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INTRODUCTION TO THE YANG-BAXTER EQUATION*

MICHIO JIMBO

Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606, Japan

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1. Introduction

For over two decades the Yang-Baxter equation (YBE) has been studied as the master equation in integrable models in statistical mechanics and quantum field theory. Recent progress in other fields—C^*-algebras, link invariants, quantum groups, conformal field theory, etc.—shed new light to the significance of YBE, and has aroused interest among many people.

In the literature YBE first manifested itself in the work of McGuire[1] in 1964 and Yang[2] in 1967. They considered a quantum mechanical many-body problem on a line having $c\sum_{i<j}\delta(x_i - x_j)$ as the potential. Using a technique—known as Bethe's Ansatz—of building exact wavefunctions, they found that the scattering matrix factorized to that of the two-body problem, and determined it exactly. Here YBE arises as the consistency condition for the factorization.

In statistical mechanics, the source of YBE probably goes back to Onsager's star-triangle relation, briefly mentioned in the introduction to his solution of the Ising model[3] in 1944. Hunt for solvable lattice models has been actively pursued since then,[4,5] culminating in Baxter's solution of the eight vertex model[6] in 1972. Another line of development was the theory of factorized S-matrix in two dimensional quantum field theory.[7] Zamolodchikov pointed out[8] that the algebraic mechanism working here is the same as that in Baxter's and others' works.

In 1978–79 Faddeev, Sklyanin and Takhtajan proposed the quantum inverse method[9,10] as a unification of the classical integrable models (= soliton theory) and the quantum ones mentioned above. In their theory the basic commutation relation of operators is described by a solution of YBE (this terminology itself is due to them). In the beginning of 1980s, the study of YBE has been actively performed in Leningrad, Moscow and other places.[11,12] These works led to the idea of introducing certain deformations of groups or Lie algebras,[13–16] as called quantum groups by Drinfeld.[17] At about the same time there appeared the discovery of new invariants of links,[18] and subsequently the aspect of YBE as the braid-type relation has been brought to

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attention. Closely related structures have also been revealed in the conformal field theory.[19–21]

The present article is aimed to be an introduction to YBE for nonspecialists. About the state of the matter up to 1982 good review papers are available.[11,12] We have tried here to include some of the more recent developments (within my limited knowledge), with the emphasis on the role of quantum groups.

The text is organized as follows. Sec. 2 is devoted to basic definitions, properties and elementary examples of solutions of YBE. In Sec. 3 the classical YBE is defined, and the structure of its solutions is depicted following Belavin-Drinfeld.[22] In Sec. 4 the quantized universal enveloping algebra $U_q\mathfrak{g}$ is introduced. Known facts about its representations and Drinfeld's universal R matrix are briefly summarized. As an application, in Sec. 5 the trigonometric solutions of YBE related to the vector representation of classical Lie algebras are described. Solutions corresponding to 'higher' representations can be obtained by the fusion procedure.[23] Sec. 6 outlines this method. In Sec. 7 the solutions of YBE of the 'face-model' type are discussed together with the braid representations induced by them.

2. The Yang-Baxter Equation

2.1. *Formulation*

Let V be a complex vector space. Let $R(u)$ be a function of $u \in C$ taking values in $\mathrm{End}_C(V \otimes V)$. The following equation for $R(u)$ is called the Yang-Baxter equation (YBE):

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u). \tag{2.1}$$

Here R_{ij} signifies the matrix on $V^{\otimes 3}$, acting as $R(u)$ on the ith and the jth components and as identity on the other component; e.g. $R_{23}(u) = I \otimes R(u)$. The variable u is called the spectral parameter. Often a solution of (2.1) is referred to as an R matrix.

In most cases we assume that $N = \dim V < \infty$. Upon taking a basis of V and writing

$$R(u) = \sum R_{ij}^{kl}(u) E_{ik} \otimes E_{jl},$$

$$E_{ij} = (\delta_{ia}\delta_{jb})_{a,b=1,\dots,N},$$

one sees that (2.1) amounts to N^6 homogeneous equations for the N^4 unknowns $R_{ij}^{kl}(u)$.

Let $P \in \mathrm{End}_C(V \otimes V)$ denote the transposition

$$Px \otimes y = y \otimes x.$$

If $R(u)$ has the property

$$R(0) = \text{const.}\, P, \tag{2.2}$$

then (2.1) is identically satisfied for $u = 0$ or $v = 0$. We call (2.2) the *initial condition.*

2.2 *Examples*

Here are some typical examples of solutions of (2.1) in the case $V = C^2$.

Example 2.1. (McGuire,[1] Yang[2])

$$R(u) = \begin{pmatrix} 1+u & & & \\ & u & 1 & \\ & 1 & u & \\ & & & 1+u \end{pmatrix} = P + uI.$$

Example 2.2

$$R(u) = \begin{pmatrix} \sin(\eta + u) & & & \\ & \sin u & \sin\eta & \\ & \sin\eta & \sin u & \\ & & & \sin(\eta+u) \end{pmatrix}.$$

Example 2.3. (Baxter[6])

$$R(u) = \begin{pmatrix} a(u) & & & d(u) \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ d(u) & & & a(u) \end{pmatrix},$$

where

$$a(u) = \theta_0(\eta)\theta_0(u)\theta_1(\eta + u)$$

$$b(u) = \theta_0(\eta)\theta_1(u)\theta_0(\eta + u)$$

$$c(u) = \theta_1(\eta)\theta_0(u)\theta_0(\eta + u)$$

$$d(u) = \theta_1(\eta)\theta_1(u)\theta_1(\eta + u).$$

The $\theta_i(u)$ denote the elliptic theta functions

$$\begin{aligned} \theta_0(u) &= \prod_{n=1}^{\infty} (1 - 2p^{n-1/2}\cos 2\pi u + p^{2n-1})(1 - p^n), \\ \theta_1(u) &= 2p^{1/8}\sin \pi u \prod_{n=1}^{\infty} (1 - 2p^n \cos 2\pi u + p^{2n})(1 - p^n). \end{aligned} \tag{2.3}$$

In all cases the initial condition (2.2) is fulfilled. In examples 2.2 and 2.3, the parameters η, p are arbitrary. In fact these three are connected by specialization:

$$\text{Example 2.3} \xrightarrow{p \to 0} \text{Example 2.2} \xrightarrow{\eta \to 0} \text{Example 2.1.}$$

Let us verify that example 2.1 solves YBE. Since both sides of (2.1) are polynomials in u of degree 2, it suffices to check (2.1) for 3 values of u.

$$u = 0 : \text{valid because of (2.2)},$$

$$u = \infty : P_{23} + vI = P_{23} + vI,$$

$$u = -v : (P_{12} - vI)P_{13}(P_{23} + vI) = (P_{23} + vI)P_{13}(P_{12} - vI).$$

The last equation reduces to the relations in the symmetric group $\mathfrak{S}_3$ (12)(13)(23) = (23)(13)(12), (12)(13) = (13)(23).

