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Interactive Aesthetic Curve Segments

Abstract To meet highly aesthetic requirements in industrial design and styling, we propose a new category of aesthetic curve segments. To achieve these aesthetic requirements, we use curves whose *logarithmic curvature histograms*(LCH) are represented by straight lines. We call such curves *aesthetic curves*. We identify the overall shapes of aesthetic curves depending on the slope of LCH α, by imposing specific constraints to the general formula of aesthetic curves. For interactive control, we propose a novel method for drawing an aesthetic curve segment by specifying two endpoints and their tangent vectors. We clarify several characteristics of aesthetic curve segments.

Keywords an aesthetic curve segment · logarithmic curvature histogram · the radius of curvature

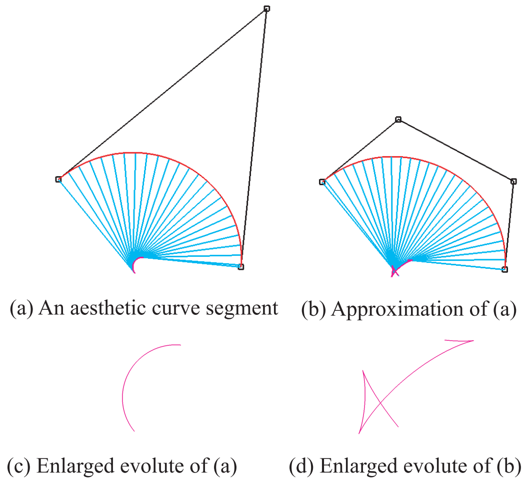

Fig. 1 (a) An aesthetic curve segment ($\alpha = 2.0$) and (b) its approximation by a cubic Bézier curve segment.

1 Introduction

Aesthetic appeal is vital for the market success of industrial products. Since the characteristic lines of a car body, for example, are very important for its aesthetic impact, curves in industrial design and styling need to meet aesthetic requirements. Most curves and surfaces used in conventional CAD systems are based on polynomial or rational parametric forms. However, these curves and surfaces, such as NURBS, are not adequate for highly aesthetic requirements. One of the reasons is the difficulty in controlling the curvature. Though the continuity of curvature can be easily satisfied, it is very hard to control the variation of curvature which dominates the distortion of reflected shapes on curved surfaces. Representation of a circular arc with NURBS is a typical example; achieving constant curvature requires special and unnatural settings of the weights and the knot vector. Fig.1 shows another example, where (a) shows one of the desirable curve segments, and (b) shows an approximation of (a) with a cubic Bézier curve. Although the shape is simple, the Bézier curve segment has undesirable curvature changes.

Harada et al. have shown that many of the aesthetic curves in artificial objects and the natural world are curves whose logarithmic curvature histograms(LCH - to be described in Section 2) can be approximated by straight lines [6,19]. They claimed that two typical aesthetically beautiful curves, the Clothoid curve (also known as a Cornu spiral, Euler Spiral or linarc) and the logarithmic spiral, have the property, where the slopes of their LCH are -1 and 1, respectively. These facts mean that the curves with linear LCH have a potential to meet the

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highly aesthetic requirements in industrial design. In this paper, we call this category of curves with linear LCH *aesthetic curves* and denote the slope of the LCH as α.

Our goal in this paper is to devise an algorithm that can draw an aesthetic curve segment interactively. Miura presented the general formula of aesthetic curves [12, 11], which is defined as the function of arc length. However, drawing aesthetic curves is not trivial. Rather, the exercise poses numerous challenges since the possible ranges of parameters, such as the arc length, are not clearly defined. Moreover, when drawing the curve, we do not know the endpoint of the curve unless we draw the curve, since this process requires numerical integration. This greatly prevents user's controllability. For practical use, the improvement of controllability of aesthetic curves is indispensable. Furthermore, the overall shape or geometric features of an aesthetic curve with an arbitrary value of α(other than 1 or -1) has not been revealed yet.

Our main contributions are the following:

Identifying the overall shapes of aesthetic curves: We identify the overall shapes of aesthetic curves by imposing specific constraints to aesthetic curves so that they become congruent under similarity transformations. By using these constraints, the formulas of aesthetic curves are derived as a function of arc length or tangential angle. The tangential angle is the angle between the tangent line to the curve and x-axis. We find that the circle involute is also included in aesthetic curves as $\alpha = 2$. We clarify the shape change of the spiral and the behavior of the point of inflection and the point of infinite curvature, depending on α.

Interactive drawing of an aesthetic curve segment: We propose a novel method for drawing an aesthetic curve segment by specifying two endpoints and their tangents. When α is specified, our aesthetic curve segment has similar controllability to a quadratic Bézier curve segment. Numerical integrations are necessary for drawing an aesthetic curve segment. However computing points on an aesthetic curve segment on the screen within the maximum error of 1×10^{-10} actually requires less than several milliseconds.

Clarifying the characteristics of aesthetic curve segments: We demonstrate the features of aesthetic curves and their evolutes for various control points and α. Generally, change of α slightly affects the shape of the curve, but drastically alters its evolute. We show that the position of control points and α dictates whether a curve segment can be drawn.

The rest of this paper is organized as follows. Section 2 reviews the relevant literature. Section 3 derives the formulas of aesthetic curves and clarifies the overall shapes and characteristics. Section 4 presents a method for interactively drawing a curve segment by specifying three control points. The final two sections present summary and discussions.

2 Related Work

There are many definitions of fairness of curves. Some of them are the curves with minimum strain energy, the curves that can be drawn with a small number of French curve segments, and the curves whose curvature plots consist of few monotone pieces[3]. See [15] for a collection of definitions. Historically, the curves that approximate *elastica*, which are idealized thin beams, were pursued [4, 9]. Since an idealized thin beam has the minimum elastic energy, the minimization of a fairness functional is widely used for fairing curves and surfaces. Such fairness functionals include the minimum strain energy and the minimum curvature variation [14, 5].

