The Distribution of Square-free Numbers*

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Abstract In this paper, we prove, under the Riemann hypothesis, that

$$\sum_{n\leqslant X}\mu^2(n)=\frac{6}{\pi^2}X+O(X^{\frac{17}{54}+\varepsilon}).$$

Keywords: square-free number, the estimation for trigonometrical sums, Vaughan's identity.

Let
$$\Delta(X)=\sum_{n\leqslant X}\mu^2(n)-6/\pi^2X. \tag{1}$$

Montgomery and Vaughan[1] showed, under the Riemann hypothesis, that
$$\Delta(X)=O(X^{\varphi+\varepsilon}), \tag{2}$$
where $\varphi=\frac{9}{28}$. Graham[2] obtained $\varphi=\frac{8}{25}$. Using Heath-Brown's method, Baker and Pintz[3] obtained $\varphi=\frac{7}{22}$. By means of the estimation for dual trigonometrical sums, Jia Chao-hua[4] got $\varphi=\frac{7}{22}$ independently. Refer to the introduction in [5].

In this paper, developing the methods in [4] further and combining Heath-Brown's method, we get

Theorem. *Assume Riemann hypothesis holds. Then*
$$\Delta(X)=O(X^{\frac{17}{54}+\varepsilon}).$$

Throughout this paper, we assume that X is sufficiently large, ε is a sufficiently small positive constant, $\delta=\varepsilon^2$ and that $c, c_1, c_2, \cdots$ are positive constants. which have different values in different places. $m\sim M$ denotes that there are positive constants c_1 and c_2 such that $c_1M<m\leqslant c_2M$.

1 Some Preliminary Lemmas

Lemma 1. *When* $N<x\leqslant cN$, $f''(x)$ *is continuous. Assume* $0<c_1\lambda_1\leqslant|f'(x)|\leqslant c_2\lambda_1$, $\frac{c_1\lambda_1}{N}\leqslant|f''(x)|\leqslant\frac{c_2\lambda_1}{N}$, $N\leqslant a<b\leqslant cN$. *Then*
$$\sum_{a<n\leqslant b}e(f(n))\ll\lambda_1^{\frac{1}{2}}N^{\frac{1}{2}}+\lambda_1^{-1}.$$

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If $c_2\lambda_1 \leqslant \frac{1}{2}$, *then*

$$\sum_{a<n\leqslant b} e(f(n)) \ll \lambda_1^{-1}.$$

Lemma 2. *Assume* $a(n)=0(1)$, $0<L\leqslant M<N\leqslant cL$, $L\gg 1, T\geqslant 1$. *Then*

$$\sum_{M<n\leqslant N} a(n) = \frac{1}{2\pi i}\int_{-T}^{T}\sum_{L<l\leqslant cL}\frac{a(l)}{l^{it}}\cdot\frac{N^{it}-M^{it}}{t}\,dt + O\left(\min\left(1,\frac{L}{T\|M\|}\right)\right)$$
$$+ O\left(\min\left(1,\frac{L}{T\|N\|}\right)\right) + O\left(\frac{L\log(1+L)}{T}\right),$$

where $\|x\|$ *denotes the nearest distance from* x *to integers.*

Proof. Using the method of Theorem 2 on p.98 of [6], we can get the proof.

Lemma 3. *Assume that when* $x\sim N$, $f(x)\ll P$, $f'(x)\gg\Delta$, *then*

$$\sum_{n\sim N}\min\left(D,\frac{1}{\|f(n)\|}\right)\ll (P+1)\left(D+\frac{1}{\Delta}\right)\log\left(2+\frac{1}{\Delta}\right).$$

Proof. Let m be an integer and let $E_m=\left\{n:\frac{m}{2}<f(n)\leqslant\frac{m+1}{2}\right\}$.

When $n_1, n_2\in E_m, n_1\neq n_2$, we have

$$\|f(n_1)-f(n_2)\| = |f(n_1)-f(n_2)|\gg\Delta.$$

Because $\Delta|n_1-n_2|\ll|f(n_1)-f(n_2)|\ll 1$, we know that the number of n belonging to E_m is $\ll 1+\frac{1}{\Delta}$. By Lemma 2 on p. 92 of [7], it is known that

$$\sum_{n\in E_m}\min\left(D,\frac{1}{\|f(n)\|}\right)\ll D+\frac{1}{\Delta}\log\left(2+\frac{1}{\Delta}\right).$$

$f(x)\ll P$, so the number of m for which E_m is not empty is $\ll P+1$. Thus

$$\sum_{n\sim N}\min\left(D,\frac{1}{\|f(n)\|}\right)=\sum_{m}\sum_{n\in E_m}\min\left(D,\frac{1}{\|f(n)\|}\right)$$
$$\ll (P+1)\left(D+\frac{1}{\Delta}\right)\log\left(2+\frac{1}{\Delta}\right). \qquad \text{Q.E.D.}$$

Lemma 4. *Suppose that* $\xi(n)$ *is an arbitrarily complex number and that* $1\leqslant Q\leqslant N$. *Then*

$$\left|\sum_{N<n\leqslant cN}\xi(n)\right|^2\leqslant\frac{c_1N}{Q}\sum_{0\leqslant q\leqslant Q}\left(1-\frac{q}{Q}\right)\mathrm{Re}\sum_{N<n\leqslant cN-q}\xi(n)\overline{\xi(n+q)}.$$

Lemma 5. *Suppose that* $f(x)$ *and* $\varphi(x)$ *are algebraic functions in* $[a,b]$ *and that* $\frac{1}{R}\leqslant|f''(x)|\ll\frac{1}{R}$, $|f'''(x)|\ll\frac{1}{RU}\,(U\geqslant 1)$, $|\varphi(x)|\ll\Phi$, $|\varphi'(x)|\ll U^{-1}\Phi$. $[\alpha,\beta]$ *is the image of* $[a,b]$ *under the transformation* $y=f'(x)$. *Then*

$$\sum_{a<n\leqslant b}\varphi(n)e(f(n)) = e\left(\frac{1}{8}\right)\sum_{\alpha<\gamma\leqslant\beta}\frac{\varphi(n_\gamma)}{\sqrt{f''(n_\gamma)}}e(f(n_\gamma)-\gamma n_\gamma)$$
$$+O\Big(\Phi\log(\beta-\alpha+2)+U^{-1}\Phi(b-a+R)$$

$$+ \Phi \min\left(\sqrt{R},\ \frac{1}{\|\alpha\|}\right) + \Phi \min\left(\sqrt{R},\ \frac{1}{\|\beta\|}\right)\Big),$$

where n_γ is the solution of $f'(x) = \gamma$.

Proof. Take $U_1 = U$ in Theorem 2.2 on p.36 of [8]. Let $b_\gamma = 1$ when $\gamma = \alpha$ or β. The errors emerging can be combined into the error term $O\left(\Phi \min\left(\sqrt{R}, \frac{1}{\|\alpha\|}\right) + \Phi \min\left(\sqrt{R},\ \frac{1}{\|\beta\|}\right)\right)$. Q.E.D.

Lemma 6. *Suppose that when $x \sim N$, $|f^{(k)}(x)| \sim \frac{\lambda_1}{N^{k-1}}$ $(k = 1, 2, \cdots)$. Then*

$$\sum_{n \sim N} e(f(n)) \ll \lambda_1^{\frac{11}{53}} N^{\frac{33}{53}} + \lambda_1^{-1}.$$

Proof. Because $\left(\frac{11}{53}, \frac{33}{53}\right)$ $(= BABA^2BA^2B(0, 1))$ is an exponent pair, by the theory of exponent pairs, the conclusion follows. Q.E.D.