Examples 2.2–2.3 can be handled in the same spirit.

2.3. *Braid relations*

Frequently YBE is also written in terms of the matrix

$$\check{R}(u) = PR(u). \tag{2.4}$$

Let $m \geq 2$, and define matrices on $V^{\otimes m}$ by $\check{R}_i(u) = I \otimes \cdots \otimes \check{R}(u) \otimes \cdots \otimes I(\check{R}(u)$ in the $(i, i+1)$th slot), $i = 1, \ldots, m-1$. One has then

$$\begin{aligned} \check{R}_i(u)\check{R}_j(v) &= \check{R}_j(v)\check{R}_i(u) \qquad \text{if} \qquad |i-j| > 1, \\ \check{R}_{i+1}(u)\check{R}_i(u+v)\check{R}_{i+1}(v) &= \check{R}_i(v)\check{R}_{i+1}(u+v)\check{R}_i(u). \end{aligned} \tag{2.5}$$

In the absence of the spectral parameters u, v, (2.5) is nothing other than Artin's braid relations. One notices that for the special values such that $u = u + v = v$, the $\check{R}_i(u)$ actually give rise to a representation of braid groups. The choice $u = v = 0$ usually leads to the trivial representation $\check{R}_i(0) = \text{const.}\, I$ (the initial condition (2.2)). In certain circumstances it makes sense to take $u = v = \infty$, leading to interesting results. See 5.3., 7.5. below.

2.4. *Generalizations*

The above formulation of YBE admits the following extensions.

(i) Instead of working with a fixed vector space V, one can equally well consider a family of vector spaces $\mathscr{F} = \{V\}$ and operators $\{R_{VV'}(u) \in \mathrm{End}_C(V \otimes V')\}_{V, V' \in \mathscr{F}}$. YBE(2.1) is then an equation in $\mathrm{End}_C(V_1 \otimes V_2 \otimes V_3)$, where $R_{ij}(u) = R_{V_iV_j}(u)$, $V_i \in \mathscr{F}$. Suppose $V_1 = V_2 = V$. Regarding $\mathrm{End}_C(V \otimes V_3) = \mathrm{End}_C(V) \otimes \mathscr{A}$, $\mathscr{A} = \mathrm{End}_C(V_3)$, let us write $R_{VV_3}(u)$ as $T(u) = \sum t_{ij}(u)E_{ij}$ with $t_{ij}(u) \in \mathscr{A}$. In this notation (2.1) becomes

$$\check{R}(u - v)T(u) \otimes T(v) = T(v) \otimes T(u)\check{R}(u - v). \tag{2.6}$$

Here $T(u) \otimes T(v) = \sum t_{ij}(u) t_{kl}(v) E_{ij} \otimes E_{kl}$, etc. (notice the ordering of the $t_{ij}(u)$ and $t_{kl}(v)$). Equation (2.6) can be viewed as giving commutation relations among the generators $t_{ij}(u)$ of an abstract algebra $\mathscr{A}$; YBE for $\check{R}(u)$ guarantees the associativity of $\mathscr{A}$ thus defined. An important feature of (2.6) is that the map

$$\Delta : \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}, \qquad \Delta\big(t_{ij}(u)\big) = \sum_k t_{ik}(u) \otimes t_{kj}(u)$$

preserves the relations (2.6) (Δ is a 'comultiplication'). The formulae (2.1) and (2.6) are the basic algebraic constituents in the quantum inverse method.[9,10]
(ii) One may consider YBE for a function of two variables $R(u, u')$:

$$R_{12}(u_1, u_2) R_{13}(u_1, u_3) R_{23}(u_2, u_3) = R_{23}(u_2, u_3) R_{13}(u_1, u_3) R_{12}(u_1, u_2). \qquad (2.7)$$

Equation (2.1) is a special case of (2.7) where the (u, u')-dependence enters only through the difference

$$R(u, u') \equiv R(u - u'). \qquad (2.8)$$

Recently Au-Yang, Baxter, McCoy, Perk and others[24,25] have found remarkable new solutions to (2.7) in which the spectral parameters live on curves of genus > 1. The difference property (2.8) does not hold for these solutions.

3. The Classical Yang-Baxter Equation

3.1 *Classical limit*

A solution of YBE is said to be *quasi-classical* if it contains an extra parameter $\hbar$ ('Planck constant') in such a way that as $\hbar \to 0$ it has the expansion

$$R(u, \hbar) = (\text{scalar}) \times \big(I + \hbar r(u) + O(\hbar^2)\big). \qquad (3.1)$$

The $r(u) \in \mathrm{End}_{\mathbf{C}}(V \otimes V)$ in (3.1) is called the *classical limit* of $R(u, \hbar)$. For instance, Examples 2.1–2.3 are all quasi-classical (take $u \to u/\hbar$ in Example 2.1, $\eta = \hbar$ in Examples 2.2–2.3). For quasi-classical $R(u, \hbar)$, YBE (2.1) implies the following *classical* Yang-Baxter equation (CYBE) for $r(u)$:

$$[r_{12}(u), r_{13}(u + v)] + [r_{12}(u), r_{23}(v)] + [r_{13}(u + v), r_{23}(v)] = 0. \qquad (3.2)$$

As for the significance of CYBE in classical integrable systems and its algebraic/geometric meaning, see Refs. 12, 26 and 27.

There are important examples of solutions of YBE which are not quasi-classical (e.g. Refs. 24, 25, 28 and 29). Nevertheless quasi-classical solutions constitute an interesting class, and we shall henceforth restrict our attention to this case.

3.2 *Universal solution*

The characteristic feature of CYBE is that it is formulated using solely the Lie algebra structure of End(V). Let $\mathfrak{g}$ be a Lie algebra, and let $r(u)$ be a $\mathfrak{g} \otimes \mathfrak{g}$-valued function. In terms of a basis $\{X_\mu\}$ of $\mathfrak{g}$, write

$$r(u) = \sum_{\mu,\nu} r^{\mu\nu}(u) X_\mu \otimes X_\nu \tag{3.3}$$

with C-valued functions $r^{\mu\nu}(u)$. Let further $r_{12}(u) = \sum r^{\mu\nu}(u) X_\mu \otimes X_\nu \otimes I \in (U\mathfrak{g})^{\otimes 3}$ and so on, where $U\mathfrak{g}$ denotes the universal enveloping algebra. One has then

$$[r_{12}(u), r_{23}(v)] = \sum r^{\mu\nu}(u) r^{\rho\sigma}(v) X_\mu \otimes [X_\nu, X_\rho] \otimes X_\sigma, \text{ etc.,}$$

so that each term in (3.2) actually lies inside $\mathfrak{g}^{\otimes 3}$. For each triplet of representations $(\pi_i, V_i)(i = 1, 2, 3)$ of $\mathfrak{g}$, $(\pi_i \otimes \pi_j)(r_{ij}(u))$ gives a matrix solution of CYBE in $V_1 \otimes V_2 \otimes V_3$. In this sense a $\mathfrak{g} \otimes \mathfrak{g}$-valued solution is a 'universal' solution of CYBE.

