GAFA, Geom. funct. anal.
Vol. 7 (1997) 535 – 560
1016-443X/97/030535-26 $ 1.50+0.20/0

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GAFA Geometric And Functional Analysis

KIRSZBRAUN'S THEOREM AND METRIC SPACES OF BOUNDED CURVATURE

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Abstract

We generalize Kirszbraun's extension theorem for Lipschitz maps between (subsets of) euclidean spaces to metric spaces with upper or lower curvature bounds in the sense of A.D. Alexandrov. As a by-product we develop new tools in the theory of tangent cones of these spaces and obtain new characterization results which may be of independent interest.

Introduction

The classical Kirszbraun theorem [K] states that every Lipschitz map $f\colon S \to \mathbb{R}^n$ defined on an arbitrary subset of $\mathbb{R}^m$ possesses a Lipschitz extension $\bar{f}\colon \mathbb{R}^m \to \mathbb{R}^n$ (i.e. $\bar{f}|S = f$) with the same Lipschitz constant. Valentine [V1] remarked that the same holds true for general Hilbert spaces. Other related results, most of them keeping the linear structure of the underlying spaces, have been obtained by a number of authors, see [A], [As], [B], [DeFl], [Gr], [IR], [JLi], [Mi], [S], and the references there. Corresponding theorems for 1-Lipschitz maps between unit spheres of equal dimensions or between spaces of constant curvature -1 have been established in [V1] and [V2], [KuSt] respectively. Fore more comments on the theorem and its history see [DGrKl, pp. 153–154]. A short and readable proof of the classical statement is given in [F, 2.10.43].

In this paper we extend Kirszbraun's theorem to geodesic metric spaces with upper or lower curvature bounds in the sense of A.D. Alexandrov [Al1,2]. These concepts include riemannian manifolds as well as many singular spaces and can roughly be described as follows (for precise definitions see section 1). A metric space X is said to be geodesic if every two points in X can be connected by a shortest curve. For $\kappa \in \mathbb{R}$ let M^2_κ denote the model surface of constant curvature κ, and let $D_\kappa = \operatorname{diam} M^2_\kappa \in (0,\infty]$ be its diameter. Then for every geodesic triangle Δ of perimeter $< 2D_\kappa$ in X there exists a comparison triangle Δ^κ in M^2_κ which is unique up to isometry, namely, a triangle with the same edge lengths. Now Δ is said to be κ-thick or κ-thin if all distances between points on the sides of Δ are at least or at most as large, respectively, as the corresponding distances for Δ^κ. Then one calls X a space of curvature $\geq \kappa$ or $\leq \kappa$ if locally, all triangles are κ-thick or κ-thin respectively. The interest in these spaces has greatly been stimulated in the last fifteen years, mainly due to new developments in geometry (Gromov-Hausdorff convergence) and geometric group theory (hyperbolic groups).

Our main result can be stated as follows.

Theorem A. (generalized Kirszbraun theorem) *Let $\kappa \in \mathbb{R}$, and let X, Y be two geodesic metric spaces such that all triangles of perimeter $< 2D_\kappa$ in X or Y are κ-thick or κ-thin respectively. Assume that Y is complete. Let S be an arbitrary subset of X and $f\colon S \to Y$ a 1-Lipschitz map with $\operatorname{diam} f(S) \leq D_\kappa/2$. Then there exists a 1-Lipschitz extension $\bar{f}\colon X \to Y$ of f.*

It also follows that $\bar{f}$ can be chosen such that $\bar{f}(X)$ belongs to the closure of the convex hull of $f(S)$ which in the given situation has diameter no larger than $f(S)$. Note that in case $\kappa = 0$, by scaling the metric on either X or Y, one obtains the statement for arbitrary Lipschitz constants. In particular, Theorem A recovers the classical result as well as its generalization to Hilbert spaces. It is obvious that the theorem fails for noncomplete target spaces. Moreover, the following example for $\kappa = 1$ shows that the bound on the diameter of $f(S)$ is optimal. For $n \geq 2$ let X be the unit sphere in $\mathbb{R}^{n+1}$ (with the canonical inner metric), $Y = X \cap \mathbb{R}^n$, S the set of vertices of a regular n-simplex in $\mathbb{R}^n$ inscribed in Y, and f the identity on S. Let $x \in X$ be one of the poles at distance $\pi/2$ of S. Since every open hemisphere of Y contains an image point of f, there is no 1-Lipschitz extension $\bar{f}\colon S \cup \{x\} \to Y$ of f. In this example, $\operatorname{diam} f(S) = \arccos(-1/n) \to \pi/2 = D_1/2$ for $n \to \infty$.

Further, the same methods of proof yield a result which may be viewed as the limit case $\kappa = -\infty$ of Theorem A. Note that for $\kappa \to -\infty$ the curvature assumption on X in Theorem A gets weaker and weaker. We show that "in the limit" it can be dropped completely and obtain the following theorem which is well-known for $Y = \mathbb{R}$ (see [M] and [G2, 3.E] for a related discussion).

Theorem B. (Lipschitz maps into trees) *Let X be an arbitrary metric space and Y a complete geodesic metric space such that every triangle in Y is κ-thin for all $\kappa \leq 0$. Let S be an arbitrary subset of X and $f\colon S \to Y$ a Lipschitz map with constant $\operatorname{Lip} f$. Then there exists a Lipschitz extension*

$\bar{f}\colon X \to Y$ *of* f *with* $\operatorname{Lip}\bar{f} = \operatorname{Lip} f$.

The main step in the proof of these results is to show that every 1-Lipschitz map $f\colon E \to Y$ defined on a finite subset $E = \{x_1, \ldots, x_n\}$ of X can be extended to an additional point $x \in X$, cf. section 5. To this end one considers the set $A(c) := \{y \in Y : d(y, f(x_i)) \leq cd(x, x_i)$ for $i = 1, \ldots, n\}$ for some constant factor c. If c is taken to be the infimum of all $b \geq 0$ with $A(b) \neq \emptyset$, it turns out that $A(c)$ consists of a single point y which is thus a canonical candidate for $\bar{f}(x)$, cf. section 4. It then remains to show that $c \leq 1$, which is not difficult for Theorem B. In the classical (euclidean) case, c is estimated by means of standard inequalities for the scalar product. In a riemannian setting this would correspond to lifting S and $f(S)$ to the tangent spaces at x and y, respectively, and carrying out the computations there. In the general situation of Theorem A, there merely exist tangent cones at x and y which are not necessarily euclidean spaces but still Alexandrov spaces of curvature ≥ 0 and ≤ 0 respectively, cf. section 3. (This is not quite true in case X has infinite Hausdorff dimension and causes some complications. The arguments needed to circumvent this difficulty are presented in an appendix.) We observe that these cones still carry "half of" the structure of a euclidean vector space. Namely, in addition to the multiplication by nonnegative numbers which comes naturally from the cone structure, we define a "vector addition" as well as a "scalar product" on (geodesic) metric cones, where the latter turns out to be sub- or superadditive in each argument if and only if the underlying cone has curvature ≥ 0 or ≤ 0 respectively, cf. section 2. Exploiting these inequalities we prove that $c \leq 1$ and thus establish Theorem A.

We further remark that in the special case where S consists of three points and one merely wants to extend f to one additional point, the curvature hypothesis on X in Theorem A can be weakend. To define curvature bounds via thick or thin triangles one needs the local existence of shortest curves in the underlying metric space. In the case of lower curvature bounds this can be avoided by using comparison angles instead and leads to a more general concept of curvature $\geq \kappa \in \mathbb{R}$. Let X be an arbitrary metric space. For three points $x, x_1, x_2 \in X$ with "perimeter" $d(x, x_1) + d(x, x_2) + d(x_1, x_2) < 2D_\kappa$ and $x_1, x_2 \neq x$ there exists an isometric triple $y, y_1, y_2 \in M^2_\kappa$ and one defines the comparison angle $\gamma^\kappa_x(x_1, x_2)$ to be the angle at the vertex y of the triangle in M^2_κ with vertices y, y_1, y_2. Note that this depends only on the distances between x, x_1, x_2. Then we say that a quadruple $(x; x_1, x_2, x_3)$ in X satisfies the γ^κ condition if the three mutual comparison angles with base point x add up to at most 2π as long as they are defined. By claiming this condition locally for all quadruples one arrives at a purely metric definition of curvature $\geq \kappa$ which in the case of local existence of shortest curves coincides with the concept described above, cf. section 1. Spaces of curvature bounded below in this sense have been investigated in [BuGP].

We prove the following result.

Theorem C. (extensions to quadruples) *Let $\kappa \in \mathbb{R}$, X a metric space with all quadruples in X satisfying the γ^κ condition, and Y a geodesic metric space such that all triangles in Y of perimeter $< 2D_\kappa$ are κ-thin. Whenever $x_1, \dots, x_4 \in X$ and $f\colon \{x_1, x_2, x_3\} \to Y$ is a 1-Lipschitz map whose image has diameter $\leq D_\kappa/2$, then there exists a 1-Lipschitz extension $\bar{f}\colon \{x_1, \dots, x_4\} \to Y$ of f.*

Note that here Y is not assumed to be complete. Theorem C fails for maps defined on more than three points. To see this consider the discrete metric space $X = \{x, x_1, \dots, x_4\}$ with $d(x, x_i) = 1/\sqrt{3}$ for $1 \leq i \leq 4$ and $d(x_i, x_j) = 1$ for $1 \leq i < j \leq 4$. The various euclidean comparison angles belong to the set $\{0, \pi/6, \pi/3, 2\pi/3\}$ and therefore all quadruples in X satisfy the γ^0 condition. Now let $f\colon \{x_1, \dots, x_4\} \to \mathbb{R}^3$ be an isometric map. Since the image of f lies at distance $> 1/\sqrt{3}$ from its barycenter, it follows that there is no 1-Lipschitz extension of f to x. The proof of Theorem C is given in section 6. Finally, we show that spaces of curvature $\geq \kappa$ or $\leq \kappa$ are actually characterized by the extensibility of 1-Lipschitz maps (defined on three points) into or from the model space M^2_κ respectively.

The present work has been motivated by the question whether a Lipschitz map f from $\mathbb{Z}^k$ into a Hadamard space Y admits an extension $\bar{f}\colon \mathbb{R}^k \to Y$ with the same Lipschitz constant (which is now answered affirmatively by Theorem A). This problem plays a certain role in the study of quasiflats in Hadamard spaces, compare [LSc].

Acknowledgments. We would like to thank M. Berger and J. Jost for stimulating discussions on the subject. The first author would also like to thank the Research Institute for Mathematics, ETH Zürich, for its hospitality and financial support.