Lemma 7 (Vaughan's Identity). *Assume $1 \leqslant U$, $V < Z$. Then*

$$\sum_{Z<n\leqslant 2Z} \mu(n)f(n) = \sum_{V<m\leqslant \frac{2Z}{U}} \mu(m) \sum_{\frac{Z}{m}<n\leqslant \frac{2Z}{m}} b(n)f(mn) - \sum_{k\leqslant UV} c(k) \sum_{\frac{Z}{k}<r\leqslant \frac{2Z}{k}} f(kr),$$

where

$$b(n) = \sum_{\substack{d|n \\ d>U}} \mu(d), c(k) = \sum_{\substack{dn=k \\ d\leqslant U, n\leqslant V}} \mu(d)\mu(n).$$

Lemma 8. *Suppose that $D(x, y, z)$ has second-order partial derivatives. Then*

$$\begin{aligned} &D(m,r,s) - D(m,0,s) - D(m,r,0) + D(m,0,0) \\ &= rs\frac{\partial^2}{\partial r\,\partial s} D(m,\xi,\eta), \end{aligned}$$

where $0 < \xi < r$, $0 < \eta < s$.

Proof. Let $E(s) = D(m, r, s) - D(m, 0, s)$.

Using Lagrange's theorem twice, we get

$$\begin{aligned} &D(m,r,s) - D(m,0,s) - D(m,r,0) + D(m,0,0) \\ &= E(s) - E(0) = E'(\eta)s \\ &= \left(\frac{\partial}{\partial s} D(m,r,\eta) - \frac{\partial}{\partial s} D(m,0,\eta)\right)s \\ &= rs\frac{\partial^2}{\partial r\,\partial s} D(m,\xi,\eta). \end{aligned}$$

Q.E.D.

2 Transformation of the Problem

Lemma 9. *Assume Riemann hypothesis holds. Then*

$$\Delta(X) = -\sum_{n\leqslant Y}\mu(n)\left(\left(\frac{X}{n^2}\right)\right) + O(X^{\frac{1}{2}+\varepsilon}Y^{-\frac{1}{2}} + Y^{\frac{1}{2}+\varepsilon}),$$

where $((\theta)) = \theta - [\theta] - \frac{1}{2}$.

See Theorem 1 in [1].

Taking $Y = X^{\frac{10}{27}}$, we want to prove, when

$$X^{\frac{17}{54}+\varepsilon} \leqslant Z \leqslant X^{\frac{10}{27}}, \tag{3}$$

$$\sum_{Z<n\leqslant 2Z}\mu(n)\left(\left(\frac{X}{n^2}\right)\right) = O(X^{\frac{17}{54}+10\delta}). \tag{4}$$

Let

$$H = ZX^{-\frac{17}{54}}. \tag{5}$$

By (8) in [3],

$$\sum_{Z<n\leqslant 2Z}\mu(n)\left(\left(\frac{X}{n^2}\right)\right) = -\frac{1}{2\pi i}\sum_{0<|h|\leqslant H}\frac{1}{h}\sum_{Z<n\leqslant 2Z}\mu(n)e\left(\frac{hX}{n^2}\right) + O\left(\sum_{h=-\infty}^{\infty}a(h)\sum_{Z<n\leqslant 2Z}e\left(\frac{hX}{n^2}\right)\right),$$

where $a(h) = O\left(\min\left(\frac{\log X}{H}, \frac{H}{h^2}\right)\right)$.

By Lemma 1, we know that

$$\sum_{h=-\infty}^{\infty}a(h)\sum_{Z<n\leqslant 2Z}e\left(\frac{hX}{n^2}\right) \ll \frac{Z\log X}{H} + \sum_{h=1}^{\infty}\min\left(\frac{\log X}{H}, \frac{H}{h^2}\right)\left(\frac{(hX)^{\frac{1}{2}}}{Z} + \frac{Z^3}{hX}\right) \ll X^{\frac{17}{54}+10\delta}.$$

In order to prove that $\Delta(X) = O(X^{\frac{17}{54}+\varepsilon})$, we need only to prove that, when

$$\frac{1}{2} < J \leqslant ZX^{-\frac{17}{54}}, \tag{6}$$

$$\sum_{J<h\leqslant 2J}\frac{1}{h}\sum_{Z<n\leqslant 2Z}\mu(n)e\left(\frac{hX}{n^2}\right) \ll X^{\frac{17}{54}+9\delta}. \tag{7}$$

3 Estimation for Trigonometrical Sums

3.1 *Estimation for Trigonometrical Sums* (1)

In the following we always assume

$$Z = 2MN. \tag{8}$$

Lemma 10. *Suppose that $a(n)$, $b(m)=O(1)$ and that (3) and (6) hold. When*

$$Z^3X^{-8/9}\ll N\ll JX^{\frac{17}{27}}Z^{-1}, \tag{9}$$

we have

$$\sum_{J<h\leqslant 2J}\frac{1}{h}\sum_{N<n\leqslant 2N}\sum_{M<m\leqslant 4M}a(n)b(m)e\left(\frac{hX}{(mn)^2}\right)\ll X^{\frac{17}{54}+6\delta}.$$

Proof. We adopt the discussion in [3] and get the proof.

Lemma 11. *Suppose that $X\leqslant W\ll ZX^{\frac{37}{54}}$ and that $a(n)=O(1)$. Suppose that (3) holds. When*

$$N\ll Z^3X^{-8/9}, \tag{10}$$

we have

$$\sum\nolimits_1=\sum_{N<n\leqslant 2N}a(n)\sum_{\frac{Z}{n}<m\leqslant\frac{2Z}{n}}e\left(\frac{W}{(mn)^2}\right)\ll X^{\frac{17}{54}+6\delta}.$$

Proof. When $N\leqslant ZX^{\frac{17}{54}}W^{-\frac{1}{2}}$, by Lemma 1, it is known that

$$\sum\nolimits_1\ll\sum_{N<n\leqslant 2N}\left|\sum_{\frac{Z}{n}<m\leqslant\frac{2Z}{n}}e\left(\frac{W}{(mn)^2}\right)\right|$$

$$\ll N((WM^{-3}N^{-2})^{\frac{1}{2}}M^{\frac{1}{2}}+M^3N^2W^{-1})\ll X^{\frac{17}{54}}.$$

Below we suppose that

$$N>ZX^{\frac{17}{54}}W^{-\frac{1}{2}}. \tag{11}$$

Let $Q=[Z^2X^{-\frac{17}{27}}]$. (12)

By (3) and (10), $1\leqslant Q\leqslant ZX^{-\delta}n^{-1}$.

$$\sum\nolimits_1\ll\left(N\sum_{N<n\leqslant 2N}\left|\sum_{\frac{Z}{n}<m\leqslant\frac{2Z}{n}}e\left(\frac{W}{(mn)^2}\right)\right|^2\right)^{\frac{1}{2}}.$$

