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USEFUL THEOREMS ON NORMAL SPACES

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We will need a few elementary facts about (perfect) normality in the course to the Nagata-Smirnov metrisation theorem. I will put them here for easy reference (as they are useful outside of this as well).

The first is a helpful characterisation of normality:

Theorem 1. *A space X is normal iff for all closed subset F and all open subsets O with $F \subset O$, there exist open subset W_n (n in $\mathbb{N}$) such that $F \subset \bigcup_n W_n$ and for all n:* $\mathrm{cl}(W_n) \subset O$.

Proof. The necessity is obvious: in a normal space we can even find one open set W with $F \subset W \subset \mathrm{cl}(W) \subset O$. (From considering F and $X \setminus O$, which are disjoint, so can be separated by open sets. The W around F is as required.)

For sufficiency: let A and B be two disjoint closed sets of X. Apply the condition to A and $X \setminus B$ to get W_i (i in $\mathbb{N}$) such that $A \subset \bigcup_i W_i$ and (for all i) $\mathrm{cl}(W_i) \cap B$ is empty. Do the same to B and $X \setminus A$ to get V_i (i in $\mathbb{N}$) such that $B \subset \bigcup_i W_i$ and (for all i) $\mathrm{cl}(V_i) \cap A$ is empty. Define $G_i := W_i \setminus (\bigcup_{j \leq i} \mathrm{cl}(V_j))$ and $H_i := V_i \setminus (\bigcup_{j \leq i} \mathrm{cl}(W_j))$. Then G_i and H_i are open subsets of X, for all i. Finally, put $U := \bigcup_i G_i$ and $V := \bigcup_i H_i$. Obviously, $A \subset U$ (A has misses all $\mathrm{cl}(V_j)$) and similarly $B \subset V$. And U and V are open (obvious) and disjoint: If we have G_i and H_j and $i \leq j$, then H_j misses V_i and hence G_i, so that $G_i \cap H_j = \emptyset$, and if $j \leq i$ then G_i misses W_j and hence H_j so that again $G_i \cap H_j = \emptyset$. So all G_i and H_j are mutually disjoint so that $U \cap V$ is empty as well. □

This characterisation has some nice uses.

Theorem 2. *A Lindelöf regular space X is normal.*

Proof. Let F be closed in X, let O be open, with $F \subset O$. For each x in X there exists, by regularity, an open neighbourhood U_x such that $x \in U_x \backslash \mathrm{cl}(U_x) \subset O$. Take a countable subcover of the $\{U_x\}$ (a closed subset of a Lindelöf space is Lindelöf) and this is the required countable family from Theorem 1. □

Corollary 1. *A second countable regular space is normal (as it is always Lindelöf). A countable regular space is normal (same reason, note that a countable regular space need* not *be second countable).*

Theorem 3. *An F-sigma subset A of a normal space X is normal.*

(F-sigma = a countable union of closed sets; most books only state this for closed subspaces).

Proof. Let $A = \bigcup_n F_n$. Let F be closed in A and O be open in A, with $F \subset O$. Let U be open in X with $U \cap A = O$ (by definition of subspace topology), and consider the $F \cap F_n$. Each $F \cap F_n$ is closed in F_n, so closed in X (closed in a closed subset is closed in the large set). So there are W_n, open in X, such that

$F \cap F_n \subset W_n \subset \mathrm{cl}(W_n) \subset U$. So letting $W'_n = W_n \cap A$, we have that $F \subset \bigcup_n W'_n$, and $\mathrm{cl}_A(W'_n) = \mathrm{cl}(W'_n) \cap A = \mathrm{cl}(W_n \cap A) \cap A \subset U \cap A = O$. So the W'_n are as required, and A is normal. $\square$

Now, recall that a space X is perfectly normal iff it is normal and every open set is an F-sigma. (So iff X is normal and every closed set is a G-delta; i.e., a countable intersection of open sets). As an aside, note that the above corollary shows that every open subset of a perfectly normal space is itself normal, and this implies that every subspace of X is then normal: let A be a subspace, F and G closed in A and disjoint. Find F' and G', closed in X, with $F' \cap A = F$ and $G' \cap A = G$. Let $O = X \setminus (F' \cap G')$, this is open, hence normal. $F' \cap O$ and $G' \cap O$ are closed there; separate them, and intersect the open sets so obtained by A. This separates F and G in A. So such spaces are hereditarily normal.

Maybe the single most important theorem on normal spaces is the Urysohn lemma: *if A and B are closed and disjoint in a normal space X, then there exists a continuous function $f\colon X \to [0,1]$ such that $f[A] = \{0\}$ and $f[B] = \{1\}$.* Note that this does *not* say that $A = f^{-1}[\{0\}]$, just $A \subset f^{-1}[\{0\}]$. A subset A such that there is a continuous function f from X to $[0,1]$ such that $A = f^{-1}[\{0\}]$, is called a zero-set.

Theorem 4. *In a normal space X, a set A is a zero-set iff A is a closed G-delta set.*

Proof. $\{0\}$ is a closed G-delta set in $[0,1]$: $\{0\} = \bigcap_n [0,1/n)$, where $[0,1/n)$ is open in $[0,1]$. So if $A = f^{-1}[\{0\}]$, with f continuous, A is closed (as $\{0\}$ is) and $A = \bigcap_n f^{-1}[[0,1/n)]$, so A is a G-delta. Now, let A be a closed G-delta in X. So $X \setminus A = \bigcup_n F_n$, where the F_n are closed in X (taking complements and applying de Morgan). Now, as A and F_n are closed and disjoint, find a Urysohn function f_n with $f_n[A] = \{0\}$ and $f_n[F_n] = \{1\}$. Define $f(x) := \sum_n 1/2^n f_n(x)$. Then f is defined by an absolutely convergent series of continuous functions, hence continuous. And $f_n(x) = 0$ for x in A, for all n, so that $f[A] = \{0\}$. And if x is *not* in A, then it is in some F_{n_0}, so that $f_{n_0}(x) = 1$ and $f(x) \geq 1/2^{n_0} \cdot f_{n_0}(x) = 1/2^{n_0} > 0$. So, $A = f^{-1}[\{0\}]$, and A is a zero-set. $\square$

This trick of using countably many functions and then summing them like this turns out to be a very useful technique in metrisation theorems. See other postings for this.