Other approaches are related to more direct control of curvature or its variation. Nutbourne et al introduced intrinsic splines with which a designer specifies the curvature over the arc length and the curve is calculated by integrating the curvature plot [16]. Roulier et al. described monotonous curvature conditions when two end points and their signed curvatures are given [17]. Meek et al. presented a guided clothoid spline passing thorough given points using clothoid segments, circular arcs, and straight line segments [8]. Higashi et al. proposed a smooth surface generation method that controls curvature distribution by determining a surface shape from the evolutes of the four boundary curves [7]. Miura introduced a unit quaternion integral curve for more direct manipulation of its curvature and variation of curvature than Bézier or B-spline curves [10]. Wang et al. described a shape control method based on sufficient monotone curvature variation conditions for planar Bézier and B-spline curves [18]. Moll et al. [13] proposed minimal energy curves that satisfy endpoint constraints for a path planning of flexible wire. It is somewhat similar to our research in that it finds curves that satisfy endpoint constraints. However, the curves and the method for finding a curve segment satisfying endpoint constraints are different from ours.

Harada et al. introduced the *logarithmic curvature histogram*(LCH) to quantitatively find a common property of many aesthetic curves [6, 19]. They assumed that the curves are planar and their curvature varies monotonously and showed that the LCH of many of aesthetic curves in artificial and the natural objects can be approximated by straight lines. Such objects included birds' eggs and butterflies' wings as well as a Japanese sword and the key lines of automobiles. They insist that there is a strong correlation between the slope of the straight line in the LCH of a curve and its impression.

The LCH can be interpreted as follows: Let s and ρ be the arc length and the radius of curvature, respectively. When a curve is subdivided into infinitesimal segments such that $\Delta\rho/\rho$ is constant, the LCH represents the relationship between ρ and Δs in a double logarithmic graph. See Fig.2. If we assume that the LCH of a

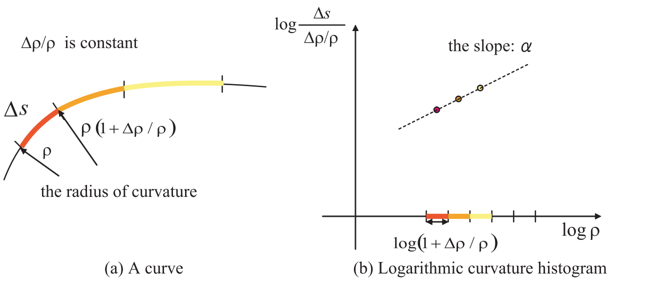


Fig. 2 (a) A curve and (b) its logarithmic curvature histogram.

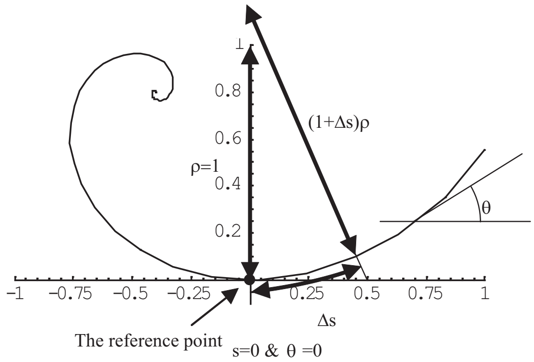


Fig. 3 The aesthetic curve in standard form I.

curve can be represented by a straight line whose slope is α, we obtain,

$$\log \frac{\Delta s}{\Delta\rho/\rho} = \alpha \log\rho + c, \tag{1}$$

where c is a constant [12,11]. Eq.(1) is the fundamental equation of aesthetic curves.

3 Formulas and Overall Shapes of Aesthetic Curves

3.1 Formulas in Standard Form

In this section, we derive the formulas of aesthetic curves in standard form for identifying the overall shapes of aesthetic curves. In order to derive the formula of an aesthetic curve with the slope of LCH α, we consider a reference point P_r on the curve. The reference point can be any point on the curve except at $\rho = 0$ or $\rho = \infty$. The following constraints are placed at the reference point. See Fig. 3.

- Scaling: $\rho = 1$ at P_r.
- Translation: P_r is placed at the origin.
- Rotation: The tangent line to the curve at P_r is parallel to x-axis.

Then the standard form can be obtained by transforming an aesthetic curve such that the above constraints are satisfied.

Using differentials, Eq.(1) can be modified as

$$\frac{ds}{d\rho} = \rho^{\alpha-1}e^{c}. \tag{2}$$

Let $\Lambda = d\rho/ds$ at the reference point P_r. In this case, $\Lambda = e^{-c}$ and $0 < \Lambda < \infty$. Using Λ, Eq.(2) can be modified as

$$\frac{ds}{d\rho} = \frac{\rho^{(\alpha-1)}}{\Lambda}. \tag{3}$$

The arc length s and the tangential angle θ are set to 0 at the reference point P_r. Integrating Eq.(3) with respect to ρ, we obtain,

$$s = \begin{cases} \frac{1}{\Lambda}\log\rho & \text{if } \alpha = 0 \\ \frac{1}{\Lambda\alpha}(\rho^{\alpha} - 1) & \text{otherwise} \end{cases} \tag{4}$$

Solving Eq.(4) with respect to ρ, we obtain

$$\rho = \begin{cases} e^{\Lambda s} & \text{if } \alpha = 0 \\ (\Lambda\alpha s + 1)^{\frac{1}{\alpha}} & \text{otherwise} \end{cases} \tag{5}$$

Eq.(5) is the Cesàro equation of aesthetic curves. A Cesàro equation is an intrinsic equation that specifies a curve in terms of s and ρ. From the fundamental theorem of the local theory of curves in differential geometry [1], curves satisfying Eq.(5) differs by a rigid motion; that is, if curves c_1 and c_2 satisfy Eq.(5), there exists an orthogonal linear map R and a vector T such that $c_2 = R\cdot c_1 + T$.