By Lemma 4,

$$\left|\sum_{\frac{Z}{n}<m\leqslant\frac{2Z}{n}}e\left(\frac{W}{(mn)^2}\right)\right|^2$$

$$\leqslant\frac{cZ}{nQ}\sum_{q=0}^{Q}\left(1-\frac{q}{Q}\right)\mathrm{Re}\sum_{\frac{Z}{n}<m\leqslant\frac{2Z}{n}-q}e\left(\frac{W}{n^2}\left(\frac{1}{m^2}-\frac{1}{(m+q)^2}\right)\right).$$

We need only to prove that

$$\sum\nolimits_2=N\sum_{q=1}^{Q}\left|\sum_{N<n\leqslant 2N}\frac{1}{n}\sum_{\frac{Z}{n}<m\leqslant\frac{2Z}{n}-q}e\left(\frac{W}{n^2}\left(\frac{1}{m^2}-\frac{1}{(m+q)^2}\right)\right)\right|\ll ZX^{10\delta}.$$

$$\sum\nolimits_3=\sum_{N<n\leqslant 2N}\frac{1}{n}\sum_{\frac{Z}{n}<m\leqslant\frac{2Z}{n}-q}e\left(\frac{W}{n^2}\left(\frac{1}{m^2}-\frac{1}{(m+q)^2}\right)\right).$$

When $N \leqslant Z^{\frac{1}{2}}$, if $qWN^2Z^{-4} \leqslant \frac{1}{50}$, by Lemma 1, we know that

$$\sum\nolimits_3 \ll Z^4q^{-1}W^{-1}N^{-2}.$$

By (11),
$$N\sum_{q=1}^{Q} Z^4q^{-1}W^{-1}N^{-2} \ll Z.$$

We can suppose that

$$qWN^2Z^{-4} > \frac{1}{50}. \tag{13}$$

In Lemma 5, let $f(m, n) = \frac{W}{n^2}\left(\frac{1}{m^2} - \frac{1}{(m+q)^2}\right)$,

$$\frac{\partial f}{\partial m}(g(n,\gamma),n) = \gamma,\ \phi(n,\gamma) = f(g(n,\gamma), n) - \gamma g(n, \gamma).$$

So

$$\sum\nolimits_3 = \sum_{N<n\leqslant 2N} \frac{1}{n}\left(e\left(\frac{1}{8}\right)\sum_{\mu_1(n)<\gamma\leqslant\mu_2(n)} \frac{e(\phi(n,\gamma))}{\sqrt{\frac{\partial^2 f}{\partial m^2}(g(n,\gamma), n)}}\right.$$
$$+ O\left(X^\delta + M^{-1}\left(M + \frac{M^5N^2}{qW}\right) + \min\left(\frac{M^{\frac{5}{2}}N}{\sqrt{qW}}, \frac{1}{\|\mu_1(n)\|}\right)\right.$$
$$\left.\left.+ \min\left(\frac{M^{\frac{5}{2}}N}{\sqrt{qW}}, \frac{1}{\|\mu_2(n)\|}\right)\right)\right),$$

where $\mu_1(n) = 2Wn(Z(+qn)^{-3} - Z^{-3})$, $\mu_2(n) = 2Wn((2Z)^{-3} - (2Z-qn)^{-3})$; $\mu_1(n)$, $\mu_2(n)$ is monotonic function in n; $\mu_1(n), \mu_2(n) \sim -qWN^2Z^{-4}$; $\mu_1 = \min \mu_1(n)$, $\mu_2 = \max \mu_2(n)$.

By Lemma 3,

$$\sum_{N<n\leqslant 2N} \frac{1}{n}\min\left(\frac{M^{\frac{5}{2}}N}{\sqrt{qW}}, \frac{1}{\|\mu_1(n)\|}\right) \ll X^\delta + (qW)^{\frac{1}{2}}N^{-\frac{1}{2}}Z^{-\frac{3}{2}}X^\delta,$$
$$\sum_{N<n\leqslant 2N} \frac{1}{n}\min\left(\frac{M^{\frac{5}{2}}N}{\sqrt{qW}}, \frac{1}{\|\mu_2(n)\|}\right) \ll X^\delta + (qW)^{\frac{1}{2}}N^{-\frac{1}{2}}Z^{-\frac{3}{2}}X^\delta.$$
$$\sum\nolimits_3 \ll \sum_{\mu_1<\gamma\leqslant\mu_2} \left|\sum_{n\in I_\gamma} \frac{e(\phi(n))}{n\sqrt{G(n)}}\right| + X^\delta + \frac{M^4N^2}{qW} + \frac{(qW)^{\frac{1}{2}}X^\delta}{N^{\frac{1}{2}}Z^{3/2}},$$

where I_γ is a subinterval of $(N, 2N]$, $G(n) = \frac{\partial^2 f}{\partial m^2}(g(n,\gamma),n)$, $\phi(n) = \phi(n, \gamma)$, $g(n) = g(n, \gamma)$.

$$N\sum_{q=1}^{Q}(X^\delta + M^4N^2q^{-1}W^{-1} + (qW)^{\frac{1}{2}}N^{-\frac{1}{2}}Z^{-\frac{3}{2}}X^\delta) \ll Z.$$

$$I_i(m) = \int_0^1 \frac{dt}{(m+qt)^{2+i}} = \frac{1}{m^{i+j}}\left(1 + O\left(\frac{Q}{M}\right)\right).$$

In $\frac{\partial f}{\partial m}(g(n),n)=\gamma$, differentiating for n, we can get

$$g'(n)=-\frac{1}{2n}\cdot\frac{I_2(g(n))}{I_3(g(n))}=-\frac{g(n)}{2n}\left(1+O\left(\frac{Q}{M}\right)\right),$$

$$G'(n)=-\frac{48qW}{n^3}I_3(g(n))-\frac{120qW}{n^2}I_4(g(n))g'(n)$$

$$=\frac{12qW}{n^3g^5(n)}\left(1+O\left(\frac{Q}{M}\right)\right).$$

Hence, $G(n)$ is monotonous and $G(n)\sim qWM^{-5}N^{-2}$.

$$\phi'(n)=\frac{\partial f}{\partial n}(g(n),n)+\left(\frac{\partial f}{\partial m}(g(n),n)-\gamma\right)g'(n)$$

$$=\frac{\partial f}{\partial n}(g(n),n)\sim-\frac{qW}{Z^3},$$

$$\phi''(n)=\frac{6qW}{n^4g^3(n)}\left(1+O\left(\frac{Q}{M}\right)\right)\sim\frac{qW}{Z^3N}.$$

By Lemma 1 and Abel summation,

$$\sum_{\mu_1<\gamma\leqslant\mu_2}\left|\sum_{n\in I_\gamma}\frac{e(\phi(n))}{n\sqrt{G(n)}}\right|$$

$$\ll qWZ^{-3}+(qW)^{-\frac{1}{2}}Z^{\frac{3}{2}}N^{-\frac{1}{2}}.$$

$$N\sum_{q=1}^{Q}(qWZ^{-3}+Z^{\frac{3}{2}}(qW)^{-\frac{1}{2}}N^{-\frac{1}{2}})\ll Z.$$

We have proved that, when $N\leqslant Z^{\frac{1}{2}}$, $\sum_1\ll X^{\frac{17}{54}+6\delta}$.

