Interval Temporal Logics and Equivalence Relations

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HIGHLIGHTS 2014

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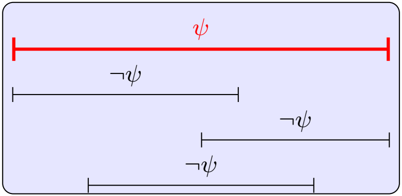
- The effects/benefits of the addition of one or more equivalence relations to a logic have been already studied in various settings, including (fragments of) first-order logic, linear temporal logic, metric temporal logic, and interval temporal logic.

- There exists a close relationship between interval temporal logics and fragments of first-order logic, that allows the transfer of results and logical tools (e.g., tableau systems) between them.

INTERVAL TEMPORAL LOGICS

- Interval temporal logics: an alternative approach to point-based temporal representation and reasoning.

Truth of formulas is defined over intervals rather than points.



- Halpern and Shoham's modal logic of intervals (*HS*)
  - *HS* features 12 modalilities, one for each possible ordering of a pair of intervals (the so-called Allen's relations);
  - decidability and expressiveness of *HS* fragments (restrictions to subsets of *HS* modalities) have been systematically studied in the last decade.
- Decidability and expressiveness depend on two crucial factors: the selected set of modalities and the class of linear orders on which they are interpreted.

IN THIS TALK

- We focus our attention on the satisfiability problem for some meaningful fragments of *HS* extended with one or more equivalence relations, interpreted over the class of finite linear orders: the interval logic of temporal neighborhood $A\bar{A}$ (aka PNL), its metric extension MPNL, and *AB*.

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- We focus our attention on the satisfiability problem for some meaningful fragments of *HS* extended with one or more equivalence relations, interpreted over the class of finite linear orders: the interval logic of temporal neighborhood $A\bar{A}$ (aka PNL), its metric extension MPNL, and *AB*.
- The original contributions can be summarized as follows :
  - decidability (*NEXPTIME*-completeness) of PNL$\sim$ (the extension of PNL with one equivalence relation);
  - decidability (*NEXPTIME*-completeness) of MPNL$\sim$ (the extension of MPNL with one equivalence relation);
  - undecidability of $AB\sim_1\sim_2$ (the extension of *AB* with two equivalence relations).

2. (Metric) PNL$\sim$

SYNTAX AND SEMANTICS OF PNL$\sim$

Formulas of PNL, built from Allen's relations *meets* and *met by*, are recursively defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle A\rangle\varphi \mid \langle \bar{A}\rangle\varphi$$

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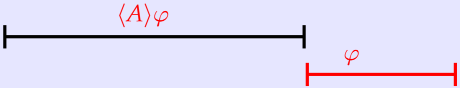

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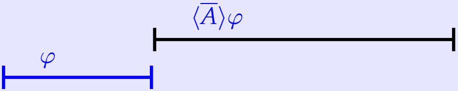

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- PNL$\sim$
  - We extend the language with a special propositional symbol $\sim$ interpreted as an equivalence relation over the points of the domain.
  - An interval $[x, y]$ satisfies $\sim$ if and only if x and y belong to the same equivalence class.

PREVIOUS RESULTS

- The satisfiability problem for PNL over finite linear orders is *NEXPTIME*-complete.
  - there is a polynomial reduction from the satisfiability problem for the two-variable fragment of first-order logic $FO^2[<]$ to the satisfiability problem for PNL, and viceversa;
  - $FO^2[<]$ is *NEXPTIME*-complete.

M. Otto. Two variable first-order logic over ordered domains. Journal of Symbolic Logic, 66(2), 2001.

D. Bresolin, V. Goranko, A. Montanari, and G. Sciavicco. Propositional interval neighborhood logics: Expressiveness, decidability, and undecidable extensions. Annals of Pure and Applied Logic, 161(3), 2009.

DECIDIBILITY OF PNL$\sim$

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The expressive completeness of PNL with respect to $FO^2[<]$ can be easily extended to PNL$\sim$ and $FO^2[<, \sim]$, and thus:

Corollary

$FO^2[<, \sim]$ *is decidable (NEXPTIME-complete) on the class of finite linear orders.*

PROOF STRUCTURE

The proof is a combination of 3 lemmas:

1. the first one provides an (exponential) upper bound to the cardinality of each equivalence class in a minimal model;
2. the second one provides a sufficient condition under which an equivalence class can be removed from the model;
3. the third one, making use of the second lemma, provides an (exponential) upper bound to the maximum number of equivalence classes in a minimal model.

The first and the third lemmas together provide an exponential upper bound to the size of a minimal model (small model theorem).

(METRIC) PNL$\sim$

Metric PNL (MPNL) is obtained from PNL by adding an infinite set of (pre-interpreted) proposition letters $len_1, \ldots, len_k, \ldots$ for length constraints, that allow one to constrain the length of the current interval to be equal to $1, 2, \ldots$

We prove the decidability of finite satisfiability problem for MPNL$\sim$ (or, equivalently, $FO^2[<, \sim, +1]$) by reducing it to the (decidable) 0-0 reachability problem for vector addition systems (VAS).