1 Alexandrov Spaces

We recall some definitions and results from the theory of metric spaces with upper or lower curvature bounds in the sense of A.D. Alexandrov

[Al1,2]. For spaces with curvature bounded from above, the general references are [AlBeN], [BeN], [Buy], or [Ba], [BaGSc], [BrH], where the emphasis is laid on nonpositive curvature. Spaces with lower curvature bounds are treated systematically in [BuGP] and [Pl].

Let (X, d) be a metric space. We denote by $B(x, r)$ and $U(x, r)$ the closed and open metric ball, respectively, with center x and radius r. The *length* of a (continuous) curve $\sigma\colon [a, b] \to X$ is given by

$$L(\sigma) := \sup \sum_{i=1}^{k} d(\sigma(t_{i-1}), \sigma(t_i)) \in [0, \infty] \ ,$$

where the supremum is taken over all positive integers k and all subdivisions $a = t_0 \leq t_1 \leq \ldots \leq t_k = b$. Then

$$d_i(x, y) := \inf \{L(\sigma) : \sigma \text{ is a curve from } x \text{ to } y\}$$

defines a metric on X which values in $[0, \infty]$. One calls d_i the *inner metric* on X induced by d. The triangle inequality makes $d_i \geq d$; if $d = d_i$ then (X, d) is said to be an *inner metric space*. A curve $\sigma\colon [a, b] \to X$ is called *minimizing* or *shortest* if $L(\sigma) = d(\sigma(a), \sigma(b))$. Then σ is said to be a *geodesic* if it additionally has constant speed, i.e. there exists $s \geq 0$ such that $L(\sigma|[a, t]) = s(t - a)$ for all $t \in [a, b]$. A metric space X is called *locally geodesic* if every $p \in X$ possesses a neighborhood U such that for all $x, y \in U$ there exists a geodesic in X from x to y. X is called a *geodesic space* if this holds for $U = X$.

A subset C of a metric space X is said to be *convex* if for all pairs of points $x, y \in C$ there exists a geodesic from x to y in X, and if all segments obtained this way lie entirely in C. The set C is called *strongly convex* if it is convex and if for all $x, y \in C$ there is a unique geodesic $\sigma\colon [0, 1] \to X$ from x to y.

For $\kappa \in \mathbb{R}$ let M^2_κ denote the (simply connected, complete) model surface of constant Gauss curvature κ, and define

$$D_\kappa := \operatorname{diam} M^2_\kappa = \begin{cases} \pi/\sqrt{\kappa} & \text{for } \kappa > 0 \ , \\ \infty & \text{for } \kappa \leq 0 \ . \end{cases}$$

Let $x_1, x_2 \in X$, $x \in X \setminus \{x_1, x_2\}$, and assume that $d(x, x_1) + d(x, x_2) + d(x_1, x_2) < 2D_\kappa$. Then there exists a distance preserving map $f\colon \{x, x_1, x_2\} \to M^2_\kappa$ and one defines the *comparison angle* $\gamma^\kappa_x(x_1, x_2) \in [0, \pi]$ to be the angle subtended by the two segments in M^2_κ connecting $f(x)$ with $f(x_1)$ and $f(x_2)$ respectively.

Definition 1.1. (γ^κ condition) *Let $\kappa \in \mathbb{R}$ and X a metric space. We say that a quadruple $(x; x_1, x_2, x_3)$ of points in X satisfies the γ^κ* condition *if $\gamma_x^\kappa(x_1, x_2) + \gamma_x^\kappa(x_1, x_3) + \gamma_x^\kappa(x_2, x_3) \le 2\pi$ as long as the three angles are defined. Then X is said to have* curvature $\ge \kappa$ *if every $p \in X$ possesses a neighborhood U such that all quadruples in U satisfy the γ^κ condition.*

This definition of curvature $\ge \kappa$ is given in [BuGP] with the additional requirement on X to be inner and locally complete. As mentioned in the introduction, if X is locally geodesic, then 1.1 can be reformulated in terms of thick triangles. We need the following definitions.

A *triangle* in X is a triple $\Delta = (\sigma_1, \sigma_2, \sigma_3)$ of geodesics $\sigma_i \colon [a_i, b_i] \to X$ whose endpoints match as usual. Assume that Δ has perimeter

$$P(\Delta) := L(\sigma_1) + L(\sigma_2) + L(\sigma_3) < 2D_\kappa \ .$$

Then there exists a *comparison triangle* Δ^κ for Δ in M_κ^2 which is unique up to isometry, namely, a triple of geodesics $\sigma_i^\kappa \colon [a_i, b_i] \to M_\kappa^2$ such that $L(\sigma_i^\kappa) = L(\sigma_i)$ for $i = 1, 2, 3$, and such that the endpoints of $\sigma_1^\kappa, \sigma_2^\kappa, \sigma_3^\kappa$ match in the same way as those of $\sigma_1, \sigma_2, \sigma_3$. Then Δ is said to be *κ-thick* or *κ-thin* if $d(\sigma_i(s), \sigma_j(t)) - d(\sigma_i^\kappa(s), \sigma_j^\kappa(t)) \ge 0$ or ≤ 0 respectively, whenever $i, j \in \{1, 2, 3\}$, $s \in [a_i, b_i]$, and $t \in [a_j, b_j]$.

The following lemma is proved in [BuGP]; spaces of curvature $\le \kappa$ are then defined analogously. These conditions go back to Alexandrov [Al1,2].

Lemma 1.2. (curvature $\ge \kappa$) *A locally geodesic metric space X has curvature $\ge \kappa$ (as defined in 1.1) if and only if every $p \in X$ possesses a neighborhood U such that all triangles in X with vertices in U and perimeter $< 2D_\kappa$ are κ-thick.*

Definition 1.3. (curvature $\le \kappa$) *A metric space X is said to have* curvature $\le \kappa$ *if it is locally geodesic and every $p \in X$ possesses a neighborhood U such that all triangles in X with vertices in U and perimeter $< 2D_\kappa$ are κ-thin.*

A riemannian manifold X has (Alexandrov) curvature $\ge \kappa$ or $\le \kappa$ if and only if the sectional curvature of X satisfies the same inequality.

In case $\kappa = 0$, 1.2 and 1.3 can be restated in a more direct way as follows. We say that $m \in X$ is a *midpoint* of $x_1, x_2 \in X$ if $d(x_1, m) = d(x_1, x_2)/2 = d(m, x_2)$. Note that for $x_1, x_2 \in \mathbb{R}^2$ the midpoint $m = (x_1 + x_2)/2$ satisfies $4\|m\|^2 = 2\|x_1\|^2 + 2\|x_2\|^2 - \|x_1 - x_2\|^2$.

Lemma 1.4. (midpoints) *A locally geodesic metric space X has curvature ≥ 0 or ≤ 0 if and only if every $p \in X$ possesses a neighborhood U such that* *for all $x_1, x_2, y \in U$ and every midpoint $m \in X$ of x_1 and x_2, $4d(m,y)^2 - 2d(x_1,y)^2 - 2d(x_2,y)^2 + d(x_1,x_2)^2 \geq 0$ or ≤ 0 respectively.*

The proof is not difficult (see [BrH]). We will also use the following global version of 1.3.

Definition 1.5. (CAT(κ) space) *A metric space X is called a* CAT(κ) space *if for all pairs of points $x, y \in X$ with $d(x,y) < D_\kappa$ there exists a geodesic from x to y, and all triangles in X of perimeter $< 2D_\kappa$ are κ-thin.*

It follows that for all $x, y \in X$ with $d(x,y) < D_\kappa$ there is actually a unique geodesic $\sigma\colon [0,1] \to X$ from x to y, and all metric balls in X with radius $< D_\kappa/2$ are strongly convex. Note that M^2_κ is a CAT(κ) space.

Finally, we mention two important globalization theorems. The first corresponds to the Toponogov comparison theorem in riemannian geometry and has been proved by Burago-Gromov-Perelman [BuGP, 2.5, 3.2] and also by Plaut [Pl]. The latter may be viewed as an analogue of the Hadamard-Cartan theorem and has been stated in different ways, cf. [G1, p. 119], [Ba], [BrH], [Buy], the main contribution being due to Alexander-Bishop [ABi].

Theorem 1.6. (thick triangles) *Let X be a complete inner metric space of curvature $\geq \kappa$ for some $\kappa \in \mathbb{R}$. Then all quadruples in X satisfy the γ^κ condition. If X is geodesic, then all triangles in X of perimeter $< 2D_\kappa$ are κ-thick.*

Theorem 1.7. (thin triangles) *Let X be a simply connected, complete inner metric space of curvature $\leq \kappa$, where $\kappa \leq 0$. Then X is geodesic and all triangles in X are κ-thin, thus X is a* CAT(κ) *space. In particular, X is contractible.*

A metric space X satisfying the assumptions of 1.7 is called a *Hadamard space*. Simple examples show that 1.7 fails for $\kappa > 0$ or for spaces that are not simply connected.

2 Metric Cones

In this section we first discuss the cone construction over metric spaces, which is essentially due to Berestovskij, cf. [AlBeN, 4.3]. For a detailed account we refer to the forthcoming [BrH]. Introducing the notions of "vector addition" and "scalar product" in this general context we establish in 2.5 and 2.6 two basic facts for complete geodesic cones of curvature ≥ 0 or ≤ 0.

Let (Ω, α) be a metric space, and let $\mathbb{R}_+ := [0, \infty)$. The *cone* $C(\Omega)$ over Ω is defined to be the quotient space $(\Omega \times \mathbb{R}_+)/\sim$, where $(\omega_1, r_1) \sim (\omega_2, r_2)$ if and only if $r_1 = r_2 = 0$ or $(\omega_1, r_1) = (\omega_2, r_2)$. For $(\omega, r) \in \Omega \times \mathbb{R}_+$ we denote by $[\omega, r]$ the corresponding element of $C(\Omega)$. The point $[\omega, 0] \in C(\Omega)$ is called the *origin* of $C(\Omega)$ and is denoted by o. One defines a metric d on $C(\Omega)$ by

$$d\big([\omega_1, r_1], [\omega_2, r_2]\big)^2 := {r_1}^2 + {r_2}^2 - 2r_1r_2 \cos\big(\min\{\alpha(\omega_1, \omega_2), \pi\}\big) ;$$

then $(C(\Omega), d)$ is called the *euclidean cone* (or 0-cone) over (Ω, α). For instance, if Ω is the unit sphere in a euclidean space and α is the induced inner metric on Ω, then the associated euclidean cone $(C(\Omega), d)$ is just the euclidean space itself. For $\lambda \geq 0$ and $v = [\omega, r] \in C(\Omega)$ we write $\lambda v := [\omega, \lambda r]$.

