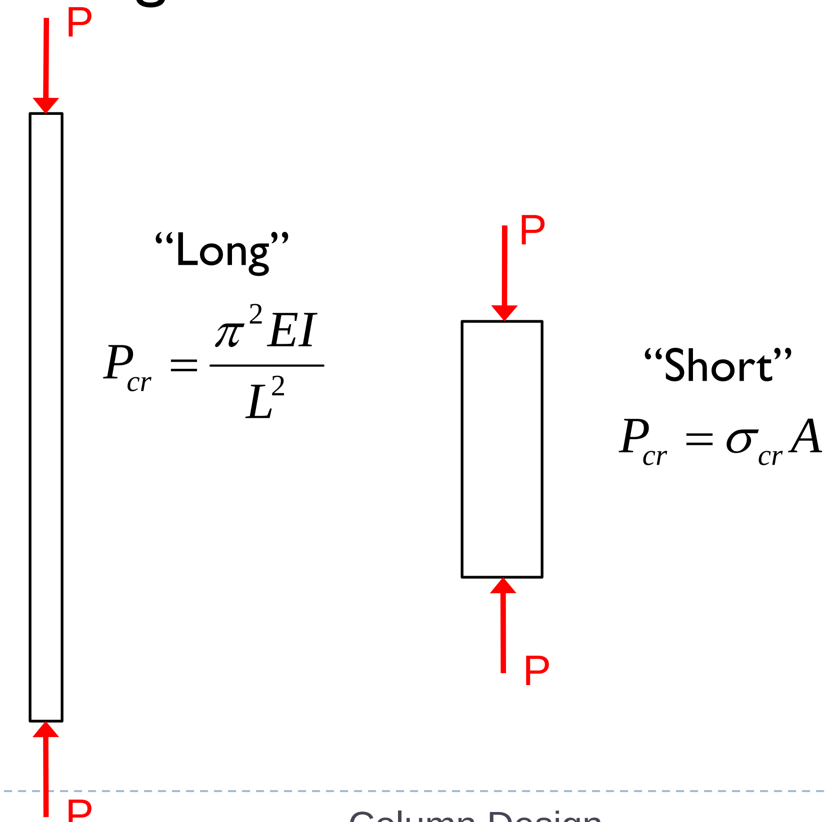
Column Design
(4.11-13)

MAE 316 – Strength of Mechanical Components

Y. Zhu

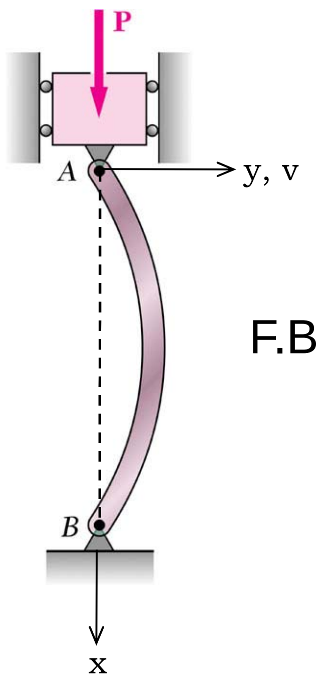
Introduction

- “Long” columns fail structurally (buckling failure)
- “Short” columns fail materially (yielding failure)
- We will examine buckling failure first, and then transition to yielding failure.

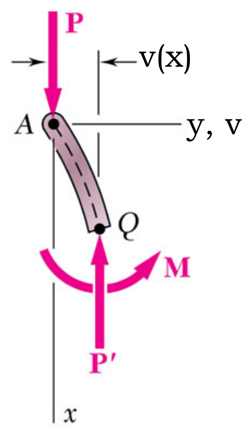


Buckling of Columns (6.2)

- For a simply supported beam (pin-pin)



F.B.D →



$$M(x) = -Pv(x)$$

$$EI\frac{d^2v}{dx^2} = M(x) = -Pv$$

$$EI\frac{d^2v}{dx^2} + Pv = 0$$

The solution to the above O.D.E is:

$$v(x) = A\sin\sqrt{\frac{P}{EI}}x + B\cos\sqrt{\frac{P}{EI}}x$$

Buckling of Columns (6.2)

$$v(x) = A\sin\sqrt{\frac{P}{EI}}x + B\cos\sqrt{\frac{P}{EI}}x$$

- Apply boundary conditions: v(0)=0 & v(L)=0

$$B = 0 \quad \& \quad A\sin\sqrt{\frac{P}{EI}}L = 0$$

- Either A = 0 (trivial solution, no displacement) or

$$\sin\sqrt{\frac{P}{EI}}L = 0 \Rightarrow \sqrt{\frac{P}{EI}}L = n\pi \quad \text{where n = 1, 2, …}$$