Using Eq.(3) and the relationship of $ds = \rho d\theta$, we obtain

$$\frac{d\theta}{d\rho} = \frac{ds}{\rho d\rho} = \frac{\rho^{\alpha-2}}{\Lambda} \tag{6}$$

By setting $\theta = 0$ at the reference point and integrating Eq.(6) with respect to ρ, we obtain

$$\theta = \begin{cases} \frac{1}{\Lambda}\log\rho & \text{if } \alpha = 1 \\ \frac{\rho^{\alpha-1}-1}{\Lambda(\alpha-1)} & \text{otherwise} \end{cases} \tag{7}$$

Since ρ changes from 0 to ∞, s and θ may have a upper or a lower bound depending on α. These bounds are shown in Table 1.

Solving Eq.(7) with respect to ρ, we obtain

$$\rho = \begin{cases} (e^{\Lambda\theta}) & \text{if } \alpha = 1 \\ ((\alpha-1)\Lambda\theta + 1)^{\frac{1}{\alpha-1}} & \text{otherwise} \end{cases} \tag{8}$$

The point on the aesthetic curve $P(\theta)$ whose tangential angle is θ is defined on the complex plane as

$$P(\theta) = \begin{cases} \int_0^{\theta} e^{(1+i)\Lambda\psi} d\psi & \text{if } \alpha = 1 \\ \int_0^{\theta} ((\alpha-1)\Lambda\psi + 1)^{\frac{1}{\alpha-1}} e^{i\psi} d\psi & \text{otherwise} \end{cases} \tag{9}$$

where i is the imaginary unit. The point at $\theta = 0$ defined by Eq.(9) goes through the origin and its tangent vector is $[1\ 0]^{\mathrm{T}}$.

Substituting Eq.(5) into $\frac{d\theta}{ds} = \frac{1}{\rho}$, and integrating it with respect to s setting $\theta = 0$ when $s = 0$, we obtain

$$\theta = \begin{cases} 1 - e^{-\Lambda s} & \text{if } \alpha = 0 \\ \frac{\log(\Lambda s+1)}{\Lambda} & \text{if } \alpha = 1 \\ \frac{(\Lambda\alpha s+1)^{\left(1-\frac{1}{\alpha}\right)} - 1}{\Lambda(\alpha-1)} & \text{otherwise} \end{cases} \tag{10}$$

Table 1 The upper and lower bounds of s and θ standard form II.

	s	
	UB	LB
$\alpha < 0$	$-\frac{1}{\Lambda\alpha}$	-
$\alpha = 0$	-	-
$\alpha > 0$	-	$-\frac{1}{\Lambda\alpha}$

	θ	
	UB	LB
$\alpha < 1$	$\frac{1}{\Lambda(1-\alpha)}$	-
$\alpha = 1$	-	-
$\alpha > 1$	-	$\frac{1}{\Lambda(1-\alpha)}$

*UB represents the upper bound and LB represents the lower bound.

Then the point on the aesthetic curve $C(s)$ whose arc length is s is defined on the complex plane as

$$C(s) = \begin{cases} \int_0^s e^{i\left(1-e^{-\Lambda u}\right)} du & \text{if } \alpha = 0 \\ \int_0^s e^{i\left(\frac{\log(\Lambda u+1)}{\Lambda}\right)} du & \text{if } \alpha = 1 \\ \int_0^s e^{i\left(\frac{(\Lambda\alpha u+1)^{(1-1/\alpha)}-1}{\Lambda(\alpha-1)}\right)} du & \text{otherwise} \end{cases} \tag{11}$$

Eq.(9) and Eq.(11) represent the same curve. The tangential angle θ and the arc length s are related by Eq.(10).

3.2 Standard form I and Overall Shapes

To identify the overall shapes of aesthetic curves and see how the curves change depending on the slope of LCH α, we consider aesthetic curves of constant Λ. In Section 3.3, aesthetic curves of arbitrary $\Lambda(\geq 0)$ is considered. On any aesthetic curves of $\alpha \neq 1$, there always exists a point such that $d\rho/ds(=\Lambda) = 1$. Setting such a point as P_r, aesthetic curves can be represented in standard form I. As for the aesthetic curves of $\alpha = 1$, assuming $d\rho/ds = 1$ restricts the possible shapes since $d\rho/ds =$ const. at any point on the curve. However, this assumption is useful for understanding the overall shapes of aesthetic curves.

We investigate the behavior of aesthetic curves for θ and s evaluated as a function of ρ where ρ ranges from 0 to ∞. Fig.4 represents the graphs of s with respect to ρ (Eq.(4)) and those of θ with respect to ρ (Eq.(7)) when $\alpha = -1, 0, 0.5, 1, 2$ and 3. s and θ are either finite or infinite at the points of $\rho = 0$ or $\rho = \infty$, which determines the behavior of the curve depending of α, as shown in Table 2.

Fig.5 represents the overall shapes of aesthetic curves with various αs using the formulas of tangential angle. By definition, $\theta = 0$ and $\rho = 1$ at the origin. From Tables 1 and 2, we can figure out the following characteristics of aesthetic curves in standard form I depending on the value of α.

The aesthetic curves of $\alpha < 0$: As θ approaches $-\infty$, the curve spirally converges to the point at $\rho = 0$. The arc length to that point is infinite. The point of $\theta = \frac{1}{1-\alpha}$, the upper bound, is the point of inflection because $\rho = \infty$. Since the arc length to that point is finite, the point of inflection exists (not at infinity). When $\alpha = -1$, the aesthetic curve is the Clothoid curve.

