Logic for Indexed Languages

Highlights of Logic, Games and Automata'14

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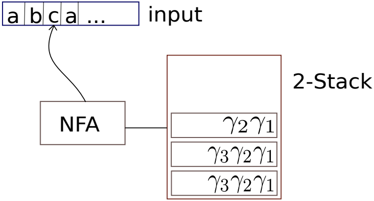
Indexed Languages ??

Definition

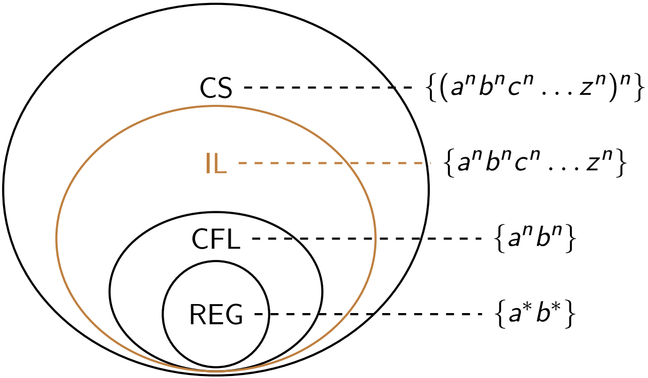
Indexed language = language recognized by a 2-PDA.

Definition

2-PDA = pushdown machine using a 2-*Stack*

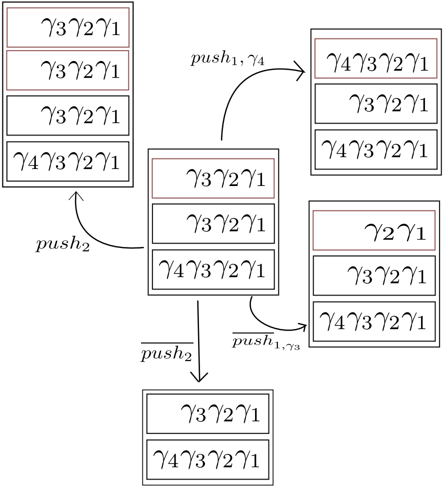


Indexed Languages ?

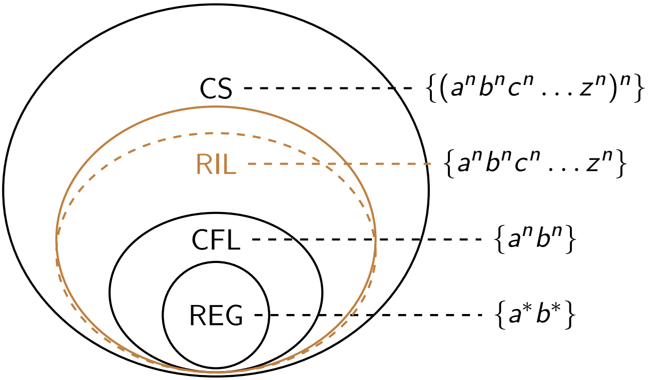


- Generalization of context-free languages
- Level 2 of the higher order languages hierarchy

2-Stack

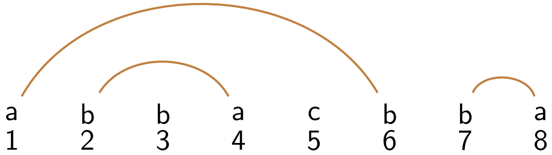


$\gamma_3\gamma_2\gamma_1$
$\gamma_3\gamma_2\gamma_1$
$\gamma_3\gamma_2\gamma_1$
$\gamma_4\gamma_3\gamma_2\gamma_1$
$push_1, \gamma_4$
$\gamma_4\gamma_3\gamma_2\gamma_1$
$\gamma_3\gamma_2\gamma_1$
$\gamma_4\gamma_3\gamma_2\gamma_1$
$\gamma_3\gamma_2\gamma_1$
$\gamma_3\gamma_2\gamma_1$
$\gamma_4\gamma_3\gamma_2\gamma_1$
$push_2$
$\gamma_2\gamma_1$
$\gamma_3\gamma_2\gamma_1$
$\gamma_4\gamma_3\gamma_2\gamma_1$
$\overline{push_2}$
$\overline{push}_{1,\gamma_3}$
$\gamma_3\gamma_2\gamma_1$
$\gamma_4\gamma_3\gamma_2\gamma_1$



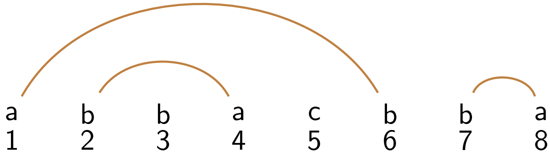
- We restrict to Realtime 2-PDA (no ε-transition)

