}{E_0 E Z^{\frac{1}{3}}}, \tag{32}$$

and ϕ_1, ϕ_2 are two functions of γ which are given in fig. 1.‡

The quantity γ is proportional to δ/α and therefore determines the effect of screening. If $\gamma = 0$, we may call the *screening* "*complete.*" Indeed, the radiation cross-section is then determined entirely by the atomic radius, *i.e.*, by α, whereas the minimum momentum transfer δ (and therefore the energy) has no longer any effect on the cross-section. The values of ϕ_1, ϕ_2 here are

$$\phi_1(0) = 4\log 183, \qquad \phi_2(0) = \phi_1(0) - \tfrac{2}{3}, \tag{33}$$

so that for very high energies, $E_0 \gg 137 mc^2 Z^{-\frac{1}{3}}$, the cross-section (31) becomes

$$\Phi(\nu)\,d\nu = \frac{Z^2}{137}\frac{r_0^2}{E_0^2}\frac{d\nu}{\nu}\,4\left[(E_0^2 + E^2 - \tfrac{2}{3}E_0 E)\log(183 Z^{-\frac{1}{3}}) + \frac{E_0 E}{9}\right]. \tag{34}$$

For a given ratio $h\nu/E_0$, this cross-section is independent of E_0. This is not so if the energy becomes smaller ($\gamma > 0$). The cross-section for a given $h\nu/E_0$

† 'Proc. Camb. Phil. Soc.,' *in press.* Referred to as C.

‡ It would have been more natural theoretically to put $\gamma = \delta/\alpha = 137\, mc^2\, h\nu/2E_0 E\, Z^{\frac{1}{3}}$. The factor 100 instead of 137/2 has been chosen for convenience in using formula (31).

then decreases, though slowly, with decreasing energy (the ϕ's decrease with increasing γ). In the limiting case $\gamma \gg 1$, *i.e.*, for energies small compared with $137mc^2 Z^{-\frac{1}{3}}$ the screening ceases to have any effect, in agreement with our considerations in the preceding section. The cross-section is, then, given by formula (16). For energies that are a little higher, more accurately for values

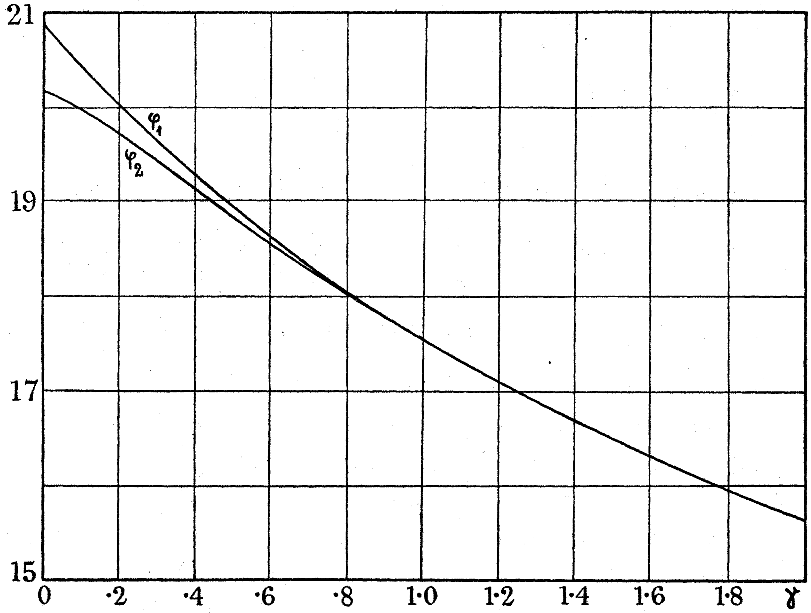


FIG. 1.—ϕ_1 and ϕ_2 (*cf.* equation (31)) as functions of $\gamma = 100mc^2 h\nu/E_0 E Z^{\frac{1}{3}}$.

of γ between 2 and 15, the screening gives a small correction to formula (16), which may be taken into account by writing instead of (16)

$$\Phi(\nu)\, d\nu = \frac{Z^2}{137} \frac{r_0^2}{E_0^2} \frac{d\nu}{\nu} 4\,(E_0^2 + E^2 - \tfrac{2}{3}E_0 E)\left(\log \frac{2E_0 E}{mc^2 h\nu} - \tfrac{1}{2} - c(\gamma)\right), \quad (35)$$

and inserting for $c(\gamma)$ the value given in Table I.

Table I.

$c(\gamma)$ =	0·21	0·16	0·13	0·09	0·065	0·05	0·03	0·02	0·01	(35A)
γ =	2	2·5	3	4	5	6	8	10	15	

For the *creation of pairs* of electrons, all formulæ are exactly similar except for the normalizing factor and for the sign of E. If, in accordance with § 2, we call the energy of the positive electron E_+, the cross-section for creation becomes

$$\Phi(E_0)\, dE_0 = \frac{Z^2}{137} r_0^2 \frac{dE_0}{(h\nu)^3} [(E_0^2 + E_+^2)(\phi_1(\gamma) - \tfrac{4}{3}\log Z) + \tfrac{2}{3}E_0 E_+ (\phi_2(\gamma) - \tfrac{4}{3}\log Z)], \quad (36)$$

$\phi_1(\gamma)$ and $\phi_2(\gamma)$ being the functions shown in fig. 1, and

$$\gamma = 100mc^2 h\nu \, Z^{-\frac{1}{3}}/E_0E_+.$$

For small energies the formula

$$\Phi(E_0)\,dE_0 = \frac{Z^2}{137} r_0^2 \frac{dE_0}{(h\nu)^2} (E_0^2 + E_+^2 + \tfrac{2}{3}E_0E_+)\, 4\left(\log\frac{2E_0E_+}{h\nu mc^2} - \tfrac{1}{2} - c(\gamma)\right), \tag{37}$$

is more convenient, $c(\gamma)$ being given in Table I (35A).

§ 4. *Radiation Probability as Function of Impact Parameter*

It is possible to get a rough idea about the probability that an electron passing at a given distance r from the nucleus emits radiation during its passage. For the main contribution to the matrix element $V(q) = \int V \exp i\,(\mathbf{qr})/\hbar c\,.\,d\tau$ arises from the region $r \backsim \hbar c/q$, since the contribution of larger r's nearly vanishes because of interference, while small r's do not contribute appreciably, because of the smallness of the corresponding volume. Therefore, the radiation emitted in the region between r and $r + dr$ will be equal to the probability of a radiation process in which a momentum between $q = \hbar c/r$ and $q + dq$ is transferred to the atom. This probability $\Phi(q)\,dq$ has been calculated in the paper (C) referred to above (§ 7), viz.,

$$\Phi(q)\,dq \backsim (1 - \mathscr{F}(q))^2\, dq/q \qquad \text{if} \ll q \ll \mu, \tag{38}$$

$$\Phi(q)\,dq \backsim (\log(q/\mu) + \text{const})\,.\,dq/q^3 \quad \text{if} \quad q \gg \mu. \tag{39}$$