Lemma 2.1. (euclidean sector) *Let $\sigma\colon [0,1] \to C(\Omega)$ be a geodesic omitting o, and let $\pi_1\colon C(\Omega) \setminus \{o\} \to \Omega$ be the canonical projection. Then the curve $\omega := \pi_1 \circ \sigma$ satisfies $L(\omega) = \alpha(\omega(0), \omega(1)) < \pi$. Moreover, the ruled surface $\Sigma := \{\lambda\sigma(t) : \lambda \geq 0,\ t \in [0,1]\}$ in $C(\Omega)$ is isometric to the convex hull of two rays in $\mathbb{R}^2$ starting at 0 and subtending the angle $L(\omega)$.*

Proof. For $t \in [0,1]$ let $\omega_t := \omega(t)$. Since σ is minimizing we have $d(\sigma(0), \sigma(s)) + d(\sigma(s), \sigma(t)) = d(\sigma(0), \sigma(t))$ whenever $0 \leq s \leq t \leq 1$. By the definition of d, and since σ omits o, this implies that either $\alpha(\omega_0, \omega_s) + \alpha(\omega_s, \omega_t) = \alpha(\omega_0, \omega_t) < \pi$, or $\alpha(\omega_0, \omega_t) \geq \pi$ and $\omega_s \in \{\omega_0, \omega_t\}$. In fact the second case is excluded by the continuity of ω. Hence, it follows that ω is a curve of length $L(\omega) = \alpha(\omega_0, \omega_1) < \pi$.

Consider the map $h\colon [\omega, r] \mapsto (r\cos\alpha(\omega_0, \omega), r\sin\alpha(\omega_0, \omega)) \in \mathbb{R}^2$ defined on Σ. Let $v_1 = [\omega_s, r_1], v_2 = [\omega_t, r_2] \in \Sigma$, where $0 \leq s \leq t \leq 1$ and $r_1, r_2 \geq 0$. Then $|h(v_1) - h(v_2)|^2 = {r_1}^2 + {r_2}^2 - 2r_1r_2\cos|\alpha(\omega_0, \omega_s) - \alpha(\omega_0, \omega_t)|$. The first part of the proof yields $|\alpha(\omega_0, \omega_s) - \alpha(\omega_0, \omega_t)| = \alpha(\omega_s, \omega_t) < \pi$, thus $|h(v_1) - h(v_2)| = d(v_1, v_2)$. □

Let $v = [\omega, r] \in C(\Omega)$ and $\lambda \geq 0$. We write $\|v\| := r = d(v, o)$; then $\|\lambda v\| = \lambda\|v\|$.

Definition 2.2. (scalar product) *For $v_1 = [\omega_1, r_1], v_2 = [\omega_2, r_2] \in C(\Omega)$ we define $\langle v_1, v_2\rangle := (\|v_1\|^2 + \|v_2\|^2 - d(v_1, v_2)^2)/2 = r_1r_2\cos(\min\{\alpha(\omega_1, \omega_2), \pi\})$.*

We have $\langle \lambda v_1, v_2\rangle = \lambda\langle v_1, v_2\rangle = \langle v_1, \lambda v_2\rangle$ for $\lambda \geq 0$ and, as usual, $\langle v, v\rangle = \|v\|^2$ for all $v \in C(\Omega)$.

Now assume that $(C(\Omega), d)$ is a geodesic space. Then for all $v, w \in C(\Omega)$ there exists a midpoint $m(v, w)$ between v and w. In general, $m(v, w)$ is not unique; in the following, statements involving $m(v,w)$ hold for all possible choices unless otherwise specified.

CONVENTION 2.3. (vector addition) *We use the symbol* $(v+w)$ *as a substitute for* $m(2v,2w)$.

As indicated by the brackets, the "vector addition" obtained this way is, in general, not associative (even if the midpoints in question are unique). The next lemma follows immediately from 2.1 and the definitions.

LEMMA 2.4. (intermediate points) *Let* $v_1, v_2 \in C(\Omega)$ *and* $t \in [0,1]$. *Then the set of possible choices of* $((1-t)v_1 + tv_2)$ *coincides with* $\{\sigma(t) : \sigma\colon [0,1] \to C(\Omega)$ *is a geodesic from* v_1 *to* $v_2\}$.

We now characterize cones of curvature ≥ 0 or ≤ 0 by means of sub- or superadditivity, respectively, of the scalar product.

PROPOSITION 2.5. (semiadditivity) *Assume that* $C(\Omega)$ *is a complete geodesic space. Then* $C(\Omega)$ *is a space of curvature* ≥ 0 *if and only if* $\langle (v_1 + v_2), w\rangle \leq \langle v_1, w\rangle + \langle v_2, w\rangle$ *for all* $v_1, v_2, w \in C(\Omega)$ *and for all choices of* (v_1+v_2). *Conversely,* $C(\Omega)$ *has curvature* ≤ 0 *if and only if* $\langle (v_1+v_2), w\rangle \geq \langle v_1, w\rangle + \langle v_2, w\rangle$ *for all* $v_1, v_2, w \in C(\Omega)$ *and for all choices of* $(v_1 + v_2)$.

Proof. Let $v_1, v_2, w \in C(\Omega)$, and let m be a choice of $m(2v_1, 2v_2)$. By definition,

$$2\langle m, w\rangle = \|m\|^2 + \|w\|^2 - d(m,w)^2 \ ,$$
$$4\langle v_i, w\rangle = \|2v_i\|^2 + \|w\|^2 - d(2v_i, w)^2 \ ,$$

$i = 1, 2$. Using 2.1 together with the formula already exploited in 1.4 we get

$$2\|2v_1\|^2 + 2\|2v_2\|^2 = 4\|m\|^2 + d(2v_1, 2v_2)^2 \, .$$

Combining these equations we see that

$$\begin{aligned} &8(\langle v_1, w\rangle + \langle v_2, w\rangle - \langle m, w\rangle) \\ &\quad = 4d(m,w)^2 - 2d(2v_1, w)^2 - 2d(2v_2, w)^2 + d(2v_1, 2v_2)^2 \ . \end{aligned}$$

Now the result follows from 1.4, 1.6, and 1.7. □

In particular, combining 2.4 and 2.5 we deduce that if $C(\Omega)$ is a complete geodesic space of curvature ≤ 0 (say), and if $\sigma\colon [0,1] \to C(\Omega)$ is a geodesic with endpoints $v_1 = \sigma(0)$, $v_2 = \sigma(1)$, then $\langle \sigma(t), w\rangle \geq (1-t)\langle v_1, w\rangle + t\langle v_2, w\rangle$ for all $w \in C(\Omega)$ and $t \in [0,1]$. We are going to iterate this procedure. Consider a third point $v_3 \in C(\Omega)$ and let $\varrho\colon [0,1] \to C(\Omega)$ be a geodesic from $\sigma(t)$ to v_3 for some $t \in [0,1]$. Let $s \in [0,1]$ and denote $\bar{s} := 1 - s$,

$\bar{t} := 1-t$. Then $\varrho(s)$ possesses a representation $(\bar{s}(\bar{t}v_1+tv_2)+sv_3)$, hence 2.5 yields

$$\langle \varrho(s), w\rangle \geq \sum_{i=1}^{3} \lambda_i \langle v_i, w\rangle \ ,$$

where $\lambda_1 = \bar{s}\bar{t}$, $\lambda_2 = \bar{s}t$, and $\lambda_3 = s$. Note that $\lambda_i \in [0,1]$ and $\sum_{i=1}^{3} \lambda_i = 1$; the point $\varrho(s)$ may thus be viewed as a "convex combination" of v_1, v_2, v_3. More generally, for a finite number of points $v_1, \dots, v_k$ in $C(\Omega)$, we want to estimate the scalar product of two points v, v' belonging to the convex hull of $\{v_1, \dots, v_k\}$ in terms of the products $\langle v_i, v_j\rangle$, $1 \leq i, j \leq k$. To this end we introduce the following formalism.

For a subset V of a geodesic metric space X let $G_1(V) \subset X$ denote the union of all geodesic segments $\sigma([0,1])$ in X with endpoints $\sigma(0), \sigma(1)$ in V. Define inductively $G_{n+1}(V) := G_1(G_n(V))$ for $n \geq 1$. Then

$$G(V) := \bigcup_{n=1}^{\infty} G_n(V)$$

is the convex hull of V, and we denote by $\overline{G}(V)$ its closure. Now let $C(\Omega)$ be a geodesic metric cone as above. For all positive integers k let Δ_{k-1} denote the standard $(k-1)$-simplex in $\mathbb{R}^k$ spanned by the unit vectors $e_1 = (1, 0, \dots, 0), \dots, e_k = (0, \dots, 0, 1)$. For $v_1, \dots, v_k \in C(\Omega)$ we define the correspondence

$$\mathcal{C}(v_1, \dots, v_k) \subset G(\{v_1, \dots, v_k\}) \times \Delta_{k-1}$$

to be the smallest set with the following two properties:

(a) $(v_i, e_i) \in \mathcal{C}(v_1, \dots, v_k)$ for $i = 1, \dots, k$.

(b) If $(v, \lambda), (v', \lambda') \in \mathcal{C}(v_1, \dots, v_k)$ and $\sigma\colon [0,1] \to C(\Omega)$ is a geodesic from v to v', then $(\sigma(t), (1-t)\lambda + t\lambda') \in \mathcal{C}(v_1, \dots, v_k)$ for all $t \in [0,1]$.

Note that the canonical projections from $\mathcal{C}(v_1, \dots, v_k)$ to $G(\{v_1, \dots, v_k\})$ or Δ_{k-1} are both surjective. We conclude this section with the following result which now follows easily from 2.4 and 2.5.

Proposition 2.6. (convex combination) *Assume that $C(\Omega)$ is a complete geodesic space. Let $v_1, \dots, v_k \in C(\Omega)$ and $(v, \lambda), (v', \lambda') \in \mathcal{C}(v_1, \dots, v_k)$, where $\lambda = (\lambda_1, \dots, \lambda_k)$, $\lambda' = (\lambda'_1, \dots, \lambda'_k)$. If $C(\Omega)$ has curvature ≥ 0 or ≤ 0, then $\langle v, v'\rangle \leq \sum_{i,j=1}^{k} \lambda_i \lambda'_j \langle v_i, v_j\rangle$ or $\langle v, v'\rangle \geq \sum_{i,j=1}^{k} \lambda_i \lambda'_j \langle v_i, v_j\rangle$ respectively.*

3 Tangent Cones

Tangent cones are particular examples of metric cones (as defined in section 2) and play a central role in our arguments.

