A partial solution of the finite spectrum problem

P. ECSEDI-TÓTH

1. Introduction

Let φ be an arbitrary first order sentence. By the finite spectrum of φ, $\mathrm{Sp}\,\varphi$, the following set is meant:

$$\mathrm{Sp}\,\varphi = \{n\in\omega \mid (\exists \underline{A})\,(\mathrm{card}\,A = n \text{ and } \underline{A} \models \varphi)\}.$$

The finite spectrum problem, due to Scholz [9], can be paraphrased in some different ways. Here are four possibilities:

FSP1 ([3], p. 512). Given φ arbitrarily, is there a first order sentence ψ such that $(\forall n\in\omega)\,(n\in \mathrm{Sp}\,\varphi \Leftrightarrow n\notin \mathrm{Sp}\,\psi)$?

FSP2 ([7], p. 269). Given φ arbitrarily, characterize $\mathrm{Sp}\,\varphi$ as a set of positive integers.

FSP3. Let $N\subset\omega$ be arbitrary. Is there a first order sentence φ such that $\mathrm{Sp}\,\varphi = N$?

FSP4. Characterize those subsets of ω which are the finite spectra of first order sentences.

The finite spectrum problem, beyond its historical interest, is closely related to some recent problems in theoretical computer science concerning NP completeness [5], [6]. Albeit a solution would be very useful, no general answer is known at the moment. For many interesting partial solutions, expecially for FSP3, see [2], [8]. As far as we know, however, no syntactically characterized non-trivial class of first order sentences has been given for which the finite spectrum problem is solvable. The purpose of the present paper is to provide one such class, the class of equality-free first order sentences, and to give answers to all of the four questions FSP 1—4 for this particular class.

We hope, that our considerations can help to attack the general problem by indicating where difficulties arise.

The class of equality-free first order sentences is by no means trivial. Indeed, it is known that computability can be formalized in a fragment of equality-free sentences (by equality-free universal Horn sentences, cf. [1], for a proof), which, in turn, play an essential role in PROLOG programming and thus, in the fifth-generation computer projects. Hence, the results presented here can have some impacts on the problems of complexity theory connected to the finite spectrum problem too.

2. Notations

Structures will be denoted by underlined capitals $\underline{A}$, $\underline{B}$; the corresponding capitals without underlining A, B stand for the universes of $\underline{A}$, $\underline{B}$, respectively. Constant, relation and function symbols are written in the lower case letters c, r, f; while their realizations in a structure, say in $\underline{A}$, will be denoted by $C^{(A)}$, $R^{(A)}$ and $F^{(A)}$.

We may suppose that there are only finitely many symbols in the first order language since our particular topic concerns simultaneously only finitely many sentences. For the sake of convenience, we shall assume that there are constant symbols, relation symbols and function symbols in the language. Thus, the universe of any structure is nonvoid. Therefore, we shall be interested in the variants of FSP 1—4 where ω is replaced by $[\omega]=\{1, 2, ...\}$. In the sequel, we shall always mean these versions when we are speaking on the finite spectrum problem.

3. An upward Löwenheim—Skolem theorem

Theorem 1. Let $\underline{A}$ be any structure of cardinality $n\in[\omega]$. Then, there is a structure $\underline{B}$ such that the cardinality of $\underline{B}$ is $n+1$ and $\underline{B}$ is elementarily equivalent to $\underline{A}$ in the equality-free sense.

Proof. Let b be a new element and define $B=A\cup\{b\}$. Let $h\colon B\to A$ be any onto mapping such that the restriction of h to A is the identity. Define $\underline{B}$ by the following items.

(i) For every constant symbol c, let $C^{(B)}=C^{(A)}$.