3.3 *Belavin-Drinfeld theory*

When $\mathfrak{g}$ is a finite-dimensional complex simple Lie algebra, solutions of CYBE have been studied in detail by Belavin and Drinfeld.[22] In the sequel we fix an orthonormal basis $\{X_\mu\}$ of $\mathfrak{g}$ with respect to a nondegenerate invariant bilinear form on $\mathfrak{g}$, and set

$$t = \sum_\mu X_\mu \otimes X_\mu .$$

By $r(u)$ we will mean a $\mathfrak{g} \otimes \mathfrak{g}$-valued meromorphic solution of (3.2) defined in a neighborhood of $0 \in C$. It is said to be *nondegenerate* if $\det(r^{\mu\nu}(u)) \not\equiv 0$ in the notation of (3.3).

Theorem.[22] Let $r(u)$ be a nondegenerate solution of (3.2). Then

(1) $r(u)$ extends meromorphically to the whole complex plane C, with all its poles being simple.

(2) $\Gamma = \{$the set of poles of $r(u)\}$ is a discrete subgroup relative to the addition of C.

(3) As a function of u there are the following possibilities for the $r^{\mu\nu}(u)$:

rank $\Gamma = 2$: elliptic function,

rank $\Gamma = 1$: trigonometric function
(i.e. a rational function in the variable $e^{\text{const.}u}$),

rank $\Gamma = 0$: rational function.

Belavin-Drinfeld show further that (i) elliptic solution exists only for $\mathfrak{g} = \mathfrak{sl}(n)$, in which case it is unique (up to certain equivalence of solutions), (ii) trigonometric solutions exist for each type, and can be classified using the data from the Dynkin diagram for affine Lie algebras.

3.4 *Examples*

Here are two typical examples in the Belavin-Drinfeld classification.

Example 3.1. The simplest rational solution is

$$r(u) = \frac{t}{u}\,.$$

Example 3.2. Let $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_\alpha \mathfrak{g}_\alpha)$ be the root space decomposition. Choose $X_\alpha \in \mathfrak{g}_\alpha$ so that $(X_\alpha, X_{-\alpha}) = 1$, and let

$$r = \sum_{\alpha>0} (X_\alpha \otimes X_{-\alpha} - X_{-\alpha} \otimes X_\alpha) \qquad \text{(sum over the positive roots)}. \tag{3.4}$$

Then

$$r(u) = r - t + \frac{2t}{1-x}\,, \qquad x = e^u\,, \tag{3.5}$$

is a trigonometric solution.

4. The Quantized Universal Enveloping Algebra

4.1 *Quantization*

Given a solution $r(u) \in \mathfrak{g} \otimes \mathfrak{g}$ of CYBE, one may ask whether there exists a quasi-classical $R(u, \hbar)$ having $r(u)$ as its classical limit.[23] One has then to decide where such an object should live. A naïve candidate is $U\mathfrak{g} \otimes U\mathfrak{g}$; as it turns out, however, it is more natural to deform (or 'quantize') the algebra $U\mathfrak{g}$ according to each $r(u)$. Motivated by this 'quantization' problem, Drinfeld developed a general theory of quantum groups.[17] In this section, we shall describe a representative class of quantum groups, the quantized universal enveloping algebra $U_q\mathfrak{g}$[15] (also called a q-analog of $U\mathfrak{g}$[16]), which is related to the trigonometric solutions of Example 3.2. As for the case related to Example 3.1, see Ref. 15.

4.2. *The algebra $U_q\mathfrak{g}$*

Hereafter $\mathfrak{g}$ will denote a Kac-Moody Lie algebra of finite or affine type. The corresponding generalized Cartan matrix $A = (a_{ij})_{1 \le i,j \le l}$ is symmetrizable in the sense that there exist nonzero integers d_i satisfying $d_i a_{ij} = d_j a_{ji}$. Fix such $\{d_i\}$. Fix also a nonzero complex number q such that $q^{2d_i} \neq 1$. We define $U_q\mathfrak{g}$ to be the associative C-algebra with 1, with $4l$ generators

$$X_i^+\,, \qquad X_i^-\,, \qquad k_i\,, \qquad k_i^{-1} \qquad (1 \le i \le l)$$

and relations

$$k_i k_j = k_j k_i\,, \qquad k_i k_i^{-1} = k_i^{-1} k_i = 1\,,$$

$$k_i X_j^{\pm} k_i^{-1} = q^{\pm d_i a_{ij}/2} X_j^{\pm}\,,$$

$$[X_i^+, X_j^-] = \delta_{ij}\frac{k_i^2 - k_i^{-2}}{q^{d_i} - q^{-d_i}},$$

$$\sum_{v=0}^{1-a_{ij}} (-)^v \begin{bmatrix} 1-a_{ij} \\ v \end{bmatrix}_{q^{d_i}} (X_i^{\pm})^{1-a_{ij}-v}(X_j^{\pm})(X_i^{\pm})^v = 0 \qquad (i \neq j).$$

Here we have used the notations from q-analysis

$$\begin{bmatrix} m \\ n \end{bmatrix}_t = \frac{[m]_t!}{[n]_t![m-n]_t!}, \qquad [m]_t! = \prod_{1 \le j \le m} \frac{t^j - t^{-j}}{t - t^{-1}}.$$

Setting formally $k_i = q^{d_i h_i/2}$ and letting $q \to 1$ one recovers the commutation relations among the Chevalley generators $\{e_i = X_i^+, f_i = X_i^-, h_i\}_{1 \le i \le l}$ of $\mathfrak{g}$.