Assume that (X, d) is a locally geodesic space. For $a, b > 0$ let $\sigma\colon [0, a] \to X$ and $\tau\colon [0, b] \to X$ be two nonconstant geodesics with common starting point $\sigma(0) = \tau(0) = x$. Then the *upper angle* between σ and τ is defined by

$$\alpha(\sigma, \tau) := \limsup_{s,t \to 0} \gamma_x^\kappa(\sigma(s), \tau(t)) \ ;$$

this is clearly independent of the choice of $\kappa \in \mathbb{R}$. Note that $\alpha(\sigma, \sigma) = 0$ since σ is minimizing, moreover α satisfies the triangle inequality, cf. [Al1]. It is easily seen that if X has curvature $\le \kappa$ or $\ge \kappa$, and if $0 < s' \le s$ and $0 < t' \le t$ are sufficiently small, then $\gamma_x^\kappa(\sigma(s'), \tau(t')) - \gamma_x^\kappa(\sigma(s), \tau(t)) \le 0$ or ≥ 0 respectively. In particular, in these cases $\alpha(\sigma, \tau)$ exists as a limit, and $\alpha(\sigma, \tau) - \gamma_x^\kappa(\sigma(s), \tau(t)) \le 0$ or ≥ 0 respectively.

We consider σ and τ to be equivalent if $\alpha(\sigma, \tau) = 0$. Then α induces a metric (which we denote α again) on the set $\Omega'_x X$ of equivalence classes of nonconstant geodesics starting at x. The metric completion $(\Omega_x X, \alpha)$ of $(\Omega'_x X, \alpha)$ is called the *space of directions* of X at x. Note that $\Omega_x X$ is empty if there is no nonconstant geodesic in X starting at x (i.e. if x is an isolated point of X). In this case, the *tangent cone* $T_x X$ of X at x is defined to be the metric space consisting of a single point (the origin o). Otherwise, $T_x X$ is defined to be euclidean cone $C(\Omega_x X)$ over $\Omega_x X$, where the metric on $T_x X$ is denoted d_x. (There are deviating definitions of $\Omega_x X$ and $T_x X$ in the literature.) If $\sigma\colon [0, a] \to X$ is a geodesic of speed $r > 0$, and if $\omega \in \Omega_{\sigma(0)} X$ is the direction determined by σ, then we define $\dot\sigma(0) := [\omega, r] \in T_{\sigma(0)} X$. In case σ is a constant geodesic, we put $\dot\sigma(0) := o \in T_{\sigma(0)} X$.

We also define $(T'_x X, d_x)$ to be the cone over $(\Omega'_x X, \alpha)$, or, if $\Omega'_x X$ is empty, $T'_x X := \{o\}$. By taking the metric completion of $T'_x X$ one gets an alternate description of the tangent cone $T_x X$. In particular, $T_x X$ is a complete metric space. Note also that for all $v \in T'_x X$ there exist $a > 0$ and a (possibly constant) geodesic $\sigma\colon [0, a] \to X$ with $\dot\sigma(0) = v$. Conversely, for every geodesic $\tau\colon [0, b] \to X$ with $b > 0$ and $\tau(0) = x$, $\dot\tau(0) \in T'_x X$.

The following result on tangent cones of spaces of curvature $\le \kappa$ or $\ge \kappa$ is due to Nikolaev [N] and Burago-Gromov-Perelman [BuGP] respectively.

PROPOSITION 3.1. (tangent cone) *If (X, d) is a locally geodesic space of curvature $\le \kappa$, then for all $x \in X$ the tangent cone $(T_x X, d_x)$ is a complete geodesic space of curvature ≤ 0 and thus a Hadamard space.*

If (X, d) is a locally geodesic space of finite Hausdorff dimension and of curvature $\geq \kappa$, then for all $x \in X$ the tangent cone $(T_x X, d_x)$ is a complete geodesic space of curvature ≥ 0.

One of the key steps in the proof of our main result is the proposition below. It generalizes the fact that if $v_1, \dots, v_k$ are elements of a euclidean vector space, then the quadratic form on $\mathbb{R}^k$ canonically associated with the Gram matrix $(\langle v_i, v_j \rangle)_{1 \leq i,j \leq k}$ is positive semidefinite.

Proposition 3.2. (Gram form) *Let X be a locally geodesic space of curvature $\geq \kappa$ for some $\kappa \in \mathbb{R}$. Let $x \in X$, $k \geq 1$, and $v_1, \dots, v_k \in T_x X$. Then $\sum_{i,j=1}^k \lambda_i \lambda_j \langle v_i, v_j \rangle \geq 0$ for all $(\lambda_1, \dots, \lambda_k) \in (\mathbb{R}_+)^k$.*

In case X has finite dimension (cf. [BuGP, 6.5]), 3.2 is an immediate consequence of 2.6 and 3.1. In the general case, where we do not know whether $T_x X$ is a nonnegatively curved geodesic space, 3.2 still holds. The proof of this is somewhat involved, and the techniques are not used any further in the rest of the paper; we therefore shift it to the appendix.

4 A General Criterion

The goal of this section is to establish Proposition 4.3 below. This result yields a general criterion for the extensibility of a 1-Lipschitz map $f \colon E \to Y$ to one additional point, where E is a finite subset of an arbitrary metric space X and Y is a complete $\mathrm{CAT}(\kappa)$ space. In 4.2 it is shown that, in some sense, there is a unique candidate for this extension (actually 4.2 applies to a more general situation). We start with a simple fact. Recall that for $F \subset Y$, $G(F)$ denotes the convex hull of F and $\overline{G}(F)$ its closure.

Lemma 4.1. (radius versus diameter) *Let Y be a $\mathrm{CAT}(\kappa)$ space and F a subset of Y with $\operatorname{diam} F \leq D_\kappa/2$. Then $\operatorname{diam} \overline{G}(F) = \operatorname{diam} F$, and $\overline{G}(F)$ is a strongly convex subset of Y. Moreover, if F is finite and $\operatorname{diam} F > 0$, then for all $y \in F$ and $\delta > 0$ there exist $z \in U(y, \delta) \cap G(F)$ and $r < \operatorname{diam} F$ such that $\overline{G}(F) \subset B(z, r)$.*

Proof. By the inductive construction of $G(F)$ described in section 2, in order to prove that $\operatorname{diam} \overline{G}(F) = \operatorname{diam} F$ it suffices to show that $\operatorname{diam} G_1(C) = \operatorname{diam} C$ whenever C is a subset of Y with $\operatorname{diam} C \leq D_\kappa/2$. This follows easily from the fact that triangles in Y with sides of length $\leq D_\kappa/2$ are κ-thin. Then $\overline{G}(F)$ is strongly convex since segments in Y of length $\leq D_\kappa/2$ are unique and depend continuously on their endpoints.

Now let $F = \{y_0, \dots, y_n\}$, where $n \geq 1$ and $y_i \neq y_j$ for $i \neq j$. We may assume that $y = y_0$ and $\delta \leq \inf\{d(y_i, y_j) : 0 \leq i < j \leq n\}$. We construct points $z_0, \dots, z_n \in G(F)$ as follows. We put $z_0 := y = y_0$, and for $i = 1, \dots, n$, assuming that z_{i-1} is already defined, we note that $d(z_{i-1}, y_i) \geq \delta/2^{i-1}$ and let z_i be the point on the segment from z_{i-1} to y_i with $d(z_{i-1}, z_i) = \delta/2^i$. Then $z := z_n$ satisfies $d(z, y) \leq \delta - \delta/2^n < \delta$, and since $\operatorname{diam} G(F) = \operatorname{diam} F$ we see that $d(z, y_i) \leq \operatorname{diam} F - \delta/2^n =: r$ for all y_i. Hence $F \subset B(z, r)$ and thus $\overline{G}(F) \subset B(z, r)$. □

The following result is a partial generalization of [F, 2.10.40].

Proposition 4.2. (unique candidate) *Let Y be a complete $\mathrm{CAT}(\kappa)$ space, $F = \{y_1, \dots, y_n\} \subset Y$ such that $\operatorname{diam} F \leq D_\kappa/2$, and $h_1, \dots, h_n \colon \mathbb{R}_+ \to \mathbb{R}_+$ continuous, nondecreasing, unbounded functions with $h_i(0) = 0$ for at least one i. Define $A(b) := \bigcap_{i=1}^n B(y_i, h_i(b))$ for $b \geq 0$. Then $c := \inf\{b \geq 0 : A(b) \neq \emptyset\} < \infty$ and $A(c) = \{y\}$ for some $y \in \overline{G}(F)$. Actually $y \in \overline{G}(F')$, where $F' := \{y_i \in F : d(y_i, y) = h_i(c)\}$.*

Proof. We may assume that $n \geq 2$ and $y_i \neq y_j$ for $i \neq j$. According to 4.1, $\operatorname{diam} \overline{G}(F) = \operatorname{diam} F$, and $\overline{G}(F)$ is contained in some ball with radius $r < \operatorname{diam} F$. In particular, choosing $b \geq 0$ such that $h_i(b) \geq r$ for all i, we have $A(b) \neq \emptyset$ and thus $c < \infty$. Moreover, we see that $h_j(c) \leq r$ for some $j \in \{1, \dots, n\}$.

We first show that $A'(b) := A(b) \cap \overline{G}(F) \neq \emptyset$ for $b > c$. Since $A'(b) \subset A'(b')$ for $b \leq b'$ it suffices to prove this for b close to c. Since $h_j(c) \leq r < D_\kappa/2$ we may thus assume that $h_j(b) < D_\kappa/2$. Let $y \in A(b)$; then $d(y, \overline{G}(F)) \leq d(y, y_j) \leq h_j(b) < D_\kappa/2$. Since Y is a complete $\mathrm{CAT}(\kappa)$ space, there exists a unique point $y' \in \overline{G}(F)$ closest to y (this is shown by adapting the argument for $\kappa = 0$ stated in [BrH]). For every $y_i \in F$ the triangle with vertices y, y', y_i has perimeter $< 2D_\kappa$ and is thus κ-thin. By the choice of y' it follows that $\gamma^\kappa_{y'}(y, y_i) \geq \pi/2$, and since $d(y', y_i) \leq D_\kappa/2$ we deduce that $d(y', y_i) \leq d(y, y_i)$. Thus $y' \in A'(b)$.

Next we show that $A(c) \subset \overline{G}(F)$ and thus $A'(c) = A(c)$. Assume the contrary and let $y \in A(c) \setminus \overline{G}(F)$. The above argument shows that there is a point $y' \in A'(c)$, and in fact $d(y', y_i) < d(y, y_i) \leq h_i(c)$ if $h_i(c) < D_\kappa/2$. Let $\delta > 0$ be such that $d(y', y_i) + \delta < h_i(c)$ for these i. By 4.1 there exists $z' \in U(y', \delta)$ with $d(z', y_i) < D_\kappa/2$ for $y_i \in F$. Then either $d(z', y_i) < d(y', y_i) + \delta < h_i(c)$ or $d(z', y_i) < D_\kappa/2 \leq h_i(c)$ for each $y_i \in F$. Since $h_i(0) = 0$ for at least one i, it follows that $c > 0$ and thus $z' \in A(b)$ for some $b < c$, a contradiction.