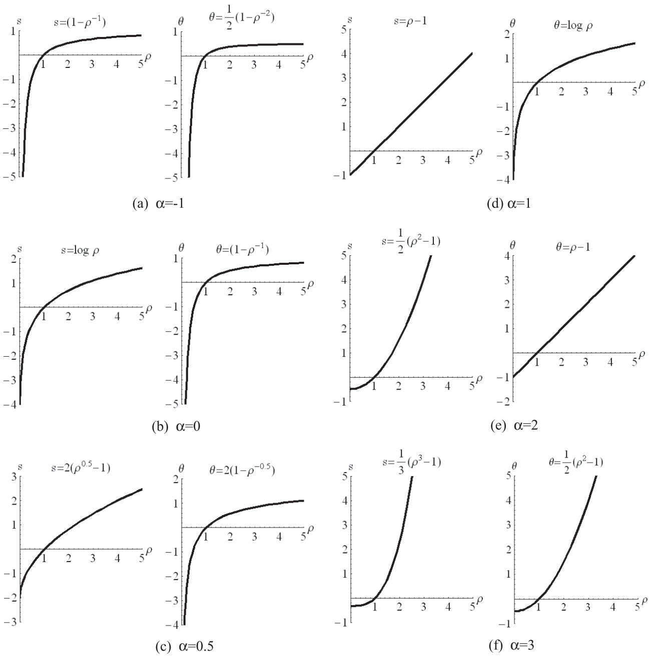


Fig. 4 The graphs of s and θ with respect to ρ are shown when $\alpha = -1, 0, 0.5, 1, 2$, and 3.

Table 2 Arc length s and tangential angle θ at $\rho = 0$ and $\rho = \infty$.

	At $\rho = 0$		At $\rho = \infty$	
	s	θ	s	θ
$\alpha < 0$	infinite*1	infinite*3	finite*5	finite*7
$\alpha = 0$	infinite*1	infinite*3	infinite*6	finite*7
$0 < \alpha < 1$	finite*2	infinite*3	infinite*6	finite*7
$\alpha = 1$	finite*2	infinite*3	infinite*6	infinite*8
$\alpha > 1$	finite*2	finite*4	infinite*6	infinite*8

*1 The arc length to the point at $\rho = 0$ is infinite.
*2 The arc length to the point at $\rho = 0$ is finite.
*3 The curve converges to the point at $\rho = 0$ swirling infinitely.
*4 The tangential direction is determined at $\rho = 0$.
*5 The point of inflection exists (not at infinity).
*6 The point of inflection is at infinity.
*7 Either the point of inflection exists or the direction of divergence is determined.
*8 The curve converges to the point at $\rho = \infty$ swirling infinitely.

The aesthetic curve of $\alpha = 0$: The characteristics of this curve are different from that of $\alpha < 0$ in the point at $\theta = \frac{1}{1-\alpha}(\rho = \infty)$. The point of inflection exists at infinity because the arc length to that point is infinite.

The aesthetic curves of $0 < \alpha < 1$: The characteristics of these curves are different from that of $\alpha = 0$ in the point at $\theta = -\infty(\rho = 0)$. As θ approaches $-\infty$, the curve spirally converges to the point at $\rho = 0$. The arc length to that point is finite.