The following comultiplication Δ, antipode S and counit ε endow $U_q\mathfrak{g}$ a Hopf algebra structure:

$$\Delta(X_i^{\pm}) = X_i^{\pm} \otimes k_i^{-1} + k_i \otimes X_i^{\pm}, \qquad \Delta(k_i) = k_i \otimes k_i,$$

$$S(X_i^{\pm}) = -q^{\mp d_i} X_i^{\pm}, \qquad S(k_i) = k_i^{-1},$$

$$\varepsilon(X_i^{\pm}) = 0, \qquad \varepsilon(k_i) = 1.$$

The Hopf algebra $U_q\mathfrak{g}$ was introduced by Kulish-Reshetikhin[13] (for $\mathfrak{g} = \mathfrak{sl}(2)$), Drinfeld[15] and the author[16] (for a Kac-Moody algebra with symmetrizable generalized Cartan matrix). Our normalization here follows Ref. 16.

4.3. *Representation theory*

It has been shown by Lusztig[30] and Rosso[31] that for generic values of q the representation theory of $U_q\mathfrak{g}$ does not change from the classical case $q = 1$.

Theorem.[30,31] Let $\dim \mathfrak{g} < \infty$, and assume that q is not a root of unity. Then an irreducible integrable $\mathfrak{g}$-module can be deformed to that of $U_q\mathfrak{g}$. The dimensionality of each weight space is the same as in the case $q = 1$.

Example. Let $\mathfrak{g} = \mathfrak{sl}(2)$, and let l be a positive integer. Setting

$$\pi(X_1^+) = \begin{pmatrix} 0 & \sqrt{[l][1]} & & & \\ & 0 & \sqrt{[l-1][2]} & & \\ & & 0 & \sqrt{[l-2][3]} & \\ & & & \ddots & \ddots \\ & & & & \sqrt{[1][l]} \\ & & & & 0 \end{pmatrix} = {}^t\pi(X_1^-),$$

$$\pi(k_1) = \begin{pmatrix} q^{l/2} & & & \\ & q^{l/2-1} & & \\ & & \ddots & \\ & & & q^{-l/2} \end{pmatrix},$$

one gets the 'quantum deformation' of the $(l+1)$-dimensional irreducible representation of $\mathfrak{sl}(2)$. Here we have set

$$[u] = \frac{q^u - q^{-u}}{q - q^{-1}}. \tag{4.1}$$

For q a root of unity, the situation resembles the modular representation. In this case Lusztig[32] developed a highest weight theory by modifying the definition of $U_q\mathfrak{g}$.

For affine Lie algebra $\mathfrak{g}$ of ADE type, Frenkel and Jing[33] constructed the vertex operator representations of $U_q\mathfrak{g}$.

4.4. *Universal R matrix*

Drinfeld constructed a 'universal R matrix' $\mathcal{R} \in U_q\mathfrak{g} \otimes U_q\mathfrak{g}$ that enjoys the following properties:[17]

$$\begin{aligned} \sigma \circ \Delta(a) &= \mathcal{R}\Delta(a)\mathcal{R}^{-1} \qquad \text{for} \qquad a \in U_q\mathfrak{g}, \\ (id. \otimes \Delta)\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{12}, \\ (\Delta \otimes id.)\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{23}, \end{aligned} \tag{4.2}$$

where $\sigma(x \otimes y) = y \otimes x$, and if $\mathcal{R} = \sum a_i \otimes b_i$ then $\mathcal{R}_{12} = \sum a_i \otimes b_i \otimes 1$, $\mathcal{R}_{13} = \sum a_i \otimes 1 \otimes b_i$, $\mathcal{R}_{23} = \sum 1 \otimes a_i \otimes b_i \in (U_q\mathfrak{g})^{\otimes 3}$. (To be precise, Drinfeld uses certain completion of $U_q\mathfrak{g}$, and $\otimes$ is to be understood in the topological sense. See Ref. 17 for details.)

Let $\hat{\mathfrak{g}}$ be affine, and let $\hat{\mathfrak{g}}' = [\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]/(\text{the center of } \hat{\mathfrak{g}})$. Let $\mathcal{R}'$ denote the analog of $\mathcal{R}$ for $\hat{\mathfrak{g}}'$. Consider the automorphism T_x of $U_q\mathfrak{g}$ given by $T_x X_0^{\pm} = x^{\pm 1} X_0^{\pm}$, $T_x X_i^{\pm} = X_i^{\pm} (i \neq 0)$, where $i = 0$ is the distinguished vertex in the Dynkin diagram of $\hat{\mathfrak{g}}$. Set

$$\mathcal{R}(x) = (T_x \otimes id.)(\mathcal{R}').$$

From (4.2) it follows that $\mathcal{R}(x)$ solves YBE in the multiplicative parametrization

$$\mathcal{R}_{12}(x)\mathcal{R}_{13}(xy)\mathcal{R}_{23}(y) = \mathcal{R}_{23}(y)\mathcal{R}_{13}(xy)\mathcal{R}_{12}(x).$$

The r-matrix (3.5) is the classical limit of $\mathcal{R}(x)$ corresponding to the nontwisted loop algebra $\hat{\mathfrak{g}}' = \mathfrak{g} \otimes C[\lambda, \lambda^{-1}]$, $\dim \mathfrak{g} < \infty$. (The classical limit for twisted loop algebras can be found in Ref. 22.)

Properties of $\mathcal{R}$ for finite dimensional $\mathfrak{g}$ are discussed also in Ref. 34.

5. *R* Matrix for the Vector Representation

5.1. *Linear equations for R*

Let us turn to the problem of constructing finite dimensional matrix solutions of YBE. Until the end of this paper we shall assume that q is not a root of unity, unless it is stated explicitly. Retaining the notations $\hat{\mathfrak{g}}' = \mathfrak{g} \otimes C[\lambda, \lambda^{-1}]$, $\dim \mathfrak{g} < \infty$ as in 4.4., let $\pi : U_q\hat{\mathfrak{g}}' \to \mathrm{End}_C(V)$ be a finite dimensional representation. Regarding $U_q\mathfrak{g}$ as a subalgebra of $U_q\hat{\mathfrak{g}}'$, we denote the restriction $\pi|_{U_q\mathfrak{g}}$ by the same letter π. We assume that at $q = 1$ the latter specializes to an irreducible representation of $\mathfrak{g}$. (An irreducible representation of $\mathfrak{g}$ always lifts to $U_q\mathfrak{g}$ (see 4.3), but not always to $U_q\hat{\mathfrak{g}}'$ unless $\mathfrak{g}$ is of type A_n, cf. Ref. 15.)