In order to prove that $A(c) = A'(c)$ is nonempty and consists of a single point, we show that $\operatorname{diam} A'(b) \to 0$ for $b \downarrow c$. If not, then there exists $\varepsilon > 0$

such that for all $b > c$ there are $p_b, q_b \in A'(b)$ with $d(p_b, q_b) \geq \varepsilon$. For $y_i \in F$ and $b > c$ the triangle with vertices y_i, p_b, q_b is κ-thin. Let $m_b \in \overline{G}(F)$ be the midpoint of p_b and q_b. Since $d(p_b, q_b) \geq \varepsilon$ we see that there exists $b > c$ such that $d(m_b, y_i) < h_i(c)$ for all i with $h_i(c) < D_\kappa/2$. Let $\delta > 0$ be such that $d(m_b, y_i) + \delta < h_i(c)$ for these i. As above we conclude that there exists $z_b \in U(m_b, \delta)$ with $d(z_b, y_i) < D_\kappa/2$ for $y_i \in F$, thus $d(z_b, y_i) < h_i(c)$ for all $y_i \in F$, contradicting the definition of c. Hence $\operatorname{diam} A'(b) \to 0$ for $b \downarrow c$. Since Y is complete and $A'(b) \neq \emptyset$ for $b > c$, it follows that $A(c)$ consists of a single point $y \in \overline{G}(F)$.

It remains to show that $y \in \overline{G}(F')$. Clearly $F' \neq \emptyset$. Suppose that $d(y, y_i) = h_i(c) = D_\kappa/2$ for all $y_i \in F'$. Then again, since $d(y, y_i) < h_i(c)$ for $y_i \in F \setminus F'$, there would exist a point z close to y with $d(z, y_i) < h_i(c)$ for all $y_i \in F$, which is impossible. Hence, there is $y_k \in F'$ with $d(y, y_k) < D_\kappa/2$, thus $d(y, \overline{G}(F')) < D_\kappa/2$. Let $y' \in \overline{G}(F')$ be the unique point closest to y, and let $\sigma\colon [0, d(y, y')] \to Y$ be the geodesic from y to y'. For every $y_i \in F$ the triangle with vertices y, y', y_i has sides of length $\leq D_\kappa/2$, and by the choice of y' it follows that $d(\sigma(s), y_i) \leq h_i(c)$ for all $s \in [0, d(y, y')]$ and $y_i \in F'$. Since $d(y, y_i) < h_i(c)$ for $y_i \in F \setminus F'$, and y is unique, we see that all points $\sigma(s)$ close to y must coincide with y, thus $y = y' \in \overline{G}(F')$. □

Now let X, Y be metric spaces, E a finite subset of X, $f\colon E \to Y$ a map, and $x \in X \setminus E$. We say that $\bar{f}\colon E \cup \{x\} \to Y$ is an *optimal extension* of f if among all extensions of f to $E \cup \{x\}$, $\bar{f}$ minimizes the number

$$c(\bar{f}) := \max_{x' \in E} \frac{d(\bar{f}(x), f(x'))}{d(x, x')} .$$

Then $d(\bar{f}(x), f(x')) \leq c(\bar{f}) d(x, x')$ for all $x' \in E$, and x' is called a *relevant point* if equality holds.

Proposition 4.3. (general criterion) *Let X be an arbitrary metric space and Y a complete $\mathrm{CAT}(\kappa)$ space. Let E be a finite subset of X, $f\colon E \to Y$ a map with $\operatorname{diam} f(E) \leq D_\kappa/2$, and $x \in X \setminus E$. Then there exists a unique optimal extension $\bar{f}\colon E \cup \{x\} \to Y$ of f, and $\bar{f}(x) \in \overline{G}(f(E'))$, where E' is the set of relevant points in E. Moreover, $\bar{f}$ is 1-Lipschitz whenever f is 1-Lipschitz and the following condition holds: If the comparison angles $\gamma_x^\kappa(x_i, x_j)$ are defined for all $i, j \in \{1, \dots, k\}$, where $x_1, \dots, x_k$ denote the elements of E', then $\sum_{i,j=1}^k \lambda_i \lambda_j \cos \gamma_x^\kappa(x_i, x_j) \geq 0$ for all $(\lambda_1, \dots, \lambda_k) \in (\mathbb{R}_+)^k$.*

Proof. Let $x_1, \dots, x_n$ be the (mutually distinct) elements of E. Then for $i = 1, \dots, n$ we let $d_i := d(x, x_i) > 0$ and $y_i := f(x_i)$. The first part of the theorem is an immediate consequence of 4.2, where we put $h_i(s) := sd_i$ for $s \in \mathbb{R}_+$. Let $y := \bar{f}(x)$ and $c := c(\bar{f})$. Then $d(y, y_i) \leq cd_i$, and by 4.1, $d(y, y_i) \leq \operatorname{diam} \overline{G}(f(E)) \leq D_\kappa/2$ for $i = 1, \ldots, n$.

Now assume that f is 1-Lipschitz. By reordering the points $x_1, \ldots, x_n$ if necessary we may assume that $E' = \{x_1, \ldots, x_k\}$ as in the theorem. Thus $d(y, y_i) = cd_i$ for $i = 1, \ldots, k$. If $\gamma_x^\kappa(x_i, x_j)$ is undefined for some $x_i, x_j \in E'$, then one of these points, x_i say, satisfies $d_i = d(x, x_i) \geq D_\kappa/2$. Since $D_\kappa/2 \geq d(y, y_i) = cd_i$ we get $c \leq 1$, thus $\bar{f}$ is 1-Lipschitz.

It remains to consider the case where $\sum_{i,j=1}^k \lambda_i \lambda_j \cos \gamma_x^\kappa(x_i, x_j) \geq 0$ for $(\lambda_1, \ldots, \lambda_k) \in (\mathbb{R}_+)^k$. We assume that $c > 1$ and derive a contradiction. For $i = 1, \ldots, k$ we let $\sigma_i \colon [0, cd_i] \to X$ be the unit speed geodesic from y to y_i and denote by $v_i := \dot{\sigma}_i(0)$ its 'initial vector' in the tangent cone T_yY.

Let o be the origin of T_yY. We claim that o belongs to the closure $\overline{G}$ of the convex hull $G = G(\{v_1, \ldots, v_k\})$. Note that T_yY is a Hadamard space by 3.1. Assume that $o \notin \overline{G}$; then there exists a unique point $w \in \overline{G}$ closest to o. Clearly $\gamma_w^0(o, v_i) \geq \pi/2$ for $i = 1, \ldots, k$ and therefore $\gamma_o^0(v_i, w) < \pi/2$. Varying w a little we may additionally assume that $w \in T'_yY$, thus there exists a geodesic $\tau \colon [0, b] \to Y$ with $\tau(0) = y$ and $\dot{\tau}(0) = w$. Then we have $\alpha(\sigma_i, \tau) = \gamma_o^0(v_i, w) < \pi/2$ for $i = 1, \ldots, k$. Since $d(y, y_i) < cd_i$ for $i = k+1, \ldots, n$, it follows that for $t > 0$ sufficiently small, $d(\tau(t), y_i) < cd_i$ for $i = 1, \ldots, n$. This contradicts the fact that $\bar{f}$ is already optimal.

Hence, for every $\varepsilon > 0$ there exists $\bar{v} \in G$ with $\|\bar{v}\| \leq \varepsilon$. Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in \Delta_{k-1}$ be such that $(\bar{v}, \lambda) \in \mathcal{C}(v_1, \ldots, v_k)$. Then by 2.6,

$$\begin{aligned} \varepsilon^2 \geq \|\bar{v}\|^2 &\geq \sum_{i,j=1}^k \lambda_i \lambda_j \langle v_i, v_j \rangle \\ &= \sum_i {\lambda_i}^2 + 2 \sum_{i<j} \lambda_i \lambda_j \langle v_i, v_j \rangle \ . \end{aligned}$$

Since $\sum_i {\lambda_i}^2 \geq 1/k$ and $\langle v_i, v_j \rangle \geq -1$ this implies that $2 \sum_{i<j} \lambda_i \lambda_j \geq (1/k) - \varepsilon^2$. Next we observe that for $1 \leq i < j \leq k$,

$$\langle v_i, v_j \rangle = \cos \alpha(\sigma_i, \sigma_j) \geq \cos \gamma_y^\kappa(y_i, y_j) \ ,$$

where we have used the fact that the triangle with vertices y, y_i, y_j has perimeter $\leq 3D_\kappa/2$ and is thus κ-thin. Consider the corresponding points x, x_i, x_j in X. Since $d(y_i, y_j) \leq d(x_i, x_j)$, $d(x_i, x_j) > 0$, and $D_\kappa/2 \geq d(y, y_i) = cd_i > d_i = d(x, x_i)$, using the geometry of M_κ^2 we deduce that $\cos \gamma_y^\kappa(y_i, y_j) > \cos \gamma_x^\kappa(x_i, x_j)$. Hence, there exists $\delta > 0$ such that

$\langle v_i, v_j \rangle \geq \cos \gamma_x^\kappa(x_i, x_j) + \delta$ for $1 \leq i < j \leq k$. We obtain

$$\begin{aligned}\varepsilon^2 &\geq \sum_i \lambda_i^2 + 2\Big(\sum_{i<j} \lambda_i \lambda_j \cos \gamma_x^\kappa(x_i, x_j)\Big) + \delta\big((1/k) - \varepsilon^2\big) \\ &= \Big(\sum_{i,j} \lambda_i \lambda_j \cos \gamma_x^\kappa(x_i, x_j)\Big) + \delta\big((1/k) - \varepsilon^2\big) \ .\end{aligned}$$

The sum in the last line is nonnegative by assumption. Since the resulting inequality $\varepsilon^2 \geq \delta((1/k) - \varepsilon^2)$ is contradictory for $\varepsilon > 0$ sufficiently small, we conclude that $c \leq 1$. Thus $\bar{f}$ is 1-Lipschitz. □

5 Main Results

In this section we prove the first two theorems stated in the introduction.

The proof of Theorem A consists of three steps. The first step is the following proposition which now follows easily from 4.3 and 3.2.