(*cf.* C. (67), (66)) since the number of incident electrons having a minimum distance from the nucleus (impact parameter) between r and $r + dr$ is proportional to $r\,dr$, we get for the probability that an electron passing at a distance r radiates

$$\Phi(r) = \Phi(q)\frac{dq}{r\,dr} \backsim \Phi(q)\, r^{-3}. \tag{40}$$

If r is smaller than the Compton wave-length $\hbar/mc$, q will be larger than mc^2. For this case we take from (39)

$$\Phi(r) = \text{const.} \log \hbar/mc\,r \qquad (\text{for } r \ll \hbar/mc). \tag{39A}$$

That means that for small r the radiation emitted at distances between r and $r + dr$ is nearly independent of r. On the other hand, for r *larger* than $\hbar/mc$,

q will be small compared with μ. If q is still large compared with α, then, according to (27), $\mathscr{K}(q)$ may be neglected in (38). Thus,

$$\Phi(r) \backsim 1/r^2 \quad (\text{if } \hbar/mc \ll r \ll a_0 Z^{-\frac{1}{3}} \quad \text{and} \quad r \ll \hbar/mc \,.\, E_0 E/h\nu mc^2). \qquad (38\text{A})$$

The second condition follows from (29). If r increases, (38A) will cease to hold. The point at which this occurs depends on whether the energy E_0 is larger or smaller than $137mc^2 Z^{-\frac{1}{3}}$. In the first case, this limit $r_{\max}$ is given by the atomic radius. Then, for distances larger than $r_{\max} = a_0 Z^{-\frac{1}{3}}$ screening becomes appreciable and causes $\Phi(q)$ to decrease like q^3, (*cf.* C (69) which means

$$\Phi(r) \sim 1/r^6 \qquad (\text{for } r \gg a_0 Z^{-\frac{1}{3}}). \qquad (41)$$

If, on the other hand, the energy is small compared with $137mc^2Z^{-\frac{1}{3}}$, formula (38A) ceases to be valid already at the point $r_{\max} = \hbar/mc \,.\, E_0E/h\nu mc^2$ (corresponding to q of the order δ).

The main contribution to the total cross-section is given by the region (38A) because the region of smaller r's (39A) has only a small volume, and for large r's the radiation probability falls off very rapidly. Therefore the total cross-section will be proportional to

$$\int_{\hbar/mc}^{r_{\max}} \Phi(r)\, r\, dr = \log \frac{r_{\max}}{\hbar/mc} = \begin{cases} \log \dfrac{a_0 Z^{-\frac{1}{3}}}{\hbar/mc} & \text{for} \quad E_0 > 137mc^2Z^{-\frac{1}{3}} \\[2ex] \log \dfrac{E_0 E}{h\nu mc^2} & \text{for} \quad E_0 < 137mc^2Z^{-\frac{1}{3}} \end{cases}, \qquad (42)$$

which agrees roughly with the results (16), (34) of our exact calculations.

The absolute radiation probability for an electron passing the nucleus at a distance smaller than $\hbar/mc$ is of the order† $Z^2/137^3$, which is very small, even for the heaviest atoms. This behaviour is very different from the result of classical electrodynamics, according to which the total energy radiated should increase as r^{-3} (*cf.* paper I § 6) if the electron passes near the nucleus, and should become equal to the primary energy of the electron for $r = r_0 Z^{\frac{2}{3}} \left(\dfrac{E_0}{mc^2}\right)^{\frac{1}{3}}$.

It would seem from this comparison that the quantum mechanical treatment yields a much smaller radiation probability; but nevertheless even this seems to be too large compared with the experiments (*cf.* § 7‡).

† These electrons form a beam of diameter $2\hbar/mc$. To obtain the radiation probability for a single electron, we have to divide the radiative cross-section for these electrons which is certainly smaller than the total cross-section, *i.e.*, $\backsim r_0^2 Z^2/137$, by the area $\pi(\hbar/mc)^2 = \pi \,.\, (137\, r_0)^3$.

‡ Similar results to those mentioned in this section have been derived by a semi-classical consideration by Weizsäcker, to whom we are indebted for the communication of his results.

II. Discussion

§ 5. *The Radiation Emitted by Fast Electrons*

1. *Intensity Distribution.*—In § 1 and 3, we have calculated the probability that an electron of energy E_0 emits a light quantum of frequency between ν and $\nu + d\nu$ (equations (15) and (31)). This probability is roughly proportional to $1/\nu$ and consequently becomes very large for the emission of quanta of low

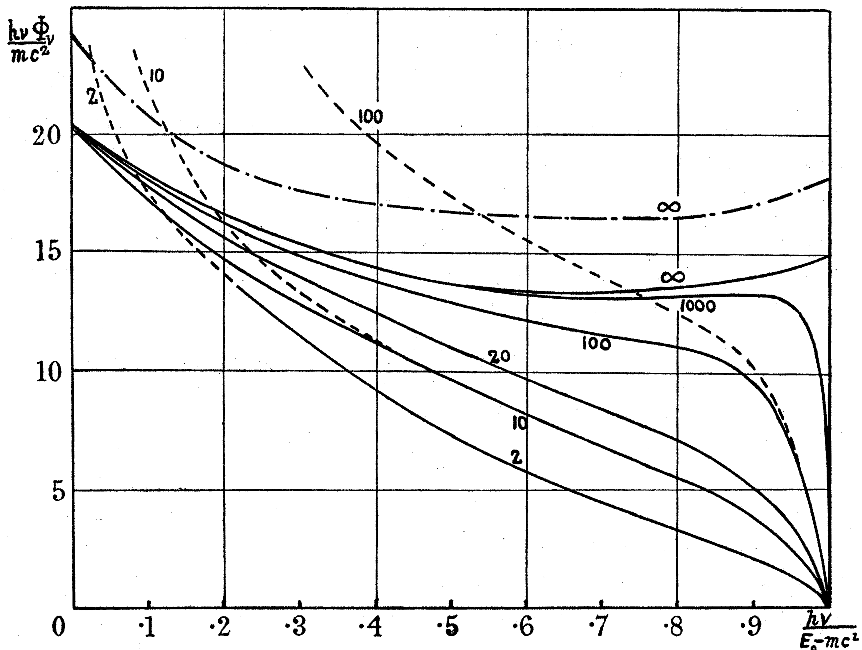


Fig. 2.—Intensity distribution $h\nu\,\Phi_\nu$ of the emitted radiation for Pb (———), H_2O (— – —), and with screening neglected (– – –). Φ_ν is the cross-section in units $Z^2 r_0^2/137$. The numbers affixed to the curves refer to the primary energy E_0 in units mc^2.

energy. We have, therefore, plotted in fig. 2 the *intensity* of the emitted radiation, *i.e.*, frequency times the probability of emission, rather than the probability itself as function of the frequency. Fig. 2 shows the intensity distribution for Pb, H_2O and for the case where the screening is neglected.†

† The general behaviour of the intensity frequency curves is similar to that which follows for the radiation of slow electrons ($v \ll c$) from the *exact* but non-relativistic Sommerfeld theory (' Ann. Physik,' vol. 11, p. 302 (1931), fig. 7). The two theories differ, however, for very small and very high frequencies : for $\nu = 0$ the intensity becomes logarithmically infinite in Sommerfeld's theory, finite in ours. This is due to the neglection of screening in Sommerfeld's calculation ; in our theory the same infinity appears if we

The cross-section Φ_ν is expressed in units $Z^2r_0^2/137$ in order to make its values for different atoms comparable.

