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Some Direct Estimates for Linear Combination of Linear Positive Convolution Operators

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Abstract: In this paper we have estimated some direct results for the even positive convolution integrals on $C_{2\pi}$, Banach space of 2π - periodic functions. Here, positive kernels are of finite oscillations of degree $2k$. Technique of linear combination is used for improving order of approximation. Property of Central factorial numbers, inverse formulas, mixed algebraic –trigonometric formula is used throughout the paper.

Keywords: Convolution Operator, Linear Combination, Positive Kernels

1. Introduction

Consider the singular positive convolution integral,

$$\left(E_{(n,\eta)}f\right)(x) = \left(f * g_{(n)}\right)(x)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(t)g_{(n)}(x-t)dt,\ n \in N \text{ and } x \in R \quad (1)$$

where, a kernel $\eta = \left(g_{(n)}\right)_{n=1}^{\infty}$, is a sequence of positive even normalized trigonometric polynomial [1].

For non-negative trigonometric polynomials $g_{(n)}(t)$ of degree atmost n and $E_{(n,\eta)}f = f * g_{(n)}$

Here, $f \in C_{2\pi}$, $C_{2\pi}$ being the Banach space of 2π – periodic functions f continuous on real axis R with usual sup norm,

$$\|f\|_c = sup\,\{\,|f(u)|: u \in R\,\}$$

Clearly, E is a bounded linear operator from $C_{2\pi}$ into itself, i.e.,

$E \in [C_{2\pi}]$, we use the notation,

$$\|E\|_{[c]} = sup\,\{\,\|Ef\|_c: \|f\|_c < 1\,\}$$

Philip C. Curtis Jr., [2] showed that,

$$\left\|E_{(n,\eta)}f - f\right\| = O(n^{-2})$$

which implies that f is identically constant provided,

$$\widehat{g_{(n)}}(0) - 1 = O(n^{-2})$$

Where, $\widehat{g_{(n)}}(k) = \frac{1}{2\pi}\int_0^{2\pi} g_{(n)}(t)e^{-kit}dt$

P. P Korovkin [3] states that there exists an arbitrary often differentiable function, $f \in C_{2\pi}$, such that,

$$\lim_{n \to \infty} sup\, n^2\left\|E_{(n,\eta)}(f,\cdot) - f(\cdot)\right\| > 0$$

Above result led to the fact that convolution integrals associated with these type of kernels has a better rate of convergence than $O(n^{-2})$.

Using extension of Korovkin Theorem [4], if we multiply our positive kernel η by a trigonometric polynomial, then approximation rate would be $O(n^{-2k-2})$, where, η is a kernel of finite oscillation of degree $2k, k \in N_0$. Here, $g_n(t)$ has $2k$ sign changes on $(0,2\pi)$ for each $n \in N$.

In this paper, we will consider linear combination of positive kernels thus of convolution integrals for improving rate of approximation. Earlier, several authors [5-9] has worked on the special cases. Here, we will introduce rather general method for obtaining better rate of approximation.

If $\eta = \left(g_{(n)}\right)_{n=1}^{\infty}$ is a positive kernel, we shall consider linear combinations, $\chi = \left\{\chi_{(n)}\right\}_{n \in N}$, given by,

$$\chi_{(n)}(x) = \sum_{v=1}^{\infty} \gamma_v g_{(n,a_v)}(x),\ x \in R \quad (2)$$

with coefficients γ_v, the a_v being certain given naturals.

Here, kernel $\eta = \{g_{(n)}\}_{n=1}^{\infty}$ be a sequence of even trigonometric polynomials of degree atmost $m(n) = O(n)$, which are normalized by,

$$\frac{1}{2\pi}\int_0^{2\pi} g_{(n)}(t)dt = 1 \quad (3)$$

i.e.,

$$g_{(n)}(x) = \sum_{k=-n}^{n} \rho_{(k,n)} e^{ikx} = 1 + 2\sum_{k=1}^{n} \rho_{(k,n)} \cos kx \quad (4)$$

Thus, $g_{(n)}(x) \geq 0$ and $g_{(n)}(x) \in \eta$, with $\rho_{(-k,n)} = \rho_{(k,n)}$ and $\rho_{(0,n)} = 1$.

