SPECTRA OF TENSOR PRODUCTS OF OPERATORS

ARLEN BROWN AND CARL PEARCY

1. **Introduction.** Let $\mathcal{H}$ be a Hilbert space, and let A and B denote operators on $\mathcal{H}$. We consider the *tensor product* $A \otimes B$ acting on the product space $\mathcal{H} \otimes \mathcal{H}$. (For a good account of tensor products of Hilbert spaces and operators, the reader may consult [**1**, pp. 22–26].) When $\mathcal{H}$ is finite dimensional, so that A and B can be regarded as matrices, it is a well known and pretty fact that the eigenvalues of $A \otimes B$ are precisely the products of the form $\alpha\beta$ where α, β are eigenvalues of A, B respectively. The purpose of the present note is to prove the following infinite dimensional generalization of this result.

THEOREM. *If A and B are bounded linear operators on an arbitrary Hilbert space $\mathcal{H}$, and if $\sigma(A)$, $\sigma(B)$ denote their respective spectra, then the spectrum of $A \otimes B$ is the set of products $\sigma(A)\sigma(B)$.*

2. In the sequel, all spaces under consideration will be complex Hilbert spaces, and all operators will be assumed to be bounded and linear. The set of all operators on a Hilbert space $\mathcal{K}$ is denoted by $\mathcal{L}(\mathcal{K})$, and the spectrum of any operator T is denoted, as above, by $\sigma(T)$.

FIRST PROOF OF THE THEOREM. We employ the fact that tensor products of operators on $\mathcal{H}$ can be identified with multiplications on the Hilbert-Schmidt class, regarded as a Hilbert space under the Schmidt norm. The procedure for making this identification is a standard one and the following abbreviated account is lifted from Dixmier [**1**, pp. 93–96]. First, construct the Hilbert space $\mathcal{H}'$ *opposed* to $\mathcal{H}$, i.e., the space whose elements and law of addition are identical with those of $\mathcal{H}$, but with the reversed inner product $(x|y) = (y, x)$ and with multiplication of a vector by a scalar redefined as $\lambda \circ x = \bar{\lambda}x$. Note that there is a one-one correspondence $A \leftrightarrow A'$ between $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H}')$ defined by $A'x = Ax$ for all x. Concerning this correspondence, we record the following fact.

LEMMA 1. $\sigma(A') = [\sigma(A)]^*$, *the set of complex conjugates.*

The proof of this lemma consists of an easy computation which we omit.

Presented to the Society, August 25, 1964 under the title *Tensor products of completely locally m-convex algebras* and November 25, 1964 under the title *Tensor products of complete commutative locally m-convex Q-algebras*; received by the editors June 29, 1964.

Next, form the tensor product $\mathcal{H}' \otimes \mathcal{H}$ and with each decomposable vector $z \otimes x \in \mathcal{H}' \otimes \mathcal{H}$ associate the operator $T_{z,x} \in \mathcal{L}(\mathcal{H})$ defined by

$$T_{z,x}(y) = (y, z)x, \qquad y \in \mathcal{H}$$

(Observe that in this formula z is regarded as an element of $\mathcal{H}$.) The following lemma is essentially [**1**, Proposition 6, page 96].

LEMMA 2. *The correspondence $z \otimes x \to T_{z,x}$ possesses a unique extension to a Hilbert space isomorphism ϕ of $\mathcal{H}' \otimes \mathcal{H}$ onto the Hilbert space $\mathfrak{N}$ of Hilbert-Schmidt operators on $\mathcal{H}$. Moreover, if $A, B \in \mathcal{L}(\mathcal{H})$, then $A' \otimes B$ is carried by ϕ onto the operator $M(B, A^*) \in \mathcal{L}(\mathfrak{N})$ defined by $M(B, A^*)X = BXA^*$ for all $X \in \mathfrak{N}$.*