The intensity decreases with increasing frequency, and falls to zero at the short wave-length limit $(E_0 - mc^2)/h$.

The intensity of the soft radiation, even in units Z^2, is dependent on the atomic number Z (increasing for small Z), but independent of the energy E_0 of the incident electron.

The intensity of the harder radiation ($\nu \backsim \nu_0$) increases slowly with increasing energy (if ν/ν_0 is kept constant) and reaches a certain asymptotic value for high E_0, which depends on the atomic number Z.

2. *The number of emitted quanta* in a given frequency interval ν to $\nu + d\nu$ is

$$n(\nu) = d\nu \int_{h\nu}^{E_i} \frac{N \Phi(\nu, E_0) dE_0}{-dE_0/dx}, \tag{43}$$

where E_i is the initial energy of the emitting electron, $-dE_0/dx$ is its energy loss per centimetre path, $\Phi(\nu, E_0)$ the cross-section for the emission of a quantum of frequency ν by an electron of energy E_0 as given by formulæ (15), (31), and N the number of atoms per unit volume. To a rough approximation, $h\nu\,\Phi(\nu, E_0) = K$ may be considered as independent of ν and E_0, fig. 2. The energy loss of the electron is for *large* energies E_0 mainly due to radiation, as is proved in § 6, if this is so, $-dE_0/dx = KE_0$. This formula holds down to a critical energy $E_c = 1600mc^2/Z$ (equation (52)), below which the energy loss due to collisions becomes important. This latter is, then, much greater than E_0K so that the contribution of $E < E_c$ to (43) may be neglected. Therefore, roughly, we have

$$\left.\begin{aligned} n(\nu) &= \frac{d\nu}{\nu}\log\frac{E_i}{E_c} \qquad \text{for} \qquad h\nu < E_c < E_i \\ n(\nu) &= \frac{d\nu}{\nu}\log\frac{E_i}{h\nu} \qquad \text{for} \qquad E_c < h\nu < E_i \end{aligned}\right\}. \tag{44}$$

use the non-screened formula (15) as we see from the dotted curves in fig. 2. On the other hand, at the short wave-length limit *our* theory is, presumably, not correct. For, we have made use of Born's first approximation which goes wrong if the energy of the electron after the radiation is small (*cf.* § 1). The exact wave-functions are, in this case, much larger near the nucleus than Born's wave-functions of first approximation. Consequently, the transition probability becomes also much larger, and the intensity of the radiation at any rate does not drop so much as shown in fig. 2. It seems plausible that actually it would tend to a finite limit for $\nu \rightarrow \nu_0$, as it does in Sommerfeld's exact theory.

If an electron of initial energy $E_i = 137mc^2$ (limit of validity of our theory, *cf.* § 7) is stopped in lead ($E_c = 20mc^2$) there will in the average be emitted :—

quanta of energy	>50	20–50	10–20	5–10	2–5	1–2 mc^2
number	0·5	1·5	1·5	1·5	2	1·5

altogether 8·5 quanta of energy greater than mc^2.

This table has been calculated from the crude formula (44), since we wanted only to show the order of magnitude ; actually there are fewer large quanta and rather more small quanta, owing to the intensity distribution in the spectrum (fig. 2).

3. *Angular Distribution of Emitted Radiation.*—The radiation of fast electrons ($E_0 \gg mc^2$) is emitted mainly in the forward direction. The average angle between the directions of motion of the electron and the emitted light is of the order $\Theta = mc^2/E_0$ (*cf.* paper C, § 8).

§ 6. *Energy Loss of Fast Electrons by Radiation.*

1. *Calculation of Energy Loss.*—The average energy radiated by an electron of energy E_0 per centimetre of its path is

$$-\left(\frac{dE_0}{dx}\right)_{\text{rad}} = N\int_0^{\nu_0} h\nu\ \Phi_\nu\, d\nu, \tag{45}$$

N being the number of atoms per cm.^{3} and Φ_ν the cross-section given by equations (15), (31). The integration over ν can, in general, only be carried out numerically. There are, however, two cases in which analytical integration is possible : (1) if E_0 is so small that screening has no effect at all

$$E_0 \ll 137mc^2Z^{-\frac{1}{3}},$$

and yet E_0 is large compared with mc^2 ; (2) if the energy E_0 is so high that the asymptotic formula (34) is valid for all values of ν (complete screening $E_0 \gg 137mc^2Z^{-\frac{1}{3}}$). In the first case, formula (16) may be used for Φ_ν. The integration yields

$$-\left(\frac{dE_0}{dx}\right)_{\text{rad}} = N\frac{Z^2}{137}r_0^2E_0\left(4\log\frac{2E_0}{mc^2} - \tfrac{4}{3}\right) \qquad (\text{for } mc^2 \ll E_0 \ll 137mc^2Z^{-\frac{1}{3}}). \tag{46}$$

This formula has already been published in a preliminary note by Heitler and Sauter.† It differs from the estimation (1) given in I by the logarithmic term, which varies only very slowly with the primary energy E_0.

† 'Nature,' vol. 132, p. 892 (1933).

For very high energies (case 2) the integration gives

$$-\left(\frac{dE_0}{dx}\right)_{\text{rad}} = N\,\frac{Z^2 r_0^{\,2}}{137}\,E_0\left(4\log 183Z^{-\frac{1}{3}} + \frac{2}{9}\right) \qquad (\text{for } E_0 \gg 137mc^2Z^{-\frac{1}{3}}), \quad (47)$$

which means a cross-section independent of E_0.

The result of the numerical integration in the intermediate range and for small energies is shown in fig. 3. For convenience of representation, we have plotted the "integrated cross-section Φ_{rad} in units $Z^2r_0^{\,2}/137$," defined by

$$-(dE_0/dx)_{\text{rad}} = NE_0Z^2r_0^{\,2}/137 \,.\, \Phi_{\text{rad}}, \qquad (48)$$

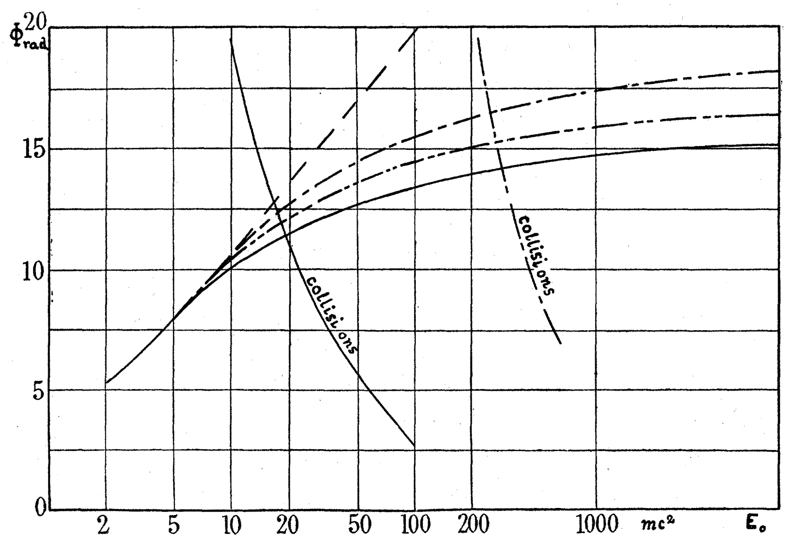


Fig. 3.—Cross-section Φ_{rad} for the energy loss by radiation (defined in equ. 48) as a function of the primary energy for Pb (———), Cu (— — — —), H_2O (— - — -) and with screening neglected (– – –). For comparison the cross-section for the energy loss by collisions for Pb and H_2O are shown on the same scale.

against $\log_{11} E_0$ for three values of Z. Φ_{rad} increases but slowly with increasing energy and decreasing atomic number. For comparison, we have plotted the cross-section for *energy-loss by collisions* in the same units $Z^2r_0^{\,2}/137$.