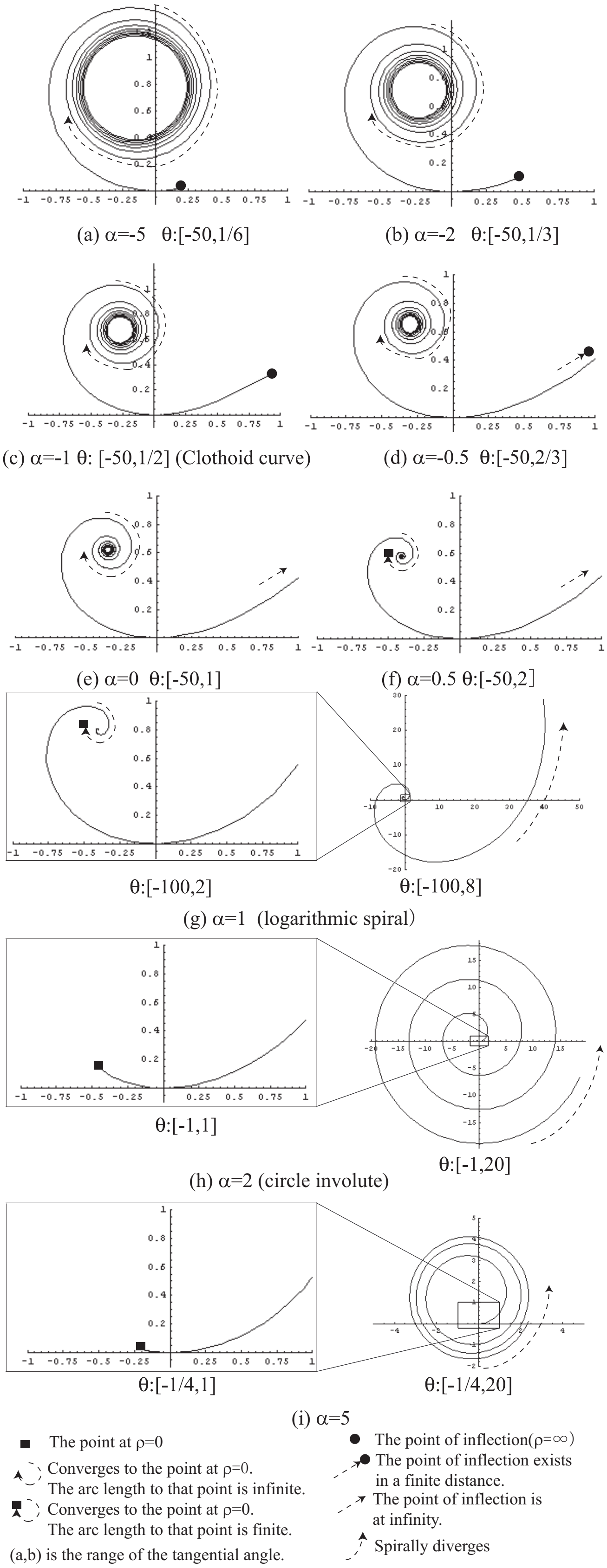


Fig. 5 The overall shape of aesthetic curves in standard form I with various αs

The aesthetic curve of $\alpha = 1$: The characteristics of this curve are different from those of $0 < \alpha < 1$ in the point at $\theta = \infty(\rho = \infty)$. As θ approaches ∞, the curve spirally diverges toward the point at $\rho = \infty$. The arc length to that point is infinite. The aesthetic curve of $\alpha = 1$ is the logarithmic spiral.

The aesthetic curves of $\alpha > 1$: The characteristics of these curves are different from those of $\alpha = 1$ in the point at $\theta = \frac{1}{1-\alpha}(\rho = 0)$. At the point of $\theta = \frac{1}{1-\alpha}$, the lower bound, the tangential direction is determined. The arc length to that point is finite. The aesthetic curve of $\alpha = 2$ is the circle involute.

The aesthetic curves of $\alpha = \pm\infty$: If we take the limit of Eq.(8) as α approaches $\pm\infty$, we get $\rho = 1$. Therefore, when $\alpha = \pm\infty$, the aesthetic curve becomes a circle with radius 1.

3.3 Standard Form II

Standard form I is useful for understanding the overall structure of aesthetic curves. Aesthetic curves in standard form II can represent all the possible shapes of the logarithmic spiral ($\alpha = 1$) and is used for drawing an aesthetic curve segment.

In standard form II, we consider aesthetic curves of arbitrary value of $\Lambda(\geq 0)$. Aesthetic curves in standard form II include those in standard form I as the case of $\Lambda = 1$. Fig.6 shows aesthetic curves in standard form II with various Λ when $\alpha = -1$,1 and 2. As shown in Fig.6(b), all the possible shapes of the logarithmic spiral ($\alpha = 1$) can be represented. When $\alpha(\neq 1)$ is specified, all the aesthetic curves are congruent under similarity transformations without depending on the value $\Lambda(\neq 0)$. From Eq.(5), it can be easily understood that when $\Lambda = 0$, ρ becomes constant. Thus, an aesthetic curve become a circle without depending on α when $\Lambda = 0$.

4 Interactive Drawing of an Aesthetic Curve Segment

4.1 Drawing Algorithm

In this section, we propose a method for interactive control of an aesthetic curve segment using standard form II. The idea of this method is to search for a curve segment in standard form II that *fits* a similar triangle defined by the three control points.

Suppose that α is specified and three control points P_a, P_b, and P_c are given as shown in Fig.7(a) on the left. Without loss in generality, we assume $|P_aP_b| \leq |P_bP_c|$. If this does not hold, the coordinates of P_a and P_c are swapped. As shown in Fig.7(a) on the right, P_0 is placed at the point whose tangential angle is 0 on the aesthetic curve in standard form II, P_2 is placed at the point whose tangential angle is θ'_d. P_1 is the intersection point of the

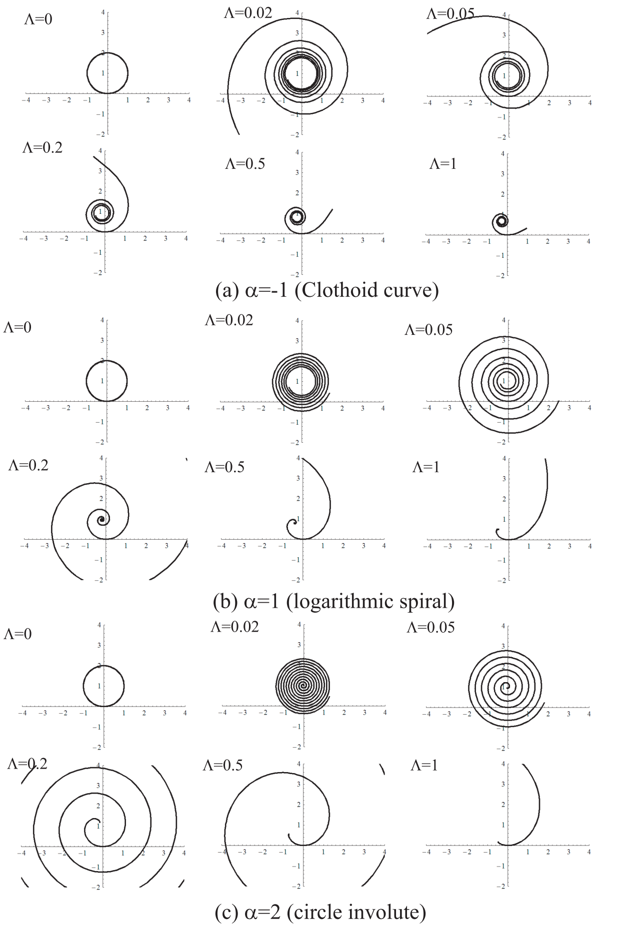


Fig. 6 Aesthetic curves in standard form II with $\alpha = -1, 1, 2$.