- The critical load is then $P_{cr} = (n\pi)^2 \frac{EI}{L^2}$

- The lowest critical load (called the Euler buckling load) occurs when n=1.

$$\boxed{P_{cr} = \frac{\pi^2 EI}{L^2}}$$

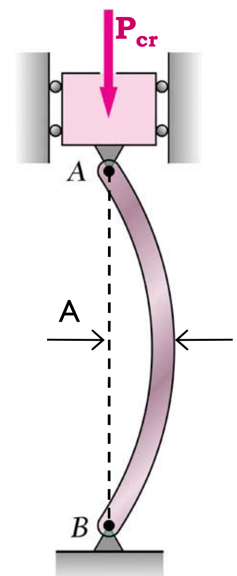
Buckling of Columns (6.2)

- The corresponding deflection at the critical load is

$$v(x) = A \sin \frac{\pi x}{L}$$

- The beam deflects in a half sine wave.

- The amplitude A of the buckled beam cannot be determined by this analysis. It is finite, however. Non-linear analysis is required to proceed further.



Buckling of Columns (6.2)

- Define the radius of gyration as

$$r_g{}^2 = \frac{I}{A}$$

- P_{cr} can be written as

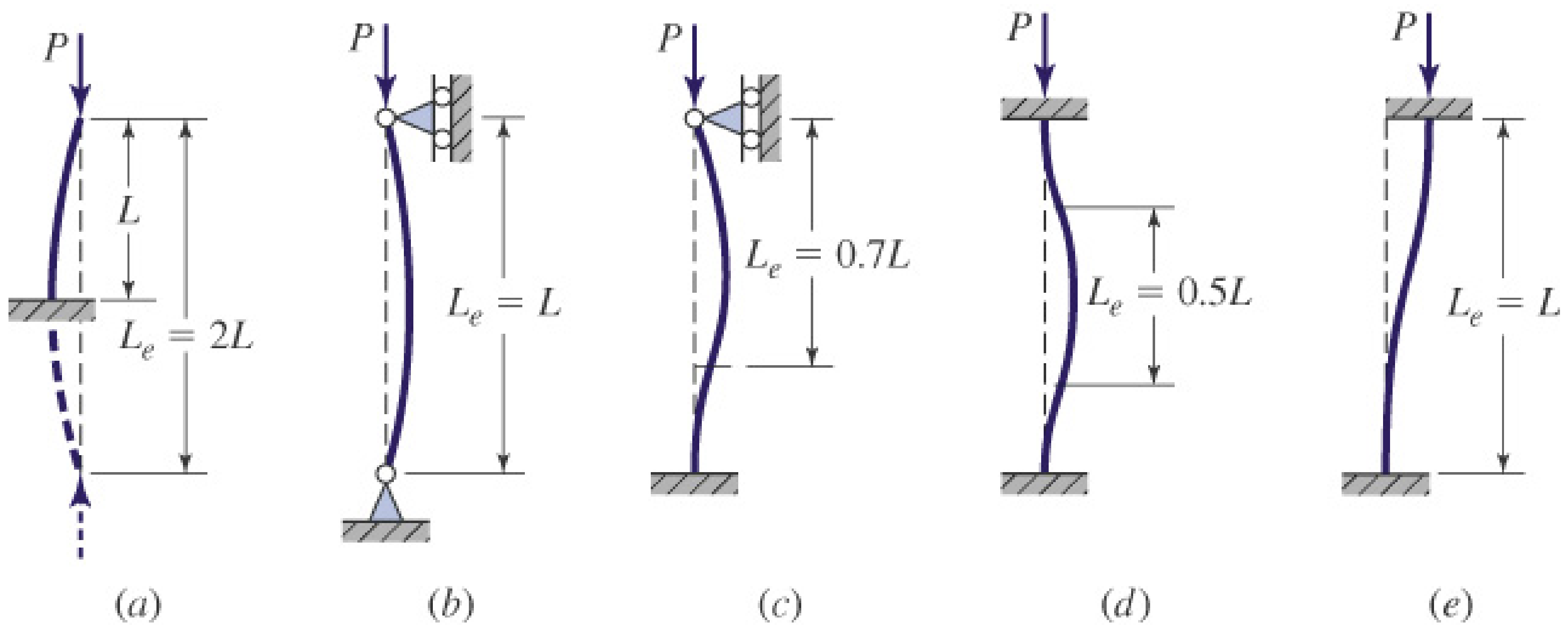
$$\frac{P_{cr}}{A} = \frac{\pi^2 E}{(L/r_g)^2}$$