Here, Fourier cosine coefficients $\rho_{(k,n)}$ are defined as usual by,

$$\rho_{(k,n)} = \begin{cases} (1/2\pi)\int_0^{2\pi} g_{(n)}(t)\cos kx\, dt, 0 \leq k \leq m(n) \\ 0, k > m(n) \end{cases} \quad (5)$$

Here, Fourier cosine coefficients are referred to as convergence factors.

The Lebesgue constants are given by,

$$L_{(n,\eta)} = \frac{1}{2\pi}\int_0^{2\pi} |g_{(n)}(t)|\, dt$$

In order (1.1) defines an approximation process on $C_{2\pi}$, i.e.,

$$\lim_{n \to \infty} \|E_{(n,\eta)}(f,\cdot) - f(\cdot)\| = 0, f \in C_{2\pi}$$

it is necessary and sufficient for the kernel η to satisfy,

$$L_{(n,\eta)} = O(1)$$

$$\lim_{n \to \infty} \rho_{(k,n)}(\eta) = 1 \quad (6)$$

This is due to the well-known theorem of Banach and Steinhaus.

In view of Bohman-korovkin theorem, for positive kernel, i.e., $g_{(n)}(x) \geq 0, n \in N, x \in R$, (1.6) reduces to,

$$\lim_{n \to \infty} \rho_{(1,n)}(\eta) = 1 \quad (7)$$

2. Some Definitions

Definition 2.1. [9] Let for $x \in R$,

$$x^{[n]} = \begin{cases} x\prod_{i=1}^{n-1}\left(\frac{2(x-1)+n}{2}\right), n \in N \\ 1, n = 0 \end{cases}$$

Here, $x^{[n]}$ denote the central factorial polynomial of degree n.

The central factorial numbers of first kind t_k^n is uniquely determined coefficients of the polynomials,

$$x^{[n]} = \sum_{k=0}^{n} t_k^n x^k$$

Similarly, central factorial numbers of second kind T_k^n is uniquely determined coefficients of the polynomials,

$$x^n = \sum_{k=0}^{n} T_k^n x^{[k]}$$

where $n \in N_0, x \in R$.

Some properties of these numbers are,

i) $t_0^n = T_0^n = \delta_{n,0}, n \in N_0$
ii) $t_k^n = T_k^n = 0, n < k$
iii) $t_{2k}^{2n+1} = t_{2k+1}^{2n} = T_{2k}^{2n+1} = T_{2k+1}^{2n} = 0, n \in N_0, k \in N_0$
iv) $\sum_{k=0}^{max\{n,m\}} t_k^n T_m^k = \sum_{k=0}^{max\{n,m\}} T_k^n t_m^k = \delta_{n,m}, n \in N_0, m \in N_0$
v) $T_k^n = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j \binom{k}{j}\left(\frac{k-2j}{2}\right)^n, 0 \leq k \leq n \in N_0$

Definition 2.2. [10] Let η be a kernel for, $\sigma \in N_0$,

$$T_{(n,\eta,2\sigma)} = \frac{1}{2\pi}\int_0^{2\pi}\left(2\sin\frac{t}{2}\right)^{2\sigma} g_{(n)}(t)\, dt$$

is called trigonometric moment of order 2σ.

We can also write,

$$T_{(n,\eta,2\sigma)} = \begin{cases} O(n^{-\tau\sigma}), 1 \leq \sigma \leq \mu \\ O\left(n^{-\tau\left(2\mu+1/2\right)}\right), \mu < \sigma \end{cases}$$

either for $\tau = 1$ or $\tau = 2$.

The algebraic moment of order $2\sigma, \sigma \in N_0$, is defined by,

$$M_{(n,\eta,2\sigma)} = \frac{1}{2\pi}\int_0^{2\pi} t^{2\sigma} g_{(n)}(t)\, dt \quad (8)$$

Here, trigonometric as well as algebraic moments of odd order vanish, since kernel is positive.