Denoting by $P \in \mathrm{End}_C(V \otimes V)$ the transposition, set

$$\check{R}(x) = P(\pi \otimes \pi)(\mathscr{R}(x)).$$

From (4.2) one then has

$$[\check{R}(x), (\pi \otimes \pi)(\Delta(a))] = 0 \qquad \text{for} \qquad a \in U_q\mathfrak{g}, \tag{5.1a}$$

$$\check{R}(x)\left(x^{\pm 1}\pi(X_0^{\pm}) \otimes \pi(k_0)^{-1} + \pi(k_0) \otimes \pi(X_0^{\pm})\right)$$
$$= \left(\pi(X_0^{\pm}) \otimes \pi(k_0)^{-1} + \pi(k_0) \otimes x^{\pm 1}\pi(X_0^{\pm})\right)\check{R}(x). \tag{5.1b}$$

In fact these linear equations uniquely determine $\check{R}(x)$ up to a scalar factor.[35]

Equation (5.1a) means that $\check{R}(x)$ belongs to the centralizer of the diagonal action of $U_q\mathfrak{g}$ in $V \otimes V$. Taking a basis $\{P_k\}$ of $\mathrm{End}_{U_q\mathfrak{g}}(V \otimes V)$ one can write

$$\check{R}(x) = \sum_k \rho_k(x) P_k. \tag{5.2}$$

The coefficients ρ_k (up to an overall factor) are to be determined from (5.1b).

5.2 *Vector representation*

As an example let us consider the vector representation (π, $V_{\Lambda_1} = C^N$) of the classical Lie algebras $\mathfrak{g} = \mathfrak{sl}(N)$, $\mathfrak{o}(N)$, or $\mathfrak{sp}(N)$ with N even. The R matrix in this representation has been calculated in Refs. 36 and 35.

As in the classical case we have the decomposition as $U_q\mathfrak{g}$-module

$$V_{\Lambda_1} \otimes V_{\Lambda_1} = V_{2\Lambda_1} \oplus V_{\Lambda_2} \qquad \text{for} \qquad \mathfrak{g} = \mathfrak{sl}(N),$$
$$= V_{2\Lambda_1} \oplus V_{\Lambda_2} \oplus V_0 \qquad \text{for} \qquad \mathfrak{g} = \mathfrak{o}(N), \mathfrak{sp}(N),$$

where $V_{2\Lambda_1}$, V_{Λ_2} or V_0 denotes the analog of the symmetric tensor, the antisymmetric tensor or the trivial representation, respectively. Let $P_{2\Lambda_1}$, P_{Λ_2}, P_0 denote the corresponding orthogonal projectors relative to a $U_q\mathfrak{g}$-invariant scalar product on V_{Λ_1}. The spectral decomposition (5.2) for the $\check{R}$ matrix then reads as follows.

$\mathfrak{g} = \mathfrak{sl}(N)$:

$$\check{R}(x) = (qx - q^{-1})P_{2\Lambda_1} - (q^{-1}x - q)P_{\Lambda_2}, \tag{5.3}$$

$\mathfrak{g} = \mathfrak{o}(N)$, $\mathfrak{sp}(N)$:

$$\check{R}(x) = (q^{N-2\varepsilon}x - 1)(qx - q^{-1})P_{2\Lambda_1} - (q^{N-2\varepsilon}x - 1)(q^{-1}x - q)P_{\Lambda_2}$$
$$+ \varepsilon(x - q^{N-2\varepsilon})(q^{-\varepsilon}x - q^{\varepsilon})P_0, \tag{5.4}$$

where for convenience we have set

$$\begin{aligned} \varepsilon &= +1 \qquad \text{for} \qquad \mathfrak{o}(N), \\ &= -1 \qquad \text{for} \qquad \mathfrak{sp}(N). \end{aligned}$$

Together with the obvious relation $1 = \sum_k P_k$, the projectors are given by the following formulae.

$\mathfrak{g} = \mathfrak{sl}(N)$:
Set

$$T = qP_{2\Lambda_1} - q^{-1}P_{\Lambda_2}. \tag{5.5a}$$

Then

$$T^{\pm 1} = q^{\pm 1} \sum E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ij} \otimes E_{ji} \pm (q - q^{-1}) \sum_{i \lessgtr j} E_{ii} \otimes E_{jj}. \tag{5.5b}$$

$\mathfrak{g} = \mathfrak{o}(N), \mathfrak{sp}(N)$:
Set

$$\begin{aligned} T &= qP_{2\Lambda_1} - q^{-1}P_{\Lambda_2} + \varepsilon q^{-N+\varepsilon}P_0, \\ S &= \frac{q^{N/2} - q^{-N/2}}{q - q^{-1}}(q^{N/2-\varepsilon} + q^{-N/2+\varepsilon})P_0. \end{aligned} \tag{5.6a}$$

Then we have

$$\begin{aligned} T^{\pm 1} = q^{\pm 1} \sum_{i \neq i'} E_{ii} \otimes E_{ii} + \sum_{\substack{i \neq j, j' \\ \text{or } i=j=j'}} E_{ij} \otimes E_{ji} + q^{\mp 1} \sum_{i \neq i'} E_{ii'} \otimes E_{i'i} \\ \pm (q - q^{-1}) \sum_{i \lessgtr j} (E_{ii} \otimes E_{jj} - \varepsilon_i \varepsilon_j q^{\bar{i} - \bar{j}} E_{j'i} \otimes E_{ji'}), \end{aligned} \tag{5.6b}$$

$$S = \varepsilon \sum \varepsilon_i \varepsilon_j q^{\bar{i} - \bar{j}} E_{j'i} \otimes E_{ji'}.$$

Here $i' = N + 1 - i$, $\varepsilon_i = 1$ for $i < i'$, $\varepsilon_i = \varepsilon$ for $i > i'$, and

$$\begin{aligned} \bar{i} &= i + \frac{1}{2}\varepsilon \qquad (i < i') \\ &= i \qquad\qquad (i = i') \\ &= i - \frac{1}{2}\varepsilon \qquad (i > i'). \end{aligned}$$

5.3. *Centralizer algebras*

As we have noted in 2.3., if an $\check{R}$ matrix is trigonometric, i.e. a polynomial in $x = e^{cu}(c \neq 0)$ up to a scalar factor, then its leading term in x gives rise to a representation of the m string braid group B_m for any $m \geq 2$. Let T be defined by (5.5) or (5.6), and set $T_i = I \otimes \cdots \otimes T \otimes \cdots \otimes I \in \mathrm{End}_{U_q\mathfrak{g}}(V^{\otimes m})$ (T in the $(i, i+1)$-th slot). One has then

$$T_i T_j = T_j T_i \qquad \text{if} \qquad |i-j| > 1\,, \qquad\qquad (5.7)$$
$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}\,.$$

In fact, the above T is the image under $\pi \otimes \pi$ of the universal R matrix corresponding to $U_q\mathfrak{g}(\subset U_q\hat{\mathfrak{g}}')$.[34]