Φ_{rad} being a slowly varying function, the radiative energy loss is approximately proportional to the initial energy of the electron E_0, whereas the energy loss by collisions is approximately constant. Further, the energy loss by radiation is proportional to Z^2, whereas the energy loss by collisions is proportional to Z. No universal mass absorption-coefficient exists, therefore, for the radiative energy loss.

Table II.—Energy Loss of Fast Electrons by Radiation and Collisions in millions of volts per centimetre path.

Substance.	Energy loss.	Energy of electron in million volts. 5	10	20	50	100	300	1000
H_2O	Radiation	0·07	0·16	0·36	0·99	2·07	6·6	22·5
	Collisions	1·98	2·15	2·32	2·55	2·72	2·99	3·29
Cu	Radiation	2·1	4·9	10·9	28·9	61	191	660
	Collisions	12·7	14·0	15·2	0·7	18·2	20·3	22·5
Pb	Radiation	6·4	14·4	31·4	85	177	550	1900
	Collisions	12·5	13·9	15·3	17·3	18·6	20·9	23·4

Table II gives the absolute value of the energy loss for various substances.

The energy loss by radiation is seen to be much greater, for high energies E_0, than the ordinary energy loss by inelastic collisions. The latter has been calculated from the usual formula for electrons with relativistic energy $> mc^2$, namely,

$$-\left(\frac{dE_0}{dx}\right)_{\text{coll}} = 2\pi r_0^2 mc^2 ZN \log \frac{E_0^3}{2mc^2 I^2}. \tag{49}$$

Following the theory of Bloch (*loc. cit.*), the average ionization potential I was assumed to be proportional to the atomic number Z, explicitly

$$I = 13 \cdot 5 \,.\, Z \text{ volts}, \tag{50}$$

the proportionality factor 13·5 being redetermined for this purpose from the observed energy loss of fast α-particles in gold.†

The ratio of radiation and collision energy loss is roughly

$$\frac{-(dE_0/dx)_{\text{coll}}}{-(dE_0/dx)_{\text{rad}}} = \frac{E_0 Z}{1600 mc^2}. \tag{51}$$

This simple formula is due to the fact that the logarithm in (49) varies with E_0 and Z in approximately the same way as that in (48). Radiation and collision become of equal importance at the "critical" energy

$$E_c = 1600\, mc^2/Z, \tag{52}$$

i.e., about $20\, mc^2 = 10$ million volts for lead, $55\, mc^2$ for copper, $200\, mc^2$ for air.

† No account was taken of the excitation of nuclear electrons, since it seems highly improbable that the probability of this excitation (if it occurs at all) can be calculated by the simple wave-mechanical formula (49), considering that the electrons presumably do not exist at all in the nucleus. We think that the excitation of nuclear electrons rarely takes place.

2. The *range of the electron* is determined by the quantity

$$-\left(\frac{d\mathrm{E}_0}{dx}\right)_{\mathrm{rad}}-\left(\frac{d\mathrm{E}_0}{dx}\right)_{\mathrm{coll}}$$

For a rough estimate we may consider Φ_{rad} in (48) as a constant and insert for it an average value Φ_0, say. Then, according to (48), (51), (52), the whole energy loss is given by

$$-\frac{d\mathrm{E}_0}{dx}=\mathrm{N}\,\frac{\mathrm{Z}^2}{137}\,r_0{}^2\,\Phi_0\,(\mathrm{E}_0+\mathrm{E}_c), \tag{53}$$

the first term (E_0) representing the radiation (E_0 being the energy after x cm. path), the second (E_c) the collisions (E_c being constant). Integrating (53) we get for the range the rough formula

$$\mathrm{R}=\frac{137}{\mathrm{NZ}^2 r_0{}^2\,\Phi_0}\log\frac{\mathrm{E}_0+\mathrm{E}_c}{\mathrm{E}_c}=\frac{137}{\mathrm{NZ}^2 r_0{}^2\,\Phi_0}\log\left(1+\frac{\mathrm{ZE}_0}{1600mc^2}\right). \tag{54}$$

The range of high energy electrons increases, according to (54), only with the logarithm of the energy and remains, therefore, very small even for the fastest electrons. Table III gives the result of an exact numerical calculation of the range in Pb, Cu, H_2O; for Pb the range calculated by the rough formula (54) is added to show the accuracy of this formula.

Table III.—Average Range of Fast Electrons in cm.

Stopping material.	Energy in million volts. 5	10	20	50	100	300	1000	10000
H_2O	2·5	4·8	8·8	18·4	30·4	58	100	195
Cu	0·37	0·67	1·12	1·96	2·78	4·25	6·0	9·4
Pb	0·33	0·54	0·81	1·23	1·68	2·25	2·88	4·08
Pb (calculated by (54))	0·25	0·43	0·68	1·12	1·50	2·14	2·87	4·30

3. *Straggling.*—The effect of radiation is to diminish the energy of an electron suddenly by rather a large fraction of its initial value. Therefore, the actual energy loss may differ very considerably from the average loss. To obtain a rough idea of the effect of this straggling, we assume for the probability of emission of a light quantum $h\nu$ a rough but convenient formula:

$$\Phi(\nu)\,d\nu=a\,\frac{d\nu}{\mathrm{E}_0\log\mathrm{E}_0/\mathrm{E}}, \tag{55}$$

where a is a constant.

(55) may be seen to represent the intensity curves $h\nu\ \Phi_\nu$ of fig. 2 fairly well. If we introduce instead of ν

$$y=\log\left[\mathrm{E}_0/(\mathrm{E}_0-h\nu)\right] \tag{56}$$

(which is convenient since on the average the log of the electronic energy decreases linearly with the distance), the probability that the electron loses the energy $h\nu$ in travelling an *infinitely short distance dl* is

$$w(y)\,dy = \Phi(\nu)\,d\nu\,dl = a\,\frac{e^{-y}\,dy}{y}\,dl. \tag{57}$$

We wish to know the probability for a decrease of the energy of the electron to e^{-y} times its initial value after traversing matter of a *finite thickness l*. For this probability we can prove the following formula to be correct:

$$w(y)\,dy = \frac{e^{-y}\,y^{al-1}}{\Gamma(al)}\,dy \tag{58}$$

(58) becomes identical with (57) for very small l. For, in this case

$$\Gamma(al) = 1/al \qquad \text{and} \qquad y^{al} = 1$$

(except for the smallest values of y, *i.e.*, for the smallest values of ν).