For, $(t/\pi) \leq \sin(t/2) \leq (t/2)$, $0 \leq t \leq \pi$,
one deduces for positive kernels immediately the estimate,

$$(2/\pi)^{2\sigma} M_{(n,\eta,2\sigma)} \leq T_{(n,\eta,2\sigma)} \leq M_{(n,\eta,2\sigma)}, \sigma \in N_0, \quad (9)$$

By the well-known inverse formulas,

$$\left(2\sin\frac{t}{2}\right)^{2\sigma} = \binom{2\sigma}{\sigma} + 2\sum_{k=1}^{\sigma}(-1)^k\binom{2\sigma}{\sigma-k}\cos kt, t \in R$$

$$\cos kt = 1 + \sum_{\sigma=1}^{k}(-1)^{\sigma}\left(2\sin\frac{t}{2}\right)^{2\sigma}\frac{1}{2\sigma!}\prod_{i=0}^{\sigma-1}(k^2 - i^2), t \in R$$

and the property (v) of the central factorial numbers, the trigonometric moments can be expressed in terms of the convergence factors and vice-versa.

In fact,

$$T_{(n,\eta,2\sigma)} = 2\sum_{k=1}^{\sigma}(-1)^{k+1}\binom{2\sigma}{\sigma-k}\left(1 - \rho_{(k,n)}(\eta)\right), \sigma \in N$$

$$\left(1 - \rho_{(k,n)}(\eta)\right) = \sum_{\sigma=1}^{k}(-1)^{\sigma+1}\frac{T_{(n,\eta,2\sigma)}}{(2\sigma)!}\sum_{i=1}^{\sigma} t_{2l}^{2\sigma} k^{2l}, 0 < k \leq m(n)$$

We can reduce our study of the asymptotic behaviour of the trigonometric moments to the asymptotic expansion of the difference $\left(1-\rho_{(k,n)}(\eta)\right)$ in the negative power of n.

In order to derive approximation theorems, we have to replace (6)(ii) by an asymptotic expansion of $\left(1-\rho_{(k,n)}(\eta)\right)$.

Definition 2.3. [11] A kernel η is said to have the expansion index $\mu \in N$ i.e, $\eta \in S^{(\tau,\mu)}$, if for all $k \in N$, there holds an expansion,

$$\left(1-\rho_{(k,n)}(\eta)\right)=\sum_{j=1}^{\mu}(-1)^{j+1} \quad {}_j(k)n^{-\tau j}+O\left(n^{-\tau\left(2\mu+1/2\right)}\right)$$

$$ {}_j(k)=\sum_{i=1}^{j} C_{ij}k^{2i} \tag{10}$$

for $C_{ij}\equiv C_{ij}(\eta)\in R$

Mostly known kernels belong to a class $S^{(\tau,\mu)}$.

3. Auxiliary Results

Lemma 3.1. [12] [13] Let $\tau=1$ or $\tau=2$ and $\mu\in N$. The following assertions are equivalent for a kernel:

i) $\eta \in S^{(\tau,\mu)}$,

ii) $T_{(n,\eta,2\sigma)}=\begin{cases}(2\sigma)!\sum_{j=\sigma}^{\mu}(-1)^{j+1}n^{-\tau j}\sum_{i=\sigma}^{j}C_{ij}T_{2\sigma}^{2i}+O\left(n^{-\tau\left(2\mu+1/2\right)}\right), 1\le\sigma\le k\\ O\left(n^{-\tau\left(2\mu+1/2\right)}\right), \mu<\sigma\end{cases}$

the C_{ij} being given as in definition 2.3.

Lemma 3.2. [14] [15] Let $s\in N$ and $a_1<a_2<\ldots\ldots\ldots<a_s$ be s different naturals. The unique solution of Vandermonde system of equations,

$\sum_{\nu=1}^{s}\gamma_\nu a_\nu^{-\tau j}=\delta_{j,0}$, where, $j=0,1,\ldots\ldots\ldots\ldots,s-1$

is given by,

$$\gamma_i=\frac{(-1)^{j+1}}{Q}\prod_{\substack{\nu=1\\ \nu\ne i}}^{s}a_\nu^{-\tau}\prod_{\substack{1\le j<\nu\le s\\ j,\nu\ne i}}\left(a_\nu^{-\tau}-a_j^{-\tau}\right)$$

where $i=1,2,\ldots\ldots\ldots,s$

Here, system-determinant Q is given by,

$$Q=\begin{vmatrix}1 & a_1^{-\tau} & \ldots\ldots\ldots & a_1^{-\tau(s+1)}\\ \vdots & \vdots & \ldots\ldots\ldots & \vdots\\ \vdots & \vdots & \ldots\ldots\ldots & \vdots\\ 1 & a_s^{-\tau} & \ldots\ldots\ldots & a_s^{-\tau(s+1)}\end{vmatrix}\ne 0$$