Below we suppose that

$$N>Z^{\frac{1}{2}}. \tag{14}$$

$$\sum\nolimits_3=\sum_{M<m\leqslant 4M-q}\ \sum_{\substack{N<n\leqslant 2N\\ \frac{Z}{m}<n\leqslant\frac{2Z}{m+q}}}\frac{1}{n}e\left(\frac{W}{n^2}\left(\frac{1}{m^2}-\frac{1}{(m+q)^2}\right)\right)$$

$$=\left(\sum_{M<m\leqslant 2M-q}\ \sum_{\frac{Z}{m}<n\leqslant 2N}+\sum_{2M-q<m\leqslant 2M}\ \sum_{\frac{Z}{m}<n\leqslant\frac{2Z}{m+q}}\right.$$

$$\left.+\sum_{2M<m\leqslant 4M-q}\ \sum_{N<n\leqslant\frac{2Z}{m+q}}\right)\frac{1}{n}e\left(\frac{W}{n^2}\left(\frac{1}{m^2}-\frac{1}{(m+q)^2}\right)\right).$$

We only estimate

$$\sum\nolimits_4=\sum_{M<m\leqslant 2M-q}\ \sum_{\frac{Z}{m}<n\leqslant 2N}\frac{1}{n}e\left(\frac{W}{n^2}\left(\frac{1}{m^2}-\frac{1}{(m+q)^2}\right)\right).$$

The other two sums can be estimated in the same way.

If $qWZ^{-3} \leqslant \frac{1}{8}$, by Lemma 1 and Abel summation, it is known that

$$\sum\nolimits_4 \ll Z^4 q^{-1} W^{-1} N^{-2},$$

$$N \sum_{q=1}^{Q} Z^4 q^{-1} W^{-1} N^{-2} \ll Z^4 W^{-1} N^{-1} X^{\delta} \ll Z.$$

We can suppose that

$$qWZ^{-3} \geqslant \frac{1}{8}. \tag{15}$$

By Lemma 5,

$$\begin{aligned}\sum\nolimits_4 = \sum_{M<m\leqslant 2M-q} \Bigg(& c_1\, e\left(\frac{1}{8}\right) \sum_{\mu_3(m)<\gamma\leqslant\mu_4(m)} \frac{e\left(c_2 \gamma^{\frac{2}{3}} W^{\frac{1}{3}} \left(\frac{1}{m^2} - \frac{1}{(m+q)^2}\right)^{\frac{1}{3}}\right)}{\gamma^{\frac{1}{3}} W^{\frac{1}{6}} \left(\frac{1}{m^2} - \frac{1}{(m+q)^2}\right)^{\frac{4}{6}}} \\ & + O\Bigg(\frac{X^{\delta}}{N} + \frac{1}{N} \cdot \frac{1}{N}\left(N + \frac{N^4}{W\left(\frac{1}{m^2} - \frac{1}{(m+q)^2}\right)}\right) \\ & + \frac{1}{N} \min\left(\frac{N^2}{\sqrt{W\left(\frac{1}{m^2} - \frac{1}{(m+q)^2}\right)}}, \frac{1}{\|\mu_3(m)\|}\right) \\ & + \frac{1}{N} \min\left(\frac{N^2}{\sqrt{W\left(\frac{1}{m^2} - \frac{1}{(m+q)^2}\right)}}, \frac{1}{\|\mu_4(m)\|}\right)\Bigg)\Bigg),\end{aligned}$$

where $\mu_3(m) = \frac{1}{4} WN^{-3}(m^{-2} - (m+q)^{-2})$, $\mu_4(m) = 2Wm^3Z^{-3}(m^{-2} - (m+q)^{-2})$; $\mu_3(m)$ and $\mu_4(m)$ are monotonous functions in m; $\mu_3(m)$, $\mu_4(m) \sim qWZ^{-3}$; $\mu_3 = \min \mu_3(m)$, $\mu_4 = \max \mu_4(m)$.

By Lemma 3,

$$\sum_{M<m\leqslant 2M-q} \min\left(\frac{N^2}{\sqrt{W\left(\frac{1}{m^2} - \frac{1}{(m+q)^2}\right)}}, \frac{1}{\|\mu_3(m)\|}\right) \ll (qW)^{\frac{1}{2}} M^{-\frac{3}{2}} N^{-1} X^{\delta} + MX^{\delta},$$

$$\sum_{M<m\leqslant 2M-q} \min\left(\frac{N^2}{\sqrt{W\left(\frac{1}{m^2} - \frac{1}{(m+q)^2}\right)}}, \frac{1}{\|\mu_4(m)\|}\right) \ll (qW)^{\frac{1}{2}} M^{-\frac{3}{2}} N^{-1} X^{\delta} + M^2 q^{-1} X^{\delta}.$$

$$\sum\nolimits_4 \ll \sum_{\mu_3<\gamma\leqslant\mu_4} \frac{1}{\gamma^{\frac{1}{3}} W^{\frac{1}{6}}} \left| \sum_{m\in I_\gamma} \frac{e\left(c_2 \gamma^{\frac{2}{3}} W^{\frac{1}{3}} \left(\frac{1}{m^2} - \frac{1}{(m+q)^2}\right)^{\frac{1}{3}}\right)}{\left(\frac{1}{m^2} - \frac{1}{(m+q)^2}\right)^{\frac{1}{6}}} \right|$$

$$+ (qW)^{\frac{1}{2}}M^{-\frac{3}{2}}N^{-2}X^{\delta} + M^2q^{-1}N^{-1}X^{\delta},$$

where I_γ is a subinterval of $(M, 2M-q]$.

$$N\sum_{q=1}^{Q}((qW)^{\frac{1}{2}}M^{-\frac{3}{2}}N^{-2}X^{\delta} + M^2q^{-1}N^{-1}X^{\delta}) \ll ZX^{2\delta}.$$

By Abel summation and Lemma 6, we obtain

$$\sum_{m\in I_\gamma}\frac{e\left(c_2\gamma^{\frac{2}{3}}W^{\frac{1}{3}}\left(\frac{1}{m^2}-\frac{1}{(m+q)^2}\right)^{\frac{1}{3}}\right)}{\left(\frac{1}{m^2}-\frac{1}{(m+q)^2}\right)^{\frac{1}{6}}} \ll q^{\frac{131}{318}}W^{\frac{11}{53}}M^{\frac{75}{106}}Z^{-\frac{22}{53}} + q^{-\frac{7}{6}}M^{\frac{5}{2}}Z^2W^{-1},$$

$$N\sum_{q=1}^{Q}\sum_{\mu_3<\gamma\leqslant\mu_4}W^{-\frac{1}{6}}\gamma^{-\frac{1}{3}}(q^{\frac{13}{318}}W^{\frac{11}{53}}M^{\frac{75}{106}}Z^{-\frac{22}{53}} + q^{-\frac{7}{6}}M^{\frac{5}{2}}Z^2W^{-1}) \ll Z.$$

Thus, the lemma follows. Q.E.D.

Lemma 12. *Suppose that* $\frac{1}{2}\leqslant J \ll ZX^{-\frac{17}{54}}$, $a(n) = O(1)$ *and that* (10) *holds. We have*

$$\sum_{J<h\leqslant 2J}\frac{1}{h}\sum_{N<n\leqslant 2N}a(n)\sum_{\frac{Z}{n}<m\leqslant\frac{2Z}{n}}e\left(\frac{hX}{(mn)^2}\right) \ll X^{\frac{17}{54}+6\delta}.$$

3.2 *Estimation for* $\max\left(\frac{1}{2}, \frac{1}{2}Z^3X^{-\frac{29}{27}}\right)\leqslant J\leqslant ZX^{-\frac{17}{54}}$

In Lemma 7, let $f(n) = \sum_{J<h\leqslant 2J}\frac{1}{h}e\left(\frac{hX}{n^2}\right)$, $V = J^{-1}Z^2X^{-\frac{17}{27}}$, $U = J^2Z^{-3}X^{\frac{34}{27}}$.