Now let ψ be any isomorphism of $\mathcal{H}$ onto $\mathcal{H}'$. (That such an isomorphism exists is assured by the fact that $\mathcal{H}$ and $\mathcal{H}'$ have the same dimension.) For each $A \in \mathcal{L}(\mathcal{H})$, denote by A_0 the element of $\mathcal{L}(\mathcal{H})$ satisfying the equation $A_0' = \psi A \psi^{-1}$. The tensor product of ψ with the identity mapping on $\mathcal{H}$ is an isomorphism of $\mathcal{H} \otimes \mathcal{H}$ onto $\mathcal{H}' \otimes \mathcal{H}$ that carries each operator $A \otimes B \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ onto the operator $A_0' \otimes B \in \mathcal{L}(\mathcal{H}' \otimes \mathcal{H})$. Thus by Lemma 2 we have

$$\sigma(A \otimes B) = \sigma(A_0' \otimes B) = \sigma(M(B, A_0^*)).$$

On the other hand, by Lemma 1

$$\sigma(A) = \sigma(A_0') = [\sigma(A_0)]^* = \sigma(A_0^*),$$

and consequently $\sigma(A)\sigma(B) = \sigma(B)\sigma(A_0^*)$. Thus the proof of the theorem reduces to showing that for arbitrary operators C and D on $\mathcal{H}$,

$$\text{(I)} \qquad \sigma(M(C, D)) = \sigma(C)\sigma(D).$$

Operators of the form $M(C, D): X \to CXD$ have been studied by Lumer and Rosenblum [**5**], and in particular they show [**5**, Theorem 10] that the equality (I) is valid when $M(C, D)$ is regarded as an operator acting on the Banach space $\mathcal{L}(\mathcal{H})$. While this is not quite our situation (our $M(C, D)$ acts on the Hilbert space $\mathfrak{N}$), it is not hard to see that the arguments of Lumer and Rosenblum remain valid in this case, except that [**5**, Lemma 2] must be given a new proof to take into account the fact that the identity operator is not in $\mathfrak{N}$. Thus we complete the first proof of the theorem by proving the following lemma.

LEMMA 3. *For $A \in \mathcal{L}(\mathcal{H})$, let L_A, $R_A \in \mathcal{L}(\mathfrak{N})$ be defined by setting for $X \in \mathfrak{N}$, $L_A(X) = AX$ and $R_A(X) = XA$, respectively. Then $\sigma(L_A) = \sigma(R_A) = \sigma(A)$.*

PROOF. Let $J(X)=X^*$ for $X\in\mathfrak{N}$. Then J is an involuntary isometry on $\mathfrak{N}$ and $R_A=JL_{A^*}J$. Since $\sigma(JUJ)=[\sigma(U)]^*$ for every $U\in\mathfrak{L}(\mathfrak{N})$, it suffices to consider L_σ. Next, since $L_{A-\lambda 1}=L_A-\lambda 1$, it suffices to prove that A is invertible on $\mathcal{H}$ when and only when L_A is invertible on $\mathfrak{N}$. Moreover, one half of this is trivial: if A is invertible on $\mathcal{H}$, then $L_{A^{-1}}L_A=L_AL_{A^{-1}}=1_{\mathfrak{N}}$. The proof will be completed by showing that if L_A is invertible, then A is invertible too. Accordingly, let L_A be invertible on $\mathfrak{N}$. For any $x\in\mathcal{H}$, the operator $T_{x,x}$ belongs to $\mathfrak{N}$, so there exists $N\in\mathfrak{N}$ such that $AN=T_{x,x}$. But then $A(Nx)=\|x\|^2x$, and it follows that A maps $\mathcal{H}$ onto itself. Hence it remains only to verify that A has trivial null space. But if $Ax=0$, then $L_A(T_{x,x})=T_{x,Ax}=0$ so that $T_{x,x}=0$ and consequently $x=0$.

SECOND PROOF OF THE THEOREM. The operator $A\otimes 1_{\mathcal{H}}$ on $\mathcal{H}\otimes\mathcal{H}$ can be regarded as an infinite diagonal matrix with the operator A in each position on the diagonal [**1**, p. 25]. It follows easily that $\sigma(A\otimes 1_{\mathcal{H}})=\sigma(A)$, and similarly that $\sigma(1_{\mathcal{H}}\otimes B)=\sigma(B)$. Since $A\otimes B=(A\otimes 1)(1\otimes B)$, and $(A\otimes 1)$ commutes $(1\otimes B)$, a standard application of the Gelfand representation theorem shows that

$$\sigma(A \otimes B) \subset \sigma(A \otimes 1)\sigma(1 \otimes B) = \sigma(A)\sigma(B).$$