Let $\mathfrak{g} = \mathfrak{sl}(N)$. From (5.5a) one has in addition to (5.7)

$$(T_i - q)(T_i + q^{-1}) = 0\,. \qquad (5.8)$$

The relations (5.7) and (5.8) mean that the braid representation factors through Iwahori's Hecke algebra[37] $H_m(q)$ for the symmetric group:

$$B_m \to H_m(q) \xrightarrow{\rho_m} \mathrm{End}(V^{\otimes m})\,.$$

Moreover ρ_m commutes with the multidiagonal action of $U_q\mathfrak{g}$ given via the $(m-1)$-fold iteration of the comultiplication $\Delta^{(m)}: U_q\mathfrak{g} \to (U_q\mathfrak{g})^{\otimes m}$,

$$\Delta^{(m)}(X_i^{\pm}) = \sum_{j=1}^{m} k_i \otimes \cdots \otimes k_i \otimes \overset{j}{X_i^{\pm}} \otimes k_i^{-1} \otimes \cdots \otimes k_i^{-1}\,.$$

Proposition.[38] For generic q, the two subalgebras of $\mathrm{End}(V^{\otimes m})$

$$\pi^{\otimes m} \circ \Delta^{(m)}(U_q\mathfrak{g}) \qquad \text{and} \qquad \rho_m(H_m(q))$$

are commutant to each other.

This is a q-version of Weyl's reciprocity concerning the action of the symmetric group and the general linear group.

In the case $\mathfrak{g} = \mathfrak{o}(N)$ or $\mathfrak{sp}(N)$, similar statement is true.[39,40] What replaces $H_m(q)$ is the Birman-Wenzl-Murakami algebra,[39,40] a q-analog of Brauer's centralizer algebra. Its quotient appears as the algebra generated by T_i and S_i defined similarly from (5.6).

6. The Fusion Procedure

6.1 *Construction of R matrices*

Many of the solution of YBE known so far have been obtained by direct methods—assuming certain symmetries or guessing the functional form and solving the cubic equation (2.1) for $R(u)$. As for the quasi-classical solutions corresponding to (3.5) and its relatives, there are alternative approaches. One is to solve the linear equations (5.1). The other is the so-called *fusion procedure* initiated by Kulish, Reshetikhin and Sklyanin.[23] This method is an analog of a standard technique to get irreducible representations of Lie algebras—form a tensor product of fundamental representations and decompose it. In this section we shall describe the idea of the construction.

6.2. *The fusion procedure*

For later use we formulate the fusion procedure for the $\check{R}$-type matrices. Thus let $\{\check{R}_{VV'}(u) \in \mathrm{Hom}_C(V\otimes V', V'\otimes V)\}_{V,V'\in\mathscr{F}}$ be a family of solutions of YBE written in the form

$$\begin{aligned}(\check{R}_{V_2V_3}(u)\otimes I)(I\otimes \check{R}_{V_1V_3}(u+v))(\check{R}_{V_1V_2}(v)\otimes I)\\ = (I\otimes \check{R}_{V_1V_2}(v))(\check{R}_{V_1V_3}(u+v)\otimes I)(I\otimes \check{R}_{V_2V_3}(u)).\end{aligned} \tag{6.1}$$

Here both sides map $V_1\otimes V_2\otimes V_3$ to $V_3\otimes V_2\otimes V_1$. We begin with the following observations.

(i) Fix u_1, u_2, and put

$$\check{R}_{V\otimes V'V''}(u) = (\check{R}_{VV''}(u+u_1)\otimes I)(I\otimes \check{R}_{V'V''}(u+u_2)). \tag{6.2a}$$

Then (6.1) remains valid by replacing $\check{R}_{V_1V_i}$ by $\check{R}_{V_1\otimes V_1'V_i}$ $(i=2,3)$. Likewise if we define

$$\check{R}_{V''V\otimes V'}(u) = (I\otimes \check{R}_{V''V'}(u+u_1))(\check{R}_{V''V}(u+u_2)\otimes I), \tag{6.2b}$$

then (6.1) holds with $\check{R}_{V_iV_3\otimes V_3'}$ in place of $\check{R}_{V_iV_3}(i=1,2)$.

(ii) Let $W_i\subset V_i$ be u-independent subspaces such that

$$\check{R}_{V_iV_j}(u)(W_i\otimes W_j)\subset W_j\otimes W_i. \tag{6.3}$$

Then (6.1) is true for the restrictions $\check{R}_{V_iV_j}(u)|_{W_i\otimes W_j}$.

In the notations of (6.2a), let us choose the subspace

$$W = \check{R}_{V'V}(u_2-u_1)(V'\otimes V)\subset V\otimes V'. \tag{6.4}$$

Using YBE (6.1) one finds

$$\begin{aligned}\check{R}_{V\otimes V'V''}(u)(W\otimes V'') &= (\check{R}_{VV''}(u+u_1)\otimes I)(I\otimes \check{R}_{V'V''}(u+u_2))\\ &\quad\times(\check{R}_{V'V}(u_2-u_1)\otimes I)(V'\otimes V\otimes V'')\\ &= (I\otimes \check{R}_{V'V}(u_2-u_1))(\check{R}_{V'V''}(u+u_2)\otimes I)\\ &\quad\times(I\otimes \check{R}_{VV''}(u+u_1))(V'\otimes V\otimes V'')\\ &\subset V''\otimes W,\end{aligned}$$

so the condition (6.3) is satisfied for $W_1 = \check{R}_{V_1'V_1}(u_2-u_1)(V_1' \otimes V_1) \subset V_1 \otimes V_1'$, $W_2 = V_2$ and $W_3 = V_3$. By the same token one has

$$\check{R}_{V''V\otimes V'}(u)(V'' \otimes W) \subset W \otimes V''.$$

With an appropriate choice of $u_2 - u_1$ the W in (6.4) becomes a proper subspace, affording nontrivial new $\check{R}$ matrices

$$\check{R}_{WV''}(u) = \check{R}_{V\otimes V'V''}(u)|_{W\otimes V''}, \tag{6.5a}$$

$$\check{R}_{V''W}(u) = \check{R}_{V''V\otimes V'}(u)|_{V''\otimes W}. \tag{6.5b}$$

6.3 *Symmetric tensors*

Let us illustrate the construction above by taking the trigonometric solution $\check{R}(x)$ (5.3) for $\mathfrak{g} = \mathfrak{sl}(N)$. We shall use the multiplicative parameter $x = e^u$. There are two cases for which this $\check{R}$ matrix degenerates, namely:

$$\check{R}(q^2) \propto P_{2\Lambda_1}, \qquad \check{R}(q^{-2}) \propto P_{\Lambda_2}.$$