To prove that (58) holds for finite l, we let the electron travel first a distance l_1 then a distance l_2. The log of the energy decreases first by y_1, then by y_2 altogether by $y = y_1 + y_2$. If (58) is assumed to be correct for the two parts of the path, the probability of the decrease y becomes:

$$w(y)\,dy = dy\int_0^y w_1(y_1)\,w_2(y-y_1)\,dy_1 = dy\,\frac{e^{-y}}{\Gamma(al_1)\,\Gamma(al_2)}\int_0^y {y_1}^{al_1-1}(y-y_1)^{al_2-1}dy_1. \tag{59}$$

The integral is evidently proportional to $y^{a(l_1+l_2)-1}$. The numerical factor follows from the fact that $\int_0^\infty w(y)\,dy = 1$. Therefore, if (58) is valid for the energy losses in the paths l_1 and l_2, it is also valid for the energy loss in the total path $l = l_1 + l_2$ and is thus proved to hold for any length of the path.

The curves in fig. 4 give the probability that an electron which loses energy only through radiation has, after travelling a certain distance l, still an energy left which is greater than $e^{-1}, e^{-2}, \ldots,$ times its initial energy. The abscissa is al. Now the *average energy loss* $\overline{y}$ is exactly equal to al, if the law (57) is accepted. (This can easily be seen by calculating $\int y\,w(y)\,dy$ from (58).) For instance, in the *average* the energy is diminished to e^{-1} times its initial value after travelling a distance $al = 1$, but, as can be seen from fig. 4, even after so great a distance as $la = 2\cdot 8$ there still remains a probability of 10% that the energy is higher than $e^{-1}\,.\,E_0$. Thus the straggling may increase the range of some electrons to twice or even three times its average value ; on the other hand, some electrons are stopped much earlier.

The straggling is characteristic of the energy loss by radiation. By this effect, the latter differs from the energy loss by collisions. The unambiguity of this distinction may be seen from Table IV, in which the distribution of energy losses is given for an electron of 50 million volts primary energy traversing ½ mm. of lead, assuming (*a*) that only collisions take place, (*b*) only radiation, (*c*) both. The last two lines give the distribution after traversing 1 mm. lead for initial energies of 50 and 10 million volts.

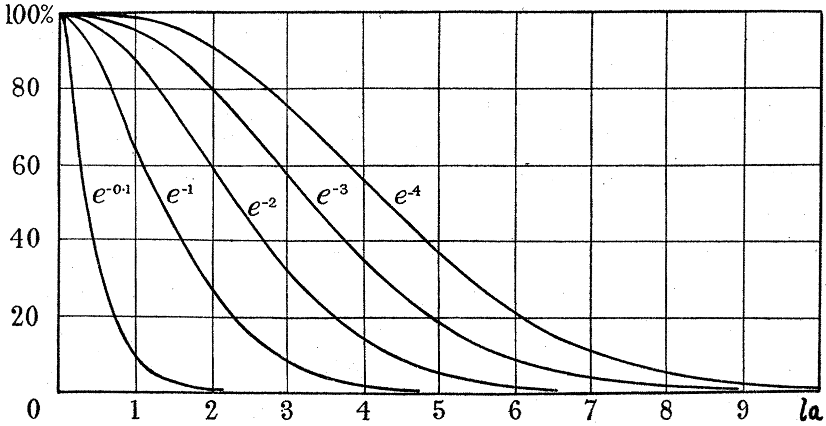


Fig. 4.—Straggling. Probability that an electron after having travelled l cm. has still an energy greater than e^{-1}, e^{-2}, ..., times its initial energy. The scale of the abscissa is chosen so that in the average the energy decreases to e^{-la}.

Table IV.—Probability of different Energy Losses for Electrons passing through Lead (%).

Initial energy.	Thickness of Pb.	Energy lost by	Energy loss in million volts. <0·1	0·1–0·5	0·5–0·7	0·7–1	1–2	2–3	3–5	5–10	10–20	20–50
50	½ mm.	Collision	0	0	58	32	7·5	1	0·7	0·4	0·2	0·2
50	½ mm.	Radiation	49	12·4	2·7	2·8	5·9	3·6	4·5	6·2	5·9	6·7
50	½ mm.	Both	0	0	29	25	14	5·0	7·3	7·0	6·1	6·6
50	1 mm.	Both	0	0	0	0	35	12	11	14	12	16
10	1 mm.	Both	0	0	0	0	57	16	13	14	—	—

§ 7. *Comparison with Experiment. Limits of the Quantum Theory.*

1. *The theoretical energy loss by radiation for high initial energy is far too large to be in any way reconcilable with the experiments of Anderson.*† He has measured the energy loss of electrons of initial energy 300 . 10^6 volts and found

† Anderson, ' Phys. Rev.,' vol. 44, p. 406 (1933).

it to be about 35 . 10^6 volts per cm. of lead as against 550 . 10^6 volts given in our theoretical table. The disagreement is definite, although there are several reasons why a more exact theory may give a lower value for the energy loss.

(1) The figures given in Table II apply to electrons having really the energy E_0. Actually, when the electron has travelled a certain distance, say, 1 mm., it has lost some energy, and therefore the energy loss in the second millimetre will be less than in the first. Thus, the actual range of the electrons will not be $300/550 = 0 \cdot 55$ cm., but about 2 cm. (*cf.* Table III). Even so, the electron should lose in 1 cm. of lead about 250 million volts, *i.e.*, nearly all its energy.

(2) It is not quite correct to base the arguments on the average loss of energy, because in each radiation process rather a large fraction of the energy is lost. Therefore, the actual energy loss may for some electrons be considerably smaller than the average (straggling, § 6, section 3). But, since Anderson has measured rather a large number of electron tracks, it seems inconceivable that all his electrons should have lost particularly small amounts of energy.

(3) Our calculations are based on Born's method which may be wrong for such heavy atoms as lead (§ 1). The error, however, should not be appreciable, since the creation of positive electrons seems to be in good accord with our calculation for energies 2–10 mc^2 (§ 8); in particular the creation probability is found experimentally to be very nearly proportional to Z^2 as required by Born's approximation. Since the calculations for creation of pairs and for emission of radiation are absolutely analogous, it is impossible that Born's approximation can be far wrong for the radiation case. Therefore, even allowing for the three corrections, it seems impossible that the theoretical energy loss can be smaller than about 150 million volts per centimetre lead for Anderson's electrons. The theory gives, therefore, quite definitely a wrong result.†

This can perhaps be understood for electrons of so high an energy. The de Broglie wave-length of an electron having an energy greater than 137 mc^2 is smaller than the classical radius of the electron, $r_0 = c^2/mc^2$. One should not expect that ordinary quantum mechanics which treats the electron as a point-charge could hold under these conditions. It is very interesting that the energy loss of fast electrons really proves this view and thus provides *the first instance*

† We do not think that the fact that the cosmic radiation reaches the bottom of Lake Constance is equally conclusive. For the highly penetrating radiation may consist of heavy particles; for instance, protons. For these the radiation probability would be almost zero, being inversely proportional to the square of the mass.

in which quantum mechanics apparently break down for a phenomenon outside the nucleus. We believe that the *radiation of fast electrons will be one of the most direct tests for any quantum-electrodynamics to be constructed.*†

2. It appears, therefore, to be of great importance indeed to test the radiation formulæ for energies for which they should be valid, *i.e.*, $E_0 < 137\ mc^2$. Even here there is a region where the *energy loss by radiation is much bigger than the energy loss by collisions.* For lead, this region lies between 10 and 50×10^6 volts. The only electrons at present available in the required energy range seem to be those in the showers of cosmic radiation. Unfortunately they always occur associated with so many other electrons that it should be rather difficult to test directly the emission of radiation by such electrons.