Proposition 5.1. (finite plus one) *Let $\kappa \in \mathbb{R}$, X a geodesic metric space such that all triangles of perimeter $< 2D_\kappa$ in X are κ-thick, and Y a complete* $\mathrm{CAT}(\kappa)$ *space. Let E be a finite subset of X, $f\colon E \to Y$ a* 1-*Lipschitz map with* $\operatorname{diam} f(E) \leq D_\kappa/2$, *and $x \in X \setminus E$. Then there exists a* 1-*Lipschitz extension $\bar{f}\colon E \cup \{x\} \to Y$ of f with $\bar{f}(x) \in \overline{G}(f(E))$.*

Proof. It remains to check that the condition given in 4.3 is satisfied. For $i = 1, \ldots, k$ we pick a unit speed geodesic $\tau_i\colon [0, d(x, x_i)] \to X$ from x to x_i and let $u_i := \dot{\tau}_i(0) \in T_x X$. Since triangles in X of perimeter $< 2D_\kappa$ are κ-thick we see that

$$\cos \gamma_x^\kappa(x_i, x_j) \geq \cos \alpha(\tau_i, \tau_j) = \langle u_i, u_j \rangle$$

for all i, j. Hence, the criterion of 4.3 is satisfied by 3.2. □

In the second step we show that 5.1 still holds if E is replaced by an arbitrary subset S of X. To this end we must prove that $\bigcap_{s \in S} B(f(s), d(x, s)) \neq \emptyset$. Let $y_s := f(s)$ and $r_s := \min\{d(s, x), D_\kappa/2\}$; we actually show that $\bigcap_{s \in S} B(y_s, r_s) \neq \emptyset$. Since $\operatorname{diam} f(S) \leq D_\kappa/2$, this holds trivially if $r_s = D_\kappa/2$ for all $s \in S$. We may thus assume that $r_z < D_\kappa/2$ for some $z \in S$. The desired conclusion is then obtained from 5.1 together with the following general Helly type theorem (compare [DGrKl]). Note that by 4.1, $\overline{G}(f(S))$ (equipped with the metric induced from Y) is a complete $\mathrm{CAT}(\kappa)$ space with $\operatorname{diam} \overline{G}(f(S)) = \operatorname{diam} f(S) \leq D_\kappa/2$.

Proposition 5.2. (nonempty intersection) *Let Y be a complete* $\mathrm{CAT}(\kappa)$ *space and S an arbitrary index set. For $s \in S$ let $y_s \in Y$ and $r_s \in [0, D_\kappa/2]$,* *and assume that* $r_z < D_\kappa/2$ *for some* $z \in S$. *If* $\bigcap_{s\in E} B(y_s, r_s) \neq \emptyset$ *for every finite subset* E *of* S, *then* $\bigcap_{s\in S} B(y_s, r_s) \neq \emptyset$.

Proof. Let $\mathcal{E} := \{E : E \text{ is a finite subset of } S\}$. For $E \in \mathcal{E}$ let $A(E) := \bigcap_{s\in E} B(y_s, r_s) \neq \emptyset$, and for $r \geq 0$ define $A^r(E) := A(E) \cap B(y_z, r)$. Note that for $r \geq r_z$, $A^r(E) \supset A(E \cup \{z\})$ and thus $A^r(E) \neq \emptyset$, moreover $A^r(E)$ is strongly convex for $r < D_\kappa/2$.

Let $\varrho := \inf\{r \geq 0 : A^r(E) \neq \emptyset \text{ for all } E \in \mathcal{E}\}$; we claim that $A^\varrho(E) \neq \emptyset$ for each $E \in \mathcal{E}$. We have $\varrho \leq r_z < D_\kappa/2$, and clearly the limit $\delta := \lim_{r\downarrow\varrho} \operatorname{diam} A^r(E)$ exists. If $\delta = 0$ then by completeness of Y we see that $A^\varrho(E)$ consists of a single point. If $\delta > 0$, then for $r \in (\varrho, D_\kappa/2)$ we choose points $p_r, q_r \in A^r(E)$ with $d(p_r, q_r) \geq \delta/2$. Since the triangle with vertices y_z, p_r, q_r is κ-thin it follows that for r sufficiently close to ϱ, the midpoint m_r of p_r and q_r belongs to $A^\varrho(E)$.

Next we claim that $\inf\{\operatorname{diam} A^\varrho(E) : E \in \mathcal{E}\} = 0$. Namely, assuming $\operatorname{diam} A^\varrho(E) \geq \varepsilon > 0$ for all $E \in \mathcal{E}$, by a midpoint construction as above we would find a number $\varrho' < \varrho$ with $A^{\varrho'}(E) \neq \emptyset$ for all $E \in \mathcal{E}$, in contradiction to the choice of ϱ.

Now choose a sequence of sets $E_i \in \mathcal{E}$, $i = 1, 2, \ldots$, such that $\operatorname{diam} A^\varrho(E_i) \to 0$ for $i \to \infty$. Let $(y_1, r_1), (y_2, r_2), \ldots$ be an enumeration of the set of all pairs (y_s, r_s) with $s \in E_i$ for some i. Let $A_k := \bigcap_{j=1}^k B(y_j, r_j)$ and $A_k^\varrho := A_k \cap B(y_z, \varrho)$. Then $\lim_{k\to\infty} \operatorname{diam} A_k^\varrho = 0$, and choosing $a_k \in A_k^\varrho$ we get a Cauchy sequence with limit $a \in Y$. For $s \in S$ we have $A_k^\varrho \cap B(y_s, r_s) \neq \emptyset$ for all k (since $A^\varrho(E) \neq \emptyset$ for all $E \in \mathcal{E}$), and thus $a \in B(y_s, r_s)$. □

For the last step of the proof of Theorem A we consider the set $\mathcal{L}$ of all 1-Lipschitz extensions $f' : S' \to Y$ of f with $S \subset S' \subset X$ and $f'(S') \subset \overline{G}(f(S))$ (and thus $\operatorname{diam} f'(S') \leq D_\kappa/2$, cf. 4.1). For $f_1', f_2' \in \mathcal{L}$ we write $f_1' \subset f_2'$ if and only if $S_1' \subset S_2'$ and $f_1' = f_2'|S_1'$, where S_1', S_2' are the domains of f_1', f_2' respectively. With respect to this partial ordering, clearly every chain in $\mathcal{L}$ has an upper bound, thus by Zorn's lemma $\mathcal{L}$ has a maximal element $\bar{f}$. By the result of the second step $\bar{f}$ is defined on all of X and therefore Theorem A holds.

The proof of Theorem B is analogous; we only have to adapt the first step. Note that by rescaling the metric on either X or Y, the Lipschitz constant of f can be normalized, thus it remains to show that the following holds.

PROPOSITION 5.3. (finite sets mapped into trees) *Let* X *be an arbitrary metric space and* Y *a complete metric space which is* $\mathrm{CAT}(\kappa)$ *for all* $\kappa \in \mathbb{R}$. *Let* E *be a finite subset of* X, $f : E \to Y$ *a* 1*-Lipschitz map, and* $x \in X \setminus E$.

Then there exists a 1*-Lipschitz extension* $\bar{f}\colon E\cup\{x\}\to Y$ *of* f *with* $\bar{f}(x)\subset\overline{G}(f(E))$.

Proof. Let $\bar{f}$ be the extension of f given by the first part of 4.3, and let $c:=c(\bar{f})$. Since every triangle in Y is κ-thin for all κ it is easily seen that $\overline{G}(f(E'))=G_1(f(E'))$. Hence, since $\bar{f}(x)\in\overline{G}(f(E'))$, there exist two points $x_1,x_2\in E'$ such that $cd(x,x_1)+cd(x,x_2)=d(\bar{f}(x),f(x_1))+d(\bar{f}(x),f(x_2))=d(f(x_1),f(x_2))\le d(x_1,x_2)\le d(x,x_1)+d(x,x_2)$, thus $c\le 1$. □

6 Quadruples

Next we prove Theorem C stated in the introduction. Let $E=\{x_1,x_2,x_3\}\subset X$. The given map $f\colon E\to Y$ can clearly be decomposed into $f_2\circ f_1$, where $f_1\colon E\to M^2_\kappa$ is a 1-Lipschitz map and $f_2\colon f_1(E)\to Y$ is isometric. Therefore it suffices to prove the two special cases of the theorem where either $X=M^2_\kappa$ and $f\colon E\to Y$ is isometric, or $Y=M^2_\kappa$.

In the first case it suffices to extend f to points contained in the convex hull of E. The existence of such extensions is ensured by (a special case of) a theorem due to Reshetnyak [Re]. (This is essentially a ruled surface construction. Note that here 5.1 does not apply since Y is not assumed to be complete.)

Now consider the case $Y=M^2_\kappa$. For given $f\colon E\to M^2_\kappa$ and $x\in X\setminus E$ let $\bar{f}\colon E\cup\{x\}$ be the optimal extension of f whose existence is asserted by the first part of 4.3. Then $\bar{f}(x)\in G(f(E'))$, where E' is the set of relevant points in E. Let $c:=c(\bar{f})$ and $y:=\bar{f}(x)$. If $y\in f(E')$, then clearly $c=0$. If y lies between two points of $f(E')$, then the same computation as in the proof of 5.3 shows that $c\le 1$. Finally, if y belongs to the interior of $G(f(E'))$, then $E'=E$ and the sum of the mutual angles spanned by the segments from y to the points of $f(E)$ is equal to 2π. Since f is 1-Lipschitz and the quadruple $(x;x_1,x_2,x_3)$ satisfies the γ^κ condition, it follows easily that $c\le 1$, thus $\bar{f}$ is 1-Lipschitz. This proves Theorem C.

Finally, using Theorem C, we deduce the following results characterizing spaces of curvature $\ge\kappa$ or $\le\kappa$ by means of extensibility of 1-Lipschitz maps (defined on three points) into or from the model space M^2_κ. We remark that, following Wald [W], Berestovskij characterized spaces of bounded curvature by means of isometric embeddings of quadruples, cf. [Be], [Pl].

Proposition 6.1. (characterizing curvature $\ge\kappa$) *A metric space* X *has curvature* $\ge\kappa$ *(as defined in 1.1) if and only if every* $p\in X$ *possesses a* *neighborhood U with the following property: Whenever $x_1, \dots, x_4 \in U$ and $f\colon \{x_1, x_2, x_3\} \to M^2_\kappa$ is a 1-Lipschitz map, then there exists a 1-Lipschitz extension $\bar{f}\colon \{x_1, \dots, x_4\} \to M^2_\kappa$ of f.*

PROPOSITION 6.2. (characterizing curvature $\leq \kappa$) *A metric space Y has curvature $\leq \kappa$ (as defined in 1.3) if and only if every $q \in Y$ possesses a strongly convex neighborhood U with the following property: Whenever $x_1, \dots, x_4 \in M^2_\kappa$ and $f\colon \{x_1, x_2, x_3\} \to U$ is a 1-Lipschitz map, then there exists a 1-Lipschitz extension $\bar{f}\colon \{x_1, \dots, x_4\} \to U$ of f.*

Besides the fact that domain and target space are interchanged, the latter statement differs from the first only by the additional requirement that U be strongly convex (meaning that for all $y_1, y_2 \in U$ there exists a unique geodesic $\tau\colon [0,1] \to Y$ from y_1 to y_2, and $\tau([0,1]) \subset U$.) Proposition 6.2 is no longer true without this assumption. Note also that both results remain true if the first occurrence of "1-Lipschitz" in each statement is replaced by "distance preserving".