Also,

$$A_s=(-1)^{s+1}\sum_{\nu=1}^{s}\gamma_\nu a_\nu^{\tau s}=\prod_{\nu=1}^{s}a_\nu^{-\tau} \tag{11}$$

Let us suppose, $\eta\in S^{(\tau,\mu)}$, with $\tau=1$ or $\tau=2$, $\mu\in N$, to be a positive kernel, we set,

$$\alpha_{(n,s)}=\begin{cases}1/n^{(\tau s+\tau)}, 1\le s\le\mu\\ 1/n^{(\tau\mu+(\tau/2))}, s=\mu\end{cases} \tag{12}$$

We consider linear combination, $\chi=\{\chi_{(n)}\}_{n\in N}$ of even trigonometric polynomials of degree (na_ν), as,

$$\chi_n(x)=\sum_{\nu=1}^{s}\gamma_\nu g_{(na_\nu)}(x), x\in R \tag{13}$$

Lemma 3.3. For linear combination χ convergence factors associated with positive kernel η admits the expansion,

$$1-\rho_{(k,n)}(\chi)=\{\ {}_s(k)/n^{\tau s}\}+O\left(\alpha_{(n,s)}\right)$$

Proof. Using (13) and lemma 2.1, we have,

$$\rho_{(k,n)}(\chi)=\sum_{\nu=1}^{s}\gamma_\nu\rho_{(k,na_\nu)}(\eta)=\sum_{\nu=1}^{s}\gamma_\nu+\sum_{j=1}^{\mu}(-1)^j\ {}_j(k)n^{-\tau j}\sum_{\nu=1}^{s}\gamma_\nu a_\nu^{-\tau s}+O\left(\alpha_{(n,s)}\right)=P+Q+O\left(\alpha_{(n,s)}\right)\text{ (say)}$$

Here, $P=1$,

$Q=0$, for, $1\le j\le(s-1)$,

Collecting all but the first non-vanishing term $(j=s)$ into the O term, we have the lemma.

Lemma 3.4. The trigonometric moments for the, $\chi=\{\chi_{(n)}\}_{n\in N}$, admits the expansion,

$$T_{(n,\chi,2\sigma)} = \begin{cases} -n^{-\tau s}(-1)^{\sigma}(2\sigma)!\,A_s \sum_{i=\sigma}^{s} C_{is} T_{2\sigma}^{2i} + O\left(\alpha_{(n,s)}\right), 1 \le \sigma \le s \\ O\left(\alpha_{(n,s)}\right), s < \sigma \end{cases}$$

Proof. Using definition 2.2,

$$T_{(n,\chi,2\sigma)} = 2\sum_{k=1}^{\sigma}(-1)^{k+1}\binom{2\sigma}{\sigma-k}\left(1-\rho_{(k,n)}(\chi)\right)$$

Now, by lemma 3.3,

$$T_{(n,\chi,2\sigma)} = 2\sum_{k=1}^{\sigma}(-1)^{k+1}\binom{2\sigma}{\sigma-k}\{\ {}_s(k)/n^{\tau s}\} + O\left(\alpha_{(n,s)}\right)$$

Again using definition 2.3, we have,

$$T_{(n,\chi,2\sigma)} = 2n^{-\tau s}A_s\sum_{k=1}^{\sigma}(-1)^{k+1}\binom{2\sigma}{\sigma-k}\ {}_s(k) + O\left(\alpha_{(n,s)}\right) = 2n^{-\tau s}A_s\sum_{i=1}^{s}C_{is}\sum_{k=1}^{\sigma}(-1)^{k+1}\binom{2\sigma}{\sigma-k}k^{2i} + O\left(\alpha_{(n,s)}\right)$$

Using property (v) of central factorial numbers, we have the lemma.