Therefore, the only means of detecting the radiation is the straggling of the energy loss of the shower electrons. To decide unambiguously whether this energy loss is due to ordinary inelastic collisions or to radiation, it would be best to investigate the loss in rather thin metal plates, say, 1 mm. thick. Then most of the electrons will undergo only collisions and will, thus, lose about 1·5 or 2 million volts while a few will lose a considerable fraction of their initial energy (*cf.* § 6, Table III).

Another test for the energy loss by radiation would be its dependence on the nuclear charge Z^2. The energy loss per gm./cm.² in Pb would, for electrons of an energy between 10 and 50 million volts, be much bigger than in Al, whereas it would be almost the same if the energy loss were due to collisions only.

The theory could, perhaps, also be proved in the region of fast β-particles for which the radiation probability is already very large. If one chooses a suitable substance which emits no γ-rays, the radiation emitted by the β-particles could be measured directly. Or one could investigate the Wilson tracks in a heavy gas such as xenon or a compound containing lead and detect directly the points where energy is lost without production of a branch track. In xenon the probability that the electron loses more than 1/10 of its energy in a radiation process is about 1 : 1000 per centimetre (electron energy 2–10 mc^2).

† A very interesting attempt has recently been made by Born to change the classical field equations so as to take the electron radius into account. He found that, for wave-lengths $\lambda < r_0$, one has to replace the electronic charge e by an "effective charge" which decreases rapidly with decreasing λ/r_0. This would, of course, immediately decrease the radiative energy loss. *Cf*, 'Nature,' vol. 132, pp. 282, 970, 1004 (1933); vol. 133, p. 63 (1934); and 'Proc. Roy. Soc.,' A, vol. 143, p. 410 (1934). An exact comparison with experiments is not yet possible, since up to the present the quantum translation of Born's theory has not been developed sufficiently.

Finally, the stopping in a thin solid plate would give a more conclusive result than for shower-electrons, because many more particles are available. But the experiments with β-particles would, of course, not make experiments on shower-electrons unnecessary, because one wants to see at what energy the quantum theory *begins* to give wrong results.

3. The lowest cosmic ray electron for which the energy loss has been measured was one of 113 million volts. This is still outside the region where quantum mechanics is expected to apply, but not very much above the limit of 137 mc^2. Anderson found the energy loss in a lead plate of 13·5 mm. thickness to be about 27 . 10^6 volts, *i.e.*, 20 . 10^6 volts per centimetre. This is only slightly more than would be expected for collisions only. If other tracks of this energy should give similar results, one would, therefore, conclude that already for this energy the quantum theory gives far too high a radiation probability. But it may be that this particular electron has, by chance, not emitted any large quantum. The question of the validity of our formulæ for the radiation of electrons with energy smaller than 137 mc^2 can thus only be decided when further experiments are available.

On the other hand, the measured energy loss of 35 . 10^6 volts for the 300 million volt electron seems to indicate that not only inelastic collisions are effective in the stopping of fast electrons.† We should like to attribute the difference to emission of radiation.

§ 8. *Creation of Positive Electrons.*

1. *Energy Distribution.*—The probability that a γ-ray-quantum of energy $h\nu$ creates a positive electron with energy between E_+ and $E_+ - dE_0$ and a negative one with energy between E_0 and $E_0 + dE_0$, is given by—

Formula (21), if $h\nu$ is of the order mc^2;

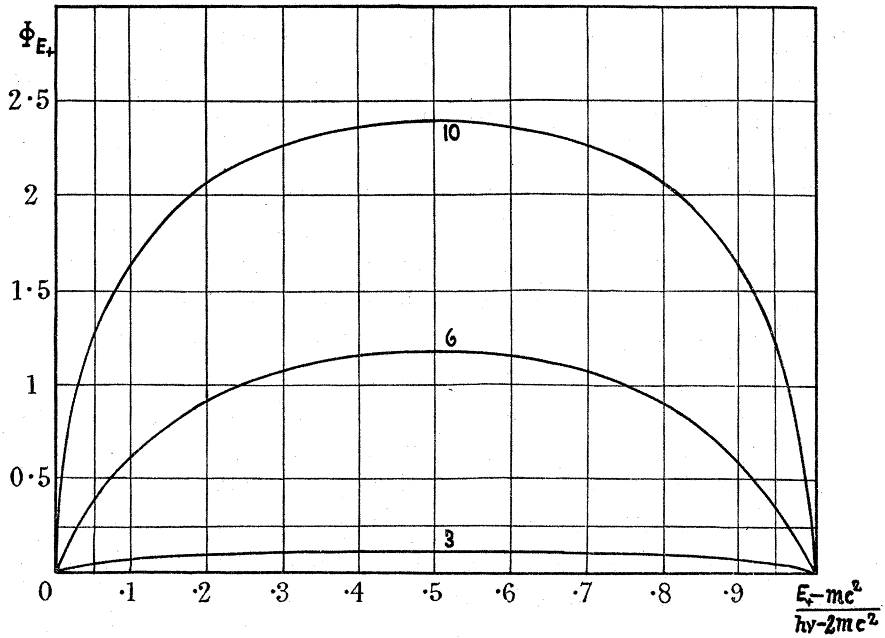
Formula (22), if $h\nu \gg mc^2$, but $h\nu \ll 137\ mc^2/Z^{\frac{1}{3}}$;

Formula (36), if $E_0E_+/h\nu\, mc^2$ is of the order $137\ mc^2/2Z^{\frac{1}{3}}$ or larger.

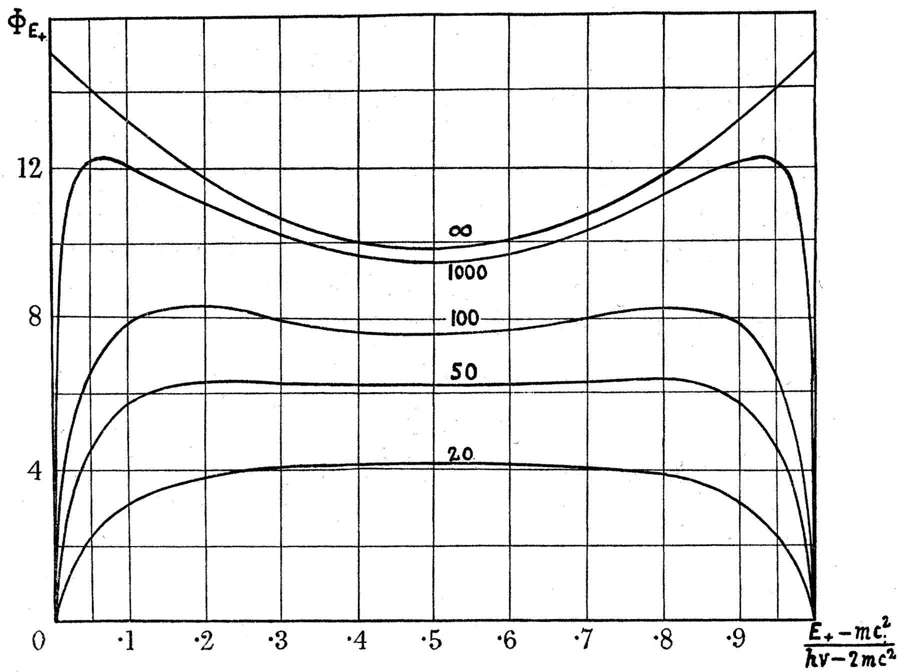
The results of these formulæ are shown in figs. 5 (*a*) and 5 (*b*). The quantity actually plotted is the cross-section Φ_{E_+} in units $r_0{}^2Z^2/137$.