Proof of 6.1. Assume that every $p \in X$ has a neighborhood U such that all quadruples in U satisfy the γ^κ condition. We may assume that $\operatorname{diam} U < D_\kappa/2$. If $x_1, \dots, x_4 \in U$ and $f\colon \{x_1, x_2, x_3\} \to M^2_\kappa$ is a 1-Lipschitz map, then by Theorem C there exists a 1-Lipschitz extension $\bar{f}\colon \{x_1, \dots, x_4\} \to M^2_\kappa$ of f.

Conversely, let U be a neighborhood of p with the property stated in 6.1, and assume without loss of generality that $\operatorname{diam} U < D_\kappa/2$. Let $E := \{x_1, x_2, x_3\} \subset U$ and $x \in U \setminus E$. Then there is a distance preserving map $f\colon E \to M^2_\kappa$. Let $\bar{f}\colon E \cup \{x\} \to M^2_\kappa$ be the optimal extension of f given by the first part of 4.3, and let $y_i := f(x_i)$, $i = 1, 2, 3$, and $y := \bar{f}(x)$. Since by assumption there exists a 1-Lipschitz extension of f, $\bar{f}$ is 1-Lipschitz as well. Hence, we have $d(y_i, y_j) = d(x_i, x_j)$ and $d(y, y_i) \leq c d(x, x_i)$ for all i, j and for some $c \leq 1$, moreover $y \in G(F')$ for $F' := \{y_i : d(y, y_i) = c d(x, x_i)\}$.

We consider three different cases. If $y \in F'$, then $c = 0$ and thus $\bar{f}$ is constant, hence $x_1 = x_2 = x_3$ and $\Gamma_x := \gamma^\kappa_x(x_1, x_2) + \gamma^\kappa_x(x_1, x_3) + \gamma^\kappa_x(x_2, x_3) = 0$. Next consider the case where $y \notin F'$ and $F' = f(E)$. Then the mutual angles $\angle_y(y_1, y_2), \angle_y(y_1, y_3), \angle_y(y_2, y_3)$ subtended by the segments from y to y_1, y_2, y_3 add up to 2π. Since f is isometric and $d(y, y_i) = c d(x, x_i) \leq d(x, x_i) < D_\kappa/2$ for $i = 1, 2, 3$, it follows that $\gamma^\kappa_x(x_i, x_j) \leq \angle_y(y_i, y_j)$ for $i < j$, thus $\Gamma_x \leq 2\pi$. It remains to consider the case where $y \notin F'$ and $F' = \{y_1, y_2\}$ say. We claim that then $\gamma^\kappa_x(x_1, x_3) + \gamma^\kappa_x(x_2, x_3) \leq \pi$ (and thus $\Gamma_x \leq 2\pi$). We choose points $y'_1, y'_2, y'_3 \in M^2_\kappa$ such that $d(y, y'_3) = c d(x, x_3) \geq d(y, y_3)$, and, for $i = 1, 2$, $d(y, y'_i) = c d(x, x_i) = d(y, y_i)$ and

$d(y_i', y_3') = d(x_i, x_3) = d(y_i, y_3)$. As in the second case it follows that $\gamma_x^\kappa(x_i, x_3) \le \angle_y(y_i', y_3')$ for $i = 1, 2$. Since $d(y, y_3') \ge d(y, y_3)$, the points y, y_1', y_2', y_3' span a convex quadrilateral, thus $\angle_y(y_1', y_3') + \angle_y(y_2', y_3') \le \pi$. □

Proof of 6.2. If Y has curvature $\le \kappa$ then every $q \in Y$ possesses a strongly convex neighborhood U such that $\operatorname{diam} U < D_\kappa/2$ and every triangle with vertices in U is κ-thin. Then by Theorem C, whenever $x_1, \dots, x_4 \in M_\kappa^2$ and $f\colon \{x_1, x_2, x_3\} \to U$ is a 1-Lipschitz map, there exists a 1-Lipschitz extension $\bar f\colon \{x_1, \dots, x_4\} \to U$ of f.

Conversely, assume that U is a strongly convex neighborhood of q with the property stated in 6.2. Let $y_1, y_2, y_3 \in U$ be the vertices of a triangle with perimeter $< 2D_\kappa$, $\tau\colon [0,1] \to Y$ the geodesic from y_2 to y_3, and $t \in [0,1]$. Then pick $x_1, x_2, x_3 \in M_\kappa^2$ such that there is a distance preserving map $f\colon \{x_1, x_2, x_3\} \to U$ with $f(x_i) = y_i$, $i = 1, 2, 3$, and let $\sigma\colon [0,1] \to M_\kappa^2$ be the geodesic from x_2 to x_3. By assumption there exists a 1-Lipschitz extension $\bar f\colon \{x_1, x_2, x_3, \sigma(t)\} \to U$ of f, and since τ is uniquely determined, $\bar f(\sigma(t)) = \tau(t)$. Thus $d(y_1, \tau(t)) \le d(x_1, \sigma(t))$. By applying this argument twice we see that all triangles of perimeter $< 2D_\kappa$ in U are κ-thin. □

Appendix A: Proof of 3.2

It remains to prove Proposition 3.2. We introduce the following notation. For $\kappa \in \mathbb{R}$ consider a triangle with sides of length $b_1, b_2 > 0$ and $c \ge 0$ in M_κ^2, where $b_1 + b_2 + c < 2D_\kappa$. Let γ denote the angle at the vertex opposite to the side of length c. We write the law of cosines implicitly as $c = c_\kappa(b_1, b_2; \gamma)$ or $\gamma = \gamma_k(b_1, b_2; c)$; besides the explicit formulas for $\kappa = 0$ we will only use the fact that $c/t = c_{t^2\kappa}(b_1/t, b_2/t; \gamma)$ and $\gamma = \gamma_{t^2\kappa}(b_1/t, b_2/t; c/t)$ for $t > 0$. Further we denote by $l_\kappa(b_1, b_2; c)$ the distance from the midpoint of the side of length c to the opposite vertex of the triangle; this is also defined if $b_1 = 0$ or $b_2 = 0$. Then $l_\kappa(b_1, b_2; c)/t = l_{t^2\kappa}(b_1/t, b_2/t; c/t)$ for $t > 0$, and

$$4 l_0(b_1, b_2; c)^2 = 2{b_1}^2 + 2{b_2}^2 - c^2 \ ,$$

cf. 1.4. Now let X be a locally geodesic space of curvature $\ge \kappa$, and let $x \in X$. For $i = 1, 2$ and $a > 0$ let $\sigma_i\colon [0, a] \to X$ be a (possibly constant) geodesic with $\sigma_i(0) = x$, and recall from section 3 that $v_i := \dot\sigma_i(0)$ is an element of $T_x'X$ whose 'norm' $\|v_i\|$ is equal to the speed of σ_i. Thus $d(x, \sigma_i(t)) = t\|v_i\|$ for $t \in [0, a]$. The squared distance between v_1 and v_2 in $T_x'X$ is given by

$$d_x(v_1, v_2)^2 = \|v_1\|^2 + \|v_2\|^2 - 2\|v_1\|\|v_2\| \cos\alpha(\sigma_1, \sigma_2) \ ,$$

and the 'scalar product' of v_1 and v_2 satisfies

$$2\langle v_1, v_2\rangle = \|v_1\|^2 + \|v_2\|^2 - d_x(v_1, v_2)^2 \ .$$

In the first equation $\alpha(\sigma_1, \sigma_2)$ is undefined if $\|v_1\| = 0$ or $\|v_2\| = 0$; then the term $\|v_1\|\|v_2\| \cos\alpha(\sigma_1, \sigma_2)$ is considered to be zero.

We start with the following observation.

LEMMA A.1. *For $\sigma_1, v_1, \sigma_2, v_2$ as above the limit $\lim_{t\downarrow 0} d(\sigma_1(t), \sigma_2(t))/t$ exists and is equal to $d_x(v_1, v_2)$.*

Proof. We may assume that $\|v_1\|, \|v_2\| > 0$. For $i = 1, 2$ and $0 < t \le a$ let $b_i(t) := d(x, \sigma_i(t)) = t\|v_i\|$. By the definition of $\gamma_x^\kappa(\sigma_1(t), \sigma_2(t)) =: \gamma_\kappa(t)$,

$$\begin{aligned} d(\sigma_1(t), \sigma_2(t))/t &= c_\kappa(b_1(t), b_2(t); \gamma_\kappa(t))/t \\ &= c_{t^2\kappa}(\|v_1\|, \|v_2\|; \gamma_\kappa(t)) \end{aligned}$$

for t sufficiently small. Since $\lim_{t\downarrow 0} \gamma_\kappa(t) = \alpha(\sigma_1, \sigma_2)$ the lemma follows. □

We also need the following partial refinement of A.1.

LEMMA A.2. *For all $\Lambda, \varepsilon > 0$ there exists $\delta > 0$ such that the following holds. Whenever σ_1, σ_2 are geodesics as above with speed $\|\dot\sigma_1(0)\|, \|\dot\sigma_2(0)\| \le \Lambda$, then $d(\sigma_1(t), \sigma_2(t))/t \le d_x(\dot\sigma_1(0), \dot\sigma_2(0)) + \varepsilon$ for $0 < t \le \delta$ (and $t \le a$).*

Proof. We use the notation of the preceding proof and assume that $\|v_1\|, \|v_2\| > 0$. Since $\gamma_\kappa(t) \le \alpha(\sigma_1, \sigma_2)$ for t sufficiently small we get

$$d(\sigma_1(t), \sigma_2(t))/t \le c_{t^2\kappa}(\|v_1\|, \|v_2\|; \alpha(\sigma_1, \sigma_2)) \ .$$

In view of $\|v_1\|, \|v_2\| \le \Lambda$ and the fact that $c_0(\|v_1\|, \|v_2\|; \alpha(\sigma_1, \sigma_2)) = d_x(v_1, v_2)$ the claim follows easily. □

The next two results are variations of the formula for l_0.

LEMMA A.3. *Let $\sigma_1, \sigma_2, \tau\colon [0, a] \to X$ be geodesics with $\sigma_1(0) = \sigma_2(0) = \tau(0) = x$ and $\dot\sigma_1(0) =: v_1$, $\dot\sigma_2(0) =: v_2$, $\dot\tau(0) =: w$. For $0 < t \le a$ let $\nu^t\colon [0, t] \to X$ be a geodesic from $\sigma_1(t)$ to $\sigma_2(t)$, and let $m_t := \nu^t(t/2)$. Then $4 \liminf_{t\downarrow 0} d(m_t, \tau(t))^2/t^2 \ge 2d_x(v_1, w)^2 + 2d_x(v_2, w)^2 - d_x(v_1, v_2)^2$.*

Proof. Let $b_i(t) := d(\sigma_i(t), \tau(t))$, $i = 1, 2$, and $c(t) := d(\sigma_1(t), \sigma_2(t))$. For t sufficiently small the triangle with vertices $\sigma_1(t), \sigma_2(t), \tau(t)$ is κ-thick, thus

$$\begin{aligned} d(m_t, \tau(t))/t &\ge l_\kappa(b_1(t), b_2(t); c(t))/t \\ &= l_{t^2\kappa}(b_1(t)/t, b_2(t)/t; c(t)/t) \ . \end{aligned}$$

Now the claim follows from A.1 and the formula for l_0. □

In particular, A.3 applies to the constant geodesic $\tau(t) \equiv x$ and thus gives a lower bound on $\liminf_{t\downarrow 0} d(x, m_t)^2/t^2$. We show that in this case actually equality holds.