Here we consider the former case. Let $V_{\Lambda_1} = C^N$, $V_{2\Lambda_1} = P_{2\Lambda_1}(V_{\Lambda_1} \otimes V_{\Lambda_1})$. Taking $V = V' = V'' = V_{\Lambda_1}$ and $W = V_{2\Lambda_1}$ in (6.5b) one has

$$\check{R}_{V_{\Lambda_1}V_{2\Lambda_1}}(x) = \frac{1}{x - q^{-2}}(I \otimes \check{R}(x))(\check{R}(xq^2) \otimes I)\Big|_{V_{\Lambda_1}\otimes V_{2\Lambda_1}}.$$

Likewise taking $V = V' = V_{\Lambda_1}$, $V'' = W = V_{2\Lambda_1}$ in (6.5a) one obtains

$$\check{R}_{V_{2\Lambda_1}V_{2\Lambda_1}}(x) = (\check{R}_{V_{\Lambda_1}V_{2\Lambda_1}}(xq^{-2}) \otimes I)(I \otimes \check{R}_{V_{\Lambda_1}V_{2\Lambda_1}}(x))|_{V_{2\Lambda_1}\otimes V_{2\Lambda_1}}. \tag{6.6}$$

By the construction, these matrices commute with the diagonal action of $U_q\mathfrak{g}$. Let

$$V_{2\Lambda_1} \otimes V_{2\Lambda_1} = V_{4\Lambda_1} \oplus V_{2\Lambda_1+\Lambda_2} \oplus V_{2\Lambda_2}$$

be the irreducible decomposition, and let $P_{4\Lambda_1}, P_{2\Lambda_1+\Lambda_2}, P_{2\Lambda_2}$ denote the corresponding orthogonal projectors. Then the spectral decomposition of (6.6) is given by

$$\begin{aligned}\check{R}_{V_{2\Lambda_1}V_{2\Lambda_1}}(x) &= (q^2x-1)(q^4x-1)P_{4\Lambda_1}\\ &\quad + (q^2x-1)(q^4-x)P_{2\Lambda_1+\Lambda_2} + (q^2-x)(q^4-x)P_{2\Lambda_2}.\end{aligned}$$

In a similar manner one can construct $\check{R}$ matrices $\check{R}_{VV'}$ where V, V' are general symmetric or antisymmetric tensors (cf. Refs. 23 and 38).

Cherednik[41] gave a prescription to get the R matrix of type $\mathfrak{sl}(N)$ for an arbitrary pair of irreducible representations. His method applies also to the elliptic extension.

7. Face Models

7.1. *Vertex* vs. *face models*

In statistical mechanics, each solution of YBE defines a two-dimensional solvable lattice model; the matrix elements R_{ij}^{kl} of an R matrix stand for the statistical weights (Boltzmann weights) of local configurations. Usually with YBE (2.1) one associates the so-called *vertex models*, where the interaction takes place among the freedom on four edges round a lattice site, or a vertex. There are models that have dual features, in the sense that the interaction takes place among the freedom on the four sites round a face. These are called the *interaction-round-a-face models*,[5] or *face models* for short. Here YBE takes a slightly different form (though the two are mathematically equivalent). An interesting feature is that the face formulation allows to treat the case $q =$ a root of unity, by restricting the range of freedom on sites. These 'restricted' face models and the braid representations they induce play important roles in statistical mechanics, conformal field theory and operator algebras (see (7.4) and (7.5)).

7.2. *Formulation*

Let $W(u)$ be a solution of YBE (2.1) on $V \otimes V$. We say that it is of face-type if the following hold.

(i) There is a direct sum decomposition $V = \bigoplus_{a,b \in \mathscr{S}} V_{ab}$ into subspaces V_{ab} indexed by some (possibly infinite) set $\mathscr{S}$.

(ii) The composition of the maps

$$V_{ab} \otimes V_{b'c'} \xrightarrow{i} V \otimes V \xrightarrow{W(u)} V \otimes V \xrightarrow{p} V_{a'd'} \otimes V_{dc}$$

vanishes unless $a = a'$, $b = b'$, $c = c'$, $d = d'$. We set

$$W\left(\begin{matrix} a & b \\ d & c \end{matrix}\middle| u\right) = p \circ W(u) \circ i \in \mathrm{Hom}_C(V_{ab} \otimes V_{bc}, V_{ad} \otimes V_{dc}).$$

In terms of these operators, YBE takes the form

$$\begin{aligned} &\sum_{g \in \mathscr{S}} W\left(\begin{matrix} f & g \\ e & d \end{matrix}\middle| u\right) W\left(\begin{matrix} b & c \\ g & d \end{matrix}\middle| u+v\right) W\left(\begin{matrix} a & b \\ f & g \end{matrix}\middle| v\right) \\ &= \sum_{g \in \mathscr{S}} W\left(\begin{matrix} g & c \\ e & d \end{matrix}\middle| v\right) W\left(\begin{matrix} a & g \\ f & e \end{matrix}\middle| u+v\right) W\left(\begin{matrix} a & b \\ g & c \end{matrix}\middle| u\right). \end{aligned} \tag{7.1}$$

Both sides map $V_{ab} \otimes V_{bc} \otimes V_{cd}$ to $V_{af} \otimes V_{fe} \otimes V_{ed}$. In the literature the case where $\dim V_{ab} = 0$ or 1 is mainly considered; the $W\left(\begin{matrix} a & b \\ d & c \end{matrix}\middle| u\right)$ are treated as numbers subject to the relations (7.1). See Ref. 5.

7.3. *A vertex-face correspondence*

As in Sec. 5, let $\hat{\mathfrak{g}}' = \mathfrak{g} \otimes C[\lambda, \lambda^{-1}]$ be a nontwisted loop algebra ($\dim \mathfrak{g} < \infty$), and let $\check{R}(x) = P(\pi \otimes \pi)(\mathscr{R}(x))$ be the trigonometric $\check{R}$ matrix associated with a fixed finite dimensional irreducible representation (π, V^π) of $U_q\hat{\mathfrak{g}}'$. Let

$$\mathscr{S} = \text{the set of dominant integral weights of } \mathfrak{g}.$$

For each $a \in \mathscr{S}$ there exists an irreducible representation (see (4.3)) $U_q\mathfrak{g} \to \mathrm{End}(V_a)$. The tensor module is completely reducible[31] with respect to $U_q\mathfrak{g}$, so that one has

$$V_a \otimes V^\pi = \bigoplus_b V_{ab} \otimes V_b\,. \tag{7.2}$$

Here V_{ab} stands for the multiplicity part. We set $V_{ab} = 0$ if V_b does not appear in $V_a \otimes V^\pi$.