The abscissæ denote $(E_+ - mc^2)/(h\nu - 2mc^2)$, *i.e.*, the kinetic energy of the positive electron as a fraction of the sum of the kinetic energies of both electrons.

† It should be expected that the exact quantum electrodynamics would give rather a lower energy loss in collisions than the quantum theory which itself gives only 20 . 10^6.



a



b

FIG. 5.—Energy distribution of pairs of positive and negative electrons. Φ_{E_+} is the cross-section (units $Z^2 r_0^2/137$) for the creation of a positive electron with kinetic energy $E_+ - mc^2$. The numbers affixed to the curves refer to the energy of the light quantum $h\nu$ in units mc^2. Fig. 5*a* is valid for any element (screening neglected), fig. 5*b* refers to lead.

The curves for small values of $h\nu$ are valid for any element, those for $h\nu > 50mc^2$ are calculated for lead, for lighter elements Φ_{E_+} would be a little larger because of the smaller effect of screening, but the general form of the curves would be the same.

For quanta of small energy the probability of creation has a broad maximum when both electrons obtain equal energy. The maximum becomes flatter with increasing energy. For higher energies the probability has a flat *minimum* for equal energy and a small maximum when one of the electrons obtains much more energy than the other. This change of the form of the curves can be seen directly from the formulæ (22) and (36).†

The energy distribution is apparently symmetrical in the energies of the two electrons. This is a consequence of the use of Born's approximation. In an exact calculation, the positive electron would be found to obtain more energy, on the average, than the negative, as has been pointed out by several authors. This is due to the repulsion of the positive electron and the attraction of the negative by the nucleus. If the electrons are generated at a distance r from the nucleus, the energy difference of the two electrons will be $2Ze^2/r$.

Now according to § 4 the main contribution to the cross-section arises from a region between $\hbar/mc$ and $(\hbar/mc)(h\nu/2mc^2)$ provided that $h\nu \ll 137\ mc^2Z^{-\frac{1}{3}}$. We, therefore, estimate that the positive electron will obtain about $2mc^2\,Z/137$ more energy than the negative for small $h\nu$; for higher $h\nu$ the difference will be smaller.

2. *Angular Distribution.*—The average angle between the direction of motion of a created electron of energy E_0 and the creating quantum is of the order $\theta \backsim mc^2/E_0$. For large energies, therefore, the electrons are emitted mainly in the forward direction. Explicitly, the number of electrons emitted at an angle θ_0 is approximately proportional to

$$\Phi(\theta_0)\,d\theta_0 = \frac{\theta_0\,d\theta_0}{(\Theta^2+\theta_0{}^2)}\quad,\quad \Theta = \frac{mc^2}{E_0}. \tag{60}$$

(*cf.* paper C, (74)). For energies of the order mc^2, the angular distribution is more complicated and the preponderance of the forward direction less marked.

3. *Total Cross-section. Comparison with Experiments.*—The total cross-section is found by integrating the cross-sections (21), (22), (36) over all possible

† For high energies $h\nu$ the minimum $E_0 = E_+$ is less marked than in the theory of Oppenheimer and Plesset, who obtain Φ_{E_+} proportional to $E_0{}^2 + E_+{}^2$ which apparently is due to an error in their calculation.

energies E_0 of the negative electron. Analytical integration is possible in two cases.

(1) If $mc^2 \ll h\nu \ll 137\, mc^2\, Z^{-\frac{1}{3}}$, then formula (22) (no screening) has to be integrated, giving

$$\Phi_{\text{pair}} = r_0^2\, \frac{Z^2}{137} \left(\frac{28}{9} \log \frac{2h\nu}{mc^2} - \frac{218}{27}\right), \quad \text{(no screening } h\nu \gg mc^2\text{)}, \qquad (61)$$

a result which has been published in the preliminary note by Heitler and Sauter (*loc. cit.*).

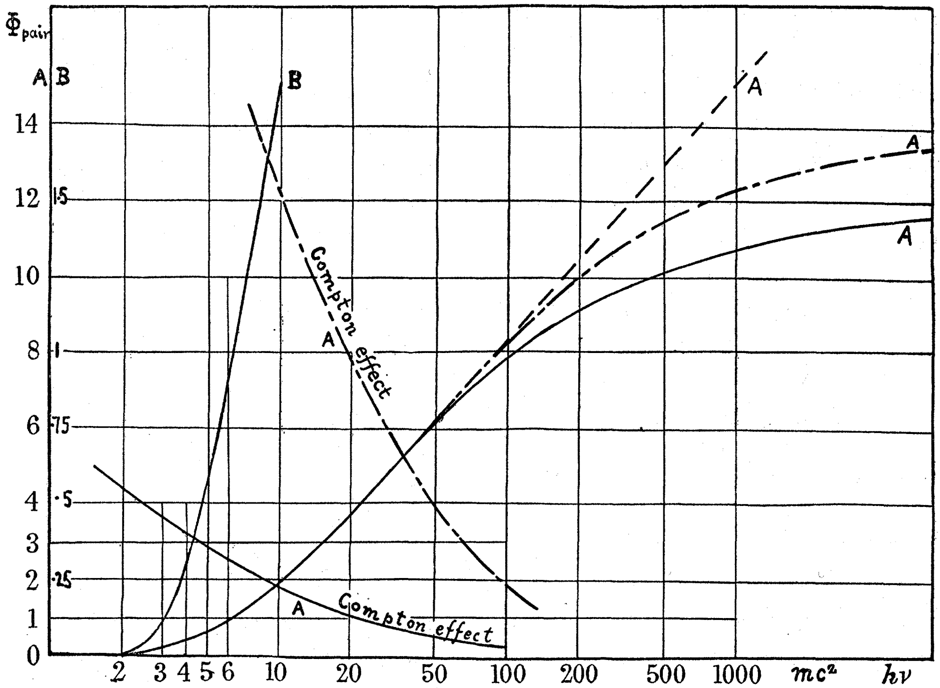


Fig. 6.—Integrated cross-section for the creation of pairs (units $Z^2 r_0^2/137$) as a function of $h\nu$ for lead (——), H_2O (— - —), and without screening (— — —). The beginning of the curve is also given with 8-fold enlargement (curve B). For comparison the cross-section for the Compton effect is given on the same scale.