(ii) For every function symbol f of arity m and for arbitrary elements $b_1, ..., b_m\in B$, put

$$F^{(B)}(b_1, ..., b_m) = F^{(A)}(h(b_1), ..., h(b_m)).$$

(iii) For every relation symbol r of arity m and elements $b_1, ..., b_m\in B$, let

$$\langle b_1, ..., b_m\rangle\in R^{(B)} \quad \text{iff} \quad \langle h(b_1), ..., h(b_m)\rangle\in R^{(A)}.$$

It is easily seen, that h is a homomorphism from $\underline{B}$ onto $\underline{A}$ in the algebraic sense and h preserves relations. It follows, that the kernel of h is a congruence on $\underline{B}$ which is invariant over relations; hence $\underline{B}$ is correctly defined.

We shall prove by a straight-forward induction that $\underline{B}$ is elementarily equivalent to $\underline{A}$ in the equality-free sense. More precisely, we prove:

For arbitrary equality-free formula φ and assignment $k\colon V\to B$ (where V is the set of variables)

$$\underline{B}\models\varphi[k] \quad \text{iff} \quad \underline{A}\models\varphi[k_h] \tag{1}$$

where $k_h\colon V\to A$ is defined by $k_h(v)=h(k(v))$.

First we notice, that for any term t,

$$h(t^{(B)}[k]) = t^{(A)}[k_h]. \tag{2}$$

(Here, e.g. $t^{(A)}[k_h]$ stands for the familiar notion "the value of t in $\underline{A}$ at k_h".) Indeed, if t is a variable or a constant symbol, then (2) trivially holds by definition. Let t be of the form $f(t_1, \ldots, t_m)$ and assume that (2) is true for t_i $(1 \leqq i \leqq m)$. Then,

$$h(t^{(B)}[k]) = h\big(F^{(B)}(t_1^{(B)}[k], \ldots, t_m^{(B)}[k]\big) =$$

$$\overset{\text{(ii)}}{=} h\big(F^{(A)}(h(t_1^{(B)}[k], \ldots, h(t_m^{(B)}[k])\big) =$$

$$\overset{(*)}{=} h\big(F^{(A)}(t_1^{(A)}[k_h], \ldots, t_m^{(A)}[k_h])\big) =$$

$$\overset{(**)}{=} F^{(A)}(t_1^{(A)}[k_h], \ldots, t_m^{(A)}[k_h]) = t^{(A)}[k_h]$$

(where the equalities denoted by (*) and (**) hold by the induction hypothesis and since h is the identity on A, respectively). Hence (2) is true.

Turning to the proof of (1), if φ is an equality-free prime formula of the form $r(t_1, \ldots, t_m)$, then

$$B \models r(t_1, \ldots, t_m)[k] \quad \text{iff} \quad A \models r(t_1, \ldots, t_m)[k_h]$$

is easily seen by using (iii) and (2).

The induction trivially passes over negation and conjunction.

Let φ be an equality-free formula of the form $\exists v\psi$, and suppose, that (1) holds for ψ. Then,

$$\begin{aligned} &\underline{B} \models \exists v\psi[k] \text{ iff} \\ &\text{There is an assignment } k'\colon V \to B \text{ such that} \\ &k(w) = k'(w) \text{ provided } w \neq v \text{ and } \underline{B} \models \psi[k']. \end{aligned} \tag{3}$$

Similarly,

$$\begin{aligned} &\underline{A} \models \exists v\psi[k_h] \text{ iff} \\ &\text{There is an assignment } k_h'\colon V \to A \text{ such that} \\ &k_h(w) = k_h'(w) \text{ if } w \neq v \text{ and } \underline{A} \models \psi[k_h']. \end{aligned} \tag{4}$$

By the induction hypothesis, (3) implies (4) for the assignment k_h', defined by $k_h'(v) = h(k'(v))$, $k_h'(w) = h(k(w))$ if $w \neq v$. Similarly, if (4) holds, then, since h is onto, there is an assignment k' such that $k'(w) = k(w)$ if $v \neq w$ and $h(k'(v)) = k_h'(v)$ and $\underline{B} \models \psi[k']$. Hence (3) and (4) are equivalent and thus,

$$B \models \exists v\psi[k] \quad \text{iff} \quad A \models \exists v\psi[k_h].$$

This completes the induction. It follows, that $\underline{B}$ is elementarily equivalent to $\underline{A}$ in the equality-free sense.