Noting that $U \gg Z^3X^{-8/9}$, $ZV^{-1}\ll JX^{\frac{17}{27}}Z^{-1}$, $UV\ll JX^{\frac{17}{27}}Z^{-1}$ and using Lemmas 2, 10 and 12, we can get (7) when $\max\left(\frac{1}{2}, \frac{1}{2}Z^3X^{-\frac{29}{27}}\right)\leqslant J\leqslant ZX^{-\frac{17}{54}}$.

3.3 *Estimation for Trigonometrical Sums* (II)

When $Z^3X^{-\frac{29}{27}}<1$, we can get the Theorem in Sec. 3.2. Below we always suppose

$$Z \geqslant X^{\frac{29}{81}} \tag{16}$$

Lemma 13. *Suppose that* $a(n)$, $b(m) = 0(1)$ *and that* (3) *and* (10) *hold. When*

$$N \gg Z^{\frac{1}{2}}, \tag{17}$$

$$N \gg WZ^2X^{-\frac{85}{54}}, \tag{18}$$

$$N \gg W^2Z^{\frac{7}{2}}X^{-\frac{85}{27}}, \tag{19}$$

$$X \leqslant W \ll Z^3X^{-\frac{2}{27}}, \tag{20}$$

we have

$$\Sigma_1 = \sum_{N<n\leqslant 2N}\sum_{M<m\leqslant 4M}a(n)b(m)e\left(\frac{W}{(mn)^2}\right) \ll X^{\frac{17}{54}+6\delta}.$$

Proof. Let $Q = [Z^2 X^{-\frac{17}{27}-2\delta}]$. (21)

By (10) and (16), $X^{10\delta} \leqslant Q \leqslant MX^{-\delta}$.

$$\Sigma_1 \ll \left(N \sum_{N<n\leqslant 2N} \left|\sum_{M<m\leqslant 4M} b(m) e\left(\frac{W}{(mn)^2}\right)\right|^2\right)^{\frac{1}{2}}.$$

By Lemma 4,

$$\left|\sum_{M<m\leqslant 4M} b(m) e\left(\frac{W}{(mn)^2}\right)\right|^2$$

$$\leqslant \frac{cM}{Q}\sum_{q=0}^{Q}\left(1-\frac{q}{Q}\right)\mathrm{Re}\sum_{M<m\leqslant 4M-q} b(m)\overline{b(m+q)}\, e\left(\frac{W}{n^2}\left(\frac{1}{m^2}-\frac{1}{(m+q)^2}\right)\right).$$

We want to prove that

$$\Sigma_2 = \sum_{q=1}^{Q}\left(1-\frac{q}{Q}\right)\sum_{M<m\leqslant 4M-q} b(m)\overline{b(m+q)} \sum_{N<n\leqslant 2N} e\left(\frac{W}{n^2}g(m,q)\right) \ll ZX^{10\delta},$$

where $g(m,q) = m^{-2} - (m+q)^{-2}$.

By Lemma 1,

$$\sum_{N<n\leqslant 2N} e\left(\frac{W}{n^2}g(m,q)\right) \ll \frac{(qW)^{\frac{1}{2}}N^{\frac{1}{2}}}{Z^{3/2}} + \frac{Z^3}{qW},$$

$$\sum_{q=1}^{Q}\left(1-\frac{q}{Q}\right)\sum_{4M-Q<m\leqslant 4M-q} b(m)\overline{b(m+q)} \sum_{N\leqslant n\leqslant 2N} e\left(\frac{W}{n^2}g(m,q)\right) \ll Z.$$

When $qWZ^{-3} \leqslant \frac{1}{64}$, by Lemma 1, we know that

$$\sum_{N<n\leqslant 2N} e\left(\frac{W}{n^2}g(m,q)\right) \ll \frac{Z^3}{qW}.$$

$$\sum_{q=1}^{Q}\sum_{M<m\leqslant 4M}\frac{Z^3}{qW} \ll Z.$$

We need only to prove, for

$$\frac{1}{128}Z^3W^{-1} < Q_1 \leqslant Q, \tag{22}$$

that

$$\Sigma_3 = \sum_{Q_1<q\leqslant 2Q_1}\left(1-\frac{q}{Q}\right)\sum_{M<m\leqslant 4M-Q} b(m)\overline{b(m+q)}$$

$$\cdot \sum_{N<n\leqslant 2N} e\left(\frac{W}{n^2}g(m,q)\right) \ll ZX^{9\delta}. \tag{23}$$

By Lemma 5,

$$\sum_{N<n\leqslant 2N} e\left(\frac{W}{n^2}g(m,q)\right) = c_1 e\left(\frac{1}{8}\right)\sum_{\frac{Wg(m,q)}{4N^3}<\gamma\leqslant\frac{2Wg(m,q)}{N^3}} \frac{(Wg(m,q))^{\frac{1}{6}}}{\gamma^{2/3}} e\left(c_2\gamma^{\frac{2}{3}}W^{\frac{1}{3}}g^{\frac{1}{3}}(m,q)\right)$$

$$+ O\left(X^{\delta} + \frac{1}{N}\left(N + \frac{N^4}{Wg(m, q)}\right) + \min\left(\frac{N^2}{\sqrt{Wg(m, q)}}, \frac{1}{\left\|\frac{Wg(m, q)}{4N^3}\right\|}\right)\right.$$

$$\left. + \min\left(\frac{N^2}{\sqrt{Wg(m, q)}}, \frac{1}{\left\|\frac{2Wg(m, q)}{N^3}\right\|}\right)\right).$$

$$\sum_{Q_1 < q \leqslant 2Q_1} \sum_{M < m \leqslant 4M} \left(X^{\delta} + \frac{N^3}{Wg(m, q)}\right) \ll Z.$$

By Lemma 3,

$$\sum_{Q_1 < q \leqslant 2Q_1} \sum_{M < m \leqslant 4M} \min\left(\frac{N^2}{\sqrt{Wg(m, q)}}, \frac{1}{\left\|\frac{2Wg(m, q)}{N^3}\right\|}\right) \ll Z.$$

$$\Sigma_3 = c_1 e\left(\frac{1}{8}\right) \sum_{Q_1 < q \leqslant 2Q_1} \left(1 - \frac{q}{Q}\right) \sum_{M < m \leqslant 4M - Q} b(m)\overline{b(m+q)}$$

$$\cdot \sum_{\frac{Wg(m,q)}{4N^3} < \gamma \leqslant \frac{2Wg(m,q)}{N^3}} \frac{(Wg(m, q))^{\frac{1}{6}}}{\gamma^{2/3}} e(c_2 \gamma^{\frac{2}{3}} W^{\frac{1}{3}} g^{\frac{1}{3}}(m,q)) + O(Z). \tag{24}$$