Matching



- non-crossing pairing, eg: $M = \{(1,6),(2,4),(7,8)\}$

Matching



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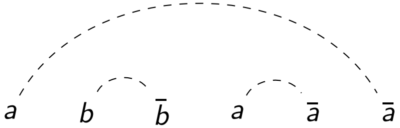
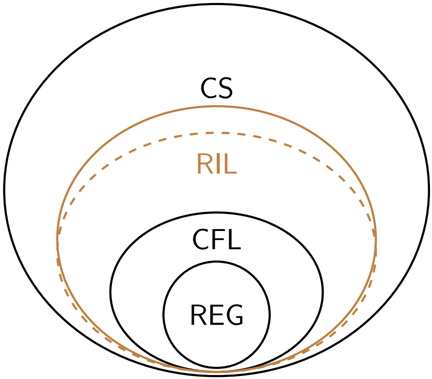
Dyck



some history of characterization



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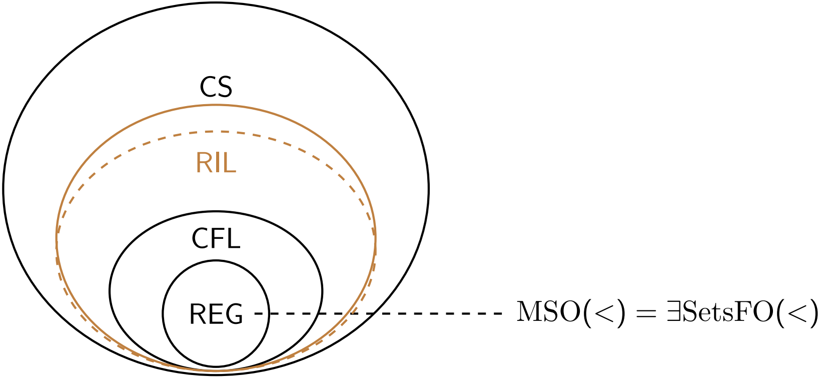

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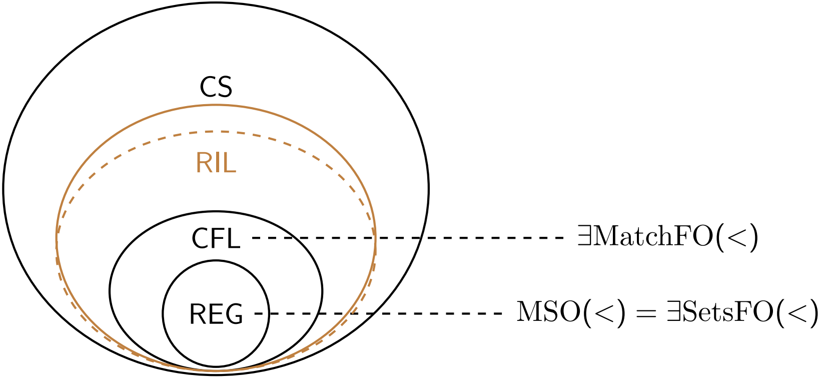


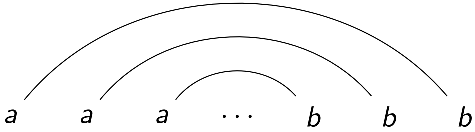
Lautemann, Schwentick & Thérien, 94:

CFL $= \exists$*MatchFO*

- *ESO* logic of the form $\exists M\phi$, ϕ : FO formula over Nested Words

Example

$\{a^n b^n, n \geq 1\}$



$$\exists x_0 : (\forall x \leq x_0 \exists y > x_0 : M(x,y) \quad \text{(Rainbow nesting)}$$
$$\wedge \forall y > x_0\, \exists x \leq x_0 : M(x,y))$$
$$\wedge \forall x, y : M(x,y) \rightarrow a(x) \wedge b(y)$$

Example

$\{a^n b^n, n \geq 1\}$

$a \quad a \quad a \quad \cdots \quad b \quad b \quad b$

$$\exists M\ \exists x_0 : (\forall x \leq x_0 \exists y > x_0 : M(x,y) \quad \text{(Rainbow nesting)}$$
$$\wedge \forall y > x_0\ \exists x \leq x_0 : M(x,y))$$
$$\wedge \forall x, y : M(x,y) \rightarrow a(x) \wedge b(y)$$

so ...

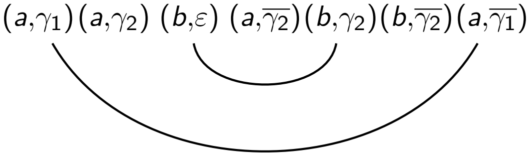
- Sets for Regulars languages
- Matching for Context-free languages
- What's for (Realtime) Indexed languages ??