  - If L/r_g is large, the beam is long and slender.
  - If L/r_g is small, the beam is short and stumpy.
- To account for different boundary conditions (other than pin-pin), use *effective length*, L_e.

$$\boxed{\frac{P_{cr}}{A} = \frac{\pi^2 E}{(L_e/r_g)^2} = \sigma_{cr}}$$

Buckling of Columns (6.2)

- Effective lengths for various boundary conditions



(a) fixed-free, (b) pin-pin, (c) fixed-pin, (d) fixed-fixed, (e) fixed-non-rotating

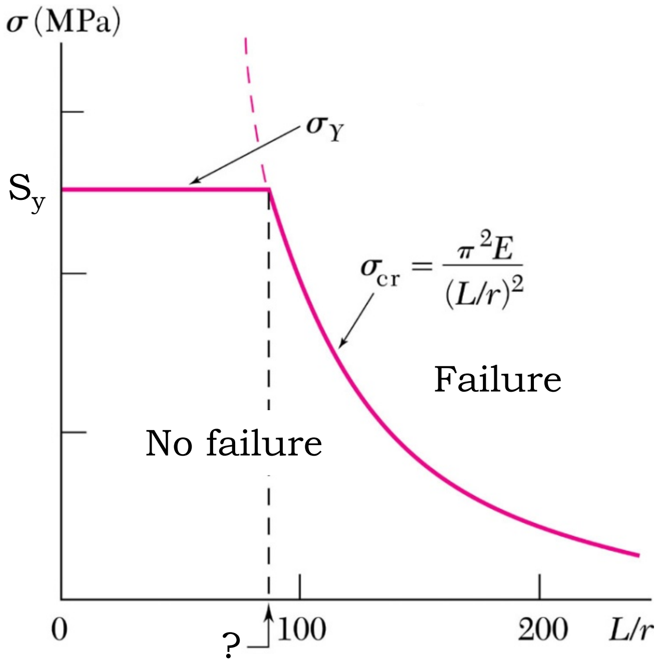
- Design values (AISI)

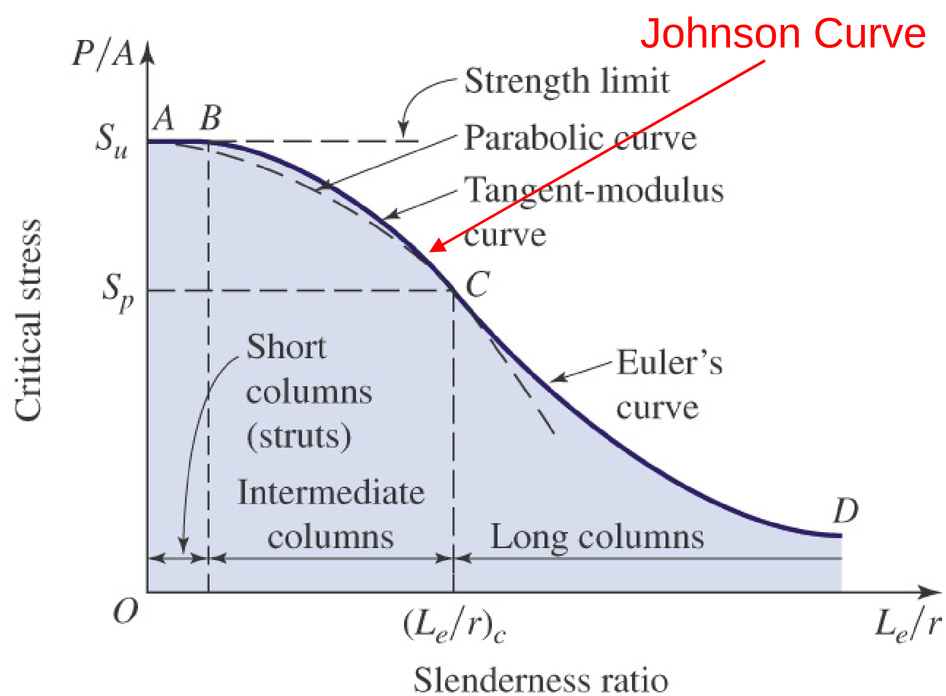
(a) L_e = 2.1L, (b) L_e = L, (c) L_e = 0.8L, (d) L_e = 0.65L, (e) L_e = 1.2L