4. Direct Results

Kernels defined by linear combination satisfy (6), so, the corresponding convolution integral defines an approximation process on $C_{2\pi}$.

Here, we will try to improve order for,

$$\lim_{n\to\infty}\left\|n^{\tau}\{E_{(n,\eta)}(f,\cdot)-f(\cdot)\}-kf^{(2)}(\cdot)\right\| = 0 \tag{14}$$

where $f \in C_{2\pi}^{(2)}$ with $k = k(\eta) \in R$
using linear combination $\chi_n(x) = \sum_{\nu=1}^{s}\gamma_\nu g_{(na_\nu)}(x), x \in R$.

Theorem 4.1. Let χ be a linear combination for the positive kernel $\eta \in S^{(\tau,\mu)}$ with $s \le \mu$. Then there holds for $f \in C_{2\pi}^{(2s)}$ the following expansion:

$$\lim_{n\to\infty}\left\|n^{\tau s}\{E_{(n,\chi)}(f,\cdot)-f(\cdot)\}+A_s\sum_{k=1}^{s}(-1)^k C_{ks} f^{(2k)}(\cdot)\right\| = 0 \tag{15}$$

Proof. A mixed algebraic-trigonometric Taylor's formula for $C_{2\pi}^{(2s)}$ is,

$$f(x+t)-f(x) = \sum_{k=0}^{s-1}\frac{f^{(2k+1)}(x)}{(2k+1)!}t^{(2k+1)} + \sum_{k=1}^{s} f^{(2k)}(x)\sum_{j=k}^{s}(-1)^k(-1)^j\frac{t_{2k}^{2j}}{(2j)!}\left(2\sin\frac{t}{2}\right)^{2j} + r_{(s)}(f,x,t) \tag{16}$$

where the remainder term is given by,

$$r_{(s)}(f,x,t) = \sum_{k=1}^{s}\frac{f^{(2k)}(x)}{(2k)!}\Theta_{(k,s)}(t)\left(\frac{1}{t^{-2s-2}}\right) + \frac{f^{(2s)}(\emptyset)}{(2s)!}t^{2s} - \frac{f^{(2s)}(x)}{(2s)!}t^{2s} \tag{17}$$

Here, $\Theta_{(k,s)}$ denotes a continuous function independent of f and $\emptyset$ lies between x and $(x+t)$.

$$\left(E_{(n,\chi)}f\right)(x) - f(x) = \sum_{k=1}^{s} f^{(2k)}(x)\sum_{j=k}^{s}(-1)^j(-1)^s\frac{t_{2k}^{2j}}{(2j)!}T_{(n,\chi,2j)} + \frac{1}{2\pi}\int_{-\infty}^{\infty}\chi_{(n)}(t)r_{(s)}(f,x,t) = H_1 + H_2 \tag{18}$$

Here,

$$H_1 = \frac{(-1)}{n^{\tau s}}A_s\sum_{k=1}^{s}f^{(2k)}(x)\sum_{j=k}^{s}(-1)^k t_{2k}^{2j}\sum_{i=j}^{s}C_{is}T_{2j}^{2i} + O\left(\alpha_{(n,s)}\right)\sum_{k=1}^{s}f^{(2k)}(x)$$

According to Landau,

$$\beta_{(n,s)} = \frac{\alpha_{(n,s)}}{n^{-\tau s}}$$

So,

$$H_1 = \frac{(-1)}{n^{\tau s}}\{A_s \sum_{k=1}^{s}(-1)^k C_{ks} f^{(2k)}(x) - O(\beta_{(n,s)}) \sum_{k=1}^{s} f^{(2k)}(x)\} \tag{19}$$

Now, with the help of (18) and (19), we can see,

$$\frac{1}{n^{\tau s}}\{(E_{(n,\chi)}f)(x) - f(x)\} + A_s \sum_{k=1}^{s}(-1)^k C_{ks} f^{(2k)}(x) = O(\beta_{(n,s)}) \sum_{k=1}^{s} f^{(2k)}(x) + \frac{H_2}{n^{-\tau s}} \tag{20}$$