Representations

Representing accepting computations of 2-PDAs:

Definition (Dyck-Nested Word)

- nested word $\langle u, M \rangle$ over an alphabet $\Sigma \times (\Gamma \cup \bar{\Gamma} \cup \{\varepsilon\})$



Representations

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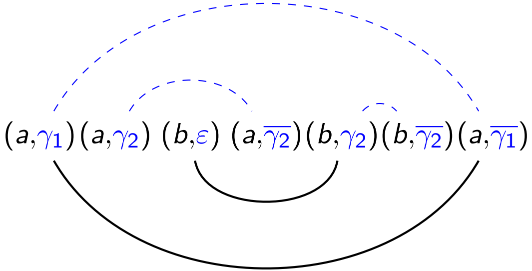
- $\pi_\Gamma(u)$ is a *Dyck*

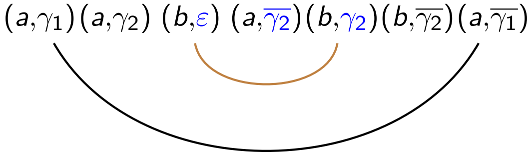


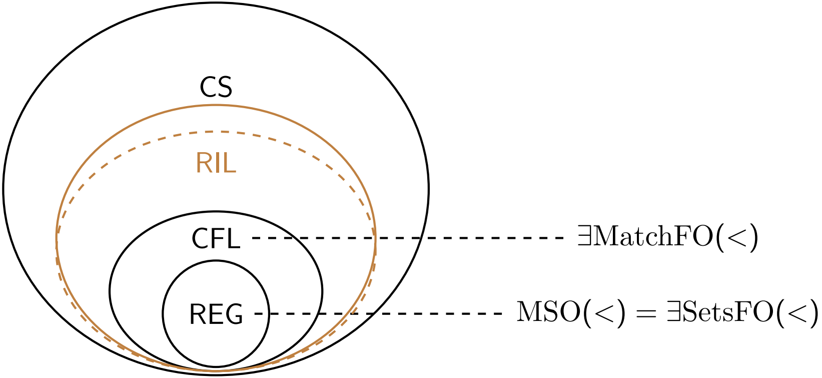
Representations

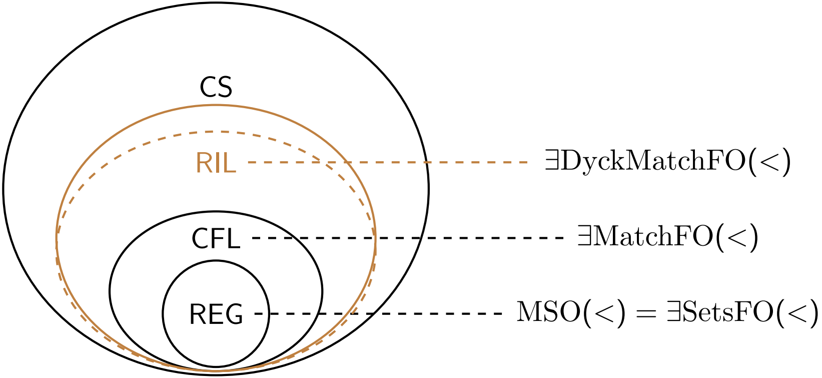
Representing accepting computations of 2-PDAs:

Definition (Dyck-Nested Word)

- nested word $\langle u, M \rangle$ over an alphabet $\Sigma \times (\Gamma \cup \bar{\Gamma} \cup \{\varepsilon\})$
- $\pi_\Gamma(u)$ is a *Dyck*
- Γ-subword within every arc $\{\gamma\bar{\gamma}, \bar{\gamma}\gamma\}$-reduces to ε




CS
RIL
CFL
REG
∃MatchFO(<)
MSO(<) = ∃SetsFO(<)



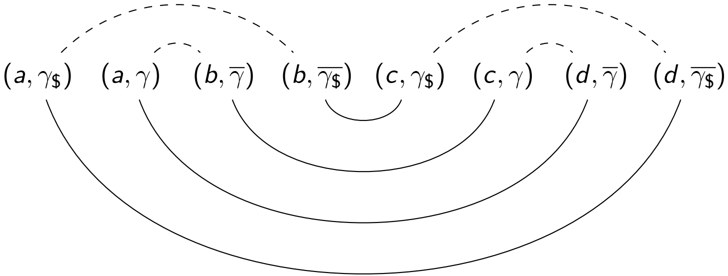
Theorem :

$RIL = \exists DyckMatchFO$

- *ESO* logic of the form $\exists(L_\Gamma, M)\phi$, ϕ : FO formula over Dyck Nested Words