Consider now $1 \otimes \check{R}(x) \in \mathrm{End}(V_a \otimes V^\pi \otimes V^\pi)$. Since this matrix commutes with the action of $U_q\mathfrak{g}$, it gives rise to a map

$$V_{ab} \otimes V_{bc} \to \bigoplus_b V_{ab} \otimes V_{bc} \xrightarrow{1 \otimes \check{R}(x)} \bigoplus_d V_{ad} \otimes V_{dc} \to V_{ad} \otimes V_{dc}$$

for each $a, d \in \mathscr{S}$. We let $W\left(\begin{matrix} a & b \\ d & c \end{matrix}\middle| u\right)(q^u = x)$ to be the composition map. In this way, starting from a 'vertex-type' R matrix, one gets a face-type solution of YBE (7.1). This construction is due to Pasquier.[42]

7.4. *Vector representation*

As an example, let us take again the $\check{R}$ matrix (5.3) for $\mathfrak{g} = \mathfrak{sl}(N)$, $\pi =$ the vector representation. Let $\mathscr{A} = \{\varepsilon_1 - \varepsilon, \ldots, \varepsilon_N - \varepsilon\}$ $(\varepsilon = (\varepsilon_1 + \cdots + \varepsilon_N)/N)$ denote the set of weights occurring in π, where the ε_i are orthonormal vectors related to the fundamental weights via $\Lambda_i = \varepsilon_1 + \cdots + \varepsilon_i - i\varepsilon$. Defining V_{ab} by (7.2) one sees that $\dim V_{ab} \le 1$. Clearly $\dim V_{ab} = 0$ unless $b - a$ has the form $\varepsilon_i - \varepsilon$. For $a \in \mathscr{S}$ we define the coordinates a_μ by

$$a_\mu = (a + \rho, \mu) \in Z\,, \qquad \mu \in \mathscr{A}\,,$$

where ρ signifies half the sum of the positive roots. Using the symbol $[u]$ in (4.1), one has the following expression[43] for the nonvanishing Boltzmann weights $W\left(\begin{matrix} a & b \\ d & c \end{matrix}\middle| u\right)$:

$$W\left(\begin{matrix} a & a+\mu \\ a+\mu & a+2\mu \end{matrix}\middle| u\right) = \frac{[1+u]}{[1]}\,,$$

$$W\left(\begin{matrix} a & a+\mu \\ a+\mu & a+\mu+\nu \end{matrix}\middle| u\right) = \frac{[a_\mu - a_\nu - u]}{[a_\mu - a_\nu]}\,, \qquad (\mu \ne \nu), \tag{7.3}$$

$$W\left(\begin{matrix} a & a+\nu \\ a+\mu & a+\mu+\nu \end{matrix}\middle| u\right) = \frac{[u]}{[1]}\left(\frac{[a_\mu - a_\nu + 1][a_\mu - a_\nu - 1]}{[a_\mu - a_\nu]^2}\right)^{1/2}, \qquad (\mu \ne \nu).$$

The case $\mathfrak{g} = \mathfrak{sl}(2)$ first appeared in Ref. 44. For the type $\mathfrak{o}(N)$ or $\mathfrak{sp}(N)$, see Ref. 45.

If we replace $[u]$ by the elliptic theta function $\theta_1(\pi u/L)$ in (2.3) ($L \neq 0$ being a parameter), the $W\left(\begin{matrix} a & b \\ d & c \end{matrix}\middle| u\right)$ above still solve YBE (7.1). Historically such elliptic solutions have been found by direct methods, and the above trigonometric one came as their degenerations. The fusion procedure for (7.3) is discussed in Ref. 46.

Let l be a positive integer. Consider the specialization of q to the root of unity

$$q = e^{\pi i/L}, \qquad L = l + g. \tag{7.4}$$

Here $g = N$ signifies the dual Coxeter number for $\mathfrak{sl}(N)$. Denoting by $\theta = \varepsilon_1 - \varepsilon_N$ the maximal root, we set

$$\mathcal{S}_l = \{a \in \mathcal{S} \mid (a, \theta) \leq l\}.$$

It can be shown[43,45] that, for the value (7.4) of q, YBE closes among the restricted set of Boltzmann weights $\left\{W\left(\begin{matrix} a & b \\ d & c \end{matrix}\middle| u\right)\right\}_{a,b,c,d \in \mathcal{S}_l}$. We call them *restricted face models.*

From the statistical mechanics point of view, the restricted face models are of particular interest. In the simplest case of $\mathfrak{sl}(2)$, their one point functions are known to be given in terms of the Virasoro characters in the minimal unitary series.[47,48] Similar results have been established for a wide range of models.[43,46,49]

7.5. *Fusion paths and braid representation*

Let us consider the braid representation arising from (7.3). Unlike the 'vertex models' of (5.3), the representation space for the 'face models' does not have the tensor structure.

Let $m \geq 2$. We call a sequence of weights $p = (a_0, a_1, \ldots, a_m)(a_i \in \mathcal{S})$ *fusion path* if for each i V_{a_i} appears in $V_{a_{i-1}} \otimes V^{\pi}$. For $a \in \mathcal{S}$, let $\mathcal{V}_m(a)$ be the vector space spanned by the fusion paths such that $a_0 = a$. Denoting by $W\left(\begin{matrix} a & b \\ d & c \end{matrix}\middle| \infty\right)$ the leading term of (7.3) in the variable $x = q^u$, we set for $i = 1, \ldots, m-1$

$$W_i p = \sum_{p'} W\left(\begin{matrix} a_{i-1} & a_i \\ a'_i & a_{i+1} \end{matrix}\middle| \infty\right) p', \qquad p \in \mathcal{V}_m(a).$$

Here the sum is over $p' = (a'_0, a'_1, \ldots, a'_m) \in \mathcal{V}_m(a)$ such that $a'_j = a_j$ for $j \neq i$. From the foregoing discussions it is clear that the W_i afford a braid representation on $\mathcal{V}_m(a)$. As in the vertex case it factorizes through the Hecke algebra.

When q is the root of unity (7.4) we define the space of restricted fusion paths $\mathcal{V}_{ml}(a)$ using $\mathcal{S}_l$ in place of $\mathcal{S}$. With the choice $a = 0$, the Hecke algebra representations on $\mathcal{V}_{ml}(0)$ coincide with the unitarizable irreducible representations of Hoefsmit[50] and Wenzl.[51] They also arise as the monodromy representations of N-point correlation functions in conformal field theory.[19,20] As for the types $\mathfrak{o}(N)$ or $\mathfrak{sp}(N)$, such face type representations of the Birman-Wenzl-Murakami algebra have been studied by Murakami.[52]

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