LEMMA A.4. *Let $\sigma_1, v_1, \sigma_2, v_2, \nu^t$, and $m_t = \nu^t(t/2)$ be given as in A.3. Then $4\lim_{t\downarrow 0} d(x, m_t)^2/t^2 = 2\|v_1\|^2 + 2\|v_2\|^2 - d_x(v_1, v_2)^2$.*

Proof. In view of A.3 it suffices to establish the corresponding upper bound on $\limsup_{t\downarrow 0} d(x, m_t)^2/t^2$. Let $b_i(t) := d(x, \sigma_i(t)) = t\|v_i\|$, $i = 1, 2$, and $c(t) := d(\sigma_1(t), \sigma_2(t))$. For $0 < t \le a$ consider the triangle with sides $\sigma_1|[0,t]$, $\sigma_2|[0,t]$, ν^t, and let $\bar\sigma_1^t \colon [0,t] \to X$ be the geodesic defined by $\bar\sigma_1^t(s) := \sigma_1(t-s)$. We may assume that each of the three sides has positive length.

We claim that the limit $\lim_{t\downarrow 0} \alpha(\bar\sigma_1^t, \nu^t)$ exists and is equal to the euclidean comparison angle $\gamma^0_{v_1}(o, v_2) = \gamma_0(\|v_1\|, d_x(v_1, v_2); \|v_2\|) =: \gamma_0$. Namely, by A.1,

$$\begin{aligned}\lim_{t\downarrow 0} \gamma^\kappa_{\sigma_1(t)}(x, \sigma_2(t)) &= \lim_{t\downarrow 0} \gamma_\kappa(b_1(t), c(t); b_2(t))\\ &= \lim_{t\downarrow 0} \gamma_{t^2\kappa}(\|v_1\|, c(t)/t; \|v_2\|)\\ &= \gamma_0\ ,\end{aligned}$$

and by [BuGP, 7.5], the difference $\alpha(\bar\sigma_1^t, \nu^t) - \gamma^\kappa_{\sigma_1(t)}(x, \sigma_2(t))$ tends to zero for $t \downarrow 0$. (For completeness we remark that the proof in [BuGP] contains a minor inconsistency. One has to use the fact that the sum of the angles in a small triangle in M^2_κ is close (not equal) to 2π.)

Now, since X has curvature $\ge \kappa$, we see that for t sufficiently small,

$$\begin{aligned}d(x, m_t)/t &\le c_\kappa(b_1(t), c(t)/2; \alpha(\bar\sigma_1^t, \nu^t))/t\\ &= c_{t^2\kappa}(\|v_1\|, c(t)/(2t); \alpha(\bar\sigma_1^t, \nu^t))\ .\end{aligned}$$

Using A.1 again, together with the above claim, we obtain

$$\begin{aligned}4\limsup_{t\downarrow 0} d(x, m_t)^2/t^2 &\le 4c_0(\|v_1\|, d_x(v_1, v_2)/2; \gamma_0)^2\\ &= 4l_0(\|v_1\|, \|v_2\|; d_x(v_1, v_2))^2\ .\end{aligned}$$

Now the lemma follows from the formula for l_0. □

LEMMA A.5. *Let $v_1, v_2 \in T'_xX$, W a bounded subset of T'_xX (i.e. $\{\|w\| : w \in W\}$ is bounded), and $\mu_1, \mu_2 \ge 0$ with $\mu_1 + \mu_2 = 1$. Then for every $\eta > 0$ there exists $v \in T'_xX$ such that $\|v\|^2 \le {\mu_1}^2\|v_1\|^2 + 2\mu_1\mu_2\langle v_1, v_2\rangle + {\mu_2}^2\|v_2\|^2 + \eta$ and $\langle v, w\rangle \le \mu_1\langle v_1, w\rangle + \mu_2\langle v_2, w\rangle + \eta$ for all $w \in W$.*

Proof. It suffices to prove the result for $\mu_1 = \mu_2 = 1/2$. By applying this iteratively one obtains the lemma for arbitrary dyadic μ_1 which is enough to establish the general claim.

For $i = 1, 2$ let $\sigma_i\colon [0, a] \to X$ be geodesics with $\dot\sigma_i(0) = v_i$. Then for $0 < t \le a$ let $\varrho^t\colon [0, t] \to X$ be a geodesic from x to a midpoint m_t of $\sigma_1(t)$ and $\sigma_2(t)$. We claim that for t sufficiently small, $v := \dot\varrho^t(0)$ does the job. To obtain the first inequality we note that by A.4,

$$\begin{aligned} 4 \lim_{t\downarrow 0} \|\dot\varrho^t(0)\|^2 &= 4 \lim_{t\downarrow 0} d(x, m_t)^2/t^2 \\ &= 2\|v_1\|^2 + 2\|v_2\|^2 - d_x(v_1, v_2)^2 \\ &= \|v_1\|^2 + 2\langle v_1, v_2\rangle + \|v_2\|^2 \ . \end{aligned}$$

Now let $w \in W$, and let $\tau\colon [0, b] \to X$ be a geodesic with $\dot\tau(0) = w$. Since W is bounded there exists $\Lambda > 0$ (independent of w) such that $\|v_1\|, \|v_2\|, \|w\| \le \Lambda$. Then by A.4, $\|\dot\varrho^t(0)\| = d(x, m_t)/t \le \Lambda$ for small t. Let $\varepsilon > 0$. Then A.2 yields $d(\varrho^t(t), \tau(t))^2/t^2 \le d_x(\dot\varrho^t(0), w)^2 + \varepsilon'$ for t sufficiently small, where $\varepsilon' := \varepsilon(4\Lambda + \varepsilon)$. Thus

$$\begin{aligned} 2\langle \dot\varrho^t(0), w\rangle &\le \|\dot\varrho^t(0)\|^2 + \|w\|^2 - d\big(\varrho^t(t), \tau(t)\big)^2/t^2 + \varepsilon' \\ &= \big(d(x, m_t)^2 - d(m_t, \tau(t))^2\big)/t^2 + \|w\|^2 + \varepsilon' \ . \end{aligned}$$

Using A.3 and A.4 we obtain

$$\begin{aligned} &\limsup_{t\downarrow 0} \big(d(x, m_t)^2 - d(m_t, \tau(t))^2\big)/t^2 + \|w\|^2 \\ &\le \big(\|v_1\|^2 + \|v_2\|^2 - d_x(v_1, w)^2 - d_x(v_2, w)^2\big)/2 + \|w\|^2 \\ &= \langle v_1, w\rangle + \langle v_2, w\rangle \ . \end{aligned}$$

Thus $v = \dot\varrho^t(0)$ does the job for t sufficiently small. □

We generalize A.5 to arbitrary convex combinations and obtain 3.2 as an immediate corollary.

LEMMA A.6. *Let $k \ge 1$, $v_1, \dots, v_k \in T'_xX$, $\lambda = (\lambda_1, \dots, \lambda_k) \in \Delta_{k-1}$, and W a bounded subset of T'_xX. Then for every $\eta > 0$ there exists $v \in T'_xX$ such that $\|v\|^2 \le \sum_{i,j=1}^k \lambda_i\lambda_j\langle v_i, v_j\rangle + \eta$ and $\langle v, w\rangle \le \sum_{i=1}^k \lambda_i\langle v_i, w\rangle + \eta$ for all $w \in W$.*

Proof. We prove the result by induction on k. For $k = 1$ it is trivial and for $k = 2$ it is contained in A.5. Now we go from $k - 1$ to k. Let $v_1, \dots, v_k$, $\lambda = (\lambda_1, \dots, \lambda_k)$, and W be given as in the lemma. Without loss of generality we may assume that $\mu_1 := \lambda_1 + \ldots + \lambda_{k-1} \ne 0$. Define $\mu_2 := \lambda_k$ and $\lambda'_i := \lambda_i/\mu_1$, $i = 1, \dots, k-1$; then $\mu_1 + \mu_2 = 1$ and $\lambda' := (\lambda'_1, \dots, \lambda'_{k-1}) \in \Delta_{k-2}$. We use the induction hypothesis for $v_1, \dots, v_{k-1} \in T'_xX$, $\lambda' \in \Delta_{k-2}$, and $W' := W \cup \{v_k\} \subset T'_xX$. Let $\eta' > 0$. Then there exists $v' \in T'_xX$ with

$$\|v'\|^2 \le \sum_{i,j=1}^{k-1} \lambda'_i\lambda'_j\langle v_i, v_j\rangle + \eta' \ ,$$

$$\langle v', w' \rangle \le \sum_{i=1}^{k-1} \lambda'_i \langle v_i, w' \rangle + \eta'$$

for all $w' \in W' = W \cup \{v_k\}$. Applying A.5 to v', v_k, W, and μ_1, μ_2 we get the existence of $v \in T'_x X$ such that

$$\|v\|^2 \le {\mu_1}^2 \|v'\|^2 + 2\mu_1\mu_2 \langle v', v_k \rangle + {\mu_2}^2 \|v_k\|^2 + \eta' \ ,$$
$$\langle v, w \rangle \le \mu_1 \langle v', w \rangle + \mu_2 \langle v_k, w \rangle + \eta'$$

for all $w \in W$. Hence,

$$\begin{aligned} \|v\|^2 &\le {\mu_1}^2 \Big(\sum_{i,j=1}^{k-1} \lambda'_i \lambda'_j \langle v_i, v_j \rangle + \eta' \Big) + 2\mu_1\mu_2 \Big(\sum_{i=1}^{k-1} \lambda'_i \langle v_i, v_k \rangle + \eta' \Big) \\ &\qquad + {\mu_2}^2 \|v_k\|^2 + \eta' \\ &= \sum_{i,j=1}^{k} \lambda_i \lambda_j \langle v_i, v_j \rangle + (\mu_1\mu_2 + \mu_1 + 1)\eta' \ , \\ \langle v, w \rangle &\le \mu_1 \Big(\sum_{i=1}^{k-1} \lambda'_i \langle v_i, w \rangle + \eta' \Big) + \mu_2 \langle v_k, w \rangle + \eta' \\ &= \sum_{i=1}^{k} \lambda_i \langle v_i, w \rangle + (\mu_1 + 1)\eta' \ . \end{aligned}$$

Since $\mu_1, \mu_2 \le 1$ this yields the desired result. □

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Submitted: June 1996
Final version: November 1996