Example

$\{a^n b^n c^n d^n\}$



$$\begin{aligned} &\exists x_0 : \mathrm{rainbow}(x_0) \wedge \mathrm{rainbowDycking}(\mathit{first}, x_0) \\ &\wedge \mathrm{rainbowDycking}(x_0 + 1, \mathit{last}) \\ &\wedge \forall x \leq x_0\ (\mathrm{opening}_\gamma(x) \rightarrow a(x) \wedge \mathrm{closing}_\gamma(x) \rightarrow b(x)) \\ &\wedge \forall x > x_0\ (\mathrm{opening}_\gamma(x) \rightarrow c(x) \wedge \mathrm{closing}_\gamma(x) \rightarrow d(x)) \end{aligned}$$

Example

$\{a^n b^n c^n d^n\}$

$a \quad a \quad b \quad b \quad c \quad c \quad d \quad d$

$$\exists (L_\gamma, M) \;\; \exists x_0 : \text{rainbow}(x_0) \wedge \text{rainbowDycking}(\mathit{first}, x_0)$$
$$\wedge \text{rainbowDycking}(x_0 + 1, \mathit{last})$$
$$\wedge \forall x \leq x_0 \; (\text{opening}_\gamma(x) \rightarrow a(x) \wedge \text{closing}_\gamma(x) \rightarrow b(x))$$
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proof idea ($\Rightarrow$)

context-free representation

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context-free representation

Proposition

for every RIL L $\in \Sigma^*$, there is a CFL K s.t. $L = \pi_\Sigma(K \cap Dyck)$

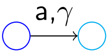
proof idea ($\Rightarrow$)

context-free representation

Proposition

for every RIL L $\in \Sigma^*$, there is a CFL K s.t. $L = \pi_\Sigma(K \cap Dyck)$

2-PDA $\rightarrow$ $CFG(\Sigma \times \Gamma)$



$\rightarrow$ $X \rightarrow (a, \gamma)YZ$

$abaa \in L(\mathcal{A}) \quad \leftrightarrow \quad S \xrightarrow{*} (a, \gamma_1)(b, \gamma_2)(a, \overline{\gamma_2})(b, \overline{\gamma_1})$

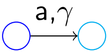
proof idea (⇒)

context-free representation

Proposition

for every RIL L $\in \Sigma^*$, there is a CFL K s.t. $L = \pi_\Sigma(K \cap Dyck)$

$$2\text{-}PDA \quad \rightarrow \quad CFG(\Sigma \times \Gamma)$$



$$\rightarrow \quad X \rightarrow (a,\gamma)YZ$$

$$abaa \in L(\mathcal{A}) \quad \leftrightarrow \quad S \xrightarrow{*} (a,\gamma_1)(b,\gamma_2)(a,\overline{\gamma_2})(b,\overline{\gamma_1})$$

$$L(\mathcal{A}) = \pi_\Sigma \quad (\quad L(G) \quad \cap \quad Dyck \quad)$$

proof idea ($\Rightarrow$)

context-free representation

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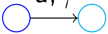
for every RIL L $\in \Sigma^*$, there is a CFL K s.t. $L = \pi_\Sigma(K \cap Dyck)$

2-PDA	$\rightarrow$	$CFG(\Sigma \times \Gamma)$
	$\rightarrow$	$X \rightarrow (a, \gamma)YZ$

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$$L(\mathcal{A}) = \pi_\Sigma \quad (\quad \underset{\substack{| \\ [\mathrm{LTS}, 94]^*}}{L(G)} \quad \cap \quad Dyck \quad)$$

proof idea (⇒)

context-free representation

Proposition

for every RIL L $\in \Sigma^*$, there is a CFL K s.t. $L = \pi_\Sigma(K \cap Dyck)$

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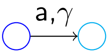

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$L(G)$: [LTS, 94]*

$Dyck$: sem. restr.(DNW)

proof idea ($\Rightarrow$)

context-free representation

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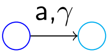

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quant. $\quad [\mathrm{LTS}, 94]^*$ $\quad$ sem. restr.(DNW)

proof idea ($\Leftarrow$)

ϕ over $\texttt{DNW}$ = Nested Word language $\cap$ DNW

Build 2-*PDA* such that:

proof idea ($\Leftarrow$)

ϕ over DNW $=$ Nested Word language $\cap$ DNW

Build 2-*PDA* such that:

- executions $\subseteq$ Nested word language

proof idea ($\Leftarrow$)

ϕ over $\texttt{DNW}$ = Nested Word language $\cap$ DNW

Build 2-*PDA* such that:

- executions $\subseteq$ Nested word language
- accepting condition = restriction to DNW

Conclusion

We have

- given a logic for (realtime) Indexed Languages.

We would like to:

- Remove the realtime restrictions
- Extend to the upper levels of the higher order hierarchy

Thank you :-)