Now, we will estimate H_2,

$$|H_2| \le \sum_{k=1}^{s} \frac{|f^{(2k)}(x)|}{(2k)!}\left(\frac{1}{2\pi}\right)\int_0^{2\pi} t^{(2s+2)}\left|\Theta_{(k,s)}(t)\chi_{(n)}(t)\right| dt + \frac{1}{(2s)!}\left(\frac{1}{2\pi}\right)\int_0^{2\pi}\left|f^{(2s)}(\emptyset) - f^{(2s)}(x)\right| \left|\chi_{(n)}(t)\right| t^{2s} dt = J + K \text{ (say)}$$

Using (8), (9) and (13), we get,

$$J \le M_{(n,\chi,2s+2)} \sum_{k=1}^{s} \frac{\|f^{(2k)}\|}{(2k)!}\|\Theta_{(k,s)}\| = O(1) T_{(n,\chi,2s+2)} \sum_{k=1}^{s}\|f^{(2k)}\|$$

So, using lemma 2.1, we see that,

$$J = \frac{O(\beta_{(n,s)})}{n^{\tau s}} \sum_{k=1}^{s}\|f^{(2k)}\| \tag{21}$$

Now,

$$K \le \frac{1}{(2s)!}\left(\frac{1}{2\pi}\right)\int_0^{2\pi} \omega(C_{2\pi}, f^{(2s)}, t) t^{2s}\left|\chi_{(n)}(t)\right| dt \le \frac{1}{(2s)!}\sum_{v=1}^{s}|\gamma_v|\left(\frac{1}{2\pi}\right)\int_0^{2\pi} \omega(C_{2\pi}, f^{(2s)}, t)\, t^{2s} g_{(na_v)}(t) dt$$

For inequality $\delta > 0$,

$$\omega(C_{2\pi}, f^{(2s)}, t) \le \left(1 + \frac{t}{\delta}\right) \omega(C_{2\pi}, f^{(2s)}, \delta) \le \left(1 + \frac{t^2}{\delta^2}\right) \omega(C_{2\pi}, f^{(2s)}, \delta)$$

Taking, $\delta = \sqrt{\beta_{(n,s)}}$, and using (9),

$$K = O(1)\left\{\omega\left(C_{2\pi}, f^{(2s)}, \sqrt{\beta_{(n,s)}}\right)\right\} \sum_{v=1}^{s} T_{(na_v,\eta,2s)} + \frac{T_{(na_v,\eta,2s)}}{\beta_{(n,s)}}$$

This implies,

$$K = O\left(\frac{1}{n^{\tau s}}\right) \omega(C_{2\pi}, f^{(2s)}, \sqrt{\beta_{(n,s)}}) \tag{22}$$

Now using (19), (21), (22), we have,

$$\left\|n^{\tau s}(E_{(n,\chi)}f)(\cdot) - f(\cdot) + A_s \sum_{k=1}^{s}(-1)^k C_{ks} f^{(2k)}(\cdot)\right\| = O(\beta_{(n,s)}) \sum_{k=1}^{s}\|f^{(2k)}\| + O(1)\omega\left(C_{2\pi}, f^{(2s)}, \sqrt{\beta_{(n,s)}}\right)$$

As, $n \to \infty, \beta_{(n,s)} \to 0$, we have the theorem.

Theorem 4.2. [16-18] Let χ be the linear combination of a positive kernel $\eta \in S^{(\tau,\mu)}$ with $S \le \mu$ as in, $\chi_n(x) = \sum_{v=1}^{s} \gamma_v g_{(na_v)}(x)$. Then there holds on estimate:

$$\|(E_{(n,\chi)}f)(\cdot) - f(\cdot)\| = O(1)\omega_{2s}(C_{2\pi}, f, n^{(-\tau/2)})$$

Proof. For $j, k \in N$ and $\in C_{2\pi}^{(k)}$, $1 \le j < k$, we have,

$$\int_{-\pi}^{\pi} f^{(j+1)}(u)\, du = f^{(j)}(\pi) - f^{(j)}(-\pi) = 0$$

There exists $\xi \in (-\pi, \pi)$ with $f^{(j+1)}(\xi) = 0$ and so,

$$\left|f^{(j)}(x)\right| = \left|\int_{\xi}^{x} f^{(j+1)}(u) du\right| \le |(x - \xi)| \|f^{(j+1)}(u)\| \le 2\pi \|f^{(j+1)}(u)\|$$