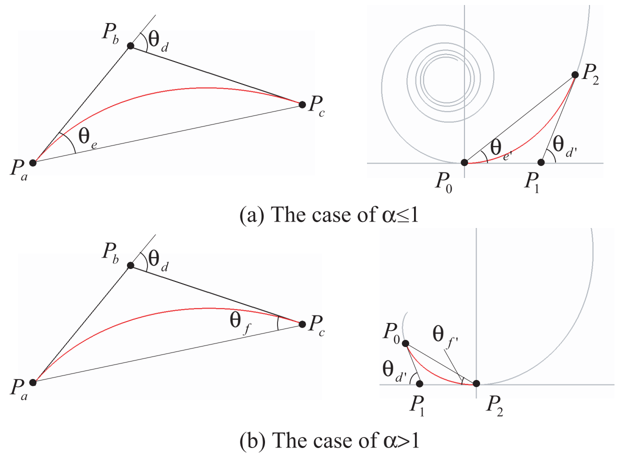


Fig. 7 Two configurations for drawing aesthetic curve segments.

tangential lines at P_0 and P_2. If the triangle $P_0P_1P_2$ is a similar triangle of the triangle $P_aP_bP_c$, then the curve segment defined by $P_aP_bP_c$ can be drawn by transforming the points on the aesthetic curve in Fig.7(a) on the right such that P_0, P_1 and P_2 are transformed to P_a, P_b and P_c. Note that there might be a case that a similar triangle is not found. Thus the position of control points and α dictate whether a curve segment can be drawn.

The similarity of two triangles can be decided by comparing two pairs of angles. Since θ_d is the change of the tangential angle from P_a to P_c, $\theta'_d = \theta_d$. Now we need to find Λ such that $\theta_e = \theta'_e$. This can be done by changing the value of Λ using the bisection method, which will be described shortly. See Fig.6 again to see how aesthetic curves in standard form II changes their shapes depending on Λ. When $\alpha > 1$, the integration range of 0 to θ_d (Fig.7(a)) may cause a problem because Λ may become infinity. To avoid this, we use the integration range of $-\theta_d$ to 0 (Fig.7(b)) when $\alpha > 1$. In this case, the bisection method is used to find Λ such that $\theta_f = \theta'_f$. Λ has a upper bound depending on the integration range when $\alpha \neq 1$. As described in Section 3.3, θ may have a upper or lower bound depending on α. When $\alpha < 1$, the integration range of $[0, \theta_d]$ is used. Since the upper bound of θ is $1/(\Lambda(1-\alpha))$, $\theta_d \leq 1/(\Lambda(1-\alpha))$ must hold. Therefore, $0 \leq \Lambda \leq 1/(\theta_d(1-\alpha))$. When $\alpha > 1$, the integration range of $[-\theta_d, 0]$ is used. Since the lower bound is $1/(\Lambda(1-\alpha))$, $-\theta_d \geq 1/(\Lambda(1-\alpha))$ must hold. Therefore, $0 \leq \Lambda \leq 1/(\theta_d(\alpha-1))$. When $\alpha = 1$, there is no upper bound for Λ, so the bisection method is extended so that $\Lambda(\geq 0)$ can be arbitrarily large. The pseudo code for the bisection method is shown in Appendix A.

For computing the points on aesthetic curves, Eq.(11) (the formula by arc length) is better than Eq.(9)(the formula by tangential angle) especially when the curve segment include the (nearby) point of inflection. At the point of inflection, ρ becomes ∞, which causes Eq.(9) numerically unstable. When the curve segment approaches the shape of a circular arc(when Λ approaches 0), $\frac{0}{0}$ may arise in Eq.(11). In this case, Eq.(9) is better. Therefore, when $\alpha <= 0.5$ and $\Lambda > 1 \times 10^{-2}$, we use Eq.(11). Otherwise, we use Eq.(9). We use an adaptive Gaussian quadrature method for numerical integration.

4.2 Curve Shapes and Drawable Regions

Fig.8(a)-(c) show aesthetic curve segments with the same control points but with different αs. It is known that two curves may look identical on the screen, yet reveal significant shape differences when plotted to full scale on a large flatbed plotter [2]. In such a situation, their evolutes (or curvature plots) reveal substantial differences. Though the curve segments in (a)-(c) look similar, their evolute are different substantially. In (d)-(f), we show three aesthetic curve segments with different αs. As the triangle formed by the control points gets closer to an isosceles triangle, aesthetic curve segments with different αs get closer (to a circular arc).

The position of control points and α dictates whether a curve can be drawn. Fig.9 shows the drawable regions

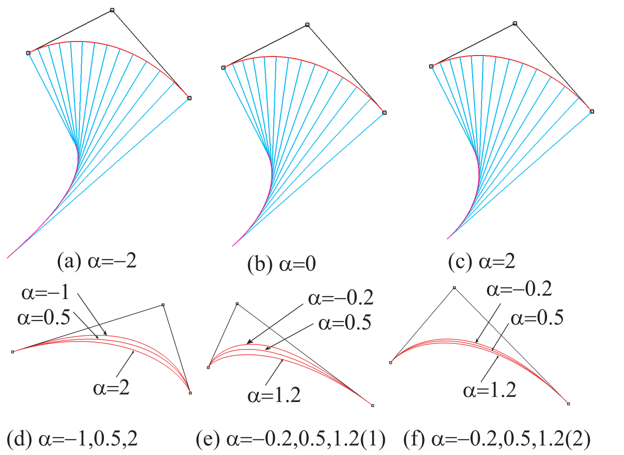


Fig. 8 In (a)-(c), aesthetic curve segments with their evolute and radii of curvature are shown. (d)-(f) show three aesthetic curve segments with different αs.