Because

$$0 < \frac{4qW}{m^3 N^3} - \frac{2Wg(m, q)}{N^3} < \frac{8q^2 W}{M^4 N^3} < X^{-\delta}, \tag{25}$$

$$\sum_{\frac{2Wg(m,q)}{N^3} < \gamma \leqslant \frac{4qW}{m^3 N^3}} 1 \leqslant \begin{cases} 1\left(\leqslant \frac{8q^2W}{M^4N^3}\left\|\frac{2Wg(m, q)}{N^3}\right\|^{-1}\right) & \text{when } \left\|\frac{2Wg(m, q)}{N^3}\right\| < \frac{8q^2W}{M^4N^3}, \\ 0 & \text{when } \left\|\frac{2Wg(m, q)}{N^3}\right\| \geqslant \frac{8q^2W}{M^4N^3} \end{cases}$$

$$\leqslant \min\left\{1, \frac{8q^2W}{M^4N^3}\left\|\frac{2Wg(m, q)}{N^3}\right\|^{-1}\right\}.$$

By Lemma 3,

$$\sum_{M < m \leqslant 4M} \sum_{\frac{2Wg(m,q)}{N^3} < \gamma \leqslant \frac{4qW}{m^3 N^3}} 1$$

$$\ll \sum_{M < m \leqslant 4M} \min\left\{1, \frac{q^2W}{M^4N^3}\left\|\frac{2Wg(m, q)}{N^3}\right\|^{-1}\right\}$$

$$\ll q^2 W Z^{-3} X^{\delta}.$$

$$\sum_{Q_1 < q \leqslant 2Q_1} \left(1 - \frac{q}{Q}\right) \sum_{M < m \leqslant 4M - Q} b(m)\overline{b(m+q)}$$

$$\cdot \sum_{\frac{2Wg(m,q)}{N^3} < \gamma \leqslant \frac{4qW}{m^3 N^3}} \frac{(Wg(m, q))^{\frac{1}{6}}}{\gamma^{2/3}} e(c_2 \gamma^{2/3} W^{\frac{1}{3}} g^{\frac{1}{3}}(m,q))$$

$$\ll \sum_{Q_1 < q \leqslant 2Q_1} \left(\sum_{M < m \leqslant 4M} \sum_{\frac{2Wg(m,q)}{N^3} < \gamma \leqslant \frac{4qW}{m^3 N^3}} 1\right) \frac{Z^2}{(qW)^{\frac{1}{2}} M^{\frac{1}{2}}} \ll Z.$$

$$\Sigma_3 = c_1 e\left(\frac{1}{8}\right)\sum_{Q_1<q\leqslant 2Q_1}\left(1-\frac{q}{Q}\right)\sum_{M<m\leqslant 4M-Q} b(m)\overline{b(m+q)}$$

$$\cdot \sum_{\frac{qW}{2m^3N^3}<\gamma\leqslant\frac{4qW}{m^3N^3}} \frac{(Wg(m,q))^{\frac{1}{6}}}{\gamma^{2/3}} e(c_2\gamma^{\frac{2}{3}}W^{\frac{1}{3}}g^{\frac{1}{3}})(m,q)) + O(Z). \qquad (26)$$

By Lemma 2, we need only to prove that

$$\Sigma_4 = \sum_{Q_1<q\leqslant 2Q_1}\left(1-\frac{q}{Q}\right)\sum_{M<m\leqslant 4M-Q} b(m)\overline{b(m+q)}\left(\frac{q}{m^3}\right)^{it}(Wg(m,q))^{\frac{1}{6}}$$

$$\cdot \sum_{\frac{Q_1W}{16Z^3}<\gamma\leqslant\frac{64Q_1W}{Z^3}} \frac{e(c_2\gamma^{2/3}W^{\frac{1}{3}}g^{\frac{1}{3}}(m,q))}{\gamma^{2/3+it}} \ll ZX^{8\delta}. \qquad (27)$$

$$\Sigma_4 \ll \sum_{\frac{Q_1W}{16Z^3}<\gamma\leqslant\frac{64Q_1W}{Z^3}} \frac{W^{\frac{1}{6}}}{\gamma^{2/3}} \sum_{M<m\leqslant 4M-Q}\left|\sum_{Q_1<q\leqslant 2Q_1} j(q)g^{\frac{1}{6}}(m,q)\right.$$

$$\left.\cdot\,\overline{b(m+q)}\,e(c_2\gamma^{2/3}W^{\frac{1}{3}}g^{\frac{1}{3}}(m,q))\right|,$$

where $j(q) = (1-qQ^{-1})q^{it}$.

$$\sum_{M<m\leqslant 4M-Q}\left|\sum_{Q_1<q\leqslant 2Q_1} j(q)g^{\frac{1}{6}}(m,q)\overline{b(m+q)}\,e(c_2\gamma^{2/3}W^{\frac{1}{3}}g^{\frac{1}{3}}(m,q))\right|$$

$$\ll\left(M\sum_{M<m\leqslant 4M-Q}\left|\sum_{Q_1<q\leqslant 2Q_1} j(q)g^{\frac{1}{6}}(m,q)\,\overline{b(m+q)}\,e(c_2\gamma^{2/3}W^{\frac{1}{3}}g^{\frac{1}{3}}(m,q))\right|^2\right)^{\frac{1}{2}}.$$

We need to prove that, for

$$\frac{1}{16}Q_1WZ^{-3} < \gamma \leqslant 64Q_1WZ^{-3}, \qquad (28)$$

$$\Sigma_5 = \sum_{M<m\leqslant 4M-Q}\left|\sum_{Q_1<q\leqslant 2Q_1} j(q)g^{\frac{1}{6}}(m,q)\,\overline{b(m+q)}\,e(c_2\gamma^{2/3}W^{\frac{1}{3}}g^{\frac{1}{3}}(m,q))\right|^2$$

$$\ll Z^4M^{-1}W^{-1}Q_1^{-\frac{2}{3}}X^{6\delta}. \qquad (29)$$

Let $R = [\max(1, MWQ_1^3Z^{-4})]$. (30)

By (17), (20) and (21), $1\leqslant R\leqslant Q_1X^{-\delta}$. By Lemma 4,

$$\Sigma_5 \ll \frac{Q_1}{R}\sum_{r=1}^{R}\left|\sum_{Q_1<q\leqslant 2Q_1-r} j(q)\overline{j(q+r)}\sum_{M<m\leqslant 4M-Q} g^{\frac{1}{6}}(m,q)g^{\frac{1}{6}}(m,q+r)\overline{b(m+q)}\right.$$

$$\left.\cdot\, b(m+q+r)e(c_2\gamma^{2/3}W^{\frac{1}{3}}(g^{\frac{1}{3}}(m,q)-g^{\frac{1}{3}}(m,q+r)))\right| + O(Z^4M^{-1}W^{-1}Q_1^{-\frac{2}{3}}).$$

$$\sum_{Q_1<q\leqslant 2Q_1-r} j(q)\overline{j(q+r)}\sum_{M<m\leqslant 4M-Q} g^{\frac{1}{6}}(m,q)g^{\frac{1}{6}}(m,q+r)\overline{b(m+q)}$$

$$\cdot\, b(m+q+r)e(c_2\gamma^{2/3}W^{\frac{1}{3}}(g^{\frac{1}{3}}(m,q)-g^{\frac{1}{3}}(m,q+r)))$$

$$= \sum_{Q_1<q\leqslant 2Q_1-r} j(q)\overline{j(q+r)}\sum_{M+q<l\leqslant 4M+q-Q} g^{\frac{1}{6}}(l-q,q)g^{\frac{1}{6}}(l-q,q+r)$$

$$\cdot\,\overline{b(l)}\,b(l+r)e(c_2\gamma^{2/3}W^{\frac{1}{3}}(g^{\frac{1}{3}}(l-q,q)-g^{\frac{1}{3}}(l-q,q+r)))$$

$$= \sum_{Q_1<q\leqslant 2Q_1-r} j(q)\overline{j(q+r)} \sum_{M<m\leqslant 4M-Q} g^{\frac{1}{6}}(m-q,\ q)g^{\frac{1}{6}}(m-q,\ q+r)$$

$$\cdot \overline{b(m)}b(m+r)e(c_2\gamma^{2/3}W^{\frac{1}{3}}k(m,q,r)) + O(Q_1^{\frac{7}{3}}M^{-1}),$$

where $k(m,q,r) = g^{\frac{1}{3}}(m-q,\ q) - g^{\frac{1}{3}}(m-q,\ q+r)$.