(2) If $h\nu \gg 137\, mc^2\, Z^{-\frac{1}{3}}$ (complete screening) we find

$$\Phi_{\text{pair}} = r_0^2\, \frac{Z^2}{137} \left(\frac{28}{9} \log \left(183\, Z^{-\frac{1}{3}}\right) - \frac{2}{27}\right) \quad \text{(complete screening)} \qquad (62)$$

For all other values of $h\nu$ the integration must be carried out numerically. The result is shown in fig. 6, which gives the total cross-section for the creation of pairs in lead and aluminium (units $Z^2 r_0^2/137$). For comparison we have plotted the cross-sections for the production of Compton-electrons.

The cross-section is proportional to the square of the atomic number. It also increases rapidly with increasing energy of the quantum $h\nu$ (for small $h\nu$). For very high energies $h\nu$ an asymptotic value is reached which is determined by the ratio of the radius of the atom to $\hbar/mc$. (In the space between $\hbar/mc$ and the atomic radius the probability for the production of pairs is appreciable.)†

The calculated cross-section is in *good agreement with experiment* as regards both the absloute number of pairs produced and the dependence on energy and atomic number. The direct experiments of Curie and Joliot,‡ Blackett and Occhialini§ and others,|| give the ratio of the number of electron pairs produced by hard γ-rays to the number of Compton plus photo-electrons the Compton effect being calculable from the Klein-Nishina formula.¶ Table V compares the theoretical and experimental values of this ratio for lead and various quantum energies. For the theoretical values an average value of $h\nu$ has been assumed of 3, 5·2, and 11 mc^2 respectively for the three sources.

Table V.—Number of Electrons Pairs produced by γ-rays.

	Pb.			Al.	Units.
Energy	2–4·4	5·2	10–12	10–12	mc^2.
Source	Ra mixture.	ThC″.	P_0 + Be.	P_0 + Be.	
I—Theoretical cross-section for production of pairs	0·12	0·6	2·0	2·0	$Z^2 r_0^2/137$
II—Theoretical cross-section for production of Compton and photo-electrons	4·0	3·0	2·1	12	$Z^2 r_0^2/137$
III—Ratio of I to II (theoretical)	0·03	0·20	0·95	0·17	—
IV—Ratio (experimental)	(0·03)	0·22	(0·67)	(0·06)	—

The agreement is better than was to be expected. The values in brackets refer to measurements made in rather thick plates of lead,** whereas the value

† *Cf.* § 4. This fact is in contrast to Oppenheimer and Plesset, who maintain that all pairs are produced at distances $\hbar/mc$.

‡ 'C. R. Acad. Sci. Paris,' vol. 196, p. 1885 (1933), and p. 1581 (1934).

§ 'Nature,' vol. 132, p. 917 (1933).

|| Grinberg, *ibid.*, vol. 197, p. 318 (1933).

¶ The number of photo-electrons is for lead roughly 10% of the number of Compton electrons, and can be calculated from Sauter's theory.

** A fairly large number of positive and negative electrons is shown to be absorbed in the plate, since very often only one positron appears in the chamber without the accompanying negative electron.

for the $5 \cdot 2\ mc^2$ γ-radiation is reduced to infinitely thin plates.† In particular the increase of the cross-section with increasing energy is very well represented by the theory.

The dependence on the atomic number has been tested by a more indirect method by Heiting.‡ He measures the " excess-scattering " of γ-rays in various elements (beside the Compton-scattering). This effect is known to be due to the recombination of a positive electron (after being stopped by collisions) with a negative electron, the rest energy of both electrons being emitted in two light quanta of energy $h\nu = mc^2$ each. Since all positive electrons die after travelling a comparatively short path, the number of the emitted quanta is just twice the number of positive electrons produced. The intensity of this " scattered " radiation is found to be almost exactly proportional to Z^2 over the whole range of Z from aluminium to lead (the primary radiation had an energy which was $h\nu = 5 \cdot 2\ mc^2$). This agreement proves the validity of Born's approximation for our calculations. This is rather surprising, since Born's approximation means an expansion in a power series in $Ze^2/\hbar c$.

4. *Absorption Coefficient for Light of Short Wave-length.*—If the energy of the quantum becomes high, the probability for the creation of pairs becomes larger than that for the Compton effect (see fig. 6), since the latter decreases with increasing energy as $1/h\nu \,.\, \log (2h\nu/mc^2)$. This fact is analogous to the energy loss of particles by radiation (*cf.* § 6). For lead, the creation of pairs is the more probable process already for $h\nu = 10\ mc^2$, for aluminium about $h\nu = 35\ mc^2$ is required to make the probability for the two processes equal.

The absorption of light of very short wave-length is, therefore, due to the creation of pairs rather than to the Compton effect. Since the *creation cross-section increases with increasing energy, the same is true for the absorption coefficient,* a behaviour which is rather unfamiliar. Table VI gives the absorption coefficient for hard γ-rays in Pb, Cu, and H_2O ; it rises, for instance, to more than 1 cm^{-1} in lead. It should, however, be considered that for $h\nu > 137\ mc^2$ the quantum theory will go wrong, as it does for the radiation of fast electrons. It is to be expected that, as a consequence, the absorption coefficient for quanta of energy greater than $137\ mc^2$ will decrease again.§

† We are indebted to Professor Blackett for his kind communication.

‡ 'Z. Physik,' vol. 87, p. 127 (1933).

§ [*Note added in proof, May* 25, 1934.—In a recently published paper v. Weizsäcker (Z. Physik,' vol. 88, p. 612 (1934), cf. footnote‡ at the end §4) came to the conclusion that the theoretical results reached in this paper should be valid also for energies $> 137\ mc^2$. If this result should be correct it would be hardly possible to reconcile it with the experiments mentioned in §7.]

Table VI.—Absorption Coefficient for Hard γ-rays in various Materials in cm^{-1}.

Material.	Absorption due to	Energy in million volts.					
		5	10	20	50	100	1000
Pb	Compton effect....	0·235	0·141	0·082	0·039	0·022	0·003
Pb	Pairs	0·24	0·46	0·68	0·96	1·15	1·43
Pb	Total	0·48	0·60	0·76	1·00	1·17	1·43
Cu	Total	0·292	0·276	0·31	0·36	0·40	0·50
H_2O	Total	0·032	0·022	0·017	0·015	0·015	0·017

Summary.

The probability for the emission of radiation by fast electrons passing through an atom is calculated by Born's method (§ 1), the calculations going beyond previous publications mainly by considering the screening of the atomic field (§ 3). The results are discussed in §§ 5 to 7. The total radiation probability (fig. 3) becomes very large for high energies of the electron, indeed the stopping of very fast electrons (energy $> 20\ mc^2$ for Pb) is mainly due to radiation, not to inelastic collisions. The theory does not agree with Anderson's measurements of the stopping of electrons of 300 million volts energy, thus showing that the quantum theory is definitely wrong for electrons of such high energy (§ 7) (presumably for $E_0 > 137\ mc^2$).

By the same formalism, the probability for the creation of a positive and a negative electron by a γ-ray is calculated (§§ 2, 3). The energy distribution of the electrons is shown in fig. 5, the total creation probability in fig. 6. For γ-rays of $h\nu$ between 3 and 10 mc^2 the theory is in very good agreement with the experiments.