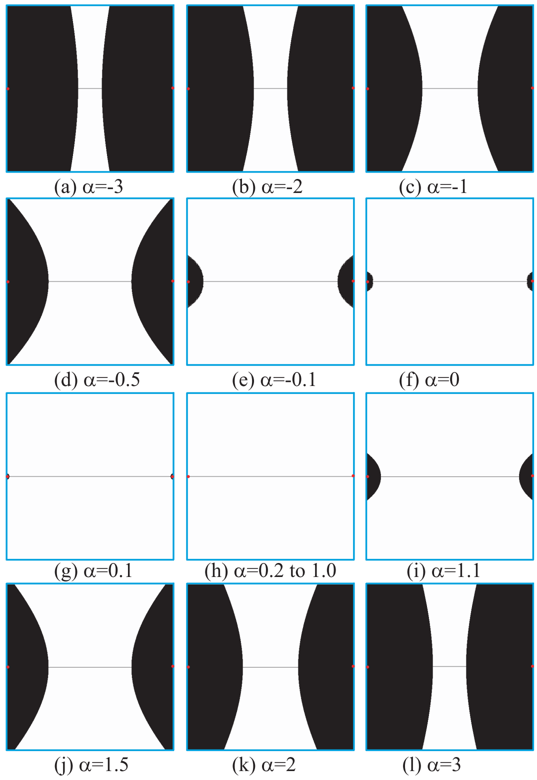


Fig. 9 The drawable regions of aesthetic curve segments with various αs.

of aesthetic curves with various αs. In (a)-(l), each rectangle is placed with their corners at $(\pm 1, \pm 1)$. The first control point is placed at $(-1, 0)$ and the third control point is placed at $(1, 0)$. The second control point is moved within the rectangle. If an aesthetic curve segment is drawable, the pixel of the second control point is drawn with white. If not, the point is drawn with black. Since the straight line is not included in aesthetic curves, the curve segment cannot be drawn when the control points are collinear.

As shown in Fig.9(g) and (h), there is little restriction for the placement of control points when α is between 0.1 and 1. As α becomes smaller than 0 or gets larger than 1, the drawable regions get smaller. This is because the shape of aesthetic curves gets closer to a circle as α gets smaller than 0 or gets larger than 1 as shown in Fig.5. The second control point can be placed outside the rectangle if it is within the drawable region.

The drawable regions of Fig.9 were experimentally constructed so that the maximum error is within 1×10^{-8}. If we allow the drawable region to get slightly smaller, we can decrease the maximum error up to 1×10^{-10}. In this case, the black (not drawable) region gets slightly larger especially when α is around 0. For practical purposes, we can use the maximum error of 1×10^{-10}. When θ_d gets very large(close to π), the desired accuracy may not be achieved or the computational efficiency is decreased since the integration range gets relatively large. However, to draw an aesthetic curve segment, it is somewhat unusual to place the second control point such that θ_d gets larger than $\pi/2$.

4.3 Computational Cost

The computation time of an aesthetic curve segment varies depending on α as well as the integration range and the number of points on the curve computed. In our implementation, the points of an aesthetic curve segment are computed with the constant step of tangential angle of around 0.02 in radian, which is close to 1 deg. More sophisticated computation using both the tangential angle and the arc length is preferable, though. Nevertheless, for drawing a curve segment on the screen, the constant step of tangential angle is satisfactory in most cases. We measured the computation time on a Pentium D 3.2GHz computer. We placed the first and the third control points at the same positions as in Fig.9 and moved the second control point randomly within the drawable region in the blue rectangle using the mouse. The total computation time of an aesthetic curve segment, not including the drawing time, is composed of the time of the bisection method and the time for computing points on the curve. Table 3 shows the total computation time with the maximum error of 1×10^{-10}, the percentage of the bisection method against the total computation time, and the standard deviation of the total computation time. As Table 3 shows, an aesthetic curve segment can be computed within less than several milliseconds. Though numerical integrations are necessary for computing the points on an aesthetic curve segment, our implementation shows that it gives fully interactive control.

Table 3 The Computation time of an aesthetic curve segments(not including drawing).

	TCT (ms)	PB (%)	σ (ms)
$\alpha = -1$	1.01	32	0.30
$\alpha = 0$	1.59	46	0.76
$\alpha = 1$	0.96	37	0.43
$\alpha = 2$	0.75	29	0.24

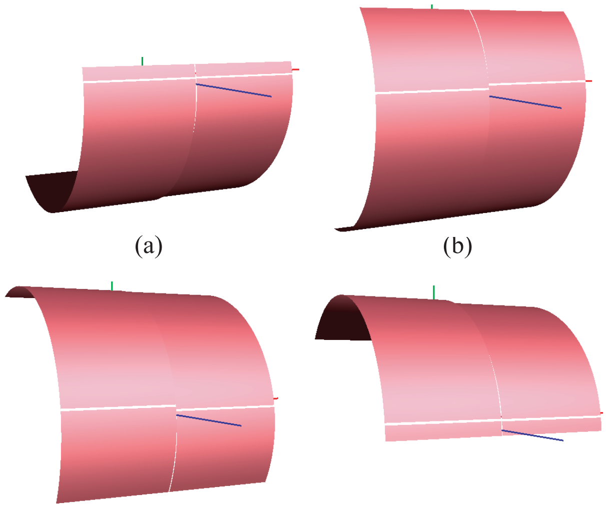
TCT, PB, σ represent the total computation time, the percentage of the bisection method against the TCT, and the standard deviation of the TCT, respectively.

5 Discussion

Aesthetic curves are very fascinating, since they can be considered as a generalization of the Clothoid, the logarithmic spiral, the circle involute, and the circle. This paper has shown the overall shapes of aesthetic curves of arbitrary α. As shown in Fig.5, the overall shapes of aesthetic curves gets closer to the shape of a circle when α gets smaller or larger than 0. Therefore, aesthetic curves are meaningful when α is near 0 for practical use. An aesthetic curve segment of $\alpha < 0$ can be connected to a straight line segment, since the point of inflection exists.