EXPSPACE-hardness immediately follows from the polynomial-time reduction from the emptiness problem for VAS to the finite satisfiability problem for $FO^2(\sim, <, +1)$ over data words (two binary relations, that is, the ordering relation $<$ and the equivalence relation $\sim$, and an arbitrary number of unary relations)

M. Bojańczyk, C. David, A. Muscholl, T. Schwentick, and L. Segoufin.
Two-variable logic on data words, ACM Transactions on Computational Logic, 12(4), 2011.

3. $AB \sim_1 \sim_2$

Syntax and Semantics of AB
Previous results
Undecidability of $AB \sim_1 \sim_2$

SYNTAX AND SEMANTICS OF *AB*

The formulas of the logic of Allen's relations *meets* and *begins*, denoted by *AB*, are recursively defined as follows:

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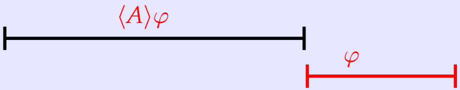

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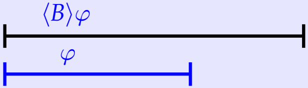

SYNTAX AND SEMANTICS OF AB

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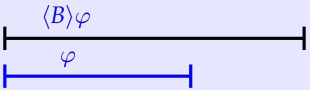

AB allows one to constrain the lenght of an interval to be equal to k ($k \in \mathbb{N}$) as well as to constrain an interval to contain exactly one point (endpoints excluded) labeled with a given proposition letter q. LTL modalities can be easily expressed in AB.

PREVIOUS RESULTS

The satisfiability problem for:

- *AB* is *EXPSPACE*-complete on the class of finite linear orders (and on $\mathbb{N}$);

A. Montanari, G, Puppis, P. Sala, and G. Sciavicco. Decidability of the Interval Temporal Logic $AB\bar{B}$ over the Natural Numbers. Proc. of the 27th STACS, 2010.

- $AB\sim$ is decidable (but non-primitive recursive hard) on the class of finite linear orders (and undecidable on $\mathbb{N}$).

A. Montanari, and P. Sala. Adding an Equivalence Relation to the Interval Logic $AB\bar{B}$: Complexity and Expressiveness. Proc. of the 28th LICS, 2013.

UNDECIDABILITY OF $AB\sim_1\sim_2$

We complete the picture by showing that the addition of two (or more) equivalence relations to *AB* makes the logic undecidable.

Theorem

The satisfiability problem for $AB\sim_1\sim_2$ on the class of finite linear orders is undecidable.

The proof relies on a reduction from the (undecidable) 0-0 reachability problem for counter machines (with two counters) to the satisfiability problem for $AB\sim_1\sim_2$ on finite linear orders.

RELATED WORK - 1

NEXPTIME-completeness of $FO^2[<]$.

M. Otto. Two variable first-order logic over ordered domains. Journal of Symbolic Logic, 66(2), 2001.

NEXPTIME-completeness of $FO^2[\sim]$.

E. Kieronski and M. Otto. Small substructures and decidability issues for first-order logic with two variables. Proc. of the 20th LICS, 2005.

2-NEXPTIME-completeness of $FO^2[\sim_1, \sim_2]$.

E. Kieronski, J. Michaliszyn, I. Pratt-Hartmann, and L. Tendera. Two-Variable First-Order Logic with Equivalence Closure. Proc. of the 27th LICS, 2012.

RELATED WORK - 2

Undecidability of $FO^2[\sim_1, \sim_2, \sim_3]$.

E. Kieronski and M. Otto. Small substructures and decidability issues for first-order logic with two variables. Proc. of the 20th LICS, 2005.

NEXPTIME-completeness of $FO^2(<, \sim)$ and decidability of $FO^2(<, \sim, +1)$ on data words (both results have been provided for both finite linear orders and $\mathbb{N}$).

M. Bojańczyk, C. David, A. Muscholl, T. Schwentick, and L. Segoufin. Two-variable logic on data words, ACM Transactions on Computational Logic, 12(4), 2011.

RESULTS AND OPEN PROBLEMS

Logic	*Complexity (on finite linear orders)*
PNL ($FO^2[<]$)	*NEXPTIME*-complete - APAL 2009
PNL$\sim$ ($FO^2[<,\sim]$)	*NEXPTIME*-complete - TIME 2014
PNL$\sim_1\sim_2$ ($FO^2[<,\sim_1,\sim_2]$)	?
MPNL$\sim$ ($FO^2[<,\sim,+1]$)	decidable (VASS-reachability) - TIME 2014
AB	*EXPSPACE*-complete - STACS 2010
$AB\sim$	non-primitive recursive hard - LICS 2013
$AB\sim_1\sim_2$	undecidable - ICTCS 2014

In addition, we would like to complete the picture for the case of $\mathbb{N}$ (we know that PNL is *NEXPTIME*-complete, *AB* is *EXPSPACE*-complete, and $AB\sim$ is undecidable over $\mathbb{N}$).