Iteratively, we get,

$$\left\|f^{(j)}(u)\right\| \leq \frac{(2\pi)^j}{(2\pi)^k}\left\|f^{(k)}(u)\right\| \quad (23)$$

for $j = k$, we can easily show (23),
Using (23) and (20), we can easily prove,

$$\left\|\left(E_{(n,\chi)}f\right)(\cdot) - f(\cdot)\right\| = O(1)n^{-\tau s}\left\|f^{(2s)}\right\|, \text{ where, } f \in C_{2\pi}^{(2s)} \quad (24)$$

Using, $L_{(n,\chi)} = O(1)$ and (24), we have the theorem.

5. Conclusion

By taking linear combination of suitable positive kernels,

$$\chi_n(x) = \sum_{\nu=1}^{s} \gamma_\nu g_{(na_\nu)}(x)\,, x \in R,$$

We have raised the approximation order of $\left(E_{(n,\chi)}f\right)$ on $C_{2\pi}$.
The trigonometric moments of η upto order 2μ grow in a linear manner, whereas, the moments of linear combination χ upto order $2s$ behave asymptotically all like $O(n^{-\tau s})$.

References

[1] R. K. S. Rathore, “On a sequence of linear trigonometric polynomial operators”, SIAM J. Math. Anal. 5(1974), 908-917.

[2] P. C. Curtis, “The degree of approximation by positive convolution operators”, Michigan Math. J., 12(1965), 155-160.

[3] P. P. Korovkin, “Linear Operator and Approximation Theory”, Hindustan Publishing Corp., Delhi, 1960

[4] E. L Stark, “An extension of a theorem of P. P. Korovkin to singular integrals with not necessarily positive kernels”, Indag. Math. 34(1972), 227-235.

[5] G. Bleimann and E. L. Stark, “Kernels of finite oscillations and convolution integrals”, Acta. Math. Acad. Sci. Hungar. 35(1980), 419-429

[6] P. L. Butzer, “Linear combination of Bernstein polynomials”, Canad. J. Math. 5(1953), 559-567.

[7] W. Engels, E. L. Stark and L. Vogt, “Optimal Kernels for a general Sampling Theorem”, J. Approx. Theory 50(1987), 69-83.

[8] O. Shisha and B. Mond, “The degree of approximation to periodic functions by linear positive operators”, J. Approx. Theory, 1(1968), 335-339.

[9] G. G. Lorentz, “Approximation of functions”, Holt, Rinehart and Winston, New York, 1968.

[10] A. Zygmund, “Trigonometric Series”, Vol. II, Cambridge university Press, Cambridge, 1959.

[11] J. Szabados, “On Convolution operators with kernels of finite oscillations”, Acta. Math. Acad. Sci. Hungar. 27(1976), 176-192.

[12] C. J. Hoff, “Approximations with Kernels of finite oscillations”, Part I, Convergence, J. Approx. Theory 3(1970), 213-218.

[13] C. J. Hoff, “Approximation with kernels of finite oscillations”, Part II, Degree of Approximation, J. Approx. Theory 12(1974), 127-145.

[14] R. Bojanic, “A note on the degree of approximation to continuous functions”, Enseign. Math., 15(1969), 43-51.

[15] R. A. DeVore, “The Approximation of Continuous Functions by Positive Linear Operators”, Springer-Verlag, New York, 1972.

[16] Gupta Vijay and Ravi P. Agarwal, “convergence estimate in approximation theory”, New York, Springer, 2014.

[17] P. Maheshwari, S. Garg, “Higher Order Iterates Of Szasz-Mirakyan-Baskakov Operators”, Stud. Univ. Babe_s-Bolyai Math. 59(1) (2014), 69-76.

[18] Tuncer Acar, Vijay Gupta, Ali Aral, Rate of convergence for generalized Szász operators, Bull. Math. Sci. (2011), 99–113.