Q.E.D.

Corollary 2. Let $\underline{A}$ be any structure of cardinality n $(n \in [\omega])$. Then for every $m \in [\omega]$, $m \geqq n$, there exists a structure $\underline{B}$ such that B has cardinality m and A is elementarily equivalent to B in the equality-free sense.

Proof. Iterate Theorem 1 $m-n$ times.

Q.E.D.

Remark. Corollary 2 can be considered as a sharpened version of the (finitary) upward Löwenheim—Skolem theorem for equality-free languages. For some generalization see [4].

The following assertion indicates, that the downward Löwenheim—Skolem theorem has no similar sharpening. Besides, it has an application in the next section.

Theorem 3. (i) For every $n \in [\omega]$, there exists a set T_n of equality-free sentences such that T_n has a model of cardinality m iff $m \geqq n$.

(ii) There exists a set T_ω of equality-free sentences such that T_ω has only infinite models.

Proof. Without loss of generality, we may suppose that there is a unary function symbol f and there is a unary relation symbol r in the language. Let c be a constant symbol.

(i) We shall use the following notation: for $k \in \omega, f^{(0)}(c)=c, f^{(k+1)}(c)=f(f^{(k)}(c))$. Let $n \in [\omega]$. We set

$$T_n = \{r(c)\} \cup \{\neg r(f^{(k)}(c)) \mid 1 \leqq k < n-1\} \cup \{r(f^{(n-1)}(c))\}.$$

A trivial induction shows that if $1 \leqq m < n$ then no model of T_n exists with cardinality m. On the other hand, it is easy to construct a structure $\underline{A}$ such that A has cardinality n and $\underline{A} \models T_n$. It follows by Corollary 2 that for each $m \geqq n$, T_n has a model with cardinality m.

(ii) The following additional notations will be used: for $k \in \omega$, $\neg_0$ is the empty sequence, $\neg_{k+1} = \langle \neg, \langle \neg_k \rangle \rangle$; and $|\log_2 k| = \max\{m \mid m \in \omega \text{ and } 2^m \leqq k\}$. Let

$$T_\omega = \{\neg_{|\log_2 k|} r(f^{(k-1)}(c)) \mid k \in [\omega]\}.$$

Again, it is easily seen by induction that T_ω has no finite models, but a model of T_ω with cardinality ω is easy to obtain.

Q.E.D.

4. The finite spectrum of equality-free sentences

Corollary 4. Let φ be an arbitrary equality-free first order sentence. Then,

(i) For every $n \in [\omega]$, if $n \in \mathrm{Sp}\, \varphi$, then for all $m \in \omega$, $m \geqq n$ implies that $m \in \mathrm{Sp}\, \varphi$.

(ii) For any equality-free first order sentence ψ, if $\mathrm{Sp}\, \varphi \neq \emptyset$ and $\mathrm{Sp}\, \psi \neq \emptyset$ then $\mathrm{Sp}\, \varphi \cap \mathrm{Sp}\, \psi \neq \emptyset$.

(iii) For any equality-free first order sentence ψ, if $\mathrm{Sp}\, \varphi \neq \emptyset$ and $\mathrm{Sp}\, \psi \neq \emptyset$, then either $\mathrm{Sp}\, \varphi \subset \mathrm{Sp}\, \psi$ or $\mathrm{Sp}\, \psi \subset \mathrm{Sp}\, \varphi$.

Proof. (i) is immediate by Corollary 2. (ii) and (iii) are entailed by (i).

Q.E.D.

According to this corollary, the answer for FSP 1 is in the negative for any equality-free φ if we restrict ourselves to searching for equality-free ψ, only. For FSP 1, however, we also have the following positive result.

Corollary 5. Let φ be equality-free. Then there exists a first order sentence ψ such that for any $n\in[\omega]$, $n\in\mathrm{Sp}\,\psi$ iff $n\notin\mathrm{Sp}\,\varphi$.