By (20) and (21),

$$Q_1^{\frac{7}{3}}M^{-1} = O(Z^4M^{-1}W^{-1}Q_1^{-\frac{5}{3}}).$$

We need only to prove that

$$\Sigma_6 = \frac{1}{R}\sum_{r=1}^{R}\Bigg| \sum_{Q_1<q\leqslant 2Q_1-r} j(q)\overline{j(q+r)} \sum_{M<m\leqslant 4M-Q} g^{\frac{1}{6}}(m-q,q)g^{\frac{1}{6}}(m-q,\ q+r)$$

$$\cdot \overline{b(m)}b(m+r)e(c_2\gamma^{2/3}W^{\frac{1}{3}}k(m,q,r))\Bigg| \ll Z^4M^{-1}W^{-1}Q_1^{-\frac{5}{3}}X^{6\delta}. \tag{31}$$

$$\Sigma_7 = \sum_{Q_1<q\leqslant 2Q_1-r} j(q)\overline{j(q+r)} \sum_{M<m\leqslant 4M-Q} g^{\frac{1}{6}}(m-q,q)g^{\frac{1}{6}}(m-q,\ q+r)$$

$$\cdot \overline{b(m)}b(m+r)e(c_2\gamma^{2/3}W^{\frac{1}{3}}k(m,q,r))$$

$$\ll \Bigg(M \sum_{M<m\leqslant 4M-Q}\Bigg| \sum_{Q_1<q\leqslant 2Q_1-r} j(q)\overline{j(q+r)}g^{\frac{1}{6}}(m-q,\ q)g^{\frac{1}{6}}(m-q,\ q+r)$$

$$\cdot e(c_2\gamma^{2/3}W^{\frac{1}{3}}k(m,q,r))\Bigg|^2\Bigg)^{\frac{1}{2}}.$$

Let $S = [\max(1, M^2W^2Q_1^6Z^{-8})]$. (32)

By (18) and (21), $1 \leqslant S \leqslant Q_1X^{-\delta}$. By Lemma 4,

$$\Sigma_8 = \sum_{M<m\leqslant 4M-Q}\Bigg| \sum_{Q_1<q\leqslant 2Q_1-r} j(q)\overline{j(q+r)}g^{\frac{1}{6}}(m-q,q)g^{\frac{1}{6}}(m-q,\ q+r)$$

$$\cdot e(c_2\gamma^{2/3}W^{\frac{1}{3}}k(m,q,r))\Bigg|^2$$

$$\ll \frac{Q_1}{S}\sum_{s=1}^{S}\sum_{Q_1<q\leqslant 2Q_1}\Bigg| \sum_{M<m\leqslant 4M-Q} g^{\frac{1}{6}}(m-q,q)g^{\frac{1}{6}}(m-q,q+r)$$

$$\cdot g^{\frac{1}{6}}(m-q-s,\ q+s)g^{\frac{1}{6}}(m-q-s,\ q+r+s)$$

$$\cdot e(c_2\gamma^{2/3}W^{\frac{1}{3}}(k(m,q,r) - k(m,q+s,r)))\Bigg| + O(Z^8M^{-3}W^{-2}Q_1^{-\frac{10}{3}}).$$

$$\begin{aligned} &k(m,q,r) - k(m,q+s,r) \\ &= g^{\frac{1}{3}}(m-q-s,\ q+r+s) - g^{\frac{1}{3}}(m-q-s,\ q+s) \\ &\quad - g^{\frac{1}{3}}(m-q,\ q+r) + g^{\frac{1}{3}}(m-q,q), \\ &g(m-q-s,\ q+r+s) = (m-q-s)^{-2} - (m+r)^{-2}. \end{aligned}$$

$$F(m,q,r,s) = \frac{\partial}{\partial m} g^{\frac{1}{3}}(m-q-s,\ q+r+s)$$

$$= \frac{2}{3}((m-q-s)^{-2} - (m+r)^{-2})^{-\frac{2}{3}}((m+r)^{-3} - (m-q-s)^{-3}).$$

$$G(m,q,r,s)=\frac{\partial^2}{\partial m^2}g^{\frac{1}{3}}(m-q-s,\ q+r+s)$$
$$=\frac{2}{9}\left((m-q-s)^{-2}-(m+r)^{-2}\right)^{-\frac{5}{3}}(5(m-q-s)^{-6}$$
$$-9(m-q-s)^{-4}(m+r)^{-2}+8(m-q-s)^{-3}(m+r)^{-3}$$
$$-9(m-q-s)^{-2}(m+r)^{-4}+5(m+r)^{-6}).$$

$$\frac{\partial^2}{\partial r\partial s}F(m,q,r,s)=\frac{8}{27}\left((m-q-s)^{-2}-(m+r)^{-2}\right)^{-\frac{8}{3}}(m-q-s)^{-3}$$
$$\cdot(m+r)^{-3}((m+r)^{-3}-9(m-q-s)^{-1}(m+r)^{-2}$$
$$+9(m-q-s)^{-2}(m+r)^{-1}-(m-q-s)^{-3})\sim m^{-2}q^{-5/3}.$$

$$\frac{\partial^2}{\partial r\partial s}G(m,q,r,s)=\frac{16}{81}\left((m-q-s)^{-2}-(m+r)^{-2}\right)^{-\frac{11}{3}}((m-q-s)^{-9}$$
$$\cdot(m+r)^{-3}+9(m-q-s)^{-8}(m+r)^{-4}$$
$$-81(m-q-s)^{-7}(m+r)^{-5}+142(m-q-s)^{-6}(m+r)^{-6}$$
$$-81(m-q-s)^{-5}(m+r)^{-7}+9(m-q-s)^{-4}$$
$$\cdot(m+r)^{-8}+(m-q-s)^{-3}(m+r)^{-9})\sim -m^{-3}q^{-5/3}.$$

By Lemma 8,

$$\frac{\partial}{\partial m}(k(m,q,r)-k(m,q+s,r))$$
$$=F(m,q,r,s)-F(m,q,0,s)-F(m,q,r,0)+F(m,q,0,0)$$
$$=rs\frac{\partial^2}{\partial r\partial s}F(m,q,\xi,\eta)\sim rsm^{-2}q^{-5/3}.$$

$$\frac{\partial}{\partial m}(c_2\gamma^{2/3}W^{\frac{1}{3}}(k(m,q,r)-k(m,q+s,r)))\sim q^{-1}WrsM^{-2}Z^{-2}.$$

$$\frac{\partial^2}{\partial m^2}(k(m,q,r)-k(m,q+s,r))$$
$$=G(m,q,r,s)-G(m,q,0,s)-G(m,q,r,0)+G(m,q,0,0)$$
$$=rs\frac{\partial^2}{\partial r\partial s}G(m,q,\xi,\eta)\sim -rsm^{-3}q^{-5/3}.$$

$$\frac{\partial^2}{\partial m^2}(c_2\gamma^{2/3}W^{\frac{1}{3}}(k(m,q,r)-k(m,q+s,r)))\sim -q^{-1}WM^{-3}Z^{-2}rs.$$