Critical Stress (6.3)

Note:
S_y = yield strength = σ_y
S_u = ultimate strength = σ_u
S_p = proportional limit

- Plot $\sigma_{cr}=P_{cr}/A$ vs. L/r_g
  - Long column: buckling occurs elastically before the yield stress is reached.
  - Short column: material failure occurs inelastically beyond the yield stress.
  - A "Johnson Curve" can be used to determine the stress for an intermediate column





Critical Stress (6.3)

- Johnson equation

$$\boxed{\frac{P_{cr}}{A} = S_y - \frac{S_y{}^2}{4\pi^2 E}\left(\frac{L_e}{r_g}\right)^2}$$

- As $L_e/r_g \rightarrow 0$, $P_{cr}/A \rightarrow S_y$
- To find the transition point $(L_e/r_g)_c$ between the Johnson and Euler regions

$$Euler = Johnson$$

$$\frac{\pi^2 E}{\left(Le/rg\right)^2} = S_y - \frac{S_y{}^2}{4\pi^2 E}\left(\frac{L_e}{r_g}\right)^2$$

$$\left(\frac{L_e}{r_g}\right)_c = \sqrt{\frac{2\pi^2 E}{S_y}}$$

Design of Columns (6.6)

- Procedure
  - Calculate $(L_e/r_g)_c$ and L_e/r_g
  - If $L_e/r_g \geq (L_e/r_g)_c$ use Euler's equation

$$\frac{P_{cr}}{A} = \frac{\pi^2 E}{(L_e / r_g)^2} = \sigma_{cr}$$

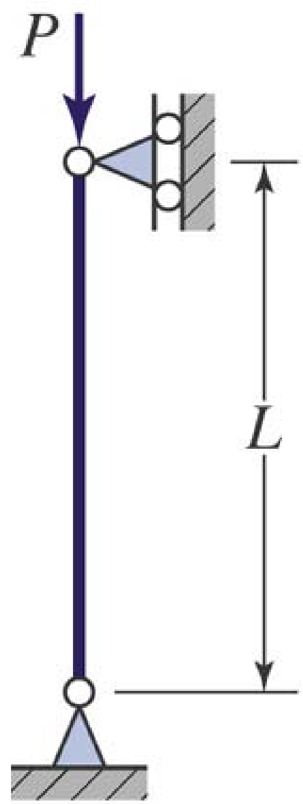
  - If $L_e/r_g \leq (L_e/r_g)_c$ use Johnson's equation

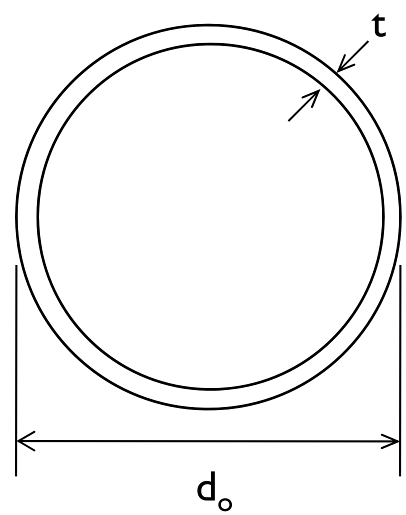
$$\frac{P_{cr}}{A} = S_y - \frac{{S_y}^2}{4\pi^2 E}\left(\frac{L_e}{r_g}\right)^2$$

- Notice we haven't included factor of safety yet…

Example

Find the required outer diameter, with F.S.=3, for the column shown below. Assume P = 15 kN, L = 50 mm, t = 5 mm, S_y = 372 MPa, E = 207,000 MPa, and the material is 1018 cold-drawn steel.





Example

Find the maximum allowable force, F_{max}, that can be applied without causing pipe to buckle. Assume L = 12 ft, b = 5 ft, d_o = 2 in, t = 0.5 in, and the material is low carbon steel.