Proof. If $\mathrm{Sp}\,\varphi=\emptyset$, then we may choose $\psi=(\forall x)\ (x=x)$. Obviously, $\mathrm{Sp}\,\psi=[\omega]$. If $\mathrm{Sp}\,\varphi\neq\emptyset$, then, by Corollary 4 (i), there exists a least number $n_0\in[\omega]$ such that

$$Sp\ \varphi=\{n|n\in\omega \quad\text{and}\quad n\geqq n_0\}.$$

Then, $[\omega]-\mathrm{Sp}\,\varphi=\{1,2,\ldots,n_0-1\}$ and we may choose for ψ the sentence $(\exists v_1\ldots\exists v_{n_0-1}\forall v)\ (v=v_1\ldots v=v_{n_0-1})$.

Q.E.D.

Let $N\subset\omega$. We say that N is a final segment of ω iff there exists an $n_0\in[\omega]$ such that $N=\{n|n\in\omega \text{ and } n\geqq n_0\}$.

For FSP 2, we have by Corollary 4, that, given the equality-free sentence φ arbitrarily, the finite spectrum of φ is a final segment of ω or else $\mathrm{Sp}\,\varphi=\emptyset$.

For the remaining two questions FSP 3 and FSP 4, we obtain:

Corollary 6. Let $N\subset[\omega]$, $N\neq\emptyset$. Then, the following two assertions are equivalent:

(i) N is the finite spectrum of an equality-free first order sentence.

(ii) N is a final segment of ω.

Proof. (i)$\Rightarrow$(ii) is immediate from Corollary 4 by definition.

(ii)$\Rightarrow$(i) Let N be a final segment of ω, $N\neq\emptyset$. Then there is a number $n_0\in[\omega]$ such that $N=\{n|n\in\omega \text{ and } n\geqq n_0\}$. Consider the set T_{n_0} for n_0 constructed in the proof of Theorem 3 (i). Obviously, T_{n_0} is finite, hence $\varphi=\bigwedge T_{n_0}$ is an equality-free sentence. By Theorem 3 (i), $\mathrm{Sp}\,\varphi=N$.

Q.E.D.

References

[1] Andréka, H., I. Németi, The generalized completeness of Horn predicate logic as a programming language, *Acta Cybernetica*, Tom. 4., Fasc. 1., pp. 3—13.

[2] Asser, G., Das Repräsentantenproblem im Prädikatenkalkül der ersten Stufe mit Identität, *Zeitschrift für Math. Logik und Grundlagen der Math.* Vol. 1. (1955) pp. 252—263.

[3] Chang, C. C., H. J. Keisler, *Model Theory* (Studies in Logic, Vol. 73) North-Holland Publ. Co., Amsterdam (1973).

[4] Ecsedi-Tóth, P., On the expressive power of equality-free languages, in *Zeitschrift für Math. Logik und Grundlagen der Math.* Vol. 32.

[5] Fagin, R., Generalized first order spectra and polynomial time recognizable sets, In: *Complexity of Computation* (ed. R. Karp), SIAM—AMS Proc. Vol. 7, (1974).

[6] Gács, P., L. Lovász, Some remarks on generalized spectra, *Zeitschrift für Math. Logik und Grundlagen der Math.* Vol. 23. (1977), pp. 547—554.

[7] Grätzer, G., *Universal Algebra*, Springer Verlag, New York (2nd ed.) (1979).

[8] Mostowski, A., Concerning a problem of H. Scholz, *Zeitschrift für Math. Logik und Grundlagen der Math.* Vol. 2. (1956), pp. 210—214.

[9] Scholz, H., *J. Symb. Log.* Vol. 17. (1952), p. 160.

Received Febr. 9, 1984

DEPT. OF COMPUTER SCIENCE
A. JÓZSEF UNIVERSITY
ARADI VÉRTANÚK TERE 1.
SZEGED, HUNGARY
H—6720