The proof of the reverse inclusion $\sigma(A)\sigma(B)\subset\sigma(A\otimes B)$ we split into cases by making a slightly unusual partition of the spectra of A and B. Write $\pi(A)$ for the approximate point spectrum of A [**3**, page 51] and write $\phi(A)$ for the balance of the spectrum $\sigma(A)\backslash\pi(A)$. Concerning the set $\phi(A)$ we shall need the facts that (i) $\phi(A)$ is an open set and (ii) if $\lambda\in\phi(A)$, then $\bar{\lambda}$ is an eigenvalue of A^*. That $\phi(A)$ is open can be seen in various ways. On the one hand, it follows from the topological fact that $\pi(A)$ is compact and contains the boundary of $\sigma(A)$ (see, e.g. [**4**, Theorem 4.11.2]). On the other hand, it can be verified directly by a slight modification of any of the standard proofs that the resolvent set is open (see, e.g. [**2**, §3]). Fact (ii) is an easy consequence of the closed graph theorem.

Case I. $\alpha\in\pi(A)$, $\beta\in\pi(B)$. There exist sequences $\{x_k\}$ and $\{y_k\}$ of unit vectors in $\mathcal{H}$ along which $A-\alpha 1$ and $B-\beta 1$ tend to zero. But then

$$\begin{aligned}[A \otimes B - \alpha\beta(1 \otimes 1)](x_k \otimes y_k) \\ &= [(A - \alpha 1) \otimes B + \alpha 1 \otimes (B - \beta 1)](x_k \otimes y_k) \\ &= (A - \alpha 1)x_k \otimes By_k + \alpha x_k \otimes (B - \beta 1)y_k\end{aligned}$$

which tends to zero as $k\to\infty$, so that $\alpha\beta\in\pi(A\otimes B)$.

Case II. $\alpha\in\phi(A)$, $\beta\in\phi(B)$. We have $\bar{\alpha}\in\pi(A^*)$ and $\bar{\beta}\in\pi(B^*)$, whence by Case I, $\bar{\alpha}\bar{\beta}\in\sigma(A^*\otimes B^*)=\sigma[(A\otimes B)^*]$. Thus $\alpha\beta\in\sigma(A\otimes B)$.

Case III. $\alpha\in\pi(A)$, $\beta\in\phi(B)$ or $\alpha\in\phi(A)$, $\beta\in\pi(B)$. We treat the case $\alpha\in\pi(A)$, $\beta\in\phi(B)$; the other case is handled similarly. Suppose first that $\alpha=0$. Let β_0 be any element of $\pi(B)$, and note that from Case I, $\alpha\beta=\alpha\beta_0=0\in\sigma(A\otimes B)$. Thus we may assume that $\alpha\neq 0$, and similarly by taking adjoints that $\beta\neq 0$. We know that $\bar{\alpha}\in\sigma(A^*)$; if $\bar{\alpha}\in\pi(A^*)$, then since $\bar{\beta}\in\pi(B^*)$, the desired result follows by appealing to Case I and taking adjoints. Thus we may even assume that $\bar{\alpha}\in\phi(A^*)$. Now introduce a real parameter t, $1\leqq t<\infty$, and consider pairs of the form $[t\bar{\alpha},\ \beta/t]$. For t sufficiently close to 1, we have $t\bar{\alpha}\in\phi(A^*)$ and $\beta/t\in\phi(B)$ since $\phi(A^*)$ and $\phi(B)$ are open sets. Hence there exists $t_0>1$ such that for $1\leqq t<t_0$, $t\bar{\alpha}\in\phi(A^*)$ and $\beta/t\in\phi(B)$ and such that $t_0\bar{\alpha}\in\pi(A^*)$ or $\beta/t_0\in\pi(B)$. Suppose $t_0\bar{\alpha}\in\pi(A^*)$ but $\beta/t_0\in\phi(B)$. Then $\bar{\beta}/t_0\in\pi(B^*)$, so that from Case I we have $(t_0\bar{\alpha})(\bar{\beta}/t_0)=\bar{\alpha}\bar{\beta}\in\sigma([A\otimes B]^*)$, and hence $\alpha\beta\in\sigma(A\otimes B)$. The case $t_0\bar{\alpha}\in\phi(A^*)$ and $\beta/t_0\in\pi(B)$ is handled similarly with the help of Case I, and the only case remaining to be dealt with is $t_0\bar{\alpha}\in\pi(A^*)$, $\beta/t_0\in\pi(B)$. In this case, let t_n be a sequence of real numbers satisfying $1<t_n<t_0$ and $t_n\to t_0$. Then for each n, $t_n\bar{\alpha}\in\phi(A^*)$, so that $t_n\alpha\in\pi(A)$, and from Case I, $(t_n\alpha)(\beta/t_0)\in\sigma(A\otimes B)$. Since $(t_n/t_0)\alpha\beta\to\alpha\beta$ and $\sigma(A\otimes B)$ is closed, $\alpha\beta\in\sigma(A\otimes B)$, and the proof is complete.