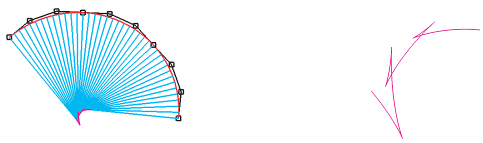
We have presented a method for interactively drawing an aesthetic curve segment by specifying three control points and α. An aesthetic curve segment has the convex hull property, but is not affinely invariant. It is invariant under similarity transformations. When connecting two segments, G^1 continuity can be easily achieved by placing the three consecutive control points such that they are collinear. For achieving G^2 continuity, more tight restriction is imposed for the placement of the control points. There is some room for improving the computation of an aesthetic curve segment. Eq.(9) and (11) can be directly integrable when $\alpha = 1$ or 2, and the bisection method might be replaced by a more sophisticated technique.

To inspect the geometric quality, we compare the swept surface of an aesthetic curve segment ($\alpha = 2$) with its approximated surface using reflection lines. For creating the approximated surface, we approximate the aesthetic curve segment by a cubic Bézier curve segment so that the positions, tangential directions, and curvatures coincide at two endpoints, and then sweep the segment by a line segment. See Fig.1 for the aesthetic curve segment and its approximation by a cubic Bézier curve segment. Fig.10 shows the swept aesthetic surface (left) and its approximated surface (right). When the two surfaces are rotated about x-axis, the reflection line of the swept aesthetic surface moves monotonously, whereas that of its approximated surface oscillates. Fig.11(a) shows the approximation of Fig.1(a) by a cubic B-spline curve that consists of three cubic Bézier segments. Uniform B-spline curve segments are created so that they interpolates four points on the aesthetic curve segments and their tangential directions at these points are colinear, and then



(a) (b) (c) (d)

Fig. 10 In (a)-(d), the reflection lines of the swept surface(left) of an aesthetic curve segment of $\alpha = 2$ and its approximated surface(right) by a cubic Bézier curve are shown. The reflection line of the approximated surface oscillates when the two surfaces are rotated.



(a) Spline curve segments (b) Enlarged Evolute of (a)

Fig. 11 Approximation of an aesthetic curve segment (Fig. 1(a)) by three Bézier curve segments. Its enlarged evolute is shown in (b).

converted to Bézier curve segments. The four points on the aesthetic curve segment are placed so that their arc lengths are separated equidistantly. As shown in Fig.11(b), the approximated curve segments exhibit undesirable curvature change and the reflection line of the swept surface oscillates similarly as in the approximated surface of Fig.10, though the amplitude is smaller. From these results, an aesthetic curve segment has a potential to meet highly aesthetic requirements in industrial design and styling.

The application of aesthetic curve segments includes designing 2-dimensional objects as well as 3-dimensional objects. In case of 3-dimensional design, aesthetic curves can be used, for example, as a guide to design the silhouettes of an object. Finding the intrinsic property of aesthetic space curves and surfaces is attractive, though. In fairing of surfaces, our aesthetic curve segment can be used as the constraints for key lines. Another application of our aesthetic curve segment is CAD systems. Since our aesthetic curves can be converted into spline curves, such as Bézier or B-spline within a specified precision, it is possible to use aesthetic curve segments in many CAD systems.

6 Conclusions

This paper has presented a new category of aesthetic curve segment. An aesthetic curve segment can be drawn by specifying two endpoints and their tangents. Thus an aesthetic curve segment has similar controllability to a Bézier curve segment of degree 2. The radius of curvature of aesthetic curves versus the arc length varies monotonously and smoothly. Therefore, a very high quality curve segment can be easily controlled by three control points and α. Decreasing α makes the curve segment become closer to its control polygon. A circular arc can be represented by placing control points so that they form an isosceles triangle. Although numerical integrations are necessary for computing the curve segment, our implementation showed that the points on aesthetic curves can be computed in less than several milliseconds on the screen within the maximum error of 1×10^{-10}. Aesthetic curves are defined such that the logarithmic curvature histograms of curves are represented by straight lines with slope α. We identified the overall shapes of aesthetic curves by considering their standard form.

Future research includes more efficient computation of curve segments, higher accuracy, the connection of several segments, and the extension to surfaces. In order to utilize various kinds of existing shape processing techniques, the approximation of the aesthetic curves with conventional curves, such as NURBS, is also an important subject. We envision that our aesthetic curve segment can be used for designing products as well as in many applications for computer graphics.

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Appendix A: The pseudo code for the bisection method

```
double bisection(double alpha, int maxIteration) {
    double lmin = 0.0,lmax = 1.0, f;
    int i = 0, enlarge = 0;
    if ( alpha == 1.0 ) enlarge = 1;
    else if ( alpha < 1.0 )
        lmax = 1 / (thetaD * (1 - alpha));
    else if ( alpha > 1.0 )
        lmax = 1 / (-thetaD * (1 - alpha));
    Lambda = (lmin + lmax) * 0.5;
    do {
        if ( alpha <= 1)
            f = thetaE -ComputeThetaEdash(Lambda);
        else f = thetaF - ComputeThetaFdash(Lambda);
        if ( fabs(f) < EPS )  return Lambda; // found
        double pLambda = Lambda;
        if ( (f < 0.0  && alpha <= 1.0) ||
             (f > 0.0 && alpha > 1.0) ) ) {
            if ( enlarge ) lmax *= 10.0;
            Lambda += (lmax - Lambda)*0.5;
            lmin = pLambda;
        }
        else  {
             enlarge = 0;
             Lambda -= (Lambda-lmin)*0.5;
             lmax = pLambda;
        }
        i++;
    } while ( i < maxIteration );
    return -1;// not found
}
```