By Lemma 1 and Abel summation, we know that

$$\sum_{M<m\leqslant 4M-Q}g^{\frac{1}{6}}(m-q,q)g^{\frac{1}{6}}(m-q,\ q+r)g^{\frac{1}{6}}(m-q-s,\ q+s)$$
$$\cdot g^{\frac{1}{6}}(m-q-s,\ q+s+r)e(c_2\gamma^{2/3}W^{\frac{1}{3}}(k(m,q,r)-k(m,q+s,r)))$$
$$\ll q^{\frac{1}{6}}W^{\frac{1}{2}}M^{-\frac{3}{2}}Z^{-1}r^{\frac{1}{2}}s^{\frac{1}{2}}+q^{5/3}Z^2W^{-1}r^{-1}s^{-1}.$$

$$\Sigma_8\ll\frac{Q_1}{S}\sum_{s=1}^{S}\sum_{Q_1<q\leqslant 2Q}(q^{\frac{1}{6}}W^{\frac{1}{2}}M^{-\frac{5}{2}}Z^{-1}r^{\frac{1}{2}}s^{\frac{1}{2}}+q^{5/3}W^{-1}Z^2r^{-1}s^{-1})+Z^8M^{-3}W^{-2}Q_1^{-\frac{10}{3}}$$

$$\ll Q_1^{\frac{13}{6}}W^{\frac{1}{2}}r^{\frac{1}{2}}S^{\frac{1}{2}}M^{-5/2}Z^{-1} + Q_1^{\frac{11}{3}}Z^2W^{-1}r^{-1}S^{-1}X^{\delta} + Z^8M^{-3}W^{-2}Q_1^{-\frac{10}{3}}.$$

$$\Sigma_7 \ll Q_1^{\frac{13}{12}}W^{\frac{1}{4}}r^{\frac{1}{4}}S^{\frac{1}{4}}M^{-\frac{3}{4}}Z^{-\frac{1}{2}} + Q_1^{\frac{11}{6}}ZM^{\frac{1}{2}}W^{-\frac{1}{2}}r^{-\frac{1}{2}}S^{-\frac{1}{2}}X^{\delta} + Z^4M^{-1}W^{-1}Q_1^{-5/3}.$$

By (19),(20),(21),(22),(30) and (32), it is known that

$$\Sigma_6 \ll Q_1^{\frac{13}{12}}W^{\frac{1}{4}}R^{\frac{1}{4}}S^{\frac{1}{4}}M^{-\frac{3}{4}}Z^{-\frac{1}{2}} + Q_1^{\frac{11}{6}}ZM^{\frac{1}{2}}W^{-\frac{1}{2}}R^{-\frac{1}{2}}S^{-\frac{1}{2}}X^{\delta} + Z^4M^{-1}W^{-1}Q_1^{-\frac{5}{3}}$$
$$\ll Z^4M^{-1}W^{-1}Q_1^{-5/3}X^{6\delta}.$$

So far, Lemma 13 is proved.

Lemma 14. *Suppose that* $a(n)$, $b(m)=0(1)$ *and that* $X^{\frac{17}{54}+\varepsilon} \leqslant Z \leqslant X^{\frac{10}{27}}$, $\frac{1}{2} < J \leqslant \frac{1}{2} Z^3X^{-\frac{29}{27}}$. *Then*

(i) *when* $N \ll Z^3X^{-8/9}$, $N \gg Z^{\frac{1}{2}}$, $N \gg JZ^2X^{-\frac{31}{54}}$, $N \gg J^2Z^{\frac{7}{2}}X^{-\frac{31}{27}}$, *we have*

$$\sum_{J<h\leqslant 2J} \frac{1}{h} \sum_{N<n\leqslant 2N} \sum_{\frac{Z}{n}<m\leqslant\frac{2Z}{n}} a(n)b(m)e\left(\frac{hX}{(mn)^2}\right) \ll X^{\frac{17}{54}+7\delta};$$

(ii) *when* $M \gg X^{8/9}Z^{-2}$, $M \ll Z^{\frac{1}{2}}$, $M \ll J^{-1}Z^{-1}X^{\frac{31}{54}}$, $M \ll J^{-2}Z^{-\frac{5}{2}}X^{\frac{31}{27}}$, *we have*

$$\sum_{J<h\leqslant 2J} \frac{1}{h} \sum_{M<m\leqslant 2M} \sum_{\frac{Z}{m}<n\leqslant\frac{2Z}{m}} a(n)b(m)e\left(\frac{hX}{(mn)^2}\right) \ll X^{\frac{17}{54}+7\delta}.$$

Proof. By Lemmas 2 and 13, the lemma is obtained.

3.4 *Estimation for* $X^{\frac{31}{54}}Z^{-\frac{3}{2}} \leqslant J \leqslant \frac{1}{2}Z^3X^{-\frac{29}{27}}$

In Lemma 7, let

$$f(n) = \sum_{J<h\leqslant 2J} \frac{1}{h} e\left(\frac{hX}{n^2}\right),\ V = J^{-1}Z^2X^{-\frac{17}{27}},\ U = J^2Z^{-3}X^{\frac{34}{27}}.$$

Noting that $U \gg \max\left(Z^{\frac{1}{2}}, JZ^2X^{-\frac{31}{54}}, J^2Z^{\frac{7}{2}}X^{-\frac{31}{27}}\right)$, $ZV^{-1} \ll JX^{\frac{17}{27}}Z^{-1}$, $UV \ll JX^{\frac{17}{27}}Z^{-1}$ and using Lemmas 10, 12 and 14, we can get (7) when $X^{\frac{31}{54}}Z^{-\frac{3}{2}} \leqslant J \leqslant \frac{1}{2}Z^3X^{-\frac{29}{27}}$.

3.5 *Estimation for* $\frac{1}{2} \leqslant J \leqslant \min\left(X^{\frac{31}{54}}Z^{-\frac{3}{2}}, \frac{1}{2}Z^3X^{-\frac{29}{27}}\right)$

In Lemma 7, let

$$f(n) = \sum_{J<h\leqslant 2J} \frac{1}{h} e\left(\frac{hX}{n^2}\right),\ U = V = J^{-1}Z^2X^{-\frac{17}{27}}.$$

Noting that $Z^{\frac{1}{2}} \gg JZ^2X^{-\frac{31}{54}}$, $Z^{\frac{1}{2}} \gg J^2Z^{\frac{7}{2}}X^{-\frac{31}{27}}$, $ZV^{-1} \ll JX^{\frac{17}{27}}Z^{-1}$ and using Lemmas 10, 12 and 14, we can get (7) when $\frac{1}{2} \leqslant J \leqslant \min\left(X^{\frac{31}{54}}Z^{-\frac{3}{2}}, \frac{1}{2}Z^3X^{-\frac{29}{27}}\right)$.

4 Conclusion

Combining the above conclusions, we can get (4). Hence, Theorem follows.

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