3. **Concluding remarks.**

(1) Two proofs of the above theorem are given because each has its own merits. The first argument shows the intimate connection between the operator $A\otimes B$ and the operator $X\to AXB$ on the Hilbert-Schmidt class. The second proof sets forth a technique for attacking spectral problems that seems fairly useful. In particular, if $\mathcal{K}$ is taken to be any of the Schatten norm ideals [**6**], the spectra of the operators $X\to AXB$ and $X\to AX\pm XB$ on $\mathcal{K}$ can be shown to be $\sigma(A)\sigma(B)$ and $\sigma(A)\pm\sigma(B)$ by arguments very similar to the second argument given above. (For $\mathcal{K}=\mathcal{L}(\mathcal{H})$ much more general results are obtained in [**5**].)

(2) It is not hard to see that the line of argument used in Lemma 3 also shows that the approximate point spectra of A and L_A (in the notation of that lemma) coincide. Indeed, even the continuous and residual spectra of A and L_A coincide.

(3) The authors have recently proved that every operator of the form $A\otimes 1_{\mathcal{H}}$ where A is nonscalar and $\mathcal{H}$ is an infinite dimensional Hilbert space is a commutator. This fact, together with the result of this note, shows that there exist commutators with arbitrarily prescribed spectra.

Bibliography

1. J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien*, Gauthier-Villars, Paris, 1957.

2. I. C. Gohberg and M. G. Kreĭn, *Fundamental theorems on defect numbers, etc.*, Uspehi Mat. Nauk **12** (1957), 43–118.

3. P. R. Halmos, *An introduction to Hilbert space*, Chelsea, New York, 1951.

4. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ. Vol. 31, Amer. Math. Soc., Providence, R. I., 1957.

5. G. Lumer and M. Rosenblum, *Linear operator equations*, Proc. Amer. Math. Soc. **10** (1959), 32–41.

6. R. Schatten, *Norm ideals of completely continuous operators*, Ergebnisse der Mathematik, Berlin, 1960.

University of Michigan

KIRZBRAUN'S THEOREM AND KOLMOGOROV'S PRINCIPLE[1]

EDWARD SILVERMAN

Let B be a Banach space. A *distance function* p on B is a non-negative valued function which is continuous, positively homogeneous of degree one and subadditive. If A is a set and if x and y map A into B then we write xpy if $p(x(a)-x(b)) \leqq p(y(a)-y(b))$ for all $a, b \in A$. If A is a k-cell, if B is Euclidean space, if p is the norm and if L is Lebesgue area, then Kolmogorov's Principle, K.P., asserts that $Lx \leqq Ly$ if xpy [**H.M.**]. Lebesgue area is a parametric integral of the type considered by McShane [**M**], for smooth enough maps. In this paper we consider other such integrals, not necessarily symmetric, for which a type of K.P. holds. We conclude with a minor application to a Plateau problem.

The proof of K.P. follows from

Kirzbraun's Theorem. *If $A \subset E^n$ and $t: A \rightarrow E^n$ is Lipschitzian, then there exists an extension T of t, $T: E^n \rightarrow E^n$, and T is Lipschitzian with the same constant as t* [**S**].

The proof of the version of K.P. in which we are interested depends upon an embedding of E^n in m, the space of bounded sequences [**B**],

Received by the editors January 11, 1965.

[1] This research was supported in part by National Science Foundation Grant No. GP634.